Kamae, M. and Takahashi, S.
Osaka J. Math.
**33** (1996), 915–925

# A SHAPE THEOREM FOR THE SPREAD OF EPIDEMICS AND FOREST FIRES IN TWO-DIMENSIONAL EUCLIDEAN SPACE

MOTOKO KAMAE and SATOSHI TAKAHASHI

(Received October 12, 1995)

## 1. Introduction and statement of results

J.T. Cox and R. Durrett (1988, [2]) considered a model of epidemics and forest fires in $\boldsymbol{Z}^2$ with nearest neighbor interactions,and have shown the shape theorem for the spread of epidemics. Y. Zhang (1993, [6]) has dealt with a model which had not only nearest neighbor interactions but also interactions between further hosts, and proved the shape theorem for the model.

We consider a model of epidemics or forest fires in $\boldsymbol{R}^2$. Hosts are distributed according to a Poisson point process $X_\lambda=\{X_i \in \boldsymbol{R}^2\}_{i=1}^{\infty}$ of intensity $\lambda(>0)$ in $\boldsymbol{R}^2$. Each individual $X_i$, $i \in \boldsymbol{N}$ can be in one of three states 1,2 or 0. In the epidemic interpretation, 1 = susceptible, 2 = infected, and 0 = immune, while for a forest fire, 1 = alive, 2 = on fire, and 0 = burnt. The state of the process is represented by a function $\eta_t(X_i) \in \{0,1,2\}$, $i \in \boldsymbol{N}$, which is the state of $X_i$ at time $t$. An individual stays infected for the amount of time 1, then it recovers and becomes immune. Once an individual becomes immune, it will be never infected. An infected individual $X_i$ emits germs after a random time $T_i$ from its infection. Germs emitted from $X_i$ go to all individuals in $S_i=\{x \in \boldsymbol{R}^2 \mid |x-X_i| \leq 1\}$, the disk of radius 1 with its center at $X_i$. (If $T_i>1$, $X_i$ really does not emit germs.) Let $T_i$, $i \in \boldsymbol{N}$ be nonnegative, i.i.d. random variables with distribution function $F$. We have $F(0)<1$ and $F(1)>0$. If a germ goes to a susceptible individual, then the individual immediately becomes infected. If the germ goes to an infected or immune individual, the individual does not change its state at all. Initially all points of $X_\lambda$ in $S_0=\{x \in \boldsymbol{R}^2 \mid |x| \leq 1\}$, the disk of radius 1 with its center at 0, are infected and all other points of $X_\lambda$ are susceptible:

$$\eta_0(X_i)=\begin{cases}2, & \text{if } \ X_i \in S_0 \\ 1, & \text{otherwise.}\end{cases}$$

We construct the probability space representing our epidemic model (or our forest fire model). Let $\Omega_\lambda$ be the Poisson point process of intensity $\lambda$ in $\boldsymbol{R}^2$ which consists of countable points valued in $[0,\infty)$ independently according to the distribution function $F$. An element of $\Omega_\lambda$ is denoted by $\{(X_i,T_i)\in\boldsymbol{R}^2\times[0,\infty)\}_{i=1}^\infty$, and $\{X_i\}_{i=1}^\infty$ by $X_\lambda$. Let $A(V,t,k)$ be the event that exactly $k$ points of $X_\lambda$ in $V$ have values less than $t$:

$$A(V,t,k)=\{\{(X_i,T_i)\}_{i=1}^\infty \mid \#\{i\in\boldsymbol{N}\mid X_i\in V,\ T_i\le t\}=k\},$$

where $V\in\mathscr{B}(\boldsymbol{R}^2)$ is bounded, $t\ge 0$, and $k\in\boldsymbol{N}\cup\{0\}$. Let

$$P_\lambda(A(V,t,k))=\frac{(\lambda F(t)\,|\,V|)^k}{k!}e^{-\lambda F(t)|V|},$$

$$\mathscr{B}(\Omega_\lambda)=\sigma(\{A(V,t,k)\mid V\in\mathscr{B}(\boldsymbol{R}^2)\text{ is bounded, } t\ge 0,\ k\in\boldsymbol{N}\cup\{0\}\}).$$

Clearly $(\Omega_\lambda,\mathscr{B}(\Omega_\lambda),P_\lambda)$ $(\lambda>0)$ is a probability space. From now on we deal with this probability space which represents our model.

Before stating our result, we define some notions. For $\{(X_i,T_i)\}_{i=1}^\infty\equiv\omega\in\Omega_\lambda$, let

$$\tau_i=\begin{cases}T_i, & \text{if } T_i\le 1\\ \infty, & \text{if } T_i>1\end{cases}\qquad i\in\boldsymbol{N}.$$

$\tau_i$ is the time lag from the infection of $X_i$ until $X_i$ emit germs. We say that $X_i$ is open if $\tau_i<\infty$, and closed otherwise. When $X_i$ is open and infected, it emits germs and infects all points of $X_\lambda$ in the disk $S_i$ of radius 1 with its center at $X_i$. When $X_i$ is closed, it cannot infect any points of $X_\lambda$ in $S_i$. For $X_i,X_j\in X_\lambda$, we call $\{X_{i_1},\cdots,X_{i_K}\}$ a path from $X_i$ to $X_j$ if the following hold:

(i) $X_{i_1}=X_i,\quad X_{i_K}=X_j$

(ii) $|X_{i_K}-X_{i_{k+1}}|\le 1$ and $X_{i_K}\ne X_{i_{k+1}},\quad \forall k=1,\cdots,K-1$

In addition to the above, we say that $\{X_{i_1},\cdots,X_{i_K}\}$ is an open path from $X_i$ to $X_j$ if the following holds:

(iii) $X_{i_K}$ is open, $\forall k=1,\cdots,K$

Let

$$C_0(\omega)=\left\{X_i\ \middle|\ \begin{array}{l}\text{There is a path from a point of } X_\lambda \text{ in}\\ S_0 \text{ to } X_i \text{ denoted by } \{X_{i_1},\cdots,X_{i_K}\}, \text{ and}\\ X_{i_1},\cdots,X_{i_{K-1}} \text{ are open.}\end{array}\right\}\tag{1.1}$$

We call $C_0(\omega)$ the cluster contaning the origin 0. In the epidemics interpretation, $C_0(\omega)$ is the set of points of $X_\lambda$ that $X_i$ will ever becomes infected if initially all points of $X_\lambda$ in $S_0$ are infected and all other points of $X_\lambda$ susceptible. Let

$$\lambda_c=\inf\{\lambda\mid P_\lambda(|C_0|=\infty)>0\}.$$

$\lambda_c$ is the critical value whether the epidemic spreads infinitely or not. For $\omega \in \Omega_\lambda$, let

$$t(X_i,X_j)=\inf\{\sum_{l=1}^{L-1} \tau_{i_l} \mid \{X_{i_1},\cdots,X_{i_L}\} \text{ is a path from } X_i \text{ to } X_j.\}, \quad X_i,X_j \in X_\lambda.$$

$t(X_i,X_j)=\infty$ if there is no path from $X_i$ to $X_j$, and $t(X_i,X_j)=0$ if $X_i=X_j$. $t(X_i,X_j)$ is the minimum time of the infection of $X_j$ if only $X_i$ is infected initially. For $x,y \in \boldsymbol{R}^2$, let

$$t'(x,y)(\omega)=\inf_{\substack{S_x \ni X_i \\ S_y \ni X_j}} t(X_i,X_j),$$

where $S_x=\{x' \in \boldsymbol{R}^2 \mid |x-x'| \le 1\}$. If there is no point of $X_\lambda$ in $S_x$ nor $S_y$, we let $t'(x,y)(\omega)=\infty$. $t'(x,y)$ is the minimum time for a point of $X_\lambda$ in $S_y$ to be infected if the points of $X_\lambda$ in $S_x$ are infected and others susceptible initially. Let $e_1=(1,0)$, then $\liminf_{n\to\infty}\frac{t'(0,ne_1)}{n}$ is almost surely constant ([2]), and we denote it by $\gamma$. $\gamma$ is the average time for the epidemic to go the unit distance. Let

$$\zeta_t(\omega)=\{X_i \in X_\lambda \mid \eta_t(X_i)=0\},$$
$$\xi_t(\omega)=\{X_i \in X_\lambda \mid \eta_t(X_i)=2\}.$$

$\zeta_t$ is the set of points of $X_\lambda$ which are immune at time $t$, and $\xi_t$ is the set of points of $X_\lambda$ which are infected at time $t$.

Here we consider in what shape it spreads out.

**Theorem 1.** *Assume $\gamma>0$. Let $D$ be the disk of radius $\frac{1}{\gamma}$ with its center at the origin. If $\lambda>\lambda_c$, then for any $\varepsilon>0$, we have*

$$P_\lambda(C_0 \cap t(1-\varepsilon)D \subset \zeta_t \subset t(1+\varepsilon)D \quad \textit{for all sufficiently large } t)=1,$$
$$P_\lambda(\xi_t \subset t(1+\varepsilon)D \backslash t(1-\varepsilon)D \quad \textit{for all sufficiently large } t)=1.$$

## 2. Probability estimates of events in the model in $\boldsymbol{Z}_n^2$

We prove Theorem 1 by approximating $\boldsymbol{R}^2$ with the lattice $\boldsymbol{Z}_n^2$:

$$\boldsymbol{Z}_n^2=\{(\frac{x}{n},\frac{y}{n}) \mid (x,y) \in \boldsymbol{Z}^2\}, \quad n=2,3,\cdots$$

We construct a site percolation in $\boldsymbol{Z}_n^2$ which corresponds to that in $\boldsymbol{R}^2$. For $z=(z_1,z_2) \in \boldsymbol{Z}_n^2$, let

$$B_n(z)=[z_1-\frac{1}{2n}, z_1+\frac{1}{2n})\times[z_2-\frac{1}{2n}, z_2+\frac{1}{2n}).$$

For $z\in \boldsymbol{Z}_n^2$ and $\omega\in\Omega_\lambda$, we define $T_z$ by

$$T_z(\omega)=\begin{cases}\inf\limits_{X_i\in B_{n(z)}} T_i, & \text{if } B_n(z)\cap X_\lambda\neq\emptyset\\ \infty, & \text{if } B_n(z)\cap X_\lambda=\emptyset\end{cases}$$

$T_z$, $z\in \boldsymbol{Z}_n^2$ are i.i.d. random variables. We say that a site $z\in \boldsymbol{Z}_n^2$ is open if $T_z\leq 1$, and closed otherwise. Then for each site $z$, $z$ is open or closed independently, and by using the distribution function $F$ of $T_i$,

$$P_\lambda(z \text{ is open})=1-e^{-\lambda F(1)n^{-2}}\equiv p_n(\lambda),$$
$$P_\lambda(z \text{ is closed})=e^{-\lambda F(1)n^{-2}}.$$

For $z_1,z_2\in \boldsymbol{Z}_n^2$, we say that $z_1$ and $z_2$ are adjacent if $|z_1-z_2|\leq 1-\frac{\sqrt{2}}{n}$. If $|z_1-z_2|\leq 1-\frac{\sqrt{2}}{n}$, then when an open point of $X_\lambda$ in $B_n(z_1)$ is infected, all healthy points of $X_\lambda$ in $B_n(z_2)$ are infected. By this adjacency relation, we define $N_z$, the neighbor of $z\in \boldsymbol{Z}_n^2$, as

$$N_z=\{z'\in \boldsymbol{Z}_n^2 \mid |z-z'|\leq 1-\frac{\sqrt{2}}{n}\}.$$

To any $z'(\neq z)\in N_z$, we give a bond oriented from $z$ to $z'$, and denote it by $\langle z,z'\rangle_n$. We let $M_n\equiv 1-\frac{\sqrt{2}}{n}$, and denote by $\boldsymbol{Z}_n^2(M_n)$ all oriented bonds $\langle z_1,z_2\rangle_n$ with $|z_1-z_2|\leq M_n$. For any oriented bond $\langle z_1,z_2\rangle_n\in \boldsymbol{Z}_n^2(M_n)$, we say that $\langle z_1,z_2\rangle_n$ is open if $z_1$ and $z_2$ are open, and closed otherwise. For $z_1,z_2\in \boldsymbol{Z}_n^2$, we call $\{z_{i_1},\cdots,z_{i_K}\}$ a path of $\boldsymbol{Z}_n^2(M_n)$ from $z_1$ to $z_2$ if the follwing hold:

(i) $z_{i_1}=z_1,\quad z_{i_K}=z_2$

(ii) $|z_{i_k}-z_{i_{k+1}}|\leq M_n$ and $z_{i_k}\neq z_{i_{k+1}},\quad \forall k=1,\cdots,K-1$

In addition to the above, we say that $\{z_{i_1},\cdots,z_{i_K}\}$ is an open path of $\boldsymbol{Z}_n^2(M_n)$ from $z_1$ to $z_2$ if the following holds:

(iii) $\langle z_{i_k},z_{i_{k+1}}\rangle_n$ is open, $\quad\forall k=1,\cdots,K-1$

For two open paths of $\boldsymbol{Z}_n^2(M_n)$ denoted by $r_1$, $r_2$, we say that $r_1$ and $r_2$ are connected in $\boldsymbol{Z}_n^2(M_n)$ if there exist open sites $z_1\in r_1$, $z_2\in r_2$ such that $z_1=z_2$ or $|z_1-z_2|\leq M_n$. Let

$$D_z^{(n)} = \{z' \in \boldsymbol{Z}_n{}^2 \mid \text{There is an open path of } \boldsymbol{Z}_n{}^2(M_n) \text{ from } z \text{ to } z'.\}$$

be the open cluster containing $z \in \boldsymbol{Z}_n{}^2$. Let $p_c(\boldsymbol{Z}_n{}^2(M_n))$ be the critical probabilty of the site percolation with the adjacency relation as mentioned above, then from $0 < p_c(\boldsymbol{Z}_n{}^2(M_n)) < 1$ ([3]), there exists $\lambda_c{}^{(n)}$, where $0 < \lambda_c{}^{(n)} < \infty$, such that

$$p_c(\boldsymbol{Z}_n{}^2(M_n)) = 1 - e^{-\lambda_c{}^{(n)} F(1) n^{-2}}.$$

We call $\lambda_c{}^{(n)}$ the critical value in $\boldsymbol{Z}_n{}^2$, as it satisfies

$$\lambda_c{}^{(n)} = \inf\{\lambda \mid P_\lambda(|D_0^{(n)}| = \infty) > 0\}.$$

**Lemma 2.1** ([7], [8]). *Let*

$$\lambda_c = \inf\{\lambda \mid P_\lambda(|C_0| = \infty) > 0\},$$
$$\lambda_T = \inf\{\lambda \mid E_\lambda |C_0| = \infty\},$$

*where* $E_\lambda|C_0| = \int_{\Omega_\lambda} |C_0| dP_\lambda$, *then* $\lim_{n\to\infty} \lambda_c{}^{(n)} = \lambda_c = \lambda_T$ *and* $0 < \lambda_c < \infty$.

Later we will use Lemma 2.1 to prove Theorem 1 in Section 3.

For $\omega \in \Omega_\lambda$, let $m(z)$, $z \in \boldsymbol{Z}_n{}^2$ be the minimum $m \in \boldsymbol{N}$ $(m > 2n)$ satisfying the following:

(i) There is an open path of $\boldsymbol{Z}_n{}^2(M_n)$ from $z + [-\frac{m}{2n}, \frac{m}{2n}]^2$ to $\infty$ in $\boldsymbol{Z}_n{}^2 \backslash \{z + [-\frac{m}{2n}, \frac{m}{2n}]^2\}$.

(ii) There is an open circuit of $\boldsymbol{Z}_n{}^2(M_n)$ in the annulus $\{z + [-\frac{3m}{2n}, \frac{3m}{2n}]^2\} \backslash \{z + [-\frac{m}{2n}, \frac{m}{2n}]^2\}$.

**Corollary 2.2** ([6]). *If* $\lambda > \lambda_c{}^{(n)}$, *then there exist positive constants* $K(\lambda)$ *and* $\beta$ *such that for any* $z \in \boldsymbol{Z}_n{}^2$, *we have*

$$P_\lambda(m(z) > 2^l K(\lambda)) \leq \beta e^{-2^l \gamma}, \quad \gamma = \log 2, \quad \forall l = 0, 1, \cdots$$

*By Corollary* 2.2, *we have*

$$P_\lambda(m(z) < \infty, \quad \forall z \in \boldsymbol{Z}_n{}^2) = 1.$$

*In Section* 3 *we let* $\Omega'_\lambda = \{\omega \mid m(z) < \infty, \forall z \in \boldsymbol{Z}_n{}^2\}$ *and consider events in* $\Omega'_\lambda$.

## 3. Proof of Theorem 1

In this section we prove Theorem 1 by using the probability estimates in Section 2.

From now on we assume $\lambda > \lambda_c$. We approximate our system in $\boldsymbol{R}^2$ by a lattice system in $Z_n^2$. By Lemma 2.1 we have that $\lambda > \lambda_c^{(n)}$ for $\lambda > \lambda_c$ with large enough $n$. Hereafter we consider $Z_n^2$ for this $n$.

DEFINITION 3.1. For $\omega \in \Omega_\lambda$, $z \in Z_n^2$, let

$$\Delta_1(z) = z + [-\frac{m(z)-1}{2n}, \frac{m(z)-1}{2n})^2,$$

$$\Delta_2(z) = z + [-\frac{m(z)+1}{2n}, \frac{m(z)+1}{2n})^2,$$

$$\Delta_3(z) = z + [-\frac{3m(z)+1}{2n}, \frac{3m(z)+1}{2n})^2.$$

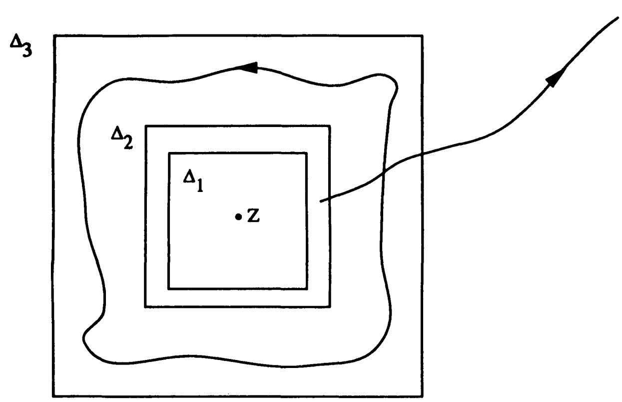


Fig. 1. Illustration of $\Delta_1$, $\Delta_2$ and $\Delta_3$.

By the definition of $m(z)$, $z \in Z_n^2$ in Section 2, we have the following:

(i) There is an open path from $\Delta_2(z)$ to $\infty$ in $R^2 \backslash \Delta_1(z)$.

(ii) There is an open circuit in the annulus $\Delta_3(z) \backslash \Delta_2(z)$.

DEFINITION 3.2. For $\omega \in \Omega_\lambda$, $z_1, z_2 \in Z_n^2$, by using $t(X_i, X_j)$ let

$$\tilde{t}(z_1, z_2)(\omega) = \inf_{\substack{\Delta_2(z_1) \ni X_i \\ \Delta_2(z_2) \ni X_j}} t(X_i, X_j).$$

$\tilde{t}(z_1, z_2)$ is the minimum time of the infection of a point of $X_\lambda$ in $\Delta_2(z_2)$ if the points of $X_\lambda$ in $\Delta_2(z_1)$ are infected and the other susceptible initially. We extend the domain of $\tilde{t}$ to all $R^2$ by letting $\tilde{t}(x,y) = \tilde{t}(\pi_n(x), \pi_n(y))$ for $x, y \in R^2$, where $\pi_n(x)$ is the element of $Z_n^2$ such that $x \in B_n(\pi_n(x))$. If $\{z_1, \cdots, z_k\}$ is a path of $Z_n^2(M_n)$

from $z_1=\pi_n(x)$ to $z_2=\pi_n(y)$, then there is an open path from $\Delta_2(\pi_n(x))$ to $\Delta_2(\pi_n(y))$ contained in $\bigcup_{i=1}^{k}\Delta_3(z_i)$, so $\hat{t}$ is finite in $\Omega'_\lambda$.

DEFINITION 3.3. For $\omega\in\Omega'_\lambda$, $z\in \boldsymbol{Z}_n{}^2$, $u(z)$ is defined as

$$u(z)=\frac{8(3m(z)+n+1)^2}{\pi n^2}.$$

By the following lemma we have that $u(z)$ is the upper bound of $t(X_i,X_j)$ for $X_i,X_j\in\Delta_3(z)\cap X_\lambda$ with $t(X_i,X_j)<\infty$.

**Lemma 3.4.** *For $\omega\in\Omega_\lambda$, $z\in \boldsymbol{Z}_n{}^2$, we have*

$$\sup_{\substack{\Delta_3(z)\cap X_\lambda\ni X_i,X_j\\ t(X_i,X_j)<\infty}} t(X_i,X_j)\le u(z)(\omega).$$

Proof. For $X_i,X_j\in\Delta_3(z)\cap X_\lambda$, assume that there is an open path from $X_i$ to $X_j$ in $\Delta_3(z)$. Now $t(X_i,X_j)<\infty$. Let $r=\{X_{i_1},\cdots,X_{i_K}\}$ be an open path from $X_i$ to $X_j$ in $\Delta_3(z)$ with the smallest $K$. Then $|X_{i_k}-X_{i_{k+1}}|\le 1$ for $k=1,\cdots,K-1$ and $|X_{i_k}-X_{i_{k'}}|>1$ for $k,k'=1,\cdots,K$ with $|k-k'|>1$. Hence we have

$$S'_{i_{2k+1}}\cap S'_{i_{2k'+1}}=\emptyset,\qquad \forall k,k'\ (k\neq k'),$$

where $S'_{i_{2k+1}}=\{x\in \boldsymbol{R}^2\mid |x-X_{i_{2k+1}}|\le\frac{1}{2}\}$. We can put at most $\frac{4(3m(z)+n+1)^2}{\pi n^2}$ disks of radii $\frac{1}{2}$ nonintersecting each other in $z+[-\frac{3m(z)+1}{2n}-\frac{1}{2},\frac{3m(z)+1}{2n}+\frac{1}{2}]^2$. From this we obtain

$$K\le\frac{8(3m(z)+n+1)^2}{\pi n^2}.$$

Therefore since $X_i$ stays infected for time interval 1,

$$t(X_i,X_j)\le\frac{8(3m(z)+n+1)^2}{\pi n^2}.$$

By the definitions and the lemma above, if $t(X_i,X_j)<\infty$, then

$$\hat{t}(X_i,X_j)\le t(X_i,X_j)\le \hat{t}(X_i,X_j)+u(\pi_n(X_i))+u(\pi_n(X_j)). \tag{3.1}$$

For $\omega\in\Omega'_\lambda$, $X_i\in X_\lambda$, let

$$t(0,X_i)=\inf_{S_0\ni X_j} t(X_j,X_i).$$

$t(0,X_i)$ is the minimum time of the infection of $X_i$ if the points of $X_\lambda$ in $S_0=\{x\in\boldsymbol{R}^2 \mid |x|\leq 1\}$ are infected and others susceptible initially. In the same way as $\hat{t}(X_i,X_j)$, if $t(0,X_i)<\infty$, then

$$\hat{t}(0,X_i)\leq t(0,X_i)\leq \hat{t}(0,X_i)+u(\pi_n(0))+u(\pi_n(X_i)). \tag{3.2}$$

For $t'(0,x)(\omega)=\inf_{\substack{S_0\ni X_i\\ S_x\ni X_j}} t(X_i,X_j)$, $x\in\boldsymbol{R}^2$, if $t'(0,x)<\infty$, then by (3.1)

$$\hat{t}(0,x)\leq t'(0,x)\leq \hat{t}(0,x)+u(\pi_n(0))+u(\pi_n(x)). \tag{3.3}$$

For $\omega\in\Omega_\lambda$, $z_1,z_2\in\boldsymbol{Z}_n{}^2$, let

$$\tilde{t}(z_1,z_2)(\omega)=\hat{t}(z_1,z_2)(\omega)+u(z_2)(\omega).$$

Then for $\omega\in\Omega_\lambda$, $z_1,z_2,z_3\in\boldsymbol{Z}_n{}^2$,

$$\tilde{t}(z_1,z_2)(\omega)\leq \tilde{t}(z_1,z_3)(\omega)+\tilde{t}(z_3,z_2)(\omega). \tag{3.4}$$

By Corollary 2.2, for $z\in\boldsymbol{Z}_n{}^2$, $k=1,2,\cdots$, we have

$$\begin{aligned} E_\lambda(u(z)^k)&=\sum_{l=0}^{\infty}\int_{2^lK(\lambda)<m(z)\leq 2^{l+1}K(\lambda)} u(z)^k dP_\lambda \\ &\leq \sum_{l=0}^{\infty}\{\frac{8(3\cdot 2^{l+1}K(\lambda)+n+1)^2}{\pi n^2}\}^k P_\lambda(m(z)>2^lK(\lambda)) \\ &\leq (\frac{8}{\pi n^2})^k\beta\sum_{l=0}^{\infty}(3\cdot 2^{l+1}K(\lambda)+n+1)^{2k}e^{-2^l\gamma}<\infty, \end{aligned} \tag{3.5}$$

and

$$E_\lambda(u(z)^k)=E_\lambda(u(0)^k) \tag{3.6}$$

**Lemma 3.5** ([2]). *If $\lambda>\lambda_c^{(n)}$, then there exists a constant $\mu(z)$ for $z\in\boldsymbol{Z}_n{}^2$, so that as $m\to\infty$ $(m\in N)$,*

$$\frac{\hat{t}(0,mz)}{m}\to\mu(z)\quad a.s..$$

By (3.3) and Lemma 3.5, we have that $\gamma=\liminf_{m\to\infty}\frac{t'(0,me_1)}{m}$ is almost surely constant.

We let $g(z)=E_\lambda(\tilde{t}(0,z))$ for $z\in\boldsymbol{Z}_n{}^2$ and extend the domain of $g$ to all of $\boldsymbol{R}^2$ by making it linear on triangles of the form $z,z+(\frac{1}{n},0)$, $z+(0,\frac{1}{n})$ and $z+(\frac{1}{n},\frac{1}{n})$, $z+(\frac{1}{n},0)$, and $z+(0,\frac{1}{n})$. Then we have the following lemma.

**Lemma 3.6** ([1]). *There is a function $\varphi$ on $\boldsymbol{R}^2$ such that $\dfrac{g(mx)}{m}$ converges uniformly to $\varphi$ on compact subsets on $\boldsymbol{R}^2$, and $\varphi(z)=\mu(z)$ for $z\in\boldsymbol{Z}_n{}^2$.*

DEFINITION 3.7. For $\omega\in\Omega_\lambda$, $z\in\boldsymbol{Z}_n{}^2$, let $C(z)=\{X_{i_1},\cdots,X_{i_L}\}$ $(X_{i_1}=X_{i_L})$ be an open circuit surrounding $z$ in $\Delta_3(z)$ with the smallest $L$ and let $\Delta(z)$ be the interior and the boundary of the polygon whose vertices are $X_{i_1},\cdots,X_{i_L}$ in $C(z)$, and whose sides $(X_{i_l},X_{i_{l+1}})$ for $l=1,\cdots,L-1$. We define $v(z)$ as

$$v(z)=\frac{|\Delta(z)|}{\pi(\frac{1}{2})^2}+|C(z)|+1,$$

where $|\Delta(z)|$ is the area of $\Delta(z)$, and $|C(z)|$ is the number of elements in $C(z)$.

If there is a path from $X_i$ to $X_j$ in $\Delta(z)$ for $X_i,X_j\in\Delta(z)\cap X_\lambda$, then $t(X_i,X_j)\le v(z)$. Proof of this fact is same as Lemma 3.4. Now to prove the following lemma, we use $c(z)$ instead of the min-circuit in [2], and use the fact above.

For $\omega\in\Omega_\lambda$, let $\hat{A}_t(\omega)=\{x\in\boldsymbol{R}^2\mid \hat{t}(0,x)\le t\}$. Let $D=\{x\in\boldsymbol{R}^2\mid \varphi(x)\le 1\}$. For any $x\in\boldsymbol{R}^2$, any $k\in\boldsymbol{Q}$ $(k=\frac{q}{p},\ p,q\in\boldsymbol{N})$, from the definition of $\varphi$, we obtain

$$\frac{\varphi(kx)}{k}=\lim_{m\to\infty}\frac{g(mp\cdot\frac{q}{p}x)}{mp\cdot\frac{q}{p}}=\varphi(x).$$

Hence for any $\alpha\in\boldsymbol{R}$,

$$\frac{\varphi(\alpha x)}{\alpha}=\varphi(x).$$

Since $\gamma=\liminf\limits_{m\to\infty}\dfrac{t'(0,me_1)}{m}=\varphi(e_1)$ from (3.3) and $\varphi(x)=\varphi(e_1)$ for any $x\in\boldsymbol{R}^2$ such that $|x|=1$, $\varphi(y)=|y|\gamma$ for any $y\in\boldsymbol{R}^2$. Therefore $D=\{x\in\boldsymbol{R}^2\mid |x|\le\frac{1}{\gamma}\}$.

**Lemma 3.8** ([1]). *Assume $\gamma>0$. If $\lambda>\lambda_c$, then for any $\varepsilon>0$, we have*

$$P_\lambda((1-\varepsilon)D\subset t^{-1}\hat{A}_t\subset(1+\varepsilon)D \textit{ for all sufficiently large } t)=1.$$

Proof of Theorem 1 ([2]). Taking $k=4$ in (3.5) and (3.6), we have $E_\lambda(u(z)^2)=E_\lambda(u(0)^2)<\infty$ for any $z\in\boldsymbol{Z}_n{}^2$, then for any $\varepsilon''>0$,

$$P_\lambda(u(z)>\varepsilon''|z|\ \text{i.o.})=0.$$

Hence for any $\varepsilon''>0$,

$$P_\lambda(\exists a>0\ \text{s.t.}\ u(z)\le\max\{a,\varepsilon''|z|\}\quad\text{for}\quad\forall z\in\boldsymbol{Z}_n{}^2)=1. \tag{3.7}$$

Taking any $\varepsilon>0$, by Lemma 3.8 we have that almost surely for all large $t$, if $(1-\varepsilon)tD\ni x$, then $\hat{t}(0,x)(\omega)\leq t$. Here let $t'=(1-\frac{\varepsilon}{2})t$. We know that if $(1-\varepsilon)t'D\ni x$, then $\hat{t}(0,x)(\omega)\leq(1-\frac{\varepsilon}{2})t$. Since $(1-\varepsilon)(1-\frac{\varepsilon}{2})=1-(\frac{3}{2}\varepsilon-\frac{\varepsilon^2}{2})$, with $\varepsilon'=\frac{3}{2}\varepsilon-\frac{\varepsilon^2}{2}(>0)$, we have that if $(1-\varepsilon')tD\ni x$, then $\hat{t}(0,x)(\omega)\leq(1-\frac{\varepsilon}{2})t$. If $X_i\in(1-\varepsilon')tD\cap C_0$, then (3.2) leads to

$$\begin{aligned} t(0,X_i)+1 &\leq \hat{t}(0,X_i)+u(0)(\omega)+u(\pi_n(X_i))(\omega)+1 \\ &\leq (1-\frac{\varepsilon}{2})t+u(0)(\omega)+u(\pi_n(X_i))(\omega)+1. \end{aligned}$$

With $d=\frac{2}{\gamma}$ (the diameter of $D$), $|X_i|\leq(1-\varepsilon')td$. By (3.7), we obtain that

$$\begin{gathered} u(0)(\omega)\leq a, \\ u(\pi_n(X_i))(\omega)\leq\max\{a,\varepsilon''\,|\,\pi_n(X_i)|\}, \end{gathered}$$

and from the definition of $\pi_n$,

$$|\pi_n(X_i)|\leq|X_i|+\frac{1}{\sqrt{2}}.$$

Hence with taking $t$ large enough, if necessary, we have

$$u(0)(\omega)+u(\pi_n(X_i))(\omega)+1\leq 3\varepsilon''(1-\varepsilon')td.$$

Taking $\varepsilon''=\dfrac{\varepsilon}{9d}$,

$$u(0)(\omega)+u(\pi_n(X_i))(\omega)+1\leq\frac{\varepsilon}{3}(1-\varepsilon')t.$$

Therefore we get

$$\begin{aligned} t(0,X_i)+1 &\leq (1-\frac{\varepsilon}{2})t+\frac{\varepsilon}{3}(1-\varepsilon')t \\ &=(1-\frac{\varepsilon}{2}+\frac{\varepsilon}{3}-\frac{\varepsilon}{3}\varepsilon')t \\ &\leq(1-\frac{\varepsilon}{6})t. \end{aligned}$$

Because $t(0,X_i)$ is the time of the infection of $X_i$, and 1 is the time lag from the infection of $X_i$ until $X_i$ is immune, we have $X_i \in \zeta_t(\omega)$. Hence for $\forall \varepsilon > 0$,

$$P_\lambda((1-\varepsilon')tD \cap C_0 \subset \zeta_t \quad \text{for all sufficiently large } t) = 1.$$

On the other hand, if $X_i \in \xi_t(\omega)$ or $X_i \in \zeta_t(\omega)$, then $t(0,X_i) \leq t$ and so $\bar{t}(0,X_i) \leq t$. Therefore for $\forall \varepsilon > 0$, Lemma 3.8 leads to $X_i \in (1+\varepsilon)tD$. Hence

$$P_\lambda(\xi_t \subset (1+\varepsilon)tD \quad \text{for all sufficiently large } t) = 1,$$

$$P_\lambda(\zeta_t \subset (1+\varepsilon)tD \quad \text{for all sufficiently large } t) = 1.$$

We have completed the proof of Theorem 1.

---

**References**


[1] J. T. Cox and R. Durrett: *Some limit theorems for percolation processes with necessary and sufficient conditions*, Ann. Probab. **9** (1981), 593–603.

[2] J. T. Cox and R. Durrett: *Limit theorems for the spread of epidemics and forest fires*, Stochastic Process Appl. **30** (1988), 171–191.

[3] G. Grimmett: Percolation, Springer-Verlag, New York, 1988.

[4] H. Kesten: Percolation Theory for Mathematicians, Birkhäuser, Boston, 1982.

[5] J. F. C. Kingman: *Subadditive ergodic theory*, Ann. Probab. **6** (1973), 833–909.

[6] Y. Zhang: *A shape theorem for epidemics and forest fires with finite range interactions*, Ann. Probab. **21** (1993), 1755–1781.

[7] S. A. Zuev and A. F. Sidorenko: *Continuous models of percolation theory I*, Theoretical and Mathematical Physics **62** (1985), 76–86.

[8] S. A. Zuev and A. F. Sidorenko: *Continuous models of percolation II*, Theoretical and Mathematical Physics **62** (1985), 253–262.



Motoko Kamae
Department of Mathematics
Osaka University
1–1 Machikaneyamachou Toyonaka,
Osaka 560, Japan
Present Address: IBM Japan, Ltd.

Satoshi Takahashi
Department of Mathematics
Osaka University
1–16 Machikaneyamachou Toyonaka,
Osaka 560, Japan