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KONTSEVICH INVARIANT FOR LINKS IN A DONUT AND LINKS OF SATELLITE FORM

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1. Introduction

In [9] Vassiliev, by investigating the complement of the discriminant of the space of knots, introduced the notion of finite type invariants (or Vassiliev invariants) of knots. Later, Birman and Lin [2] showed that the Vassiliev invariants can be characterized by the nilpotency with respect to the crossing changes. It turned out that the coefficients of many polynomial invariants, including quantum invariants, are of finite type. Later, Kontsevich [5], [1] modified the construction [4] of invariants for braids, which uses monodromy representations of formal connections on the configuration space, to get an invariant of knots. This invariant is called the universal Vassiliev-Kontsevich (VK) invariant, since one can get all the finite type invariants from it.

In this paper, we construct an isotopy invariant $\hat{Z}_f^{donut}$: {framed oriented links in a donut} $\to \mathscr{A}(F)$. For an arbitrary surface F, we denote by $\mathscr{A}(F)$ the space of chord diagrams on F in the sense of N.E. Reshetikhin [8]; $\mathscr{A}(F)$ is a $\boldsymbol{C}$-vector space formally generated by all the homotopy types of Feynman diagrams drawn on F with the relation called the four-term relation. Such an invariant is constructed independently in [3].

By using $\hat{Z}_f^{donut}$, we shall give a formula for satellite links of framed oriented links in $\boldsymbol{R}^3$. Let L_1 be a framed link in $\boldsymbol{R}^3$, K its component, $\hat{Z}_f(L_1)$ the VK invariant of L_1, and $\hat{Z}_f((L_1,K))$ the VK invariant of L_1 with specified circle corresponding to K. Let L_2 be a framed link in a donut, $(L_1,K)\diamond L_2$ the satellite obtained by substituting the donut containing L_2 into a tubular neighborhood of the component K of the framed link L_1 in $\boldsymbol{R}^3$ by a standard way. Let $\hat{Z}_f((L_1,K))\diamond\hat{Z}_f^{onut}(L_2)$ be the satellite obtained by modifying the distinguished component of $\hat{Z}_f((L_1,K))$ by $\hat{Z}_f^{donut}(L_2)$, which is introduced later. Then we have the following formula.

$$\hat{Z}_f((L_1,K)\diamond L_2)=\hat{Z}_f((L_1,K))\diamond\hat{Z}_f^{donut}(L_2).$$

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1) The author passesd away on August 14, 1995.

For the detail of the theory of the VK invariant for (unframed) knots, see [5] and [1]. For the VK invariant of framed oriented links, see [6].

1. Kontsevich invariant of a link in a donut

Let F be an arbitrary oriented 2-mainfold.

DEFINITION 1.1. A mapping μ from {oriented links in $F \times \boldsymbol{R}^3$} to an additive group is called a *Vassiliev invariant* of order n if it satisfies the following two conditions.

1. μ is an isotopy invariant.
2. For an arbitrary oriented link L in $F \times \boldsymbol{R}^3$ with arbitrary specified $n+1$ crossings $x_1, x_2, \cdots, x_{n+1}$ of the link projection to F, we have

$$\sum_{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1} = \pm 1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n+1} \mu(L_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n+1}}) = 0,$$

where $L_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n+1}}$'s are the same with L outside the neighborhoods of $x_1, x_2, \cdots, x_{n+1}$, and are obtained by crossing changes at x_i so that the sign of the crossing at x_i is equal to ε_i.

Moreover, an invariant μ is called of *finite type* if it is of order n for some n.

We construct the universal invariant of the isotopy type of a framed oriented link in a donut.

DEFINITION 1.2. Let D be a one-dimensional finite CW-complex such that

$$D = \overbrace{S^1 \cup \cdots \cup S^1}^{n} \cup \overbrace{[0,1] \cup \cdots \cup [0,1]}^{m},$$

where $D \backslash (\overbrace{S^1 \cup \cdots \cup S^1}^{n}) = \overbrace{(0,1) \cup \cdots \cup (0,1)}^{m}$. Let φ be the homotopy class of a map from D ro F. Then the pair (D, φ) is called *a chord diagram of degree m* on F. The set of chord diagrams of degree m on F is denoted by $\mathscr{C}_m(F)$. The image of each S^1 in F is denoted by solid lines and the image of each $[0,1]$ is denoted by dashed lines.

EXAMPLE 1.3. Let F be an annulus. Two chord diagrams on F in Figure 1 are both of degree 1 but are different.

DEFINITION 1.4. Let $\mathscr{A}(F)$ denote the $\boldsymbol{C}$-vector space of chord diagrams on F subjected to the four-term relation in [1].

In the rest of this paper, F denotes an annulus.

Proposition 1.5. *There is an invariant $\hat{Z}_f^{donut}$ of framed oriented links in a donut which takes values in $\mathcal{A}(F)$ and satisfies the following.*

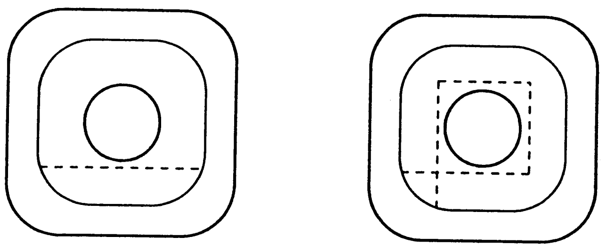

Figure 1. Examples of a chord diagram on an annulus

1. *It can be expressed as a finite sum in each degree as:*

$$\hat{Z}_f^{donut}(L)=\sum_{k=0}^{\infty}\sum_{d\in\mathscr{C}_k(F)} c_d d. \quad (c_d\in C)$$

2. *Let $L_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n}$ be as in the last section, and let $\hat{Z}_f^{donut}(L)^{(k)}$ be the degree k part of $\hat{Z}_f^{donut}(L)$. Then, for $k<n$, we have*

$$\sum_{\varepsilon_i=\pm 1}\varepsilon_1\varepsilon_2\cdots\varepsilon_n\hat{Z}_f^{donut}(L_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n})^{(k)}=0.$$

In other words, $\hat{Z}_f^{donut}(L)^{(k)}$ is a Vassiliev invariant of order d for $k\leq d$.

Proof. For an oriented framed link L in the donut, we express L as the one obtained from an (n,n)-tangle T in $D^2\times[0,1]$ such that the n endpoints lie in $D^2\times\{1\}$ to the lower side $D^2\times\{0\}$ by the identity map from D^2 to D^2. We can give a universal invariant $\hat{Z}_f(T)$ of T by [6]. We connect upper endpoints and lower endpoints of $\hat{Z}_f(T)$, which is easily seen to be independent of presentation of L as a closing of T. So we put

$$\hat{Z}_f^{donut}(L)=\text{closure of } \hat{Z}_f(T).$$

Then it can be seen that this is an invariant of a framed link in the donut with values in $\mathcal{A}(F)$. This satisfies the above properties (1) and (2). □

2. Parallel and satellite operations

Next, we consider the value of the universal invariant $\hat{Z}_f^{donut}$ for links which are obtained by parallelizing a component of some oriented link in the donut.

DEFINITION 2.1. Let L be an oriented framed link in a donut and K be a component of L. Then we put

$\Delta_K(L)$ = link obtained by duplication (taking parallel) of the component K of L.

Here, L is given as a link diagram on the annulus, its framing is given by the blackboard framing of the diagram, and we draw the parallel string of K on the annulus.

DEFINITION 2.2. For a chord diagram d with a distinguished component s, let $\Delta_s(d)$ be the chord diagram by duplicating s by the rule below. The above rule

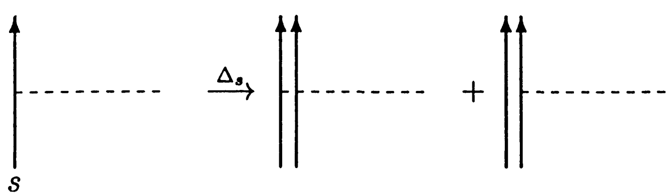

Figure 2. Parallel of a chord diagram

is applied at every endpoint of chords on s and so, if s contains k endpoints, then $\Delta_s(d)$ is a sum of 2^k chord diagrams.

Proposition 2.3. *For an oriented framed link L with a distinguished component K in a donut,*

$$\hat{Z}_f(\Delta_K L) = \Delta_{s(K)}(\hat{Z}_f(L)),$$

where $s(K)$ is the component of $\hat{Z}_f(\Delta_K L)$ corresponding to K.

Proof of this proposition is similar to the proof for any (r,r)-tangle in [7].

We introduce here some notation. Let F be an annulus. For a framed oriented link L_1 in $\boldsymbol{R}^3$ given by a diagram on $\boldsymbol{R}^2$, its comonent K and a framed oriented link L_2 in the donut given by a diagram on the annulus, we denote by $(L_1,K)\diamond L_2$ the satellite link obtained by putting the annullus containing the diagram of L_2 in K of L_1. In the following, we define a satellite operation for the chord diagram and show that these satellite operations are compatible with respect to $\hat{Z}_f$. For a framed oriented link L with a distinguished component K in $\boldsymbol{R}^3$, let $\hat{Z}_f(L,K)$ be the value of $\hat{Z}_f(L)$ in $\mathscr{A}(\boldsymbol{R}^2)$ with the distinguished component corresponding to K. For any chord diagram in $\hat{Z}_f^{donut}(L_2)\in\mathscr{A}(F)$, every chord is contained in a horizontal plane with respect to the height map $D^2\times S^1\to S^1$. On the horizontal plane D_0^2 corresponding to the origin of S^1, no minimal or maximal point is present, every string crosses D_0^2 transversally and the intersection points of strings are distinct. Let D_0^2 be the horizontal plane corresponding to the origin of S^1. Then we may assume that D_0^2 does not contain any minimal nor maximal point and D_0^2 intersects transversally with the strings (solid lines) of c. Cutting c by D_0^2, we get a chord diagram c' of (r,r)-tangle in $D^2 \times I$. For a chord diagram $d \in \mathscr{A}(F)$ with a distinguished component s, we mark a point p on s. We apply Δ_s^{r-1} and the orientation changes so that the r strings coming from s by $(\Delta_s)^{r-1}$ have the compatible orientation with the above (r,r)-tangle corresponding to c'. We cut $(\Delta_s)^{r-1}(d)$ at the points corresponding to p and insert c'. Then we get a new chord diagram. This one is written as $c \diamond d$. According to the above construction and the four-term relation, we have the following.

Proposition 2.4. *The above $c \diamond d$ does not depend on the place of p on s and the choice of c'.*

Our construction yields the following.

Theorem 2.5. *For an oriented framed link L_1 with a distinguished component K in $\boldsymbol{R}^3$ or in the donut, we have*

$$\hat{Z}_f((L_1,K)\diamond L_2)=\hat{Z}_f((L_1,K))\diamond \hat{Z}_f(L_2).$$

Here we extend the definition of the operator $\diamond$ linearly.

This theorem is proved by an argument similar to the one used in the proof of Theorem 2.3.1 in [6].

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