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HYPOELLIPTICITY OF SECOND ORDER OPERATORS IN R^2 OF THE FORM fX^2+Y+g

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1. Introduction

Let Ω be an open set in $\boldsymbol{R}^2$ and L be a second order partial differential operator defined in Ω of the form

$$L=f(x,y)\left(a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}\right)^2+c(x,y)\frac{\partial}{\partial x}+d(x,y)\frac{\partial}{\partial y}+g(x,y). \tag{1.1}$$

In this paper we give necessary and sufficient conditions for hypoellipticity of L under the following assumptions:

(H.1) f, a, b, c, d and g are real valued analytic functions defined in Ω;

(H.2) the operators $a\partial/\partial x+b\partial/\partial y$ and $c\partial/\partial x+d\partial/\partial y$ are independent in Ω, that is, $ad-bc\neq 0$ in Ω.

We recall that L is said to be hypoelliptic in Ω if for any open subset ω of Ω and any $u\in D'(\omega)$, $Lu\in C^\infty(\omega)$ implies $u\in C^\infty(\omega)$.

Set

$$X=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y},\quad Y=c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}. \tag{1.2}$$

Then we have the following theorem.

Theorem. *Suppose that* (H.1) *and* (H.2) *hold. Then, L is hypoelliptic in Ω if and only if*

(A) $Xf(x,y)=0$ *for any* $(x,y)\in\Omega$ *such that* $f(x,y)=0$,

(B) *f does not vanish identically on any integral curve of Y,*

(C) *f does not change sign from plus to minus along any integral curve of Y,*

where we consider Y as a vector field in Ω.

The necessity of (A) and (B) follows from Theorem II.I (iii) and Theorem II.I

(ii) of [14] respectively, and the necessity of (C) follows from Theorem 1.2 of [2]. For details see §2.

It is already proved in [11], [5], [9] and [4] that (A), (B) and (C) are sufficient for hypoellipticity of L in Ω if one of the following four conditions is satisfied:

(i) $f(x,y)\geqq 0$ in Ω or $f(x,y)\leqq 0$ in Ω (cf. [11]);

(ii) $X=a(x,y)\partial/\partial x$, $Y=\partial/\partial y$ and $f(x,y)=y\alpha(x,y)$ in Ω, where $\alpha(x,y)$ is a real valued analytic function defined in Ω and $\alpha(x,y)\geqq 0$ there (cf. [5]);

(iii) $f(x,y)=\phi(x,y)^p h(x,y)$ in Ω with some real valued functions $\phi,h\in C^\infty(\Omega)$ and an integer $p\geqq 3$, where $h(x,y)\geqq 0$ in Ω and $\phi^{-1}(0)$ is a finite union of C^1-curves (cf. [9]);

(iv) $Yf(x,y)>0$ for any $(x,y)\in\Omega$ such that $f(x,y)=0$ (cf. [4]).

With regard to the condition (iv) see also [3].

Our proof of the sufficiency of (A), (B) and (C) will be given in §3 and §4 by considering the above cases (i), (ii) and (iii). In case 3 of §3 we shall adopt the reasoning of [9] with several auxiliary lemmas, and among them the results of Lemma 3.2 relating to (iii) will play an essential role.

2. Necessity of (A), (B) and (C)

In this section we shall prove that (A), (B) and (C) hold if L is hypoelliptic in Ω. We write

$$
\begin{aligned}
(2.1)\qquad L&=f(a^2\partial^2/\partial x^2+2ab\partial^2/\partial x\partial y+b^2\partial^2/\partial y^2)+(faa_x+fba_y+c)\partial/\partial x\\
&\quad+(fab_x+fbb_y+d)\partial/\partial y+g\\
&=\partial/\partial x(fa^2\partial/\partial x+fab\partial/\partial y)+\partial/\partial y(fab\partial/\partial x+fb^2\partial/\partial y)\\
&\quad+[c-\{(fa)_x+(fb)_y\}a]\partial/\partial x+[d-\{(fa)_x+(fb)_y\}b]\partial/\partial y+g.
\end{aligned}
$$

By the assumption (H.2), $|a|+|b|\neq 0$ and $|c|+|d|\neq 0$ in Ω. Hence

$$
\begin{aligned}
&|fa^2|+|2fab|+|fb^2|+|faa_x+fba_y+c|+|fab_x+fbb_y+d|\\
&=|f|(|a|+|b|)^2+|f(aa_x+ba_y)+c|+|f(ab_x+bb_y)+d|\neq 0 \text{ in } \Omega.
\end{aligned}
$$

This shows that the N. T. D condition of [14] is fulfilled and we can apply Theorem II.I of [14] to the operator L.

Proof of (A). (A) follows immediately from (2.1) and Theorem II.I (iii) of [14].

Proof of (B). The proof is by contradiction. Suppose that there exists an integral curve Γ of Y where f vanishes identically. Let p be a point on Γ. Then we have

$$(2.2)\qquad Y^n f(p)=0 \text{ for all non-negative integers } n.$$

Now we set $Q_0 = [c - \{(fa)_x + (fb)_y\}a]\partial/\partial x + [d - \{(fa)_x + (fb)_y\}b]\partial/\partial y$, $Q_1 = fa^2\partial/\partial x + fab\partial/\partial y$ and $Q_2 = fab\partial/\partial x + fb^2\partial/\partial y$. Since $f=0$ on Γ, it follows from (A) and (1.2) that $(fa)_x+(fb)_y = Xf + fa_x + fb_y = 0$ on Γ. Therefore $Y = Q_0$ on Γ and we have by (2.2)

(2.3) $Q_0^n f(p) = 0$ for all non-negative integers n.

On the other hand, according to Theorem II.I(ii) of [14], the hypoellipticity of L implies that

(2.4) rank Lie $[Q_0, Q_1, Q_2](p) = 2$,

where Lie $[Q_0, Q_1, Q_2]$ is the Lie algebra generated by Q_0, Q_1 and Q_2.

REMARK. Theorem II.I(ii) of [14] states that rank Lie $[Q, Q_1, Q_2](p) = 2$ with $Q = (faa_x + fba_y + c)\partial/\partial x + (fab_x + fbb_y + d)\partial/\partial y$. But its proof indicates that (2.4) holds. Compare two expressions of L in (2.1).

Successive use of the formula: $[W, \phi Z] = \phi[W, Z] + (W\phi)Z$, where W and Z are first order operators with smooth coefficients, ϕ is a smooth function and $[\ ,\ ]$ denotes the Lie bracket, yields that any element of Lie $[Q_0, Q_1, Q_2]$ is of the form: $hQ_0 + fM_0 + (Q_0 f)M_1 + \cdots + (Q_0^k f)M_k$, where h is a real analytic function in Ω, k is a non-negative integer, and $M_i, i=0,\cdots,k$, are first order operators in Ω with real analytic coefficients. Hence, in virtue of (2.3), Lie $[Q_0, Q_1, Q_2](p)$ is generated by $Q_0(p)$ and so rank Lie $[Q_0, Q_1, Q_2](p) \leqq 1$ which contradicts to (2.4). Thus we obtain (B).

Proof of (C). The proof is by contradiction. Suppose that f changes sign from plus to minus along an integral curve $\Gamma: (x(t), y(t))$, $t_1 < t < t_2$, of Y. We set

(2.5) $$F(t) = f(x(t), y(t)), \quad t_1 < t < t_2.$$

$F(t)$ is real analytic on (t_1, t_2) and changes sign from plus to minus when t increases. Therefore there exist $t_o, t_1 < t_o < t_2$, a constant $c<0$ and an odd integer $q>0$ such that

(2.6) $$F(t) = c(t-t_o)^q + O((t-t_o)^{q+1}).$$

Set $(x_o, y_o) = (x(t_o), y(t_o))$. It follows from the hypothesis (H.2) that $|a(x_o, y_o)| + |b(x_o, y_o)| \neq 0$. Without loss of generality we may suppose that

(2.7) $$a(x_o, y_o) \neq 0.$$

Let $x = \phi_1(u, v)$ and $y = \phi_2(u, v)$ be the solutions of the initial value problem

(2.8) $$\frac{dx}{du}=a(x,y),\quad \frac{dy}{du}=b(x,y),\quad x|_{u=0}=x_o,\quad y|_{u=0}=y_o+v.$$

Then it is obvious that $\phi_1(u,v)$ and $\phi_2(u,v)$ are real analytic functions defined in an open neigborhood of (0,0). Since $\partial(\phi_1,\phi_2)/\partial(u,v)|_{u=v=0}=a(x_o,y_o)\neq 0$ by (2.8) and (2.7), we can introduce the coordinate transformation

$$\Phi:\begin{cases} x=\phi_1(u,v) \\ y=\phi_2(u,v) \end{cases}$$

from an open neighborhood $\tilde{\omega}_o$ of (0,0) in the uv-plane to an open neighborhood ω_o of (x_o,y_o) in the xy-plane. The operator L is transformed by Φ to the operator

$$\tilde{L}=\tilde{f}(u,v)\frac{\partial^2}{\partial u^2}+\tilde{c}(u,v)\frac{\partial}{\partial u}+\tilde{d}(u,v)\frac{\partial}{\partial v}+\tilde{g}(u,v),$$

where $\tilde{f}(u,v)=f(\phi_1(u,v),\phi_2(u,v))$, $\tilde{g}(u,v)=g(\phi_1(u,v),\phi_2(u,v))$, $\partial/\partial u=(\Phi^{-1})_*X$ and $\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v=(\Phi^{-1})_*Y$. From the hypothesis (H.2) it follows that $\partial/\partial u$ and $\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v$ are independent in $\tilde{\omega}_o$, that is,

(2.9) $$\tilde{d}(u,v)\neq 0,\quad (u,v)\in\tilde{\omega}_o.$$

Let $\tilde{\Gamma}$ be the image of Γ by Φ^{-1}. Then $\tilde{\Gamma}$ is the integral curve of $\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v$ through (0,0), and we have

(2.10) $$\tilde{f}|_{\tilde{\Gamma}}=f(x(t),y(t)),\quad t_1'<t<t_2',$$

where $t_1<t_1'<t_o<t_2'<t_2$. Now we consider the operator

$$\frac{1}{\tilde{d}}\tilde{L}=\frac{\tilde{f}}{\tilde{d}}\frac{\partial^2}{\partial u^2}+\frac{\tilde{c}}{\tilde{d}}\frac{\partial}{\partial u}+\frac{\partial}{\partial v}+\frac{\tilde{g}}{\tilde{d}}\quad\text{in}\quad\tilde{\omega}_o.$$

It is clear that

(2.11) $$\frac{1}{\tilde{d}}\tilde{L}\ \text{ is hypoelliptic in }\ \tilde{\omega}_o.$$

Let $\tilde{\Gamma}'$ be the integral curve of $\tilde{c}/\tilde{d}\partial/\partial u+\partial/\partial v$ through (0,0). Then $\tilde{\Gamma}'$ coincides with $\tilde{\Gamma}$ except for parametrization. From (2.9) we see that $\tilde{d}>0$ in $\tilde{\omega}_o$ or $\tilde{d}<0$ in $\tilde{\omega}_o$ if we shrink $\tilde{\omega}_o$ to (0,0). In the former case $\tilde{\Gamma}'$ has the same direction as $\tilde{\Gamma}$, and in the latter case the opposite one. Therefore, by (2.6) and (2.10), $\tilde{f}/\tilde{d}$ changes sign from plus to minus along $\tilde{\Gamma}'$ in a neighborhood of (0,0). Hence, denoting $\tilde{\Gamma}'$ by $(u(v),v)$, $|v|<\varepsilon_o$ ($\varepsilon_o>0$ is small), we see that there exist a constant $c'<0$ and an odd integer $q'>0$ such that $\tilde{f}(u(v),v)/\tilde{d}(u(v),v)=c'v^{q'}+O(v^{q'+1})$, $|v|<\varepsilon_o$, because $\tilde{f}(u,v)/\tilde{d}(u,v)$ is real analytic in $\tilde{\omega}_o$ and $u(v)$ is real analytic on $(-\varepsilon_o,\varepsilon_o)$. Then, according to

Theorem 1.2 of [2], $\tilde{L}/\tilde{\partial}$ is not hypoelliptic in $\tilde{\omega}_o$ which contradicts to (2.11). Thus we obtain (C).

3. Sufficiency of (A), (B) and (C): special case

In this section we shall prove that the conditions (A), (B) and (C) are sufficient for L to be hypoelliptic in Ω when

(3.1) $$b(x,y)=c(x,y)=0 \quad \text{and} \quad d(x,y)=1, \ (x,y)\in\Omega.$$

Then we can write

(3.2) $$L=f\left(a\frac{\partial}{\partial x}\right)^2+\frac{\partial}{\partial y}+g=\frac{\partial}{\partial x}Q_1+Q_0+g,$$

where $Q_0=(-f_x a^2-faa_x)\partial/\partial x+\partial/\partial y, \quad Q_1=fa^2\partial/\partial x.$

Here we list up the properties that f, a and g have. By the assumptions (H.1) and (H.2) it holds respectively that

(3.3) f, a and g are real valued and analytic in Ω,

(3.4) $a(x,y)\neq 0, \quad (x,y)\in\Omega,$

and by (A) with (3.4) above, (B) and (C) it holds respectively that

(3.5) $f_x(x,y)=0$ for any $(x,y)\in\Omega$ such that $f(x,y)=0$,

(3.6) for any $x\in \boldsymbol{R}^1$ and any interval I such that $\{x\}\times I\subset\Omega$, the function $y\to f(x,y)$ does not vanish identically on I,

(3.7) for any $x\in \boldsymbol{R}^1$ and any interval I such that $\{x\}\times I\subset\Omega$, the function $y\to f(x,y)$ does not change sign from plus to minus when y increases on I.

Lemma 3.1. *Let Q_0 and Q_1 be the first order operators introduced in* (3.2). *Then* rank Lie $[Q_0,Q_1](p)=2$, $p\in\Omega$, *where* Lie $[Q_0,Q_1]$ *is the Lie algebra generated by* Q_0 *and* Q_1.

Proof. Let p be an arbitrary point in Ω and let $\Gamma:(x(t),y(t))$, $t_1<t<t_2$ $(t_1<0<t_2)$, be the integral curve of Q_0 such that $p=(x(0),y(0))$. Suppose that $f=0$ on Γ. Then, by (3.5), $-f_x a^2-faa_x=0$ on Γ. Hence Γ is the straight line parallel to the y-axis and $f=0$ there, which contradicts to (3.6). Thus it has been shown that f does not vanish identically on Γ. Therefore, since $f(x(t),y(t))$ is real analytic on (t_1,t_2), it holds that there exists an integer $n\geqq 0$ such that

$$
\begin{aligned}
&Q_0^k f(p) = d^k/dt^k f(x(t),y(t))|_{t=0} = 0, \quad k=0,\cdots,n-1, \quad \text{and} \\
&Q_0^n f(p) = d^n/dt^n f(x(t),y(t))|_{t=0} \neq 0.
\end{aligned} \tag{3.8}
$$

By using the formula: $[W,\phi Z]=\phi[W,Z]+(W\phi)Z$ repeatedly, where W and Z are first order operators with smooth coefficients and ϕ is a smooth function, we have

$$(\operatorname{ad} Q_0)^n Q_1 = \sum_{i=0}^{n-1} Q_0^i(fa^2)Z_i + Q_0^n(fa^2)\partial/\partial x,$$

where $(\operatorname{ad} A)B = AB - BA$ for any operators A and B, and Z_i, $i=0,\cdots,n-1$, are first order operators with smooth coefficients. Hence, in virtue of (3.4) and (3.8), $(\operatorname{ad} Q_0)^n Q_1 = c\partial/\partial x$ at p with a constant $c \neq 0$, and so Q_0 and $(\operatorname{ad} Q_0)^n Q_1$ are linearly independent at p which proves the Lemma.

Here we remark that hypoellipticity is a local property, that is, L is hypoelliptic in Ω if and only if for any $p \in \Omega$ there exists an open neighborhood ω_p of p such that L is hypoelliptic in ω_p.

Let p be an arbitrary point of Ω. For the sake of simplicity we let $p=(0,0)$. Setting for $r_1,r_2>0$

$$D_{r_1,r_2} = \{(x,y) \mid |x|<r_1,\ |y|<r_2\} \tag{3.9}$$

we must show that

(3.10) L is hypoelliptic in D_{r_1,r_2} for sufficiently small r_1, r_2.

In virtue of (3.3) and (3.6) we can write with a constant $\alpha \neq 0$ and an integer $k \geqq 0$

(3.11) $f(0,y) = \alpha y^k + O(y^{k+1})$, $|y| \leqq r$, for sufficiently small $r>0$.

Furthermore, by the Weierstrass preparation theorem, we can write for sufficiently small $r>0$

(3.12) $f(x,y) = q(x,y)y^l F(x,y)$, $(x,y) \in D_{r,r}$,

where

(3.13) $q(x,y)$ is a real valued analytic function in $D_{r,r}$ and $q(x,y) \neq 0$, $(x,y) \in D_{r,r}$,

(3.14) $F(x,y) = y^m + a_{m-1}(x)y^{m-1} + \cdots + a_0(x)$, $(x,y) \in D_{r,r}$,

(3.15) l and m are non-negative integers and $l+m=k$,

(3.16) $a_i(x)$, $i=0,\cdots,m-1$, are real valued analytic functions on $(-r,r)$ and $a_i(0)=0$, $i=0,\cdots,m-1$, (3.17) $a_0(x)\not\equiv 0$ on $(-r,r)$.

We shall divide the proof of (3.10) into three parts: Case 1, Case 2 and Case 3.

Case 1: k is even. Let $\alpha>0$ in (3.11). Then we can take r_1 and r_2, $0<r_1,r_2<r$, so small that $f(x,-r_2)>0$, $|x|\leqq r_1$. Therefore it follows from (3.7) that $f(x,y)\geqq 0$, $(x,y)\in D_{r_1,r_2}$. Combining this with (3.3) and Lemma 3.1, we see from Theorem 2.8.2 of [11] that L is hypoelliptic in D_{r_1,r_2}.

Next let $\alpha<0$ in (3.11). Then we can take r_1 and r_2, $0<r_1,r_2<r$, so small that $f(x,r_2)<0$, $|x|\leqq r_1$. Therefore it follows from (3.7) that $f(x,y)\leqq 0$, $(x,y)\in D_{r_1,r_2}$. By the change of variables: $x'=-x$, $y'=-y$, L is transformed to the operator $L'=f(-x',-y')(a(-x',-y')\partial/\partial x')^2-\partial/\partial y'+g(-x',-y')$ and $-L'$ is hypoelliptic in D_{r_1,r_2} by the previous argument. Hence L is hypoelliptic in D_{r_1,r_2}.

Case 2: k is odd and l is odd. Since k is odd, it follows from (3.11) and (3.7) that $\alpha>0$. On the other hand, it follows from (3.11)–(3.16) that $q(0,y)y^k=\alpha y^k+O(y^{k+1})$. Therefore $q(0,0)=\alpha>0$ and so we have by (3.13)

$$q(x,y)>0,\quad (x,y)\in D_{r,r}. \tag{3.18}$$

Since $a_0(x)$ is analytic on $|x|<r$ and $a_0(x)\not\equiv 0$ there by (3.16) and (3.17), there exists r_1, $0<r_1<r$, such that $a_0(x)\neq 0$, $0<|x|<r_1$. Suppose that $a_0(x_o)<0$ for some x_o, $0<|x_o|<r_1$. Then $F(x_o,y)<0$ for sufficiently small y and so $f(x_o,y)=q(x_o,y)y^lF(x_o,y)$ changes sign from plus to minus when y increases near 0, because $q(x_o,y)>0$ by (3.18) and l is odd by the hypothesis. This contradicts to (3.7) and so $a_0(x_o)>0$ which implies that $a_0(x)>0$, $0<|x|<r_1$. Hence, for any fixed x_o, $0<|x_o|<r_1$, there exists r', $0<r'<r$, such that $F(x_o,y)>0$, $|y|<r'$ (r' may depend on x_o). Hence $f(x_o,y)=q(x_o,y)y^lF(x_o,y)>0$ on $0<y<r'$ and $f(x_o,y)=q(x_o,y)y^lF(x_o,y)<0$ on $-r'<y<0$, because l is odd by the hypothesis and $q(x_o,y)>0$ on $|y|<r'$ by (3.18). Then it follows from (3.7) that $f(x_o,y)\geqq 0$, $0<y<r$ and $f(x_o,y)\leqq 0$, $-r<y<0$. This implies that $F(x_o,y)\geqq 0$, $|y|<r$. Since x_o, $0<|x_o|<r_1$, is arbitrary we obtain

$$F(x,y)\geqq 0,\quad (x,y)\in D_{r_1,r}. \tag{3.19}$$

Taking into account that l is odd, we see from the Example 2 of [5] that (3.12)–(3.14), (3.18) and (3.19) yield that L is hypoelliptic in D_{r_1,r_2} with $r_2=r$.

Case 3: k is odd and l is even (hence m is odd). As in the Case 2 it holds that $\alpha>0$ and

$$q(x,y)>0,\quad (x,y)\in D_{r,r}. \tag{3.20}$$

Since $\alpha>0$ and k is odd by the hypothesis, it follows from (3.11) that there exist

ρ_1 and ρ_2, $0<\rho_1,\rho_2<r$, such that $f(x,-\rho_2)<0$ on $|x|\leqq\rho_1$ and $f(x,\rho_2)>0$ on $|x|\leqq\rho_1$. Hence, from (3.12), (3.20) and the fact that l is even by the hypothesis, we have

(3.21) $\quad F(x,-\rho_2)<0$ on $|x|\leqq\rho_1$ and $F(x,\rho_2)>0$ on $|x|\leqq\rho_1$

and, moreover, in virtue of (3.7) it holds that

(3.22) for any $x\in[-\rho_1,\rho_1]$, the function $y\to F(x,y)$ does not change sign from plus to minus when y increases on $[-\rho_2,\rho_2]$.

Then it is not difficult to see that

(3.23) there exists a unique continuous function $\lambda(x)$ defined on $[-\rho_1,\rho_1]$ such that $\lambda(0)=0$, $|\lambda(x)|<\rho_2$ on $|x|\leqq\rho_1$, $F(x,y)\leqq 0$ in $\{(x,y)|\ |x|\leqq\rho_1,\ -\rho_2\leqq y\leqq\lambda(x)\}$ and $F(x,y)\geqq 0$ in $\{(x,y)|\ |x|\leqq\rho_1,\ \lambda(x)\leqq y\leqq\rho_2\}$.

The uniqueness follows from the fact that the function $y\to F(x,y)$ does not vanish identically on any sub-interval of $[-\rho_2,\rho_2]$. We define $\lambda(x)$ as $\sup\{y_o\in[-\rho_2,\rho_2]|\ F(x,y)\leqq 0$ on $-\rho_2\leqq y\leqq y_o\}$.

Lemma 3.2. *There exist* r_1, r_2 $(0<r_1<\rho_1,\ 0<r_2<\rho_2)$ *and real valued analytic functions* $\phi(x,y)$, $h(x,y)$ *in* D_{r_1,r_2} *such that*

(3.24) $f(x,y)=\phi(x,y)^3h(x,y),\quad (x,y)\in D_{r_1,r_2}$;

(3.25) $h(x,y)\geqq 0,\quad (x,y)\in D_{r_1,r_2}$;

(3.26) $|\lambda(x)|<r_2,\quad x\in(-r_1,r_1)$;

(3.27) $\phi(x,y)\leqq 0$ *in* $\{(x,y)\in D_{r_1,r_2}\ |y\leqq\lambda(x)\}$ *and* $\phi(x,y)\geqq 0$ *in* $\{(x,y)\in D_{r_1,r_2}\ |y\geqq\lambda(x)\}$;

(3.28) $\lambda(x)$ *is real analytic on* $0<|x|<r_1$;

(3.29) *for any fixed* $y\in[-r_2,r_2]$, *the number of* x*'s on* $[-r_1,r_1]$ *satisfying* $\lambda(x)=y$ *is less than or equal to* M, *where* M *is the order of zero of the function* $a_0(x)$ *at* $x=0$.

Proof. We consider a factorization of $F(x,y)$. Let A_o be the ring of germs of real valued analytic functions of x at $x=0$, and let $A_o[y]$ be the polynomial ring of A_o. It is well-known that A_o and $A_o[y]$ are unique factorization domains. We regard $F(x,y)$ as an element of $A_o[y]$. Then there exist irreducible polynomials $P_1,\cdots,P_N\in A_o[y]$, $P_i\neq P_j$ $(i\neq j)$, and positive integers $m_1,\cdots,m_N$ such that $F=P_1^{m_1}\cdots P_N^{m_N}$. Since F is a monic polynomial of y, we may suppose that P_i, $i=1,\cdots,N$, are also monic polynomials of y of degree $\mu_i\geqq 1$. Since P_i and $\partial P_i/\partial y$ are relatively prime, their resultant ω_i, $\omega_i\in A_o$, is not equal to 0, and there exist G_i, $H_i \in A_o[y]$ such that $G_iP_i+H_i\partial P_i/\partial y=\omega_i$. Furthermore, since P_i and P_j $(i\neq j)$ are relatively prime, their resultant $\omega_{i,j}$, $\omega_{i,j}\in A_o$, is not equal to 0, and there exist $G_{i,j}$, $H_{i,j}\in A_o[y]$ such that $G_{i,j}P_i+H_{i,j}P_j=\omega_{i,j}$.

We choose r_1, $0<r_1<\rho_1$, so small that ω_i, $\omega_{i,j}$ and all coefficients of F, P_i, G_i, H_i, $G_{i,j}$ and $H_{i,j}$, $1\leqq i\neq j\leqq N$, are real valued analytic functions defined on $(-r_1,r_1)$ and they can be extended analytically to the complex domain $\{z\in \boldsymbol{C}|\ |z|<r_1\}$. Then we can regard F, P_i, G_i, H_i, $G_{i,j}$ and $H_{i,j}$, $1\leqq i\neq j\leqq N$, as analytic functions defined in $D=\{(z,w)\in \boldsymbol{C}^2|\ |z|<r_1,\ |w|<\infty\}$, and ω_i, $\omega_{i,j}$, $1\leqq i\neq j\leqq N$, as analytic functions defined in $|z|<r_1$. Of course, $F(z,w)$, $P_i(z,w)$, $G_i(z,w)$, $H_i(z,w)$, $G_{i,j}(z,w)$ and $H_{i,j}(z,w)$, $1\leqq i\neq j\leqq N$, are polynomials of w. Then we have by choosing r_1 smaller if necessary

(3.30) $F(z,w)=P_1(z,w)^{m_1}\cdots P_N(z,w)^{m_N}$ in $D=\{(z,w)\in \boldsymbol{C}^2|\ |z|<r_1,\ |w|<\infty\}$;

(3.31) $G_i(z,w)P_i(z,w)+H_i(z,w)\partial P_i(z,w)/\partial w=\omega_i(z)$ in D, and $\omega_i(z)\neq 0$ in $0<|z|<r_1$, $1\leqq i\leqq N$;

(3.32) $G_{i,j}(z,w)P_i(z,w)+H_{i,j}(z,w)P_j(z,w)=\omega_{i,j}(z)$ in D, and $\omega_{i,j}(z)\neq 0$ in $0<|z|<r_1$, $1\leqq i\neq j\leqq N$.

(3.31) and (3.32) imply respectively that

(3.33) for any fixed z, $0<|z|<r_1$, the equation $P_i(z,w)=0$ has no multiple roots, $1\leqq i\leqq N$;

(3.34) for any fixed z, $0<|z|<r_1$, the equations $P_i(z,w)=0$ and $P_j(z,w)=0$ have no common roots, $1\leqq i\neq j\leqq N$.

We set $\Delta^+=\{z\in \boldsymbol{C}|\ |z|<r_1,\ \mathrm{Re}\,z>0\}$ and fix $x_o\in(0,r_1)$. In virtue of (3.33) the equation $P_i(x_o,w)=0$ has distinct roots $\alpha_{i,1},\cdots,\alpha_{i,\mu_i}$. Since the coefficients of $P_i(x_o,w)$ as a polynomial of w are real, it is possible to choose ν_i, $0\leqq\nu_i\leqq\mu_i$, so that $\alpha_{i,1},\cdots,\alpha_{i,\nu_i}$ are real; $\alpha_{i,\nu_i+1},\cdots,\alpha_{i,\mu_i}$ are not real and $\alpha_{i,j}=\overline{\alpha_{i,j+1}}$, $j=\nu_i+1$, $\nu_i+3,\cdots,\mu_i-1$. Here we let $\nu_i=0$ if the equation $P_i(x_o,w)=0$ has no real roots. It follows from (3.33) and the implicit function theorem that there exist r_o, $0<r_o<\min(x_o,r_1-x_o)$, and analytic functions $w_{i,1}(z),\cdots,w_{i,\mu_i}(z)$ defined in $B_o=\{z\in \boldsymbol{C}|\ |z-x_o|<r_o\}$ such that $w_{i,k}(x_o)=\alpha_{i,k}(1\leqq k\leqq\mu_i)$, $P_i(z,w_{i,k}(z))=0$ in B_o $(1\leqq k\leqq\mu_i)$, and $w_{i,k}(z)\neq w_{i,j}(z)$ in B_o $(1\leqq k\neq j\leqq\mu_i)$. It is obvious that the functions $w_{i,k}(x)$, $k=1,\cdots,\nu_i$, are real valued on (x_o-r_o,x_o+r_o). In virtue of (3.33) we can extend $w_{i,k}(z)$, $k=1,\cdots,\mu_i$, to analytic functions $\lambda_{i,k}(z)$, $k=1,\cdots,\mu_i$, defined in Δ^+ such that $\lambda_{i,k}(z)\neq\lambda_{i,j}(z)$ in Δ^+ $(1\leqq k\neq j\leqq\mu_i)$. This fact is well-known in the theory of analytic functions. Hence we have

$$P_i(x,y)=(y-\lambda_{i,1}(x))\cdots(y-\lambda_{i,\mu_i}(x)),\quad (x,y)\in(0,r_1)\times(-\infty,\infty).$$

The functions $\lambda_{i,k}(x)$, $k=1,\cdots,\nu_i$, are real valued on $(0,r_1)$, because $\lambda_{i,k}(z)$,

$k=1,\cdots,\nu_i$, are the analytic extensions of $w_{i,k}(z)$, $k=1,\cdots,\nu_i$, and $w_{i,k}(x)$, $k=1,\cdots,\nu_i$, are real valued on (x_o-r_o, x_o+r_o). On the other hand, it is easy to see that $\overline{\lambda_{i,j}(\bar{z})}$, $j=\nu_i+1,\cdots,\mu_i$, are analytic in Δ^+. Since $\overline{\lambda_{i,j+1}(x_o)}=\overline{\alpha_{i,j+1}}=\alpha_{i,j}=\lambda_{i,j}(x_o)$ and $P_i(x,\overline{\lambda_{i,j+1}(x)})=\overline{P_i(x,\lambda_{i,j+1}(x))}=0=P_i(x,\lambda_{i,j}(x))$ on $(0,r_1)$, $j=\nu_i+1,\nu_i+3,\cdots,\mu_i-1$, it follows from (3.33) and the implicit function theorem that $\overline{\lambda_{i,j+1}(x)}=\lambda_{i,j}(x)$, $j=\nu_i+1,\nu_i+3,\cdots,\mu_i-1$, on an open interval containing x_o. Therefore, by the coincidence theorem, $\overline{\lambda_{i,j+1}(\bar{z})}=\lambda_{i,j}(z)$, $j=\nu_i+1,\nu_i+3,\cdots,\mu_i-1$, in Δ^+, and so $\overline{\lambda_{i,j+1}(x)}=\lambda_{i,j}(x)$, $j=\nu_i+1,\nu_i+3,\cdots,\mu_i-1$, on $(0,r_1)$.

We note that $\operatorname{Im}\lambda_{i,j}(x)\neq 0$, $j=\nu_i+1,\cdots,\mu_i$, on $(0,r_1)$, since $\overline{\lambda_{i,j+1}(x)}=\lambda_{i,j}(x)$ and $\lambda_{i,j+1}(x)\neq\lambda_{i,j}(x)$, $j=\nu_i+1,\nu_i+3,\cdots,\mu_i-1$, on $(0,r_1)$. Then $(y-\lambda_{i,j}(x))(y-\lambda_{i,j+1}(x))$ $=(y-\lambda_{i,j}(x))(y-\overline{\lambda_{i,j}(x)})>0$, $j=\nu_i+1,\nu_i+3,\cdots,\mu_i-1$, in $(0,r_1)\times(-\infty,\infty)$. Hence

$$P_i(x,y)=(y-\lambda_{i,1}(x))\cdots(y-\lambda_{i,\nu_i}(x))Q_i(x,y) \quad \text{in} \quad (0,r_1)\times(-\infty,\infty), \tag{3.35}$$

where

$$Q_i(x,y)\equiv\prod_{\nu_i+1\leqq j\leqq\mu_i}(y-\lambda_{i,j}(x))>0 \quad \text{in} \quad (0,r_1)\times(-\infty,\infty). \tag{3.36}$$

Here we take $(y-\lambda_{i,1}(x))\cdots(y-\lambda_{i,\nu_i}(x))$ to be equal to 1 in $(0,r_1)\times(-\infty,\infty)$ if $\nu_i=0$, and $Q_i(x,y)$ to be equal to 1 in $(0,r_1)\times(-\infty,\infty)$ if $\nu_i=\mu_i$.

It follows from (3.30), (3.35) and (3.36) that

$$\begin{aligned}\prod_{1\leqq i\leqq N}&\{(y-\lambda_{i,1}(x))\cdots(y-\lambda_{i,\nu_i}(x))\}^{m_i}\\ &=F(x,y)\prod_{1\leqq i\leqq N}Q_i(x,y)^{-m_i} \quad \text{in} \quad (0,r_1)\times(-\infty,\infty),\end{aligned} \tag{3.37}$$

and from (3.22) and (3.36) that

(3.38) for any fixed $x\in(0,r_1)$, the left-hand side of (3.37), as a function of y, does not change sign from plus to minus when y increases on $[-\rho_2,\rho_2]$.

We have by (3.33) and (3.34)

$$\lambda_{i,k}(x)\neq\lambda_{i',k'}(x) \quad \text{for all } x\in(0,r_1) \text{ if } (i,k)\neq(i',k'). \tag{3.39}$$

Since $\lim_{x\to 0}F(x,w)=w^m$ uniformly on $|w|=\rho_2$ by (3.14) and (3.16), it follows from Rouché's theorem that for sufficiently small $r_1>0$

$$|\lambda_{i,k}(x)|<\rho_2 \text{ for all } x\in(0,r_1),\ 1\leqq i\leqq N,\ 1\leqq k\leqq\mu_i. \tag{3.40}$$

Combining (3.38) with (3.39) and (3.40) we see that

(3.41) there exists at most one i, $1\leqq i\leqq N$, such that m_i is odd and $\nu_i\geqq 1$;

(3.42) $\nu_i=1$ if m_i is odd and $\nu_i\geqq 1$.

Since $m=\mu_1 m_1+\cdots+\mu_N m_N$ by (3.14), (3.30) and the definition of μ_i, and m is odd by the assumption of Case 3, there exists i_o, $1\leqq i_o\leqq N$, such that $\mu_{i_o}m_{i_o}$ is odd. Then μ_{i_o} and m_{i_o} are odd, and moreover, $\nu_{i_o}\geqq 1$ because $\mu_{i_o}-\nu_{i_o}$ is even from the definition of ν_{i_o}. Hence $\nu_{i_o}=1$ by (3.42). On the other hand, suppose that $\mu_i m_i$ is odd with $i\neq i_o$. Then m_i is odd and $\nu_i=1$ by the previous argument, which contradicts to (3.41). Thus it has been proved that

(3.43) there exists a unique i_o, $1\leqq i_o\leqq N$, such that $\mu_{i_o}m_{i_o}$ is odd;

$$\nu_{i_o}=1. \tag{3.44}$$

Let $i\neq i_o$. If m_i is odd, then $\nu_i=0$ by (3.41), (3.43) and (3.44), and so $P_i(x,y)^{m_i}=Q_i(x,y)^{m_i}>0$ in $(0,r_1)\times(-\infty,\infty)$. If m_i is even, it is clear that $P_i(x,y)^{m_i}\geqq 0$ in $(0,r_1)\times(-\infty,\infty)$. Thus we have

$$P_i(x,y)^{m_i}\geqq 0 \quad \text{in} \quad (0,r_1)\times(-\infty,\infty) \quad \textit{if } i\neq i_o. \tag{3.45}$$

By using (3.30), (3.35) with $i=i_o$, (3.36) with $i=i_o$, (3.44) and (3.45), we can write

$$F(x,y)=(y-\lambda_{i_o,1}(x))^{m_{i_o}}\tilde{F}(x,y), \quad (x,y)\in(0,r_1)\times(-\infty,\infty), \tag{3.46}$$

where

$$\tilde{F}(x,y)\equiv Q_{i_o}(x,y)^{m_{i_o}}\prod_{\substack{1\leqq i\leqq N\\ i\neq i_o}} P_i(x,y)^{m_i}\geqq 0, \quad (x,y)\in(0,r_1)\times(-\infty,\infty).$$

Since m_{i_o} is odd by (3.43), we see from the definition of $\lambda(x)$ in (3.23) that $\lambda_{i_o,1}(x)=\lambda(x)$ on $(0,r_1)$. Hence

(3.47) $\lambda(x)$ is real analytic on $(0,r_1)$,

and by (3.35), (3.36) with $i=i_o$, and (3.44)

(3.48) $P_{i_o}(x,y)\leqq 0$ if $0<x<r_1$ and $y\leqq\lambda(x)$;

$P_{i_o}(x,y)\geqq 0$ if $0<x<r_1$ and $y\geqq\lambda(x)$.

Now we shall show that

$$m_{i_o}\geqq 3. \tag{3.49}$$

Suppose that $m_{i_o}=1$. We have by (3.12) and (3.46)

$$f(x,y)=q(x,y)y^l(y-\lambda_{i_o,1}(x))\tilde{F}(x,y), \quad (x,y)\in(0,r_1)\times(-r,r).$$

Since $|\lambda_{i_o,1}(x)|<\rho_2$ on $(0,r_1)$ by (3.40), and $(0,r_1)\times(-\rho_2,\rho_2)\subset D_{r,r}\subset\Omega$, it follows from (3.5) that $f_x(x,\lambda_{i_o,1}(x))=0$ on $(0,r_1)$, that is, $q(x,\lambda_{i_o,1}(x))(\lambda_{i_o,1}(x))^l(-d\lambda_{i_o,1}(x)/dx)\tilde{F}(x,\lambda_{i_o,1}(x))=0$ on $(0,r_1)$. By (3.20), $q(x,\lambda_{i_o,1}(x))\neq 0$ on $(0,r_1)$; furthermore, $\tilde{F}(x,\lambda_{i_o,1}(x))\neq 0$ on $(0,r_1)$, because $Q_{i_o}(x,\lambda_{i_o,1}(x))\neq 0$ on $(0,r_1)$ by (3.36), and $P_i(x,\lambda_{i_o,1}(x))\neq 0$ on $(0,r_1)$, $i\neq i_o$, by (3.34). Therefore $(\lambda_{i_o,1}(x))^l d\lambda_{i_o,1}(x)/dx=0$ on $(0,r_1)$, and so $(\lambda_{i_o,1}(x))^{l+1}$ is constant on $(0,r_1)$. This constant is equal to 0, because $\lambda_{i_o,1}(x)=\lambda(x)$ on $(0,r_1)$ and $\lim_{x\to+0}\lambda(x)=0$ by (3.23). Hence $\lambda_{i_o,1}(x)=0$ on $(0,r_1)$ which implies from (3.46) and (3.14) that $a_0(x)=F(x,0)=0$ on $(0,r_1)$. Since $a_0(x)$ is analytic on $(-r,r)$ by (3.16), this shows that $a_0(x)=0$ on $(-r,r)$ which contradicts to (3.17). Thus we have proved that $m_{i_o}\geqq 2$. Since m_{i_o} is odd by (3.43), we obtain (3.49).

In the case $x<0$, we adopt the same reasoning as in the case $x>0$. Then, from the uniqueness of i_o such that $\mu_{i_o}m_{i_o}$ is odd, we obtain for sufficiently small $r_1>0$

(3.50) $P_i^{m_i}(x,y)\geqq 0$ in $(-r_1,0)\times(-\infty,\infty)$ if $i\neq i_o$;

(3.51) $\lambda(x)$ is real analytic on $(-r_1,0)$;

(3.52) $P_{i_o}(x,y)\leqq 0$ if $-r_1<x<0$ and $y\leqq\lambda(x)$;

$P_{i_o}(x,y)\geqq 0$ if $-r_1<x<0$ and $y\geqq\lambda(x)$.

Combining (3.45), (3.47) and (3.48) with (3.50)–(3.52) we have

(3.53) $P_i^{m_i}(x,y)\geqq 0$ in $(-r_1,r_1)\times(-\infty,\infty)$ if $i\neq i_o$;

(3.54) $\lambda(x)$ is real analytic on $(-r_1,0)\cup(0,r_1)$;

(3.55) $P_{i_o}(x,y)\leqq 0$ if $|x|<r_1$ and $y\leqq\lambda(x)$;

$P_{i_o}(x,y)\geqq 0$ if $|x|<r_1$ and $y\geqq\lambda(x)$.

We take r_2 such that $0<r_2<\rho_2$ and set

$$\phi(x,y)=P_{i_o}(x,y),\quad (x,y)\in D_{r_1,r_2};$$

$$h(x,y)=q(x,y)y^l P_{i_o}(x,y)^{m_{i_o}-3}\prod_{\substack{1\leqq i\leqq N\\ i\neq i_o}} P_i(x,y)^{m_i},\quad (x,y)\in D_{r_1,r_2}.$$

Since l is even by the hypothesis of Case 3, and $m_{i_o}\geqq 3$ is odd, it follows from (3.20) and (3.53) that $h(x,y)\geqq 0$ in D_{r_1,r_2}. It is clear that ϕ and h are real valued analytic functions defined in D_{r_1,r_2}, and it follows from (3.12) and (3.30) that $f=\phi^3 h$ in D_{r_1,r_2}. Thus we obtain (3.24) and (3.25). From (3.54) and (3.55) we obtain (3.28) and (3.27) respectively.

Finally we shall prove (3.29) and (3.26) by taking $r_1,r_2>0$ sufficiently small. In virtue of (3.17) we can take $r_1>0$ so small that the function $a_0(z)$, $z\in \boldsymbol{C}$, has zero of order M only at $z=0$ when $|z|\leqq r_1$. Since $\lim_{y\to 0} F(z,y)=a_0(z)$ uniformly on $|z|=r_1$, it follows from Rouché's theorem that if $r_2>0$ is sufficiently small, then for any fixed y, $|y|\leqq r_2$, $F(z,y)$, as a function of z, has M zeros in $|z|\leqq r_1$. We fix y, $|y|\leqq r_2$, arbitrarily. Let x, $|x|\leqq r_1$, satisfy $\lambda(x)=y$. Then $F(x,y)=F(x,\lambda(x))=0$ by (3.30) and (3.55). Therefore the number of x's satisfying $\lambda(x)=y$ is less than or equal to M, which proves (3.29). Since $\lambda(x)$ is continuous at $x=0$ and $\lambda(0)=0$, we obtain (3.26) by taking $r_1>0$ smaller. Q.E.D.

Let D_{r_1,r_2} be the open set determined in the above lemma. As was proved in Theorem 2.1 of [9], to show that L is hypoelliptic in D_{r_1,r_2} it is sufficient to prove the following:

$$(3.56)\quad \begin{cases} \text{for any } p\in D_{r_1,r_2} \text{ there exist positive constants } C,\ \varepsilon,\ \delta \text{ and } \phi_1,\ \phi_2,\ \phi_3 \\ \in C_0^\infty(D_{r_1,r_2}) \text{ such that} \\ \text{(i)}\quad \sum_{i=1,2} \|L^{0(i)}u\|_0^2 \leqq C\{|(Lu,\phi_1 u)|+\|u\|_0^2\}, \\ \text{(ii)}\quad \sum_{i=1,2} \|L^0_{(i)}u\|_0^2 \\ \qquad \leqq C\sum_{i=1,2}\{|(L(D_i u),\phi_2 D_i u)|+|(L(D_i u),\phi_3 D_i u)|+\|u\|_1^2\}, \\ \text{(iii)}\quad \|u\|_\varepsilon^2 \leqq C\{\|Lu\|_0^2+\|u\|_0^2\}, \quad \text{for all } u\in C_0^\infty(S(p,\delta)), \end{cases}$$

where $L^{0(1)}=fa^2\partial/\partial x$, $L^{0(2)}=0$, $L^0_{(1)}=(fa^2)_x\partial^2/\partial x^2$, $L^0_{(2)}=(fa^2)_y\partial^2/\partial x^2$; $D_1=\partial/\partial x$, $D_2=\partial/\partial y$; $(,)$ denotes the inner product in $L^2(\boldsymbol{R}^2)$, and $\|\cdot\|_s$, $s\in\boldsymbol{R}$, denotes the H^s norm; $S(p,\delta)=\{(x,y)\in\boldsymbol{R}^2|\ |(x,y)-p|<\delta\}$.

We obtain (3.56)(i) by the same argument as in the proof of Lemma 3.1 of [9]. We obtain (3.56)(ii) by Lemma 3.2 (3.24) of this paper and the same argument as in the proof of Lemma 3.2 of [9].

To prove (3.56)(iii) we use the following notations introduced in [9]. We set $D^+_{r_1,r_2}=\{(x,y)\in D_{r_1,r_2}|\ y>\lambda(x)\}$, $D^-_{r_1,r_2}=\{(x,y)\in D_{r_1,r_2}|\ y<\lambda(x)\}$ and we set for $u,v\in C_0^\infty(D_{r_1,r_2})$

$$(u,v)^+=\int_{D^+_{r_1,r_2}} u\bar{v}\,dxdy,\quad (u,v)^-=\int_{D^-_{r_1,r_2}} u\bar{v}\,dxdy.$$

Then the following two lemmas hold.

Lemma 3.3. *We have*

$$|(fa^2u_x,v_x)^\pm|\leqq C\{|(Lu,u)^\pm|+|(Lv,v)^\pm|+\|u\|_0^2+\|v\|_0^2\}$$

for all $u,v\in C_0^\infty(D_{r_1,r_2})$, *where* $C>0$ *is a constant independent of* u *and* v.

Lemma 3.4. *Let* $0<s<1/2$ *and for every* $v\in C_0^\infty(D_{r_1,r_2})$ *set* $v_0(x,y)=v(x,y)$ *if* $(x,y)\in D^+_{r_1,r_2}$, $v_0(x,y)=0$ *if* $(x,y)\notin D^+_{r_1,r_2}$. *Then we have*

$$\|v_0\|_s \leqq C\|v\|_s \quad \textit{for all } v\in C_0^\infty(D_{r_1,r_2}),$$

where $C>0$ *is a constant independent of* v.

REMARK. If $\lambda(x)$ is continuously differentiable in a neighborhood of $x=0$, then the proof of Lemma 3.3 is contained in that of Lemma 3.4 of [9] and Lemma 3.4 is a consequence of Theorem 11.4 and Theorem 9.2 of [10].

Proof of Lemma 3.3. We follow the way of proof of Lemma 3.4 of [9] with slight modifications. We have

$$\text{(3.57)}\quad |(fa^2u_x,v_x)^\pm| \leqq \int_{D^\pm_{r_1,r_2}} |f|a^2(|u_x|^2+|v_x|^2)dxdy$$
$$= \pm\int_{D^\pm_{r_1,r_2}} fa^2(|u_x|^2+|v_x|^2)dxdy,$$

since $f\geqq 0$ in $D^+_{r_1,r_2}$ and $f\leqq 0$ in $D^-_{r_1,r_2}$ by (3.24), (3.25) and (3.27). We shall estimate the right-hand side of (3.57).

For every $\varepsilon, 0<\varepsilon<r_1$, we set $m(\varepsilon)=\max_{|x|\leqq\varepsilon}|\lambda(x)|$, $K_\varepsilon=\{(x,y)\in \boldsymbol{R}^2|\ |x|\leqq\varepsilon,\ |y|\leqq m(\varepsilon)\}$, $D^+_\varepsilon=D^+_{r_1,r_2}\backslash K_\varepsilon$, $D^-_\varepsilon=D^-_{r_1,r_2}\backslash K_\varepsilon$. Since $\lambda(x)$ is continuous at $x=0$ and $\lambda(0)=0$, $\lim_{\varepsilon\to 0} m(\varepsilon)=0$ and so $K_\varepsilon\subset D_{r_1,r_2}$ for sufficiently small $\varepsilon>0$. In the rest of the proof we shall take $\varepsilon>0$ small.

By (3.28), ∂D^+_ε and ∂D^-_ε are piecewise smooth curves, and by (3.24) and (3.27), $f(x,y)=0$ on $y=\lambda(x)$. Hence we have

$$\int_{D^\pm_\varepsilon} fa^2|u_x|^2\,dxdy = -\int_{D^\pm_\varepsilon}(fa^2u_x)_x\bar{u}\,dxdy + \int_{\partial D^\pm_\varepsilon} fa^2u_x\bar{u}\,dy$$
$$= -\int_{D^\pm_\varepsilon}(fa^2u_x)_x\bar{u}\,dxdy + \int_{\gamma^\pm_\varepsilon} fa^2u_x\bar{u}\,dy,$$

where γ^+_ε is the polygonal line with vertices $(-\varepsilon,\lambda(-\varepsilon))$, $(-\varepsilon,m(\varepsilon))$, $(\varepsilon,m(\varepsilon))$ and $(\varepsilon,\lambda(\varepsilon))$; γ^-_ε is the polygonal line with vertices $(\varepsilon,\lambda(\varepsilon))$, $(\varepsilon,-m(\varepsilon))$, $(-\varepsilon,-m(\varepsilon))$ and $(-\varepsilon,\lambda(-\varepsilon))$. By (3.2) we can write $(fa^2u_x)_x = Lu-\alpha u_x-u_y-gu$ where $\alpha = -f_xa^2-faa_x$. Hence, taking into account that $fa^2|u_x|^2$ is real valued, we have

$$(3.58)\quad \int_{D_\varepsilon^\pm} fa^2|u_x|^2\,dxdy = -\mathrm{Re}\int_{D_\varepsilon^\pm}(Lu)\bar{u}\,dxdy + \mathrm{Re}\int_{D_\varepsilon^\pm}\alpha u_x\bar{u}\,dxdy$$
$$+\mathrm{Re}\int_{D_\varepsilon^\pm} u_y\bar{u}\,dxdy + \mathrm{Re}\int_{D_\varepsilon^\pm} g|u|^2\,dxdy + \mathrm{Re}\int_{\gamma_\varepsilon^\pm} fa^2u_x\bar{u}\,dy.$$

Since $f(x,y)=0$ on $y=\lambda(x)$, we see from (3.5) that $f_x(x,y)=0$ on $y=\lambda(x)$. Therefore $\alpha(x,y)=0$ on $y=\lambda(x)$. Hence

$$\int_{D_\varepsilon^\pm}\alpha u_x\bar{u}\,dxdy = -\int_{D_\varepsilon^\pm} u(\alpha\bar{u})_x\,dxdy + \int_{\partial D_\varepsilon^\pm}\alpha|u|^2\,dy$$
$$= -\int_{D_\varepsilon^\pm}(u\alpha\bar{u}_x + \alpha_x|u|^2)\,dxdy + \int_{\gamma_\varepsilon^\pm}\alpha|u|^2\,dy,$$

and so we have

$$(3.59)\qquad 2\mathrm{Re}\int_{D_\varepsilon^\pm}\alpha u_x\bar{u}\,dxdy = -\int_{D_\varepsilon^\pm}\alpha_x|u|^2\,dxdy + \int_{\gamma_\varepsilon^\pm}\alpha|u|^2\,dy.$$

On the other hand

$$\int_{D_\varepsilon^\pm} u_y\bar{u}\,dxdy = -\int_{D_\varepsilon^\pm} u\bar{u}_y\,dxdy - \int_{\partial D_\varepsilon^\pm}|u|^2\,dx,$$

and so, noting that $u=0$ on $\partial D_{r_1,r_2}$, we have

$$(3.60)\qquad \pm 2\mathrm{Re}\int_{D_\varepsilon^\pm} u_y\bar{u}\,dxdy = \mp\int_{\partial D_\varepsilon^\pm}|u|^2\,dx \leqq 0.$$

Combining (3.58)–(3.60) we have

$$\pm\int_{D_\varepsilon^\pm} fa^2|u_x|^2\,dxdy \leqq \mp\mathrm{Re}\int_{D_\varepsilon^\pm}(Lu)\bar{u}\,dxdy \mp\frac{1}{2}\int_{D_\varepsilon^\pm}\alpha_x|u|^2\,dxdy$$
$$\pm\frac{1}{2}\int_{\gamma_\varepsilon^\pm}\alpha|u|^2\,dy \pm \int_{D_\varepsilon^\pm} g|u|^2\,dxdy \pm \mathrm{Re}\int_{\gamma_\varepsilon^\pm} fa^2u_x\bar{u}\,dy.$$

Hence, letting $\varepsilon\to 0$, we have

$$(3.61)\qquad \pm\int_{D_{r_1,r_2}^\pm} fa^2|u_x|^2\,dxdy \leqq \mp\mathrm{Re}\int_{D_{r_1,r_2}^\pm}(Lu)\bar{u}\,dxdy$$
$$\mp\frac{1}{2}\int_{D_{r_1,r_2}^\pm}\alpha_x|u|^2\,dxdy \pm \int_{D_{r_1,r_2}^\pm} g|u|^2\,dxdy$$

$$\leqq C\{|(Lu,u)^{\pm}|+\|u\|_0^2\}.$$

In the same way we have

$$\pm\int_{D^{\pm}_{r_1,r_2}} fa^2|v_x|^2\,dxdy \leqq C\{|(Lv,v)^{\pm}|+\|v\|_0^2\}. \tag{3.62}$$

From (3.57), (3.61) and (3.62) we obtain Lemma 3.3. Q.E.D.

Proof of Lemma 3.4. From the hypothesis that $0<s<1/2$, it follows that

(3.63) $u\in H^s(\boldsymbol{R}^2)$ if and ony if $u\in H^0(\boldsymbol{R}^2)$ and $\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}(|u(x+t,y)-u(x,y)|^2+|u(x,y+t)-u(x,y)|^2)dxdy<\infty$; the norms $\|u\|_s$ and $\left\{\|u\|_0^2+\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}(|u(x+t,y)-u(x,y)|^2+|u(x,y+t)-u(x,y)|^2)dxdy\right\}^{\frac{1}{2}}$ are equivalent,

and

(3.64) there exists a constant $C>0$ such that

$$\int_0^\infty x^{-2s}|\phi(x)|^2dx \leqq C\int_0^\infty t^{-(2s+1)}dt\int_0^\infty|\phi(x+t)-\phi(x)|^2dx$$

for any $\phi\in C^\infty([0,\infty))$ with a bounded support.

(3.63) is, for example, due to Theorem 10.2 of [10]. The proof of the inequality (11.24) of [10] indicates that (3.64) holds whether $\phi(0)=0$ or not.

Let $\chi(x,y)$ be the characteristic function of the set $\{(x,y)\in\boldsymbol{R}^2|\ |x|<r_1, y>\lambda(x)\}$. Then $v_0=\chi v$ and in virtue of (3.63), to prove Lemma 3.4 it is sufficient to show that

$$\begin{aligned}\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}&(|\chi(x+t,y)v(x+t,y)-\chi(x,y)v(x,y)|^2\\&+|\chi(x,y+t)v(x,y+t)-\chi(x,y)v(x,y)|^2)dxdy\\ \leqq C\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}&(|v(x+t,y)-v(x,y)|^2\\&+|v(x,y+t)-v(x,y)|^2)dxdy\end{aligned} \tag{3.65}$$

for all $v\in C_0^\infty(D_{r_1,r_2})$, where $C>0$ is a constant independent of v. In the rest of the proof we shall denote by C positive constants independent of $v\in C_0^\infty(D_{r_1,r_2})$.

We have

$$
\begin{aligned}
&|\chi(x+t,y)v(x+t,y)-\chi(x,y)v(x,y)|^2\\
&=|\chi(x+t,y)(v(x+t,y)-v(x,y))+v(x,y)(\chi(x+t,y)-\chi(x,y))|^2\\
&\leqq 2|v(x+t,y)-v(x,y)|^2+2|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&|\chi(x,y+t)v(x,y+t)-\chi(x,y)v(x,y)|^2\\
&\leqq 2|v(x,y+t)-v(x,y)|^2+2|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2.
\end{aligned}
$$

Hence

(3.66) the left-hand side of (3.65)

$$
\begin{aligned}
&\leqq 2\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}(|v(x+t,y)-v(x,y)|^2\\
&\qquad\qquad\qquad +|v(x,y+t)-v(x,y)|^2)dxdy\\
&\quad +2I_1+2I_2,
\end{aligned}
$$

where

$$
I_1=\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2dxdy,
$$

$$
I_2=\int_0^\infty t^{-(2s+1)}dt\int_{\boldsymbol{R}^2}|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2dxdy.
$$

First we estimate I_1. Since $\operatorname{supp} v\subset D_{r_1,r_2}$, we can write

$$
I_1=\int_{-r_2}^{r_2}dy\int_{-r_1}^{r_1}dx\int_0^\infty t^{-(2s+1)}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2dt.
$$

Fix any $y\in(-r_2,r_2)$. In view of (3.29) we let $x_1,\cdots,x_m$ be the points on $(-r_1,r_1)$ such that $x_1<x_2<\cdots<x_m$ and $y=\lambda(x_i)$, $i=1,\cdots,m$. We let $m=0$ if there exists no $x\in(-r_1,r_1)$ such that $y=\lambda(x)$. Then, setting $x_0=-r_1$ and $x_{m+1}=r_1$, we have

$$
\begin{aligned}
&\int_{-r_1}^{r_1}dx\int_0^\infty t^{-(2s+1)}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2dt\\
&=\sum_{i=0}^m\int_{x_i}^{x_{i+1}}dx\int_0^\infty t^{-(2s+1)}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2dt\\
&=\sum_{i=0}^m\int_{x_i}^{x_{i+1}}dx\int_{x_{i+1}-x}^\infty t^{-(2s+1)}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2dt,
\end{aligned}
$$

because $(y-\lambda(x))(y-\lambda(x+t))>0$ if $x_i<x<x_{i+1}$ and $x_i<x+t<x_{i+1}$, and so, by the

definition of χ, $|\chi(x+t,y)-\chi(x,y)|=0$ if $x_i<x<x_{i+1}$ and $x_i<x+t<x_{i+1}$. Therefore

$$\begin{aligned}
&\int_{-r_1}^{r_1} dx\int_0^\infty t^{-(2s+1)}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2 dt\\
&\leqq \sum_{i=0}^{m}\int_{x_i}^{x_{i+1}} dx\int_{x_{i+1}-x}^{\infty} t^{-(2s+1)}|v(x,y)|^2 dt\\
&= \sum_{i=0}^{m}\int_{x_i}^{x_{i+1}} \frac{1}{2s}(x_{i+1}-x)^{-2s}|v(x,y)|^2 dx\\
&= \sum_{i=0}^{m}\int_{0}^{x_{i+1}-x_i} \frac{1}{2s}|v(x_{i+1}-x,y)|^2 x^{-2s} dx\\
&\leqq \sum_{i=0}^{m}\int_{0}^{\infty} \frac{1}{2s}|v(x_{i+1}-x,y)|^2 x^{-2s} dx.
\end{aligned}$$

Since $v(x_{i+1}-x,y)\in C^\infty([0,\infty))$ and it has a bounded support as a function of x, we have by (3.64)

$$\begin{aligned}
&\int_0^\infty |v(x_{i+1}-x,y)|^2 x^{-2s} dx\\
&\leqq C\int_0^\infty t^{-(2s+1)} dt\int_0^\infty |v(x_{i+1}-x-t,y)-v(x_{i+1}-x,y)|^2 dx\\
&= C\int_0^\infty t^{-(2s+1)} dt\int_{-\infty}^{x_{i+1}-t} |v(x,y)-v(x+t,y)|^2 dx\\
&\leqq C\int_0^\infty t^{-(2s+1)} dt\int_{-\infty}^{\infty} |v(x+t,y)-v(x,y)|^2 dx.
\end{aligned}$$

Therefore, taking into account that $m\leqq M$ (constant) by (3.29), we have

$$\begin{aligned}
&\int_{-r_1}^{r_1} dx\int_0^\infty t^{-(2s+1)}|v(x,y)|^2|\chi(x+t,y)-\chi(x,y)|^2 dt\\
&\leqq C\int_0^\infty t^{-(2s+1)} dt\int_{-\infty}^{\infty} |v(x+t,y)-v(x,y)|^2 dx.
\end{aligned}$$

Hence

$$I_1\leqq C\int_0^\infty t^{-(2s+1)} dt\int_{R^2} |v(x+t,y)-v(x,y)|^2 dxdy. \tag{3.67}$$

Secondly we estimate I_2. Since $\operatorname{supp} v \subset D_{r_1,r_2}$, we can write

$$I_2=\int_{-r_1}^{r_1} dx\int_{-r_2}^{r_2} dy\int_0^{\infty} t^{-(2s+1)}|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2dt.$$

Fix any $x\in(-r_1,r_1)$. Then we have

$$\begin{aligned}&\int_{-r_2}^{r_2} dy\int_0^{\infty} t^{-(2s+1)}|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2dt\\ &=\int_{-r_2}^{\lambda(x)} dy\int_0^{\infty} t^{-(2s+1)}|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2dt,\end{aligned}$$

because, by the definition of χ, $\chi(x,y+t)=\chi(x,y)$ if $y>\lambda(x)$ and $t\geqq 0$. On the other hand, $\chi(x,y+t)=\chi(x,y)$ if $y\leqq\lambda(x)$ and $y+t\leqq\lambda(x)$. Therefore

$$\begin{aligned}&\int_{-r_2}^{r_2} dy\int_0^{\infty} t^{-(2s+1)}|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2dt\\ &=\int_{-r_2}^{\lambda(x)} dy\int_{\lambda(x)-y}^{\infty} t^{-(2s+1)}|v(x,y)|^2|\chi(x,y+t)-\chi(x,y)|^2dt\\ &\leqq\int_{-r_2}^{\lambda(x)} dy\int_{\lambda(x)-y}^{\infty} t^{-(2s+1)}|v(x,y)|^2dt\\ &=\int_{-r_2}^{\lambda(x)} |v(x,y)|^2\frac{1}{2s}(\lambda(x)-y)^{-2s}dy\\ &=\int_0^{\lambda(x)+r_2} \frac{1}{2s}|v(x,\lambda(x)-y)|^2y^{-2s}dy\\ &\leqq\int_0^{\infty} \frac{1}{2s}|v(x,\lambda(x)-y)|^2y^{-2s}dy.\end{aligned}$$

Since $v(x,\lambda(x)-y)\in C^{\infty}([0,\infty))$ and it has a bounded support as a function of y, we have by (3.64)

$$\begin{aligned}&\int_0^{\infty} |v(x,\lambda(x)-y)|^2y^{-2s}dy\\ &\leqq C\int_0^{\infty} t^{-(2s+1)}dt\int_0^{\infty} |v(x,\lambda(x)-y-t)-v(x,\lambda(x)-y)|^2dy\\ &=C\int_0^{\infty} t^{-(2s+1)}dt\int_{-\infty}^{\lambda(x)-t} |v(x,y)-v(x,y+t)|^2dy\\ &\leqq C\int_0^{\infty} t^{-(2s+1)}dt\int_{-\infty}^{\infty} |v(x,y+t)-v(x,y)|^2dy.\end{aligned}$$

Hence

$$I_2 \leqq C\int_0^\infty t^{-(2s+1)}dt\int_{R^2}|v(x,y+t)-v(x,y)|^2dxdy. \tag{3.68}$$

From (3.66)–(3.68) we obtain (3.65). Thus Lemma 3.4 has been proved. Q.E.D.

Lemma 3.3 corresponds to Lemma 3.4 of [9]. Then, as was shown in Lemma 3.5 of [9], it holds that

(3.69) for any $p \in D_{r_1,r_2}$ there exist positive constants C, δ such that $\|Q_0u\|^2_{-1/2} \leqq C\{\|Lu\|_0^2+\|u\|_0^2\}$, $u \in C_0^\infty(S(p,\delta))$.

For the definition of Q_0 see (3.2). On the other hand, as was shown in Lemma 3.6 of [9], we have by Lemma 3.4

$$|(u,v)^+|=|(u,v_0)| \leqq \|u\|^2_{-s}+\|v_0\|_s^2 \leqq C(\|u\|^2_{-s}+\|v\|_s^2),\ u,v \in C_0^\infty(D_{r_1,r_2}) \tag{3.70}$$

and so

$$|(u,v)^-|=|(u,v-v_0)| \leqq C(\|u\|^2_{-s}+\|v\|_s^2),\ u,v \in C_0^\infty(D_{r_1,r_2}). \tag{3.71}$$

(3.70) and (3.71) correspond to Lemma 3.6 of [9].

By using the same reasoning as in Lemma 3.7 and Lemma 3.8 of [9], we see from Lemma 3.1, (3.56)(i)(ii) and (3.69)–(3.71) that (3.56)(iii) holds. Thus L is hypoelliptic in D_{r_1,r_2}.

4. Sufficiency of (A), (B) and (C): general case

We shall prove that under the assumptions (H.1) and (H.2), the operator L defined by (1.1) is hypoelliptic in Ω if (A), (B) and (C) hold. To this end, it is sufficient to show that for any $p \in \Omega$ there exists an open neighborhood ω_p of p such that L is hypoelliptic in ω_p.

Let p be any point of Ω. As in the proof of necessity of (C), we can introduce an analytic coordinate transformation Φ from an open neighborhood $\tilde{\omega}_o$ of (0,0) in the uv-plane to an open neighborhood of p in the xy-plane such that $\Phi(0)=p$ and $\Phi_*(\partial/\partial u)=a\partial/\partial x+b\partial/\partial y$. L is transformed by Φ to the operator

$$\tilde{L}=\tilde{f}(u,v)\frac{\partial^2}{\partial u^2}+\tilde{c}(u,v)\frac{\partial}{\partial u}+\tilde{d}(u,v)\frac{\partial}{\partial v}+\tilde{g}(u,v),$$

where $\Phi_*(\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v)=c\partial/\partial x+d\partial/\partial y$, $\tilde{f}(u,v)=f(x,y)$ and $\tilde{g}(u,v)=g(x,y)$. Then, in virtue of (H.2), $\tilde{d}\neq 0$ in $\tilde{\omega}_o$. Furthermore, from (A), (B) and (C) it follows respectively that

$(A)_1$ $\tilde{f}_u(u,v)=0$ for any $(u,v)\in\tilde{\omega}_o$ such that $\tilde{f}(u,v)=0$;

$(B)_1$ $\tilde{f}$ does not vanish identically on any integral curve of $\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v$;

$(C)_1$ $\tilde{f}$ does not change sign from plus to minus along any integral curve of $\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v$.

We consider the operator $(1/\tilde{d})\tilde{L}=(\tilde{f}/\tilde{d})\partial^2/\partial u^2+(\tilde{c}/\tilde{d})\partial/\partial u+\partial/\partial v+\tilde{g}/\tilde{d}$. Then, by $(A)_1$

$(A)_2$ $(\tilde{f}/\tilde{d})_u(u,v)=0$ for any $(u,v)\in\tilde{\omega}_o$ such that $(\tilde{f}/\tilde{d})(u,v)=0$.

An integral curve of $(\tilde{c}/\tilde{d})\partial/\partial u+\partial/\partial v$ through a point of $\tilde{\omega}_o$ coincides with that of $\tilde{c}\partial/\partial u+\tilde{d}\partial/\partial v$ through the same point except for parametrization. If $\tilde{d}>0$ in $\tilde{\omega}_o$, both integral curves have the same directions and if $\tilde{d}<0$ in $\tilde{\omega}_o$, the opposite ones. Hence, from $(B)_1$ and $(C)_1$, it follows respectively that

$(B)_2$ $\tilde{f}/\tilde{d}$ does not vanish identically on any integral curve of $(\tilde{c}/\tilde{d})\partial/\partial u+\partial/\partial v$;

$(C)_2$ $\tilde{f}/\tilde{d}$ does not change sign from plus to minus along any integral curve of $(\tilde{c}/\tilde{d})\partial/\partial u+\partial/\partial v$.

Let $u=\psi_1(s,t)$ and $v=\psi_2(s,t)$ be the solutions of the initial value problem

$$\frac{du}{dt}=\frac{\tilde{c}(u,v)}{\tilde{d}(u,v)},\ \frac{dv}{dt}=1,\ u|_{t=0}=s,\ v|_{t=0}=0.$$

Then $\psi_2(s,t)=t$, and $\psi_1(s,t)$ is real analytic in $W_r=\{(s,t)|\ |s|<r,\ |t|<r\}$ for some $r>0$. Since $\partial(\psi_1,\psi_2)/\partial(s,t)|_{s=t=0}=1$, we can introduce the coordinate transformation

$$\Psi:\begin{cases}u=\psi_1(s,t)\\ v=\psi_2(s,t)=t\end{cases}$$

from W_r to an open neighborhood $\tilde{\omega}_o'\subset\tilde{\omega}_o$ of (0,0) by taking $r>0$ small. Then, the operators $\partial/\partial u$ and $(\tilde{c}/\tilde{d})\partial/\partial u+\partial/\partial v$ are transformed by Ψ to the operators $(\partial\psi_1/\partial s)^{-1}\partial/\partial s$ and $\partial/\partial t$ respectively, and so the operator $(1/\tilde{d})\tilde{L}$ to the operator

$$\tilde{\tilde{L}}=\tilde{\tilde{f}}(s,t)\left(\tilde{\tilde{a}}(s,t)\frac{\partial}{\partial s}\right)^2+\frac{\partial}{\partial t}+\tilde{\tilde{g}}(s,t)$$

where $\tilde{\tilde{f}}(s,t)=\tilde{f}(\psi_1(s,t),t)/\tilde{d}(\psi_1(s,t),t)$, $\tilde{\tilde{a}}(s,t)=(\partial\psi_1(s,t)/\partial s)^{-1}$, $\tilde{\tilde{g}}(s,t)=\tilde{g}(\psi_1(s,t),t)/\tilde{d}(\psi_1(s,t),t)$ and all of them are real analytic in W_r. Here we note that $\tilde{\tilde{a}}\neq 0$ in W_r.

From $(A)_2$, $(B)_2$ and $(C)_2$ it follows respectively that

$(A)_3$ $\tilde{\tilde{f}}_s(s,t)=0$ for any $(s,t)\in W_r$ such that $\tilde{\tilde{f}}(s,t)=0$;

$(B)_3$ for any fixed $s\in(-r,r)$, the function $t\to\tilde{\tilde{f}}(s,t)$ does not vanish identically on any sub-interval of $(-r,r)$;

(C)$_3$ for any fixed $s\in(-r,r)$, the function $t\to\tilde{\tilde{f}}(s,t)$ does not change sign from plus to minus when t increases on $(-r,r)$.

Hence, from the result of §3, $\tilde{\tilde{L}}$ is hypoelliptic in W_r. This implies that L is hypoelliptic in $\omega_p\equiv\Phi(\Psi(W_r))\ni p$.

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