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CLOSURES OF ORBITS OF *C* AND *C** ACTIONS

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1. Introduction and statement of results

Let G be a 1-dimensional complex linear algebraic group. In [7] Horrocks has shown that when G is the additive group $\boldsymbol{C}$ of complex numbers acting regularly on a normal (complex) projective variety, say X, or an algebraic variety X which is algebraically locally factorial, then the closure C of each G-orbit of is nonsingular. Moreover, Mabuchi [9] has shown that, if C touches a 1-codimensional component in X of the set X^G of fixed points, then C is nonsingular and it intersect X^G transversally for any complex manifold X.

On the other hand, when $G=\boldsymbol{C}^*$, the multiplicative group of complex numbers, Horrocks [7] showed that on a variety X as above the closure of each orbit is locally irreducible. If x is any point of X, this is equivalent to saying that either $x \in X^G$, or $0(x) \neq \infty(x)$, where $0(x)=\lim_{t \to 0} t(x)$ and $\infty(x)=\lim_{t \to \infty} t(x)$ for $t \in \boldsymbol{C}^*$. Furthermore, this fact was generalized as follows (cf. [3], [4]). Let X be a complex manifold on which $G=\boldsymbol{C}^*$ acts biholomorphically and meromorphically. Then a sequence of points $x_1, \cdots, x_s$, $s \geq 1$, of X is said to *generate a quasi-cycle* if $x_i \in X-X^G$ for each i, and $\infty(x_i)$ and $0(x_{i+1})$ are contained in one and the same connected component of X^G for $1 \leq i \leq s$, where i is counted modulo s, so that $s+1=1$ by definition. Then the result is: If X is a compact Kähler manifold, then there exists no sequence of points on X which generates a quasi-cycle.

Here, we say that a biholomorphic action of G on X is *meromorphic* if the morphism of complex spaces $G \times X \to X$ defining the action extends to a meromorphic map $\boldsymbol{P} \times X \to X$ with respect to the natural inclusion $G \hookrightarrow \boldsymbol{P}$, where $\boldsymbol{P}$ denotes the complex projective line. The purpose of this note is then to generalize the above results in the following two theorems:

Theorem 1.1. *Let X be a normal compact Kähler space* (*cf.* §4) *on which a 1-dimensional complex linear algebraic group G acts biholomorphically and meromorphically. Then the following hold*:
1) *Suppose that $G=\boldsymbol{C}^*$. Then there exists no sequence of points $x_1, \cdots, x_s \in X-X^G$ which generates a quasi-cycle.*
2) *Suppose that $G=\boldsymbol{C}$. Then the closure C of any orbit is nonsingular, and C intersect X^G transversally at a* (*single*) *point x, where the transversality means that*

$T_xX^G \cap T_xC = \{0\}$, T_x *being the Zariski tangent space at* x.

Theorem 1.2. *Let* X *be a compact complex manifold in the class* $\mathscr{C}$ *(cf.* §4*) on which a* 1*-dimensional complex linear algebraic group* G *acts biholomorphically and meromorphically. Then the following hold:*
1) *Suppose that* $G = \boldsymbol{C}^*$. *Let* $x_1, \cdots, x_s$ *be a sequence of points in* X *which generates a quasi-cycle. Let* C_i *be the closure of the orbit of* x_i. *Then the* 1*-cycle* $C = C_1 + \cdots + C_s$ *is* $\boldsymbol{Q}$*-homologous to zero in* X.
2) *Suppose that* $G = \boldsymbol{C}$. *Let* C *be a closure of some orbit such that either* C *is singular or* (C *is nonsingular but*) *the intersection of* C *with* X^G *is not transversal. Then* C *is* $\boldsymbol{Q}$*-homomologous to zero in* X.

Note that if a given $\boldsymbol{C}^*$-action admits no quasi-cycle in Theorem 1.2, then the associated Biyalynicki-Birula decomposition has a good property and the Hodge version of the Frankel equality holds true on X (cf. [3][4]). In [7] the main ingredient of the proof was the study of the induced G-action on the Picard group $\mathrm{Pic}\, Z = H^1(Z, \mathcal{O}_Z^*)$ of a certain G-variety Z. In the Kähler case, we consider instead the G-action on the real vector space $H^1(Z, \mathscr{P}_Z)$, where $\mathscr{P}_Z$ is the sheaf of germs of pluriharmonic functions on Z. This will be treated in §3. As preliminaries for this, in §2 we prove some general results on pluriharmonic functions on a complex space, which are of some independent interest. The proof of the theorems will then be given in §4; we also prove there a weaker form of Theorem 1.1 when X is noncompact. Finally, in §5 we give an example which shows that the conclusion of 2) of Theorem 1.1 is in general not true when X is a normal non-kählerian Moishezon surface.

2. Sheaf of pluriharmonic functions

Let X be a (reduced) complex space. We denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X. Then we have the following standard exact sequence of sheaves of real vector spaces on X;

$$0 \to \boldsymbol{R} \to \mathcal{O}_X \xrightarrow{\mathrm{Im}} \mathscr{P}_X \to 0$$

where $\boldsymbol{R}$ is considered as the constant sheaf and Im denotes taking the imaginary part. We consider the associated long exact sequence

$$(1) \qquad \to H^1(X, \boldsymbol{R}) \xrightarrow{\alpha} H^1(X, \mathcal{O}_X) \xrightarrow{\beta} H^1(X, \mathscr{P}_X) \xrightarrow{\delta} H^2(X, \boldsymbol{R}) \to$$

of real vector spaces. We first note the following result on pluriharmonic functions on a normal complex space.

Proposition 2.1. *Let X be a normal complex space and $r:\tilde{X}\to X$ a resolution of the singularity of X. Then we have a natural isomorphism $\mathscr{P}_X \cong r_*\mathscr{P}_{\tilde{X}}$.*

Proof. Fix any point x of X. Then $A:=r^{-1}(x)$ is compact and since X is normal it is connected. Take any pluriharmonic function h defined on a neighborhood U of A. Then it suffices to show that h descends to a pluriharmonic function defined in a neighborhood of x. Fix a point $o\in A$. Write $h=\mathrm{Re}f$ for some holomorphic function f in a neighborhood of o. Then f can be analytically continued along any closed path γ contained in U with base point o. It suffices to show that the result of any such analytic continuation, say f_1, coincides with the original branch f at o. In fact, then f would descend to a holomorphic function $\bar{f}$ on X by the normality of X, and h would be the pullback of a pluriharmonic function $\mathrm{Re}\bar{f}$.

Now restricting U if necessary, we may assume that $\pi_1(A)\cong\pi_1(U)$, and hence that γ as above is contained in A. But since A is connected and compact, h is constant on A; hence $f_1=f$ when restricted to A. On the other hand, the real parts of f and f_1 coincide, being equal to h, and hence f_1-f is a pure imaginary constant. Together with the equality $f_1=f$ on A, this implies that $f_1=f$ as a germ at o. □

The following general extension result of a pluriharmonic function seems noteworthy though it is not used in the sequel.

Corollary 2.2. *Let X be a normal complex space and A a nowhere dense analytic subset of X. Set $U=X-A$. Then any bounded pluriharmonic function u on U extends to a pluriharmonic function on the whole X.*

Proof. Let V be the set of somooth points of X. It is well-known that u extends across $A\cap V$. Hence, we may assume from the beginning that $U=V$. Take a resolution $r:\tilde{X}\to X$ of X whose exceptinal set E of r has at worst normal crossings. Then the pullback r^*h, which is bounded on $\tilde{U}:=r^{-1}(U)$, extends across E to a pluriharmonic function on $\tilde{X}$ by the smoothness of $\tilde{X}$. Then by the lemma above, h descends to a pluriharmonic function on X. □

In general for a (reduced) complex space Z we denote by $\mathscr{E}_Z$ the sheaf of germs of real valued C^∞ functions on Z; i.e., those functions on Z which are induced locally, with respect to a local holomorphic embedding of X into a domain D of $\boldsymbol{C}^n$, by a smooth function on the domain. Here, a real C^∞ function u on D is regared to be identically zero on Z if it is identically zero on the smooth part of Z. We have a natural inclusion $\mathscr{P}_Z\hookrightarrow\mathscr{E}_Z$. Denote by Φ_Z the cokernel of this inclusion so that we have an obvious exact sequence

(2) $$0 \to \mathscr{P}_Z \to \mathscr{E}_Z \xrightarrow{q} \Phi_Z \to 0$$

of sheaves of real vector spaces. Note that when Z is smooth, Φ_Z is identified with the sheaf of germs of real d-closed (1,1)-forms on Z with $q=\sqrt{-1}\partial\bar{\partial}$.

We shall prove the following result by using a theorem of Spallek [12].

Proposition 2.3. *Let X be a complex space and $n:\tilde{X}\to X$ the normalization of X. Then a C^∞ function u on X is pluriharmonic if and only if its pull-back $\tilde{u}:=n^*u$ to $\tilde{X}$ is pluriharmonic.*

Proof. The necessity is clear. We show the sufficiency. The problem is local. So we take any point $o\in X$ and identify X with the analytic germ it defines at o. Let $n^{-1}(o)=\{o_1,\cdots,o_m\}$. Accordingly, $\tilde{X}$ (as a germ along $n^{-1}(o)$) consists of m connected components $\tilde{X}_i$ with $o_i\in\tilde{X}_i$. By assumption we may write $\tilde{u}=\mathrm{Re}\tilde{f}$ for some holomorphic function $\tilde{f}$ on $\tilde{X}$ with $\tilde{f}(o_i)=\tilde{u}(o_i)=u(o)$ for any i, $1\le i\le m$. We claim that $\tilde{f}$ descends to a weakly holomorphic function on X.

First we show that $\tilde{f}$ descends to a well-defined continuous function on X. Let $R:=\tilde{X}\times_X\tilde{X}$ be the natural fibered product with the first and the second projections $p_i:R\to\tilde{X}$, $i=1,2$. We have to show that $\tilde{f}_1=\tilde{f}_2$, where $\tilde{f}_i=p_i^*\tilde{f}$. Set $\tilde{u}_i=p_i^*\tilde{u}$. For $g:=\tilde{f}_1-\tilde{f}_2$ we have

$$\mathrm{Re}\,g=\mathrm{Re}(\tilde{f}_1-\tilde{f}_2)=\tilde{u}_1-\tilde{u}_2=p^*u-p^*u=0,$$

where $p:R\to X$ is the natural projection. Hence, the holomorphic function g is locally constant on R. In fact, if we take any locally finite decomposition $R=\bigcup_\alpha R_\alpha$ of R into complex submanifolds R_α, then g is clealy locally constant on each R_α, and then so is it also on the whole R by the continuity. Now note that R consists of m^2 connected components R_{ij} with $(o_i,o_j)\in R_{ij}$. Then g is constant on each R_{ij} with the corresponding value equal to $g(o_i,o_j)=\tilde{f}(o_i)-\tilde{f}(o_j)=0$. Therefore, g is identically zero on the whole R as desired. Hence, we can find a contimuous function f on X such that $n^*f=\tilde{f}$. Then we clearly have $\mathrm{Re}f=u$. Moreover, it is clear that f is holomorphic at smooth points of X. Thus, f is a weakly holomorphic function on X whose real part is a C^∞ function on X. Then by a theorem of Spallek [12, Satz 4.2] f is holomorphic on X, and hence u is pluriharmonic. □

Corollary 2.4. *Let X be a germ of complex spaces. Then a C^∞ function u on X is pluriharmonic if so is the restriction of u to each irreducible component X_i of X.*

Proof. Let $\hat{X}$ be the disjoint union of X_i and $m:\hat{X}\to X$ the natural morphism. By assumption m^*u is pluriharmonic on X. Let $n:\tilde{X}\to X$ be the normalization of X. Then we have a natural morphism $q:\tilde{X}\to\hat{X}$ such that

$n=mq$. Thus, $n^*u=q^*m^*u$, and hence by Proposition 2.3 u also, is pluriharmonic. □

Let $p:Z\to X$ be a morphism of complex spaces and $p_\Phi^*:H^0(X,\Phi_X)\to H^0(Z,\Phi_Z)$ a natural linear map. The results of the above proposition and corollary can then be stated in terms of p_Φ^* as follows.

Lemma 2.5. *Suppose that either p is the normalization of X or that Z is the disjoint union of the irreducible components of X and p is the natural map. Then the map p_Φ^* is injective.*

3. Action on $H^1(X,\mathscr{P}_X)$

3.1. Case of normal complex spaces

Suppose that a connected complex Lie group G acts biholomorphically on a compact complex space X. Then it acts naturally on the exact sequence (1) by real linear transformations, which is trivial on the terms $H^i(X,\boldsymbol{R})$ because G is connected. We are interested in the action of G on $H^1(X,\mathscr{P}_X)$ when G is an algebraic group acting meromorphically on X and X is a compact complex space in $\mathscr{C}$, i.e., is bimeromorphic to a compact Kähler manifold.

Lemma 3.1. *Let X be a normal compact complex space in $\mathscr{C}$ on which a connected complex linear algebraic group G acts biholomorphically and meromorphically. Then the induced action on the real vector space $H^1(X,\mathscr{P}_X)$ is trivial.*

Proof. When X is nonsingular, by the Hodge theory we know that $H^i(X,\boldsymbol{R})\to H^i(X,\mathcal{O}_X)$ is necessarily surjective, and hence $H^i(X,\mathscr{P}_X)$ is injectively mapped into $H^{i+1}(X,\boldsymbol{R})$. Thus by the above remark the induced actions on $H^i(X,\mathscr{P}_X)$ are trivial for all i. In the general case we take a resolution $r:\tilde{X}\to X$ of X and consider the following commutative diagram of real vector spaces

$$\begin{array}{ccc} H^1(X,\mathscr{P}_X) & \xrightarrow{\delta_X} & H^2(X,\boldsymbol{R}) \\ {\scriptstyle r_{\mathscr{P}}^*}\downarrow & & \downarrow{\scriptstyle r_{\boldsymbol{R}}^*} \\ H^1(X,\mathscr{P}_{\tilde{X}}) & \xrightarrow{\delta_{\tilde{X}}} & H^2(\tilde{X},\boldsymbol{R}). \end{array}$$

By Proposition 2.1 we have $r_*\mathscr{P}_{\tilde{X}}=\mathscr{P}_X$, and hence by the Leray spectral sequence $r_{\mathscr{P}}^*$ is injective. Hence, the diagram shows that δ_X is injective as well as $\delta_{\tilde{X}}$. Thus, the action on $H^1(X,\mathscr{P}_X)$ is again trivial. □

REMARK 3.1. We can prove the injectivity of δ_X also as follows. According to [5, Lemma 8] $\mathrm{Pic}_0X\to\mathrm{Pic}_0\tilde{X}$ is a closed embedding, which in turn implies that

$\mathrm{Pic}_0 X$ is a complex torus, or equivalenty, the natural homomorphism $\alpha: H^1(\tilde{X}, \boldsymbol{R}) \to H^1(X, \mathcal{O}_X)$ is surjective. Here $\mathrm{Pic}_0 X$ denotes the identity component of Pic X. Thus, the exact sequence (1) shows that δ_X is injective. Conversely, from the injectivity of δ_X in the proof of the lemma follows the surjectivity of α. This gives an alternative direct proof of the compactness of $\mathrm{Pic}_0 X$ for a compact complex space in $\mathscr{C}$.

3.2. Case of certain non-normal complex spaces

Next we consider some non-normal spaces. Namely, let B be one of the following four types of compact connected nonnormal complex spaces.

Case 1

a) A rational curve with one node

b) A weakly normal compact connected complex space with irreducible components B_i, $1 \leq i \leq m$, such that the following conditions are satisfied:

i) Each B_i is nonsingular and

ii) for each $1 \leq i \leq m$, B_i and B_{i+1} intersect transversally at a unique point p_i, where i is considered modulo m, and $B_i \cap B_k = \emptyset$ if $|i-k| > 1$.

Case2

a) A rational curve with an ordinary cusp

b) A union of a nonsingular rational curve C and a compact connected complex space Y such that C and Y have a unique point p in common and that $C \cap Y$ is defined in C by m^2, where m is the maximal ideal of $\mathcal{O}_{C,p}$.

Suppose that a 1-dimensional connected commutative algebraic group G acts nontrivially on B biholomorphically and meromorphically. We assume that G is the multiplicative group $\boldsymbol{C}^*$ in Case 1 and is the additive group $\boldsymbol{C}$ in Case 2. Furthermore, we make the following additional assumptions:

In Case 1, b): Let J be the set of indices j such that the action is nontrivial on B_j. Then B_j is a nonsingular rational curve for any $j \in J$. We assume further that for $j \in J$ and $x \in B_j - \{p_{i-1}, p_i\}$, we have $0(x) = p_i$ and $\infty(x) = p_{i+1}$, where $p_0 = p_m$.

In Case 2, b): We assume that the action of G on Y is trivial.

In these situations we consider the induced action of G on the real vector space $H^1(B, \mathscr{P}_B)$ and on the Picard group Pic $B = H^1(B, \mathcal{O}_B^*)$ of B. Since the possible intersections of two irreducible components are of *zero*-dimension, we have a natural isomorphism $\oplus_i H^2(B_i, \boldsymbol{R}) \cong H^2(B, \boldsymbol{R})$, where B_i in general denote the irreducible components of B.

In the above four cases we define the notion of a degree of an element η of $H^1(B, \mathscr{P}_B)$ as a real number as follows. Define a subspace B' of B by

$$B' = \begin{cases} B & \text{in Cases 1,a) and 2,a)} \\ \bigcup_{j \in J} B_j & \text{in Case 1,b)} \\ C & \text{in Case 2,b).} \end{cases} \tag{3}$$

Then the *degree* of η is the image of the composite map (cf.(1))

$$(4)\qquad H^1(B,\mathscr{P}_B)\to H^1(B',\mathscr{P}_{B'})\overset{\delta}{\to} H^2(B',\boldsymbol{R})\overset{\mathrm{tr}}{\to}\boldsymbol{R},$$

where the first arrow is the restriction map, δ is as in (1) and finally tr is the natural isomorphism in Case 1,a) and Case 2, and in Case 1,b) it is given by $(a_j)\to\Sigma_j a_j$ with $(a_j)\in\oplus_{j\in J}H^2(B_j,\boldsymbol{R})\cong\oplus_j\boldsymbol{R}$. The degree of an element L of Pic B is then defined to be that of its refined chern class $\hat{c}(L)\in H^1(B,\mathscr{P}_B)$ (cf.(13) below).

Proposition 3.2. *Let the notations and assumptions be as above. In either of the above situations the following hold.*
1) *Let η be any element of $H^1(B,\mathscr{P}_B)$. Then η is G-invariant with respect to the natural action of G on $H^1(B,\mathscr{P}_B)$ if and only if the degree of η is zero.*
2) *Let L be any holomorphic line bundle on B. Then $L\in H^1(B,\mathcal{O}_B^*)$ is G-invariant if and only if its degree* $\deg L=0$.

Note that the assertion 2) in Cases 1,a) and 2,a) is due to Horrocks [7]. For the proof of the proposition we first compute the effect of standard G-action on the Fubini-Study form of a complex projective line $\boldsymbol{P}$. Write $\boldsymbol{P}=\boldsymbol{C}(z)\cup\{\infty\}$ once and for all, and consider the action of $G=\boldsymbol{C}^*=\boldsymbol{C}^*(t)$ (resp. $\boldsymbol{C}=\boldsymbol{C}(t)$) of the form

$$(5)\qquad z\to tz\qquad(\text{resp. } z\to z+t)\ \text{on } \boldsymbol{P}.$$

We consider the real (1,1)-form $\tau=\sqrt{-1}\partial\bar{\partial}\log(1+|z|^2)$ on $\boldsymbol{P}$. We write $w=1/z$. Then the following is a simple verification.

Lemma 3.3. *For any $t\in G$ the form $\lambda=t^*\tau-\tau$ is written in the form $\lambda=\sqrt{-1}\partial\bar{\partial}h$, where h is a C^∞ function on $\boldsymbol{P}$ defined by the formula;*

$$(6)\qquad h(z)=\log\{(1+|tz|^2)/(1+|z|^2)\}=\log\{(|t|^2+|w|^2)/(1+|w|^2)\},$$

when $G=\boldsymbol{C}^*$, and

$$(7)\qquad h(z)=\log\{(1+|z+t|^2)/(1+|z|^2)\}=\log\{(|w|^2+|1+tw|^2)/(1+|w|^2)\},$$

when $G=\boldsymbol{C}$.

Proof of Proposition 3.2. Define $\tilde{B}$ to be the normalization of B in Cases 1,a),b) and 2,a), and to be the disjoint union $B_1=C$ and $B_2=Y$ in Case 2,b). Let

$$(8)\qquad n:\tilde{B}\to B$$

be the natural morphism. The G-action on B naturally lifts to a biholomorphic and meromorphic G-action on $\tilde{B}$ making n a G-morphism. We start with the assertion 1).

Let K be the kernel of $n_{\mathscr{P}}^*: H^1(B,\mathscr{P}_B) \to H^1(\tilde{B},\mathscr{P}_{\tilde{B}})$. Take any element g of G and set $\xi = g^*\eta - \eta$. We claim that the image $\tilde{\xi}$ of ξ in $H^1(\tilde{B},\mathscr{P}_{\tilde{B}})$ vanishes. Let $\tilde{\eta} = (\eta_i)$ be the image of η in $H^1(\tilde{B},\mathscr{P}_{\tilde{B}}) = \oplus_i H^1(\tilde{B}_i,\mathscr{P}_{\tilde{B}_i})$ with $\tilde{\eta}_i \in H^1(\tilde{B}_i,\mathscr{P}_{\tilde{B}_i})$, where $\tilde{B}_i$ are connected components of $\tilde{B}$. (Similar notations will be used in the sequel). Then we have $\tilde{\xi} = (\tilde{\xi}_i)$ with $\tilde{\xi}_i = g^*\tilde{\eta}_i - \tilde{\eta}_i$. But $g^*\tilde{\eta}_i - \tilde{\eta}_i = 0$ for all i; for if $\tilde{B}_i$ is not a nonsingular rational curve this follows from our assumption in the Case 1,b) and Case 2,b), and if $\tilde{B}_i$ is a nonsingular rational curve, this is clear (cf. Lemma 3.1). Thus $\tilde{\xi} = 0$, and ξ is contained in K.

Now since $\mathscr{E}_Z$ is a fine sheaf, we have the following commutative diagram of exact sequences of cohomology groups associated to the sheaf sequence (2);

$$\begin{array}{ccccccccc}
H^0(B,\mathscr{P}_B) & \to & H^0(B,\mathscr{E}_B) & \to & H^0(B,\Phi_B) & \to & H^1(B,\mathscr{P}_B) & \to & 0 \\
 & & \downarrow n_{\mathscr{E}}^* & & \downarrow n_{\Phi}^* & & \downarrow n_{\mathscr{P}}^* & & \\
H^0(\tilde{B},\mathscr{P}_{\tilde{B}}) & \xrightarrow{d} & H^0(\tilde{B},\mathscr{E}_{\tilde{B}}) & \to & H^0(\tilde{B},\Phi_{\tilde{B}}) & \to & H^1(\tilde{B},\mathscr{P}_{\tilde{B}}) & \to & 0 \\
\| & & \| & & \| & & \| & & \\
\oplus_i H^0(\tilde{B}_i,\mathscr{P}_{\tilde{B}_i}) & \to & \oplus_i H^0(\tilde{B}_i,\mathscr{E}_{\tilde{B}_i}) & \to & \oplus_i H^0(\tilde{B}_i,\Phi_{\tilde{B}_i}) & \to & \oplus_i H^1(\tilde{B}_i,\mathscr{P}_{\tilde{B}_i}) & \to & 0.
\end{array}$$

Since n_{Φ}^* is injective by Lemma 2.5, we have the naturally induced injection

$$c: K \quad Q := H^0(\tilde{B},\mathscr{E}_{\tilde{B}})/(\mathrm{Im}\, d + \mathrm{Im}\, n_{\mathscr{E}}^*),$$

where Im denotes the image. Note that the given action of G induces a natural G-action on the above commutative diagram. It then suffices to show the following:

Claim. $c(\xi)$ is zero in Q for any element g of G if and only if the degree of η is zero.

First we describe $c(\xi)$. For this purpose we lift η to an element $v \in H^0(B,\Phi_B)$ in the above exact sequence. Then ξ is represented by $\omega = g^*v - v$. Let $\tilde{\omega} = (\tilde{\omega}_i)$ be the image of ω in $H^0(\tilde{B},\Phi_{\tilde{B}})$. Then $\tilde{\omega}$ is mapped to zero in $H^1(\tilde{B},\mathscr{P}_{\tilde{B}})$, and hence we can find an element

$$\tilde{f} = (\tilde{f}_i) = H^0(\tilde{B},\mathscr{E}_{\tilde{B}})$$

which is mapped to $\tilde{\omega}$. Then $c(\xi)$ is nothing but the image q of $\tilde{f}$ in Q by the natural projection.

Thus, what we have to show is that q is zero for any element of g if and only if the degree of η is zero. For this purpose we shall specify appropriate choices of $\tilde{f}_i$. First we note that for $i \notin J$ (with J defined in all cases as in Case 1,b)) we have $\tilde{\omega}_i = g^*\tilde{v}_i - \tilde{v}_i = 0$ since G acts on $\tilde{B}_i$ trivially, so that we may take $\tilde{f}_i = 0$, where $n_{\Phi}^* v = (\tilde{v}_i)$;

(9) $$\tilde{f}_i = 0, \quad j \notin J.$$

We next assume that $i \in J$; in particular, we may identify $\tilde{B}_i$ with a complex projective line $\boldsymbol{P} = \boldsymbol{C}(z) \cup \{\infty\}$ and may assume that the action of $G = \boldsymbol{C}^*$ (resp. $\boldsymbol{C}$) on $\tilde{B}_i$ is of the form (5) with respect to this identification. We denote an element

g of G by the corresponding complex number t. Now after adding a suitable element in the image of $H^0(\tilde{B}_i,\mathscr{E}_{\tilde{B}_i})$ we may assume that $\tilde{v}_i$ is of the form $(a_i/2\pi)\tau$, where a_i is the degree of the restriction of η to $\tilde{B}_i$ $(=\int_{\tilde{B}_i}\tilde{v}_i)$ and τ is as in Lemma 3.3. (Recall that $\Phi_{\tilde{B}_i}$ is identified with the sheaf of germs of real d-closed (1,1)-forms on $\tilde{B}_i$ with the quotient map $\mathscr{E}_{\tilde{B}_i}\to\Phi_{\tilde{B}_i}$ given by $\tilde{f}_i\to\sqrt{-1}\partial\bar{\partial}\tilde{f}_i$. Then by Lemma 3.3 we can take

$$\tilde{f}_i=(a_i/2\pi)h,$$

where h is the C^∞ function on $\tilde{B}_i=\boldsymbol{P}$ given in (6).

We shall now identify the image $q\in Q$ of $\tilde{f}$ for the above choices. Consider the diagram

$$(10)\qquad \begin{array}{ccccc} & & & & H^0(B,\mathscr{E}_B)\\ & & & & \downarrow n_{\mathscr{E}}^*\\ \oplus_i\boldsymbol{R} & \cong & \oplus_iH^0(\tilde{B}_i,\mathscr{P}_{\tilde{B}_i}) & \xrightarrow{d} & \oplus_iH^0(\tilde{B}_i,\mathscr{E}_{\tilde{B}_i})\\ & & & & \downarrow p\\ & & & & N \end{array}$$

where N is the cokernel of $n_{\mathscr{E}}^*$ with p the natural projection. We then describe p in each of the four cases. We begin with:

Case 1,b)

In this case we may identify B_i with $\tilde{B}_i$. We have $N\cong\oplus_i\boldsymbol{R}^{(i)}$ with a copy $\boldsymbol{R}^{(i)}$ of $\boldsymbol{R}$ one for each $p_i, 1\le i\le m$, and for any $\hat{f}=(\hat{f}_i)\in\oplus_iH^0(\tilde{B}_i,\mathscr{E}_{\tilde{B}_i})$, $p(\hat{f})=(\hat{f}_i(p_i)-\hat{f}_{i+1}(p_i))$. Thus, $pd(r)$ for $r=(r_i)\in\oplus_iH^0(\tilde{B}_i,\mathscr{P}_{\tilde{B}_i})\to\oplus_i\boldsymbol{R}$ is given by $pd(r)=((r_i-r_{i+1}))$. Hence, we have $Q\cong\boldsymbol{R}$ and the induced map $\tilde{p}:\oplus_iH^0(\tilde{B}_i,\mathscr{E}_{\tilde{B}_i})\to\boldsymbol{R}$ is identified with the map $\tilde{p}((\hat{f}_i))=\Sigma_{1\le i\le m}(-\hat{f}_i(p_{i-1})+\hat{f}_i(p_i))$. Thus, for $\hat{f}=\tilde{f}$ with $\tilde{f}$ above we have

$$(11)\qquad \tilde{p}(\tilde{f})=\sum_{i\in J}(-\tilde{f}_i(p_{i-1})+\tilde{f}_i(p_i))=1/2\pi\left(\sum_{i\in J}a_i\right)(-h(0)+h(\infty))$$

$$=1/2\pi\left(\sum_{i\in J}a_i\right)\log|t|^2=(\deg v)/2\pi\log|t|^2.$$

Thus as long as $|t|\neq 1$, $p(\tilde{f})=0$ if and only if $\deg v=\deg\eta=0$.

Case 1,a)

In this case $m=1$, and we omit the index i. We have $N\cong\boldsymbol{R}$ corresponding to the node of B and $p(\tilde{f})=(a/2\pi)(-h(0)+h(\infty))=(a/2\pi)\log|t|^2$. Since the image of $H^0(\tilde{B},\mathscr{P}_{\tilde{B}})\cong\boldsymbol{R}$ in N is zero, we have $p(\tilde{f})=c(\xi)=0$ if and only if the degree a of η is zero provided that $|t|\neq 1$.

Next we consider Case 2. We start with

Case 2,a)

Again $m=1$ and we omit the suffix i. The image of d is contained in that of

$n_{\mathscr{E}}^{*}$, and hence, $N=Q$, which is nonzero by our assumption. By Lemma 3.3 we get that

$$c(\xi)=p(\tilde{f})\equiv ah=a(2\mathrm{Re}(tw)+\cdots) \qquad \text{mod the image of } n_{\mathscr{E}}^{*}, \tag{12}$$

where h is given by (7) and $\cdots$ denotes the terms which are of order ≥ 2 in w and $\bar{w}$, the complex conjugate of w. Since this contains a nonzero linear term for $t\neq 0$, it follows that $p(\tilde{f})=c(\xi)=0$ if and only if the degree a of η is zero.

It remains to treat the final:

Case 2,b)

Recall that $B=C\cup Y$ and we set $B_1=C$ and $B_2=Y$. We have a direct sum decomposition $H^0(Z,\mathscr{E}_Z)=\boldsymbol{R}\oplus H^0(Z,m_Z\mathscr{E}_Z)$, where m_Z denotes the maximal ideal of $\mathscr{E}_{Z,p}$ for $Z=B,Y$ or C. In the diagram (10) applied to this case, the image of d is exactly $\boldsymbol{R}\oplus\boldsymbol{R}$. Hence, we have an exact sequence

$$H^0(B,m_B\mathscr{E}_B)\overset{n_{\mathscr{E}}^{*}}{\rightarrow} H^0(C,m_C\mathscr{E}_C)\oplus H^0(Y,m_Y\mathscr{E}_Y)\rightarrow Q\rightarrow 0$$

On the other hand, we have a natural surjection

$$b: H^0(C,m_C\mathscr{E}_C)\oplus H^0(Y,m_Y\mathscr{E}_Y)\rightarrow V$$

where V is the vector space $V=V_C\oplus V_Y$ with $V_Z=m_Z/m_Z^2$, $Z=C,Y$. We show that for our $\tilde{f}$ the image $q_V:=b(\tilde{f}-\tilde{f}(p))$ is not in the image of $H^0(B,m_B\mathscr{E}_B)$ (and hence q is nonzero), if the degree of η is nonzero. First note that q_V is of the form $q_V=(2a\mathrm{Re}(tw),0)$, where a is the degree of η in the sense of the proposition (cf. (12)). Now suppose that $b(\tilde{f}-\tilde{f}(p))$ comes from some element f of $H^0(B,m\mathscr{E}_B)$. Then since f is identically zero on Y as well as $\tilde{f}$ and since T_xC is contained in T_xY by our assumption, we must have that the first component of its image in V must also vanish identically. Hence $a\mathrm{Re}(tw)=0$ for any w. This is impossible unless $a=0$, and the assertion is proved.

Conversely, suppose that $a=0$. Then in $f=(f_1,f_2)$ we have $f_1=f_C=0$ by (12) and $f_2=f_Y=0$ by (9). Hence, $c(\xi)=q=0$.

Proof of the assertion 2). From the short exact sequence

$$0\rightarrow S^1\rightarrow \mathscr{O}_B^{*}\rightarrow \mathscr{P}_B\rightarrow 0$$

we get the associated long exact sequence

$$\rightarrow H^1(B,S^1)\rightarrow H^1(B,\mathscr{O}_B^{*})\overset{\hat{c}}{\rightarrow} H^1(B,\mathscr{P}_B)\rightarrow H^2(B,S^1) \tag{13}$$

on which the group G naturally acts. Then by what we have proved above, it turns out that if $\deg L=\deg\hat{c}(L)\neq 0$, then L is not G-invariant. The necessity follows.

We show the sufficiency. We have only to consider Cases 1,b) and 2,b)(cf. [7]). In Case 1,b) we shall give a direct proof of both directions. Define H to be the algebraic torus $\Pi_{1\leq i\leq m}\boldsymbol{C}^{*(i)}$, where $\boldsymbol{C}^{*(i)}$ are copies of $\boldsymbol{C}^{*}$. Then, from the exact sheaf sequence

$$0 \to \mathcal{O}_B^* \to n_* \mathcal{O}_{\tilde{B}}^* \to \mathcal{R} \to 0$$

where $\mathcal{R}$ is the quotient sheaf, we get the exact sequence of connected algebraic groups

$$(14) \qquad 0 \to C^* \to H \xrightarrow{u} H \xrightarrow{v} \operatorname{Pic} B \to \operatorname{Pic} \tilde{B} \to 0$$

with the induced C^*-action. Here the homomorphism u is defined by $u((s_i)) = ((s_i \cdot s_{i+1}^{-1}))$, $s_i \in C^{*(i)}$; thus an element $(t_i) \in H$ is in the image of u if and only if $\Pi_i t_i = 1$. We also see that $\operatorname{Pic} B$ is a principal C^*-bundle over $\operatorname{Pic} \tilde{B}$.

In order to detect the C^* action on $\operatorname{Pic} B$ we represent an element L of $\operatorname{Pic} B$ by a pair $\{(L_i), (h_i)\}$, where $L_i = L|_{B_i} \in \operatorname{Pic} B_i$ and h_i are the canonical isomorphisms $h_i : L_i|p_i \cong L_{i+1}|p_i$, $1 \le i \le m$. When we describe the C^* action on such a representative we may assume that for $i \in J$ $L_i = N^{k_i}$, where k_i is the degree of L_i on B_i. Here, N_k is the line bundle defined by

$$N_k := (U_0 \times C(\zeta_0)) \cup (U_1 \times C(\zeta_1)), \quad \text{with } \zeta_0 = w^{-k}\zeta_1 \text{ over } U_0 \cap U_1,$$

where $U_0 = C(z)$ and $U_1 = C(w)$. For $t \in C^*$ the line bundle $t^* N_k$ has the new transition function $t^{-k} w^{-k}$ instead of w^{-k}. Then the formula $\zeta_0 \to \zeta_0$, $\zeta_1 \to t^{-k}\zeta_1$ gives an isomorphism $t^* N_k \cong N_k$.

Now for any $t \in C^*$ we are interested in the map $f = f_t : \operatorname{Pic} B \to \operatorname{Pic} B$ defined by $f(L) = t^* L \otimes L^{-1}$. From the above and the fact that C^* acts trivially on B_i for $i \notin J$ we infer that $t^*(L)$ is represented by the pair $\{(L_i, h_i t^{k_i})\}$, and then, $f(L)$ by the pair $\{(1_{(i)}), (t^{k_i})\}$, which is nothing but the image by v in (14) of $(t^{k_i}) \in H$, where we have put $k_i = 0$ if $i \notin J$, and $1_{(i)}$ is the trivial line bundle on B_i. Hence, by what we have noted above, $f(L)$ is trivial (for all t) if and only if $\Pi_i(t^{k_i}) = 1$, or equivalently, $\deg L = \Sigma k_i = 0$.

Finally, consider Case 2,b). Since $\deg L = 0$, $L|C$ is trivial by the definition of degree in this case. Then $f(L)$ defined as above is trivial when restricted to both C and to Y. Since $\operatorname{Pic} B \xrightarrow{\sim} \operatorname{Pic} \tilde{B}$ in this case as follows from (14), we get that $f(L)$ is trivial on B, which proves that L is G-invariant.

In the above arguments we have used the C^∞-objects on singular complex spaces, but we could have also used the corresponding real analytic objects as in [2].

4. Proof of theorems

First we recall some definitions: A complex space X is said to be a *Kähler space* if there exist an open covering $\{U_\mu\}$ of X and a system of C^∞ strictly plurisubharmonic functions $\{u_\mu\}$ such that each u_μ is defined on U_μ and $\varphi_{\mu\nu} := u_\mu - u_\nu$ is pluriharmonic on $U_\mu \cap U_\nu$. In this case, $\omega := \sqrt{-1}\partial\bar{\partial} u_\mu$ on U_μ defines a global real positive d-closed (1,1)-form on X, called a *Kähler form* on X. Moreover, the class $[\omega]$ in $H^1(X,\mathscr{P}_X)$ (well-)defined by the cocycle $\{\varphi_{\mu\nu}\}$ is called the (refined) *Kähler class* of ω. On the other hand, a compact complex manifold is said to be *in the class* $\mathscr{C}$ if it is bimeromorphic to a compact Kähler manifold.

In **4.1** we prove the theorems in the introduction and in **4.2** discuss the noncompact case also.

4.1 Compact case

Proof of Theorem 1.1. Suppose that the statement of the theorem is not true. Then we define a G-invariant complex subspace $\bar{B}$ of X as follows. First consider the case $G=\boldsymbol{C}^*$ and assume that we have a sequence of points $x_1,\cdots,x_s$, $s\geq 1$, generating a quasi-cycle. Let C_i be the closure of Gx_i and F_i the connected component of X^G which contains both $\infty(x_i)$ and $0(x_{i+1})$, $1\leq i\leq s$, $s+1=1$. Then $\bar{B}$ is by definition the union of complex subspaces $\bar{B}_i$ which is defined by: $\bar{B}_1=C_1$, $\bar{B}_2=F_2$, $\bar{B}_3=C_2,\cdots,\bar{B}_{2s-1}=C_s$, $\bar{B}_{2s}=F_s$.

Suppose next that $G=\boldsymbol{C}$ and assume that there exists a closure C of an orbit such that either C is not smooth or C does not intersect X^G transversally. We define, in the first case, $\bar{B}=C$ and in the second case $\bar{B}=C\cup X^G$.

We then define a complex space B in respective cases as follows. First in the case $G=\boldsymbol{C}^*$ we define B to be the weak normalization of $\bar{B}$. In the first case for $G=\boldsymbol{C}$, we define B to be an irreducible rational curve with an ordinary cusp as in Case 2,a) in **3.2**. In the second case for $G=\boldsymbol{C}$ let m be the maximal ideal of $\mathcal{O}_{C,p}$, where $p=x$ as in Theorem 1.1, and denote by E the subspace of C defined by the ideal m^2. By our assumption we have a natural embedding $k:E \quad Y$. Then we define B to be the pushout $C\cup_E Y$ with respect to the natural embeddings $\iota:E \quad C$ and k (cf. [8,(1.8)]). In all the cases B is a meromorphic G-space and we have a natural G-morphism $B\to\bar{B}$. Composed by the inclusion $\bar{B} \quad X$ we get a G-morphism $h:B\to X$.

Clearly, B is G-equivariantly isomorphic to one of the G-spaces defined at the beginning of **3.2**. Thus by Lemma 3.1 and Proposition 3.2 the image of the induced map $h^*:H^1(X,\mathscr{P}_X)\to H^1(B,\mathscr{P}_B)$ consists of elements of degree zero in the sense defined before that proposition. On the other hand, the pullback of any Kähler class on X to B is necessarily of positive degree, which is a contradiction.

Proof of Theorem 1.2. As in the above proof our assumption implies that we can find a G-morphism $h:B\to X$ such that the image of the induced map $h^*:H^1(X,\mathscr{P}_X)\to H^1(B,\mathscr{P}_B)$ consists of elements of degree zero. On the other hand, since X is a manifold in $\mathscr{C}$, we have

$$H^1(X,\mathscr{P}_X)=H^{1,1}(X)_{\boldsymbol{R}}:=H^2(X,\boldsymbol{R})\cap H^{1,1}(X), \tag{15}$$

where $H^{1,1}(X)$ is the Hodge (1,1)-component. Let the subspace B' of B be defined by (3) before Proposition 3.2 with the induced map $h': B' \to X$. Then the image of $h'^*: H^2(X,\boldsymbol{R}) \to H^2(B',\boldsymbol{R})$ coincides with $h^{*\prime}(H^{1,1}(X)_{\boldsymbol{R}})$ since B' is of dimension 1. Then by the definition of the degree this image is contained in the kernel of $\mathrm{tr}: H^2(B',\boldsymbol{R}) \to \boldsymbol{R}$ (cf. (4)). This implies then the conclusion of the theorem by the definition of the map tr.

REMARK 4.1. 1) As the above proof shows Theorem 1.2 holds true for any compact complex manifold X for which the Hodge decomposition theorem holds for $H^i(X,\boldsymbol{R})$, $i=1,2$.
2) We can consider X^G also in the category of general complex spaces which are not necessarily reduced. It is in fact more natural to do so in the case $G=\boldsymbol{C}$. Then we may ask if Theorems 1.1 and 1.2 are still true when X^G is given with this refined structure.

4.2 Noncompact case

In this subsection we prove a weaker form of Theorem 1.1 when X is not necessarily compact. We say that a sequence of points $x_1,\cdots,x_s$, $s \geq 1$, of X *generates a cycle* if $x_i \in X - X^G$ for each i and $\infty(x_i) = 0(x_{i+1})$ for $1 \leq i \leq s$, where $s+1=1$. Then we prove

Theorem 4.1. *Let X be a connected (not necessarily compact) Kähler manifold on which a 1-dimensional complex linear algebraic group G acts biholomorphically and meromorphically. Then the following hold:*
1) Suppose that $G=\boldsymbol{C}^*$. Then there exists no sequence of points $x_1,\cdots,x_s$ of X which generates a cycle.
2) Suppose that $G=\boldsymbol{C}$. Then the closure C of any orbit is nonsingular.

Recall the complex space B considered in **3.2**. According to the situation of the above theorem, we consider only Cases 1,a),b) and Case 2,a) there: moreover, in Case 1,b) we assume that all the irreducible components B_i of B are a nonsingular rational curve on which G acts effectively, namely we assume $J=\{1,\cdots,m\}$ there. Then we shall not distingush Cases 1,a) and 1,b) and call it simply Case 1. We also call Case 2,a) simply Case 2 in what follows. In particular, we set $m=1$ in Case 2.

Let D be a compact Riemann surface. More precisely, we take $D=\boldsymbol{P}$ when $G=\boldsymbol{C}^*$ (Case 1), and take D to be of genus ≥ 1 when $G=\boldsymbol{C}$ (Case 2). Let $\xi: P \to D$ be a nontrivial principal G-bundle on D. Let $p: Y \to D$ be the fiber bundle with typical fiber B associated to ξ and to the natural action of G on B. By our choice of D we have a canonical exact sequence

$$0 \to H^2(D,\boldsymbol{R}) \to H^2(Y,\boldsymbol{R}) \xrightarrow{b} H^2(B,\boldsymbol{R}) \to 0.$$

We set $A=\boldsymbol{R}^m$ and $A_{\boldsymbol{Z}}=\boldsymbol{Z}^m$. Define $\mathrm{tr}:A\to\boldsymbol{R}$ by $\mathrm{tr}((a_i))=\Sigma_i a_i$. We then define maps $\alpha:H^1(Y,\mathcal{O}_Y^*)\to A_{\boldsymbol{Z}}$ and $\beta:H^1(Y,\mathcal{P}_Y)\to A$ by the composite maps in the following commutative diagram

$$\begin{array}{ccccccccc} H^1(Y,\mathcal{O}_Y^*) & \xrightarrow{c_1} & H^2(Y,\boldsymbol{Z}) & \xrightarrow{b} & H^2(B,\boldsymbol{Z}) & \cong & \oplus H^2(B_i,\boldsymbol{Z}) & \cong & A_{\boldsymbol{Z}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(Y,\mathcal{P}_Y) & \xrightarrow{\delta} & H^2(Y,\boldsymbol{R}) & \xrightarrow{b} & H^2(B,\boldsymbol{R}) & \cong & \oplus H^2(B_i,\boldsymbol{R}) & \cong & A. \end{array}$$

We call an element (a_i) of A *positive* if $a_i>0$ for all i.

Lemma 4.2. *There exists no element of $H^1(Y,\mathcal{P}_Y)$ which is mapped to a positive element by β.*

Proof. We first note that the image of $\alpha:H^1(Y,\mathcal{O}_Y^*)\to A_{\boldsymbol{Z}}$ is precisely the kernel of $\mathrm{tr}:A_{\boldsymbol{Z}}\to\boldsymbol{Z}$. In fact, when B is irreducible, this is due to Horrocks [7]. By 2) of Proposition 3.2 the same proof works also in the general case. Then since $H^1(Y,\mathcal{P}_Y)$ is a real vector space, the image of β contains the hyperplane Ker tr in A, where Ker denotes the kernel. Suppose now that there exists an element, say γ, which is mapped to a positive element of A. Then by what we have noted above β becomes surjective. This implies that $\delta:H^1(Y,\mathcal{P}_Y)\to H^2(Y,\boldsymbol{R})$ also is surjective since the subspace $H^2(D,\boldsymbol{R})$ is clearly in its image. Then from the commutative diagram of exact sequences

$$\begin{array}{ccccccc} \to H^1(Y,\mathcal{O}_Y^*) & \xrightarrow{c_1} & H^2(Y,\boldsymbol{Z}) & \to & H^2(Y,\mathcal{O}_Y) & \to \\ \downarrow & & \downarrow & & \| & \\ \to H^1(Y,\mathcal{P}_Y) & \xrightarrow{\delta} & H^2(Y,\boldsymbol{R}) & \to & H^2(Y,\mathcal{O}_Y) & \to \end{array}$$

we get that c_1 is surjective. Hence α also is surjective. This contradicts our first remark. □

Let $\xi:P\to D$ be the principal G-bundle on D as before. Let $f:E\to D$ be the fiber bundle with typical fiber X associated to ξ and to the given action of G on X. Let $E_d=f^{-1}(d)$ denote the fiber over $d\in B$.

Lemma 4.3. *There exists an element of $H^1(E,\mathcal{P}_E)$ which restricts to a Kähler class in $H^1(E_d,\mathcal{P}_{E_d})$ for general d.*

We first need some construction. Adding 0-section and ∞-section (resp. just ∞-section) we may compactify ξ to a $\boldsymbol{P}$-bundle $\xi:Q\to D$, in case $G=\boldsymbol{C}^*$ (resp.

C). We consider the pull-back $f_Q : \xi^*E = Q \times_D E \to Q$ of f to Q via ξ. The restriction of f_Q to P is canonically isomorphic to the product $P \times X$ over P. Moreover, since the action of G on X is meromorphic, the isomorphism extends to a bimeromorphic map $u : \xi^*E \to Q \times X$; namely, the closure of the graph of the above isomorphism (as a subspace of $(P \times_D E) \times_P (P \times X)$) is an analytic subspace, say Λ, of $(Q \times_D E) \times_Q (Q \times X)$ which is mapped *properly* and bimeromorphically onto both factors via the natural projections. In fact, in this case Λ is itself a fiber bundle over D whose fiber is isomorphic to the graph $\Gamma \subseteq \boldsymbol{P} \times X \times X$ of the meromorphic action $\boldsymbol{P} \times X \to X$ if we identify a fiber of $(Q \times_D E) \times_Q (Q \times X)$ over any point d of D with $(\boldsymbol{P} \times X) \times_{\boldsymbol{P}} (\boldsymbol{P} \times X) = \boldsymbol{P} \times X \times X$ via a fixed isomorphism $\xi^{-1}(d) \cong \boldsymbol{P}$. (Note here that the projections $p_{12} : \Gamma \to \boldsymbol{P} \times X$ and $p_{13} : \Gamma \to \boldsymbol{P} \times X$ onto respective factors are both proper and bimeromorphic.) Now we are ready to start:

Proof of Lemma 4.3. By taking a resolution of Λ, we get a complex manifold T and proper bimeromorphic morphisms $a : T \to \xi^*E$ and $b : T \to Q \times X$. Take a Kähler form ω_X (resp. ω_Q) on X (resp. Q) so that $\omega := \pi_Q^*\omega_Q + \pi_X^*\omega_X$ is a Kähler form on $Q \times X$, where π_Z is the projection to the factor Z chosen to be Q or X. Then define a d-closed positive current ω' (of type (1.1)) on ξ^*E by $\omega' := a_* b^*\omega$, where a_* is the direct image of the form $b^*\omega$ considered as a current on T. Since ξ^*E is nonsingular, ω' then defines an element $\tilde{\gamma}$ of $H^1(\xi^*E, \mathscr{P}_{\xi^*E})$ as usual.

Now take any holomorphic section $s : D \to Q$ such that the image is not contained in $Q - P$. Since $s^*\xi^*E \cong E$, by restricting the class $\tilde{\gamma}$ over $s(D)$ we obtain the induced class γ of $H^1(E, \mathscr{P}_E)$. On the other hand, since u is isomorphic over P, ω' is a Kähler form over P. Thus, for any $d \in D$ with $s(d) \in P$, γ restricts to a Kähler class on the fiber E_d. □

Proof of Theorem 4.1 (cf.[7, Proof of Cor.]). Let $\xi : P \to D$ be the principal G-bundle as above. By Lemma 4.3 there exists an element γ of $H^1(E, \mathscr{P}_E)$ which restricts to a Kähler class in $H^1(E_d, \mathscr{P}_{E_d})$ for a general $d \in D$. Suppose now that the statement of the theorem is not true. Define the subspace $\bar{B}$ of X and the G-morphism $h : B \to X$ with image $\bar{B}$ as in the proof of Theorem 1.1. (In Case 1, $\bar{B}$ now consists of the union of the clousre of orbits of points x_i generating a cycle.) Associated to h we obtain a bundle map $\kappa : Y \to E$ which is a finite morphism of complex spaces. By the property of γ we see that $\kappa^*\gamma$ is mapped to a positive element of A by β, which contradicts Lemma 4.3. Thus, the proof of the theorem is complete.

5. Examples

In this section we shall construct examples of C-actions on some compact normal non-algebraic and non-kählerian Moishezon surfaces X such that the closures of some (or all the) 1-dimensional orbits are singular. We also give examples, where the closures of all the orbits are nonsingular, but the intersection of X^G with any of them is not transversal. Thus Theorem 1.1,2) does not hold in general for Moishezon spaces at least when we allow singularities. For the corresponding couterexamples in the case of $\boldsymbol{C}^*$-actions see [11].

We use the examples of the surfaces of the above type constructed by Grauert in [6]. Let C be a compact Riemann surface of genus $g \geq 2$. We fix a holomorphic line bundle L of degree d on C such that $h^0(L)>0$ and $h^1(L)>0$, where $h^i(L)=\dim H^i(C,L)$, $i=0,1$. For any element ξ of $H^1(C,L)$ we can construct naturally a holomorphic $\boldsymbol{P}$-bundle over C as follows.

Fix a finite Stein open covering $\mathscr{U}=\{U_i\}$ of C with respect to which L is defined by a system of transition functions $\{a_{ij}\}$. We may then represent ξ by an alternating cocycle $\{\xi_{ij}\}$ with respect to $\mathscr{U}$ and then define an affine $\boldsymbol{C}$-bundle $f_0: V \to C$ by $V=\bigcup_i (U_i \times \boldsymbol{C})$, where $(x,\zeta_i) \in U_i \times \boldsymbol{C}$ and $(x,\zeta_j) \in U_j \times \boldsymbol{C}$ with $x \in U_{ij} := U_i \cap U_j$ are identified if and only if

$$\zeta_i = a_{ij}(x)\zeta_j + \xi_{ij}(x). \tag{16}$$

Then adding a point at infinity to each fiber we obtain a $\boldsymbol{P}$-bundle $f: Y \to C$. We set $E=Y-V$, which is a subspace of Y mapped isomorphically onte C via f. Further, the normal bundle N of E in Y is isomorphic to L^*, the dual of L. By our assumption on L, N is thus negative, and hence E can be blown down to a point o of a (compact) normal Moishezon surface X. Let $\pi: Y \to X$ be the natural map. X is known to be non-algebraic (in the sense of Weil) (cf.[6, §4, Sect.8]) and even to be non-kählerian (cf.[10, §2]) as long as $\xi \neq 0$, which we shall assume in the sequel. Note that X depends on L and the choice of $\xi \in H^1(C,L)$.

We now describe $\boldsymbol{C}$-actions on the surface X. Let η be any nonzero element of $H^0(C,L)$ represented by a cochain $\{\eta_i\}$ with respect to $\mathscr{U}$. Then it defines a natural $\boldsymbol{C}$-action on Y by the formula;

$$\zeta_i \to \zeta_i + t\eta_i, \quad t \in \boldsymbol{C}, \quad \text{on } U_i \times \boldsymbol{C}(\zeta_i).$$

Let $Z=\{p_1,\cdots,p_m\}$ be the zero set of η. Then the fixed point set Y^G of this action is given by

$$Y^G = E \bigcup (\bigcup_i \tilde{F}_{p_i}).$$

(In general for any point $p \in C$, $\tilde{F}_p$ denotes the fiber over $f^{-1}(p)$ over p.) In particular, the $\boldsymbol{C}$-action on Y descends to one on X. We write this actions on X and on Y by σ_η when we emphasize its dependence on η.

Now our purpose is to show that by suitable choices of C, L, ξ and η as above the action σ_η on $X=X(C,L,\xi)$ have the properties mentioned at the beginning of this section. Here, we also note that X (resp. σ_η) depends on ξ (resp. η) only up to multiplicative constants.

For any point p of C we write $F_p = \pi(\tilde{F}_p)$. Then the fixed point set X^G on X is written as

$$X^G = \bigcup_i F_{p_i}. \tag{17}$$

The closures of 1-dimensional orbits on X are then precisely the curves F_p for $p \notin Z$. We are thus interested in the smoothness of F_p in general for $p \in C$. We identify E with C via f in the sequel. Let $\mathscr{I}$ be the ideal sheaf of E in Y. Then we have a natural isomorphism $\mathcal{O}_E(L^k) \cong \mathscr{I}^k / \mathscr{I}^{k+1}$ as $\mathcal{O}_E$-modules, which we consider as an identification in what follows. We then get a long exact sequence

$$\to H^0(E, L^2) \to H^0(E_{(2)}, \mathscr{I}/\mathscr{I}^3) \to H^0(E, L) \overset{\delta}{\to} H^1(E, L^2) \to \tag{18}$$

associated to the short exact sequence

$$0 \to \mathcal{O}_E(L^2) \to \mathscr{I}/\mathscr{I}^3 \to \mathcal{O}_E(L) \to 0,$$

of $\mathcal{O}_{E_{(2)}}$-modules, where in general $E_{(k)}$ is the subspace of Y defined by the ideal sheaf $\mathscr{I}^{k+1}$. More generally, for any $k \geq 3$ we may similarly consider the long exact sequence

$$\to H^0(E_{(k)}, \mathscr{I}^2/\mathscr{I}^{k+1}) \to H^0(E_{(k)}, \mathscr{I}/\mathscr{I}^{k+1}) \to H^0(E, L) \to . \tag{19}$$

We shall examine the smoothness of F_p through the following:

Lemma 5.1. *Let p be a point of $C = E$.*
1) *If F_p is nonsingular, then there exists an element, say λ, of $H^0(C, L)$ with $\lambda(p) \neq 0$ which is liftable to an element of $H^0(E_{(2)}, \mathscr{I}/\mathscr{I}^3)$ in* (18).
2) *Conversely, if there exists an element μ of $H^0(C, L)$ with $\mu(p) \neq 0$ which is liftable to $H^0(E_{(k)}, \mathscr{I}/\mathscr{I}^{k+1})$ for any positive integer k in the sequence* (19), *then F_p is nonsingular.*

Proof. Note first that F_p is nonsingular if and only if there exists a holomorphic function, say β, on X defined in a neighborhood of o such that $\beta(o) = 0$ and $\tilde{\beta} := \pi^*\beta$ vanishes at $\infty_p := \tilde{F}_p \cap E$ to the first order when restricted to $\tilde{F}_p$. Now we show 1). Supposing that F_p is nonsingular, take any function β as above. Since $\tilde{\beta}$ vanishes along E, we may take its image λ in $H^0(E, \mathscr{I}/\mathscr{I}^2) = H^0(E, L)$. The above condition then implies that $\lambda(p) \neq 0$ and clearly λ is liftable to an element of $H^0(E_{(2)}, \mathscr{I}/\mathscr{I}^3)$.
2) Let M^k be the line bundle on Y defined by $M^k = [E]^{k+1} \otimes K_Y$, where K denotes the canonical bundle. Then the restriction $M^k|E$ to E is isomorphic to $N^k \otimes K_C$ and hence is negative if $kd > 2g - 2$. Fix any such k. Then, M^k is negative also in a small strongly pseudoconvex neighborhood W of E in Y, and hence, by the

Grauert-Riemenschneider vanishing theorem $H^1(W,\mathscr{I}^{k+1})=0$. Now let μ be an element as in the condition of 2). Take any lift of μ to an element μ_k of $H^0(E_{(k)},\mathscr{I}/\mathscr{I}^{k+1})$ which extends, by the above vanishing result, to a holomorphic function $\tilde{\beta}$ defined on W. This $\tilde{\beta}$ descends to a holomorphic function β on X such that $\tilde{\beta}$ $(=\pi^*\beta)$ fulfills the condition stated at the beginning of the proof; hence F_p is nonsingular. □

In view of the above lemma it is important to identify the kernel of δ in (18). This will be given in the next lemma. We denote by

$$\varphi: H^0(E,L)\times H^1(E,L)\rightarrow H^1(E,L^2)$$

be the natural bilinear pairing induced by the cup product. Then:

Lemma 5.2. *Let $\psi=\psi_\xi: H^0(E,L)\rightarrow H^1(E,L^2)$ be the linear map induced by φ and the given element $\xi\in H^1(E,L)=H^1(C,L)$. Then, $\delta=\psi$. In particular, the kernels of δ and ψ coincide.*

Proof. Take any element $\eta=\{\eta_i\}$ of $H^0(E,L)=H^0(\mathscr{U},L)$, where $\eta_i=a_{ij}\eta_j$ on U_{ij}. It suffices to show that $\delta(\eta)$ is represented by the 1-cocycle $\{\eta_i\xi_{ij}\}$ with respect to $\mathscr{U}$. For this purpose we first describe the structure of $E_{(2)}$. Set $\tau_i=1/\zeta_i$. Then τ_i is a defining equation of E on $f^{-1}(U_i)$. From (16) we compute easily the relation of coordinates on each $f^{-1}(U_{ij})$ and find

$$\tau_j\equiv b_{ji}\tau_i-b_{ji}^2\xi_{ji}\tau_i^2 \mod \mathscr{I}^3, \tag{20}$$

where $b_{ij}=a_{ij}^{-1}$. Now, regarding η_i as a section of $\mathscr{I}/\mathscr{I}^2$ consider $\tau_i\eta_i \mod \mathscr{I}^3$ as a function on $E_{(2)}$ over U_i; then

$$\{\alpha_{ij}\} \quad \text{with} \quad \alpha_{ij}=\tau_i^{-2}(\tau_i\eta_i-\tau_j\eta_j) \mod \mathscr{I}^3$$

is a 1-cocycle representing the class $\delta(\eta)\in H^1(E,L^2)$. By substituting (20) we compute

$$\alpha_{ij}\equiv\tau_i^{-1}\eta_i-\tau_i^{-2}(b_{ji}\tau_i-b_{ji}^2\xi_{ji}\tau_i^2)b_{ij}\eta_i\equiv\xi_{ij}\eta_i \mod \mathscr{I}^3.$$

Hence the assertion is proved. □

In order to get concrete examples we specializes to the case where $L^2\cong K$, where K is the canonical bundle of C. Namely, L is a theta characterestic of C. In this case the pairing φ above becomes

$$\varphi: H^0(C,L)\times H^1(C,L)\rightarrow H^1(C,K)\cong \boldsymbol{C},$$

which is perfect by the Serre duality. We assume moreover that the base locus $B=Bs|L|$ of the linear system $|L|$ defined by $H^0(C,L)$ is nonempty.

The kernel of $\psi=\psi_\xi$ is now a hyperplane $V=V_\xi$ of $H^0(C,L)$. Let B_ξ be the (set-theoretic) base locus of the sublinear system of $|L|$ defined by the elements of V. Obviously, we have two cases depending on the choice of ξ. Case A: $B \neq B_\xi$ and Case B: $B = B_\xi$. It is easy to see that both cases occur if $h^0(L) \geq 3$, while if $h^0(L) = 1$ or 2, only Case A occurs. Here, in case $h^0(L) = 1$, we understand that $V = \{0\}$ and $B_\xi = C$.

Lemma 5.3. 1) *If $B \neq B_\xi$, then for any point $p \in B_\xi - B$, F_p is not smooth and there exists an element $\eta \in H^0(C,L)$ such that the associated* **C**-*action σ_η on X admits F_p as a closure of an orbit.*

2) *If $B = B_\xi$, then for any element η of $H^0(C,L)$, the closure F_p of any* 1-*dimensional orbit for the associated* **C**-*action σ_η on X is smooth. However, if $\eta \notin V_\xi$ and the zeroes of η are all simple, then the intersection of the fixed point set X^G and F_p is not transversal.*

Corollary 5.4. *If $h^0(L) = 1$, then with respect to the unique* **C**-*action on X the closure F_p of any* 1-*dimensional orbits are singular.*

Proof. 1) Every element of $V = V_\xi$ vanishes at p by our assumption. Hence, by Lemmas 5.2 and 5.1 and the sequence (18), F_p must be singular. Moreover, there exists an element $\eta \in H^0(C,L)$ with $\eta(p) \neq 0$. Then the associated action σ_η has F_p as a closure of an orbit (cf. (17)). The assertion is proved.

2) By our assumption, for any point $p \notin B$ there exists an element $\lambda \in V$ such that $\lambda(p) \neq 0$. By the definition of V and Lemma 5.2 λ is liftable to $H^0(E, \mathscr{I} / \mathscr{I}^3)$. Then since $H^1(E, L^m) = 0$ for $m \geq 3$, we can successively lift λ to $H^0(E_{(k)}, \mathscr{I} / \mathscr{I}^{k+1})$ for any k. Hence, by Lemma 5.1 F_p is nonsingular. On the other hand, if η is any nonzero element of $H^0(C,L)$, the closures of 1-dimensional orbits of the associated action σ_η is of the form F_p with $p \notin B$ by (17), and hence is nonsingular.

Now suppose that η satisfies the condition of the second statement. Take any holomorphic function g defined in a neighborhood of o in X vanishing identically along X^G. Let Z be the zero set of η on C $(=E)$. Then $\tilde{g} := \pi^* g$ vanishes identically along $Y^G = E \cup (\cup_{p \in Z} \tilde{F}_p)$, and hence its image $\lambda \in H^0(E,L)$ vanishes on Z. Since λ and η are both sections of L and η has only simple zeroes, it follows that $\lambda = c\eta$ for some constant c. But since $\eta \notin V$, it is not liftable even to $H^0(E_{(2)}, \mathscr{I} / \mathscr{I}^3)$. Hence, we must have $c = 0$ so that $\tilde{g}$ is a section of $\mathscr{I}^2$. Therefore, for any $p \in C$, $\tilde{g}$ vansihes at ∞_p at least to the second order when restricted to $\tilde{F}_p$; and hence so does g at o when restricted to F_p. This implies that F_p does not intersect transversally with X^G. □

Note that the conclusion of the first assertion of 2) is true for any line bundle L with degree $g-1$ which is not necessarily a theta characterestic, as the above proof clearly shows.

Example. Concrete examples which fall under either of the assumptions of the above lemma are given as follows. Take C to be a hyperelliptic curve with the canonical double covering $\gamma: C \rightarrow \boldsymbol{P}$ with branch points $p_1, \cdots, p_{2g+2}$. Let m be any integer with $0 \leq m < (g-1)/2$. Then we may take L to be the line bundle associated to the divisor $p_1 + \cdots + p_l + m\gamma^*[\infty]$, where $l = g-1-2m > 0$. Then we have $L^2 \cong K$, $h^0(L) = m+1$ and $B = \{p_1, \cdots, p_l\}$. (See [1, p.288].) Moreover, the general members of $H^0(C,L)$ have only simple zeroes. Thus the various assumptions of Lemma 5.3 are realized for suitable choices of ξ and η and the corresponding $\boldsymbol{C}$-actions give examples of the type mentioned at the beginning of this section.

For instance, suppose that $g=2$. We fix C and let L vary among all the line bundles of degree $g-1=1$ and whith $h^0(L)=1$. These are naturally parametrized by $C; L = L_q$, $q \in C$. Here L is a theta characterstic if and only if q is one of the haperelliptic branch points p_i, $1 \leq i \leq 6$. The corresponding Moishezon surfaces X_q, which is independent of ξ in this case, form a flat family parametrized by C. (In fact, one can check that the geometric genus p_g of the normal surface germs (X_q, o_q) is equal to four, independently of q.) The natural $\boldsymbol{C}$-action on X_q forms a holomorphic family of $\boldsymbol{C}$-actions, which are *good* for $q \neq p_i$ and degenerates to a *bad* one at each p_i. (This is legitimate since the closures of orbits do not form a flat family over C.)

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