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ON THE $\bar{\partial}$-COHOMOLOGY GROUPS OF STRONGLY q-CONCAVE MANIFOLDS

KAZUHISA MIYAZAWA

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0. Introduction

Let X be a paracompact complex manifold of dimension n and $\pi: E \to X$ be a holomorphic vector bundle. We denote by $\Omega^p(E)$ the germ of E-valued holomorphic p-forms, and by $H^q(X, \Omega^p(E))$ the sheaf cohomology group of X of degree q with coefficients in $\Omega^p(E)$. In 1955, Serre showed the following basic theorem with respect to complex analysis.

Theorem (Serre duality, cf: [15]). *If $H^{q+i}(X, \Omega^p(E))$ $(i=0, 1)$ are Hausdorff, then $H^q(X, \Omega^p(E))$ is a Fréchet space, and its dual space and $H_k^{n-q}(X, \Omega^{n-p}(E^*))$ are isomorphic. Here, E^* denotes the dual of E, and $H_k^{\cdot}(X, \Omega^{\cdot}(E))$ denotes the compactly supported sheaf cohomology group of X with coefficients in $\Omega^{\cdot}(E)$.*

If $H^q(X, \Omega^p(E))$ is finite dimensional, then it is Hausdorff (cf: [15]). But, in general $H^q(X, \Omega^p(E))$ is not Hausdorff (cf: [8], [15]).

The cohomology groups of open manifolds were studied by Grauert [5] for solving Levi's problem, and his result played a fundamental role in the theory of singularities and hyperfunctions. As a natural extension of Grauert's work, it has been known that the finiteness of the cohomology groups results from on the convexity of manifolds:

X is called strongly q-convex (resp. strongly q-concave) if there exists an exhaustion function $\Phi: X \to \boldsymbol{R}$ of class C^∞ whose Levi form has at least $n-q+1$ positive (resp. $n-q+1$ negative) eigenvalues outside a compact subset K of X. We call K an exceptional set. In 1962, Andreotti and Grauert established finiteness theorems for cohomology groups which include the following theorem as a special case.

Theorem A-G (cf : Thēorēme 14 in [1]). *Let X be a strongly q-convex (resp. strongly q-concave) manifold of dimension n, and let E be a holomorphic vector bundle over X. Then*

$$\dim H^s(X, \Omega^r(E)) < \infty \text{ for } s \geq q \text{ (resp. } s < n-q).$$

They showed this theorem, using homological algebra and sheaf theory. Andreotti and Vesentini [3] showed this theorem for q-complete manifolds (i.e, q-convex manifolds with $K=\phi$), using so-called "Bochner-technique". Moreover, they proved that when X is strongly q-convex (resp. strongly q-concave), $H_K^s(X, \Omega^r(E))$ is finite dimensional for $s \leq n-q$ (resp. $s > q$) by using the method of [1]. At almost the same time, Hörmander [7] generalized the method for the $\bar{\partial}$-Neumann problem by J.J. Kohn, and proved Theorem A-G. Ohsawa [10] generalized the method of [3], [7] and gave an alternative proof of Theorem A-G. For further results, see [12], [13].

Andreotti and Vesentini [3] stated the following.

Theorem A-V. *Let X be a strongly q-concave manifold of dimension n, and let $\pi : E \to X$ be a holomorphic vector bundle over X. Then*

$$H^{n-q}(X, \Omega^r(E)) \text{ is Hausdorff.}$$

This theorem has been extended by Andreotti and Kas [2], and Ramis [14] in the case where X is a complex space and E is a coherent analytic sheaf, by using homological algebra and sheaf theory. In 1988, Henkin and Leiterer [6] gave a proof of Theorem A-V in case X is a q-concave domain of a compact complex manifold by integral formula.

In this paper, we use the method of L^2 estimate for $\bar{\partial}$ and give a straightforward proof of Theorem A-V. Moreover we show Hausdorffness of a certain cohomology group of a compact complex space by using the method. Particularly, we utilize not the basic estimate for differential forms satisfying $\bar{\partial}$-Neumann condition on a relatively compact q-concave domain with a smooth boundary, but one with respect to a complete hermitian metric on a strongly q-concave manifold. Application of such a method has not been well known since [10].

The L^2 method seems to have advantages since infinite dimensional cohomology groups seem to be better understood in the L^2 context. For instance, Takegoshi showed a harmonic representation theorem for some cohomology group by using an L^2 estimate for the $\bar{\partial}$-operator, and proved the torsion freeness theorem for higher direct image sheaves of semipositive vector bundle in [16].

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1. Preliminaries

(1) Hermitian Geometry

Let X be a paracompact complex manifold of dimension n and let E be a holomorphic vector bundle over X with a C^∞ fiber metric h. Canonically, h induces metrics along the fibers of E^*, $\bar{E}$, $\wedge^m E$, $\otimes^m E$. We also denote by $\langle\cdot,\cdot\rangle$ (resp. $|\cdot|$) the pointwise inner product (resp. norm) with respect to the induced metrics. Let ds^2 be a hermitian metric on X and let ω be the fundamental form associated to ds^2 and we denote the volume element by $dv=\frac{1}{n!}\overbrace{\omega\wedge\cdots\wedge\omega}^{n\text{-times}}$. Let Λ be the adjoint of the multiplication $L: u\longmapsto\omega\wedge u$ with respect to ds^2. We call L the Lefschetz operator with respect to ds^2.

We denote by $C^{p,q}(X, E)$ the space of E-valued (p, q)-forms of class C^∞ on X and by $C_0^{p,q}(X, E)$ the space of the forms in $C^{p,q}(X, E)$ with compact supports. As usual we denote the exterior differentiation by d and the $(1, 0)$ part of d by ∂ and the $(0, 1)$ part of d by $\bar\partial$. We set $D_E:=\bar\partial+h^{-1}\partial h$, $D'_E:=h^{-1}\partial h=\partial+h^{-1}(\partial h)$, $\vartheta_E:=-\overline{*\,\overline{h^{-1}\partial h}\,*}$, $\bar\vartheta:=-\overline{*\partial*}$.

Theorem 1.1 (cf: [10], [4]). *We set $\tau=[\Lambda, \partial\omega]$, and denote its adjoint by τ^*. Then*

$$[D'_E, \Lambda]=-\sqrt{-1}(\vartheta_E+\overline{\tau^*}),\ [\bar\partial, \Lambda]=\sqrt{-1}(\bar\vartheta+\tau^*).,$$
$$\sqrt{-1}[D_E^2, \Lambda]=[\bar\partial, \vartheta_E]-[D'_E, \bar\vartheta]+[\bar\partial, \overline{\tau^*}]-[\tau^*, D'_E].$$

We set $T_1=\overline{\tau^*}$ and $T_2=\tau^*$. T_i and the adjoints T_i^* of T_i $(i=1, 2)$ are called the torsions of ds^2. D_E^2 is a multiplication of a $\mathrm{Hom}(E, E)$-valued $(1, 1)$-form. We set $D_E^2=e(\Theta_h)$, $\Theta_h\in C^{1,1}(X, \mathrm{Hom}(E, E))$. Θ_h is called the curvature form of E with respect to h.

(2) Basic estimate

Let H_1 and H_2 be two Hilbert spaces and $T: H_1\to H_2$ a closed linear operator with dense domain. We denote its domain, range and nullity by D_T, R_T, N_T, and the adjoint of T by T^*. We set $(f, g)=\int_X\langle f, g\rangle dv$ for f, $g\in C_0^{p,q}(X, E)$. $C_0^{p,q}(X, E)$ is provided with the structure of a pre-Hilbert space with a norm $\|f\|=\sqrt{(f, f)}$. $L^{p,q}(X, E, h, ds^2)$ denotes the space of integrable E-valued (p, q)-forms with respect to ds^2 and h on X. We denote by $\bar\partial: L^{p,q}(X, E, h, ds^2)\to L^{p,q+1}(X, E, h, ds^2)$ the maximal closed extension of the original $\bar\partial$. Other operators are naturally extended to closed linear operators on $L^{p,q}(X, E, h, ds^2)$, we denote $D_{\bar\partial}$ by $D_{\bar\partial}^{p,q}$ and so on. In general, $D_{\bar\partial^*}^{p,q}\subset D_{\vartheta_E}^{p,q}$. But it has been known due to Gaffney and Andreotti-Vesentini (cf: [3]) that if the hermitian metric ds^2 is complete, then $\bar\partial^*=\vartheta_E$ and $D_E'^*=\bar\vartheta$.

We say that the basic estimate holds at bi-degree (p, q) if ds^2 is a complete hermitian metric on X and there exists a compact subset K of X and a constant C_0, satisfying

$$\|f\|^2 \leq C_0\left(\|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2 + \int_K \langle f, f\rangle dv\right) \text{ for all } f \in D_{\bar{\partial}^*}^{p,q} \cap D_{\bar{\partial}}^{p,q}$$

Proposition 1.2 (cf: [7], [10]). *Assume that the basic estimate holds at bi-degree (p, q). Then $R_{\bar{\partial}}^{p,q}$ and $R_{\bar{\partial}}^{p,q+1}$ are closed and* $\dim N_{\bar{\partial}}^{p,q}/R_{\bar{\partial}}^{p,q} < \infty$.

2. L^2 estimate on strongly q-concave manifolds

DEFINITION 2.1. *Let X be a complex manifold of dimension n, and let q be a positive integer. X is said to be strongly q-concave if there is a real valued C^∞ function Φ on X satisfying* 1) $X_c := \{x \in X \mid \Phi(x) < c\} \subset\subset X$ *or* $= X$ *for any* $c \in \boldsymbol{R}$, 2) *the Levi form of Φ has at least $n-q+1$ negative eigenvalues outside a compact subset K of X.*

We call Φ an exhaustion function and K an exceptional set. A strongly q-concave manifold admits a bounded exhaustion function. In fact, if Φ is an unbounded satisfying 1) and 2), then $\tilde{\Phi} = -\exp(-\Phi)$ is a bounded exhaustion function satisfying 1) and 2). From now on let X be strongly q-concave, and Φ be an exhaustion function, and we assume $\sup_{x \in X} \Phi(x) =: d < +\infty$, and $\inf_{x \in X} \Phi(x) = 0$. Moreover we assume that at least $n-q+1$ negative eigenvalues of the Levi form of Φ are smaller than $-N$, and positive eigenvalues of the Levi form of Φ are smaller than $1/N$ for a positive integer $N \geq q+3$ with respect to ds^2 (cf: [10]).

Lemma 2.2. *Let μ be a C^∞ function on $[0, d)$ with $\mu(0)=0$, $\mu'(t)>0$, and* $\lim_{t \to d} \mu(t) = \infty$, $\int_0^{d_1} \mu(s) ds \geq C_1$, $\lim_{t \to d}\left\{\int_0^t \mu(s) ds / \mu(t)\right\} = 0$. *Then one can find a C^∞ function μ_i on $[0, d)$ satisfying $\mu_1(t) \geq \mu(t)$ on $[0, d)$, $\mu_2(t) \geq \mu_1(t)$ on $[d_1, d)$, $\mu_i'(t) \leq \frac{1}{C_1}\{\mu_i(t)\}^2$ on $[d_1, d)$ for $i=1, 2$. Here, we can take* $\mu_2(t) = \frac{1}{C_1}\int_0^t \{\mu_1(s)\}^2 ds + L$, *where L is a constant.*

Proof. See [9].

(1) Basic estimate for E-valued $(0, p)$-form

Given a C^∞ function μ on $[0, d)$ satisfying the conditions of Lemma 2.2, we set

$$\begin{cases} \lambda(t):=\mu_1(t) \\ ds_\lambda^2:=ds^2+\lambda(\Phi)\partial\Phi\otimes\bar{\partial}\Phi \\ h_\lambda:=h\exp\left\{-B\int_0^{\Phi(x)}\lambda(t)^2dt\right\} \end{cases}$$

where B is a positive number. We denote by ω_λ the fundamental form associated to ds_λ^2. As for the curvature form with respect to h_λ, we have $\Theta_{h_\lambda}=\Theta_h+B\{\lambda(\Phi)^2\partial\bar{\partial}\Phi+2\lambda(\Phi)\lambda(\Phi)'\partial\Phi\wedge\bar{\partial}\Phi\}$.

Lemma 2.3. *Let $\Gamma_1\geq\cdots\geq\Gamma_n$ be the eigenvalues of $\lambda(\Phi)\partial\bar{\partial}\Phi+2\lambda(\Phi)'\partial\Phi\otimes\bar{\partial}\Phi$ at $x\in X$ with respect to ds_λ^2. We assume $X_{d_1}\supset K$, $C_1=2$ in* Lemma 2.2. *Then*

1) $\sum_{i=1}^{q+1}\Gamma_i\leq-\lambda(\Phi(x))$ *for* $x\in X-X_{d_1}$.
2) $e(\Theta_h)$ *and* $e(d\omega_\lambda)$ *are locally bounded with respect to* h_λ *and* ds_λ^2.

Proof. 1): See [10], Theorem 4.2.
2): By Schwartz's inequality,

$$|e(\Theta_h)\varphi|_{h_\lambda,\,ds_\lambda^2}\leq|\Theta_h|_{h_\lambda,\,ds_\lambda^2}|\varphi|_{h_\lambda,\,ds_\lambda^2} \text{ for } \varphi\in T_{x,X}^{p,q}\otimes E_x.$$

As induced metric on $\mathrm{Hom}(E, E)\simeq E\otimes E^*$ with respect to ds_λ^2 is $h_\lambda\otimes{}^th_\lambda^{-1}=h\otimes{}^th^{-1}$, we have $|\Theta_h|_{h_\lambda,\,ds_\lambda^2}=|\Theta_h|_{h,\,ds_\lambda^2}\leq|\Theta_h|_{h,ds^2}$. Therefore, we have

$$\sup_{x\in K}\frac{|e(\Theta_h)\varphi|_{h_\lambda,\,ds_\lambda^2}}{|\varphi|_{h_\lambda,\,ds_\lambda^2}}\leq\sup_{x\in K}|\Theta_h|_{h,ds^2}<\infty \text{ for any compact set } K\subset X.$$

On the other hand, we have for any $\varphi\in T_{x,X}^{p,q}\otimes E_x$,

$$\begin{aligned}|e(d\omega_\lambda)\varphi|^2_{h_\lambda,\,ds_\lambda^2}&=|\{d\omega+\sqrt{-1}\lambda(\Phi)\,\bar{\partial}\partial\Phi(\bar{\partial}\Phi-\partial\Phi)\}\cdot\varphi|^2_{h_\lambda,\,ds_\lambda^2}\\&\leq\{|d\omega|^2_{h,ds^2}+2n\lambda(\Phi)|\,\bar{\partial}\partial\Phi|^2_{h,ds^2}\}\cdot|\varphi|^2_{h_\lambda,\,ds_\lambda^2}\end{aligned}$$

Therefore, we have

$$\sup_{x\in K}\frac{|e(d\omega_\lambda)\varphi|^2_{h_\lambda,\,ds_\lambda^2}}{|\varphi|^2_{h_\lambda,\,ds_\lambda^2}}\leq\sup_{x\in K}\{|d\omega|^2_{h,ds^2}+2n\lambda(\Phi)|\,\bar{\partial}\partial\Phi|^2_{h,ds^2}\}<\infty.$$

q.e.d.

The following proposition is basic for our purpose.

Proposition 2.4. *Given a C^∞ function μ on $[0, d)$ such that*

1) *μ satisfies the conditions of Lemma* 2.3
2) $\sup_{x\in\Phi^{-1}(t)}\frac{|e(\Theta_h)\varphi|_{h_\lambda,\,ds_\lambda^2}}{|\varphi|_{h_\lambda,\,ds_\lambda^2}}<\mu(t)^2$ *for any* $t\in[d_1, d)$
3) $\sup_{x\in\Phi^{-1}(t)}\frac{|e(d\omega_\lambda)\varphi|^2_{h_\lambda,\,ds_\lambda^2}}{|\varphi|^2_{h_\lambda,\,ds_\lambda^2}}<\mu(t)^2$ *for any* $t\in[d_1, d)$

4) $\int_0^d \sqrt{\mu(t)}\,dt=\infty$

then the basic estimate holds at $(0, p)$ $(p\leq n-q-1)$ *with respect to* h_λ *and* ds_λ^2 *for sufficiently large* B, *where* $\lambda(t)=\mu_1(t)$ *in* Lemma 2.2.

Proof. See [10], Theorem 4.2.

(2) L^2 convergence for E-valued $(0, p)$-forms

We denote by $L_{loc}^{p,q}(X, E)$ the space of locally square integrable E-valued (p, q)-forms on X. $L_{loc}^{p,q}(X, E)$ is a Frêchet space under the ordinary topology.

Proposition 2.5. *Given* f_j, $f\in L_{loc}^{p,q}(X, E)$ *with* $f_j\to f$ *in* $L_{loc}^{p,q}(X, E)$, *we can find a real valued* C^∞ *function* $\nu(t)$ *on* $[0, d)$ *such that there exists a subsequence* $\{f_{j_k}\}$ *with* $f_{j_k}\to f$ *in* $L^{p,q}(X, E, ds^2, \bar{h}_\nu)$. *Here* $\bar{h}_\nu=h\exp(-\nu(\Phi))$.

Proof. For any measurable set $Y\subset X$, we denote by $\|\cdot\|_{Y,\nu}$ the norm with respect to ds^2 and $h\exp(-\nu(\Phi))$ on Y. We fix any sequence $\{d_l\in \boldsymbol{R}\,|\,l=1, 2, \cdots\}$ with $d_l\nearrow d$.

We can find a real valued C^∞ function ν_j on $[0, d)$ with $\|f_j-f\|_{X,\nu_j}<\frac{1}{j}$.

Consider a real valued C^∞ function ν on $[0, d)$ with $\nu(x)\geq\max_{1\leq j\leq l}\{\nu_j(x)\}$ on $[0, d_l)$ $(l=1, 2, \cdots)$, and $\|f\|_{X,\nu}<\infty$. We can select subsequences $\{f_i\}\supset\{f_{1,i}\}\supset\cdots\{f_{k-1,i}\}\supset\{f_{k,i}\}\supset\cdots$ such that

$$\|f_{k,i}-f\|_{X_{d_k},\,\nu}<\frac{1}{i},\quad \|f_{k,k}-f\|_{X\setminus\overline{X_{d_k}},\,\nu}<\frac{1}{k}.$$

Then we have

$$\|f_{k,k}-f\|_{X,\nu}<\frac{2}{k}.$$

Therefore, $f_{j_k}:=f_{k,k}\to f$ in $L^{p,q}(X, E, ds^2, \bar{h}_\nu)$.

q.e.d.

We set $\bar{h}_\mu=h\exp(-\mu(\Phi(x)))$, and $ds_\mu^2=ds^2+\mu(\Phi(x))\partial\Phi\otimes\bar{\partial}\Phi$. By the diagonalization for ds^2 and ds_μ^2, one can choose a basis $\{\sigma_1, \cdots, \sigma_n\}$ of $T_{x,X}^{1,0}$, which denotes the holomorphic cotangent space to X at x, so that

$$ds^2=\sum_{i=1}^n\sigma_i\otimes\bar{\sigma}_i \text{ and } ds_\mu^2=ds^2+\mu(\Phi)\beta(x)\sigma_1\otimes\bar{\sigma}_1 \text{ at } x\in X$$

where $\{\sigma_1, \cdots, \sigma_n\}$ are the orthonormal basis of $T_{x,X}^{1,0}$ with respect to ds^2, and $\beta(x)$ is the non-negative C^∞ function on X.

Proposition 2.6. *Let* $\mu(t)$, $\mu_1(t)$, $\mu_2(t)$ *be as* Lemma 2.2, *satisfying* $\mu(\Phi(x)) \geq \max\{1, \beta(x)\}$. *Then we have* $\|f\|^2_{ds^2(\mu_1),\ \bar{h}(2\mu_2)} \leq 2 \sup_{0\leq t<d}\{\mu_1(t)^2 \cdot \exp(-\mu_1(t))\} \cdot \|f\|^2_{ds^2,\ \bar{h}_\mu} < \infty$. *Here,* $\|\cdot\|_{ds^2(\mu_1),\ \bar{h}(2\mu_2)}$ *denotes the norm with respect to* $ds^2_{\mu_1}$ *and* $\bar{h}_{2\mu_2}$

Proof. For $f \in L^{0,p}(X, E, ds^2, \bar{h}_\mu)$ with $f \equiv 0$ on $\overline{X_{d_1}}$,

$$\begin{aligned}
\|f\|^2_{ds^2(\mu_1),\ \bar{h}(2\mu_2)} &= \int_X \langle f, f\rangle_{ds^2_{\mu_1},\ h} \exp(-2\mu_2(\Phi(x)))dv_{\mu_1} \\
&\leq \int_X \{1+\mu_1(\Phi(x))\beta(x)\}\langle f, f\rangle_{ds^2,h} \exp(-2\mu_2(\Phi(x)))dv \\
&\leq 2 \sup_{x\in X}\{\mu_1(\Phi(x))^2 \cdot \exp(-\mu_2(\Phi(x)))\} \cdot \\
&\qquad \int_X \langle f, f\rangle_{ds^2,h} \exp(-\mu_2(\Phi(x)))dv
\end{aligned}$$

where dv (resp. dv_{μ_1}) is the volume element with respect to ds^2 (resp. $ds^2_{\mu_1}$).

In view of Lemma 2.2, $\mu_2(t) \geq \mu_1(t) \geq \mu(t)$ for $[d_1, d)$. Therefore,

$$\|f\|^2_{ds^2(\mu_1),\ \bar{h}(2\mu_2)} \leq 2 \sup_{0\leq t<d}\{\mu_1(t)^2 \cdot \exp(-\mu_1(t))\} \cdot \|f\|^2_{ds^2,\ \bar{h}_\mu} < \infty.$$

q.e.d.

3. Proof of Theorem A-V

For $u \in L^{p,q}_{loc}(X, E)$ and $v \in L^{p,q+1}_{loc}(X, E)$, we denote $\bar{\partial} u = v$ if the equation $(u, \vartheta_E \varphi) = (v, \varphi)$ holds for any $\varphi \in C_0^{p,q+1}(X, E)$. In view of Chapter 2, Proposition 3.1 in [10] and Proposition 4.5 in [17], we have only to show that for any $g \in L^{n-q}_{loc}(X, E)$ such that there exists a sequence $\{f_j\} \subset L^{0,n-q-1}_{loc}(X, E)$ with $\bar{\partial} f_j \to g$ in $L^{0,n-q}_{loc}(X, E)$, there exists $f \in L^{0,n-q-1}_{loc}(X, E)$ such that $\bar{\partial} f = g$.

In view of Proposition 2.5, there exists a real valued C^∞ function ν on $[0, d)$ such that there exists a sequence $\{\bar{\partial} f_{j_k}\}$ with $\bar{\partial} f_{j_k} \to g$ in $L^{0,n-q}(X, E, ds^2, \bar{h}_\nu)$, where $\bar{h}_\nu = h \cdot \exp(-\nu(\Phi(x)))$.

Consider a real valued C^∞ function μ on $[0, d)$ with

$$\begin{aligned}
&\{\mu(t)\}^2 \geq \sup_{x\in\Phi^{-1}(t)} |\Theta_h|_{h,ds^2} \text{ for } t\in[d_1, d) \\
&\{\mu(t)\}^2 \geq \sup_{x\in\Phi^{-1}(t)} \{|d\omega|^2_{h,ds^2} + 2n\mu(t)|\partial\bar{\partial}\Phi|^2_{h,ds^2}\} \text{ for } t\in[d_1, d) \\
&\mu(t) \geq \max\{1, \beta(x)\} \text{ for } x\in X \\
&\mu(t) \geq \nu(t) \text{ for } t\in[0, d) \\
&\int_0^{d_1} \mu(s)ds \geq 2,\ \int_0^d \sqrt{\mu(s)}\,ds = \infty,\ \lim_{t\to d}\left\{\int_0^t \mu(s)ds/\mu(t)\right\} = 0
\end{aligned}$$

where $\beta(x)$ is the eigenvalue of $\partial\Phi \otimes \bar{\partial}\Phi$ with respect to ds^2 at x.

We set $\lambda(t) := \mu_1(t)$, and $ds^2_\lambda := ds^2 + \lambda(\Phi)\partial\Phi \otimes \bar{\partial}\Phi$. $h_\lambda := h \exp(-B\mu_2(\Phi(x))) = \bar{h}_{B\mu_2}$, where $\mu_1(t)$ and $\mu_2(t)$ are in Lemma 2.2 and B is a constant.

Then there exists a subsequence $\{f_{j_k}\}$ with $\bar\partial f_{j_k} \to g$ in $L^{0,n-q}(X, E, ds_\lambda^2, h_\lambda)$ in view of Proposition 2.6, and the basic estimate holds at $(0, n-q-1)$ with respect to ds_λ^2 and h_λ for sufficiently large B in view of Proposition 2.4. Therefore, there exists $f \in L^{0,n-q-1}(X, E, ds_\lambda^2, h_\lambda)$ with $\bar\partial f = g$ by Proposition 1.2.

q.e.d.

4. Application

In this section, by using the basic estimate with respect to a complete hermitian metric we show Hausdorffness of a certain cohomology group.

Let X be a compact complex space of pure dimension $\geq n$ whose singular points are isolated and X^* be the nonsingular part of X. Let $\pi: E \to X^*$ be a holomorphic vector bundle over X^* with a C^∞ fiber metric h. We denote the canonical bundle of X^* by K_{X^*}. Suppose that the singular points consist of nonempty sets S_1 and S_2. Let Φ be a family of closed subsets of X^* defined by $\Phi = \{C \subset X^*;$ there exists a neighborhood U of S_1 such that $U \cap C = \phi\}$.

For any $p_k \in S_1$ $(1 \leq k \leq l)$ we have a holomorphic embedding of the germ $(X, p_k) \hookrightarrow (\boldsymbol{C}^N, 0)$. We fix in the followings a holomorphic coordinate $z (= z^{(p_k)}) = (z_1, \ldots, z_N) \in \boldsymbol{C}^N$ and the euclidian norm $|z|$ of z. We set $X_c^* = X \cap \cup_{1 \leq k \leq l} \{0 < |z^{(p_k)}| < c\}$ for $0 < c \leq 1$. We set $F_c(z) = -\log((c - |z|^2) \cdot (\log(c/|z|)))$ and $F(z) = F_1(z)$. Then X_c^* is a complete Kähler manifold with respect to $\partial\bar\partial F_c$.

We set $L_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h) := \{f \in L_{loc}^{p,q}(X^*, E);$ there exists a neighborhood U of S_1 such that $f|_{U \setminus S_1}$ is square integrable with respect to $\partial\bar\partial F$ and h$\}$. A sequence $\{f_j\} \subset L_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h)$ converges to $f \in L_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h)$ if $f_j \to f$ in $L_{loc}^{p,q}(X^*, E)$ and there exists a neighborhood U of S_1 such that $f_j \to f$ in $L^{p,q}(U \setminus S_1, E, \partial\bar\partial F, h)$, and we write $f_j \to f$ in $L_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h)$. We set $H_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h) := \mathrm{Ker}\,\bar\partial \cap L_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h) / \mathrm{Im}\,\bar\partial \cap L_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h)$. $H_\Phi^{p,q}$ denotes the cohomology with supports in Φ. Then the sequence

$$\to H_\Phi^{p,q}(X^*, E) \to H_{S_1}^{p,q}(X^*, E, \partial\bar\partial F, h) \to \varinjlim H_{S_1}^{p,q}(U \setminus S_1, E, \partial\bar\partial F, h) \to$$

is exact. We set

$H_{(2)}^{p,q}(X_c^*, E, \partial\bar\partial F_c, h) := \mathrm{Ker}\,\bar\partial \cap L^{p,q}(X_c^*, E, \partial\bar\partial F_c, h) / \mathrm{Im}\,\bar\partial \cap L^{p,q}(X_c^*, E, \partial\bar\partial F_c, h)$.

Proposition 4.1 (cf: [11]). *Assume that $K_{X^*}^{-1} \otimes E$ is extendable to $X \setminus S_2$ as a holomorphic vector bundle. Then*

$$\varinjlim H_{S_1}^{0,q}(U \setminus S_1, E, \partial\bar\partial F, h \exp(-mF)) = 0 \text{ for } q \geq 1$$

and for sufficiently large m.

Proof. We may assume $S_1=\{p\}$. We set $c=1$. As the curvature form of $K_{X^*}^{-1} \otimes E$ is, by the assumption, bounded with respect to the euclidian induced metric and h, there exists an integer m_1 such that the curvature form of $K_{X^*}^{-1}\otimes E$ is Nakano positive with respect to $h\exp(-m_1F)$. We set $m=m_1+1$. Then

$$(e(\Theta_{hm})\Lambda u,\ u)_m \geq \|u\|_m^2 \text{ for } u\in C_0^{n,q}(X_1^*,\ K_{X^*}^{-1}\otimes E).$$

Here, Θ_{hm} denotes the curvature form of $K_{X^*}^{-1}\otimes E$ and $h_m:\ =h\exp(-mF)$ and $\|\cdot\|_m$, $(\cdot,\ \cdot)_m$, Λ denote the norm, the inner product, the adjoint of Lefschetz operator with respect to $\partial\bar{\partial}F$ and h_m. Then we have

$$\|\bar{\partial}u\|_m^2+\|\vartheta_{h_m}u\|_m^2 \geq \|u\|_m^2 \text{ for } u\in C_0^{n,q}(X_1^*,\ K_{X^*}^{-1}\otimes E)$$

by Kodaira-Nakano inequality. Therefore $H_{(2)}^{0,q}(X_1^*,\ E,\ \partial\bar{\partial}F,\ h\exp(-mF))=0$ for $q\geq 1$. Similarly we have $H_{(2)}^{0,q}(X_c^*,\ E,\ \partial\bar{\partial}F_c,\ h\exp(-m_cF_c))=0$ for $q\geq 1$ and any $0<c<1$ and sufficiently large integer m_c. Since $\partial\bar{\partial}F_c$ and $h\exp(-m_cF_c)$ is quasi-isometric near S_1 to $\partial\bar{\partial}F$ and h_m, we obtain $\varinjlim H_{S_1}^{0,q}(X_c^*,\ E,\ \partial\bar{\partial}F,\ h\exp(-mF))=0$ for $q\geq 1$.

q.e.d.

Theorem 4.2. *Assume that $K_{X^*}^{-1}\otimes E$ is extendable to $X\setminus S_2$ as a holomorphic vector bundle. Then $H_{\Phi}^{0,n-1}(X^*,\ E)$ is Hausdorff.*

Proof. By Proposition 4.1, we have only to show that for any $g\in L_{S_1}^{0,n-1}(X^*,\ E,\ \partial\bar{\partial}F,\ h_m)$ such that there exists a sequence $\{f_j\}\subset L_{S_1}^{0,n-2}(X^*,\ E,\ \partial\bar{\partial}F,\ h_m)$ with $\bar{\partial}f_j\to g$ in $L_{S_1}^{0,n-1}(X^*,\ E,\ \partial\bar{\partial}F,\ h_m)$, there exists $f\in L_{S_1}^{0,n-2}(X^*,\ E,\ \partial\bar{\partial}F,\ h_m)$ such that $\bar{\partial}f=g$.

Let ρ be a C^∞ function such that $\rho=1$ on $X_{\frac{1}{2}}^*$ and $\rho=0$ on $X^*\setminus X_1^*$. Then there exists a complete hermitian metric $d\bar{s}^2$ on X^* and a fiber metric $\bar{h}$ of E such that

1) $\bar{\partial}f_{jk}\to g$ in $L^{0,n-1}(X^*,\ E,\ d\bar{s}^2,\ \bar{h})$.
2) there exists a compact subset K of X^* and a constant C_0, satisfying

$$\|(1-\rho)u\|^2\leq C_0(\|u\|_K^2+\|\bar{\partial}u\|^2+\|\vartheta_{\bar{h}}u\|^2) \text{ for } u\in C_0^{0,q}(X^*,\ E),\ n-2\geq q$$

3) there exists a constant C_1, satisfying

$$\|\rho u\|^2\leq C_1(\|u\|_{X_1^*\setminus X_{\frac{1}{2}}^*}^2+\|\bar{\partial}u\|^2+\|\vartheta_{\bar{h}}u\|^2) \text{ for } u\in C_0^{0,n-2}(X^*,\ E)$$

where $\|\cdot\|$ denotes the norm with respect to $d\bar{s}^2$ and $\bar{h}$.

Indeed, we set $d\bar{s}^2=\partial\bar{\partial}F$ and $\bar{h}=h_m$ near S_1 in Proposition 4.1. Then we define $d\bar{s}^2$ and $\bar{h}$ near S_2 in the same way as in the construction of ds_λ^2 and h_λ in Section 3 because X^* is strongly 1-concave (cf: [1]). By patching these metrics defined near S_1 and S_2 with any metric inbetween we obtain a complete hermitian metric $d\bar{s}^2$ on X^* and a fiber metric $\bar{h}$ of E enjoying the above mentioned properties. In fact , condition 1) and 2) are satisfied because $\text{supp}[(1-\rho)u]\subset X^*\setminus X^*_{\frac{1}{2}}$ and, by the assumption and Proposition 4.1 in [11], condition 3) is satisfied, too.

Therefore the basic estimate holds at bi-degree $(0, n-2)$.

q.e.d.

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Department of Mathematics
Nagoya University
Nagoya 464-01, Japan