In this note, developing our previous work [8] with S. Takashima, we will characterize rings $R$ for which every finitely generated submodule of the injective envelope $E(R)$ is torsionless. Those characterizations would yield recent results of Gómez Pardo and Guil Asensio [6, Theorems 1.5 and 2.2]. Also, we will provide a necessary and sufficient condition for an extension ring $Q$ of a ring $R$ to be a quasi-Frobenius maximal two-sided quotient ring of $R$.

Throughout this note, $R$ stands for an associative ring with identity, modules are unitary modules, and torsion theories are Lambek torsion theories. Sometimes, we consider right $R$-modules as left $R^{op}$-modules, where $R^{op}$ denotes the opposite ring of $R$, and we use the notation $_RX$ (resp. $X_R$) to stress that the module $X$ considered is a left (resp. right) $R$-module. We denote by $\text{Mod} R$ the category of left $R$-modules and by $(-)^*$ both the $R$-dual functors. For a module $X$, we denote by $E(X)$ its injective envelope and by $\varepsilon_X: X \to X^{**}$ the usual evaluation map. A module $X$ is called torsionless (resp. reflexive) if $\varepsilon_X$ is a monomorphism (resp. an isomorphism). For an $X \in \text{Mod} R$, we denote by $\tau(X)$ its Lambek torsion submodule. Namely, $\tau(X)$ is a submodule of $X$ such that $\text{Hom}_R(\tau(X), E(R)) = 0$ and $X/\tau(X)$ is cogenerated by $E(R)$. A module $X$ is called torsion (resp. torsionfree) if $\tau(X) = X$ (resp. $\tau(X) = 0$). A submodule $Y$ of a module $X$ is called a dense (resp. closed) submodule if $X/Y$ is torsion (resp. torsionfree).

Here we recall some definitions. Let $Y$ be a submodule of a module $X$. Then $X$ is called a rational extension of $Y$ if $\text{Hom}_R(X/Y, E(X)) = 0$. Let $Q$ be an extension ring of $R$, i.e., $Q$ is a ring containing $R$ as a subring with common identity. Then $Q$ is called a left (resp. right) quotient ring of $R$ if $_RQ$ (resp. $Q_R$) is a rational extension of $R$ (resp. $R_R$). A left quotient ring $Q$ of $R$ is called a maximal left quotient ring of $R$ if $E(\tau Q)/Q$ is torsionfree. As an extension ring of $R$, a maximal left quotient ring of $R$ is isomorphic to the biendomorphism ring of $E(R)$ (see, e.g., Lambek [10] for details). An extension ring $Q$ of $R$ is called a maximal two-sided quotient ring of $R$ if it is both a maximal left quotient ring of $R$ and a maximal right quotient ring of $R$. A ring homomorphism $R \to Q$ is called a left (resp. right) flat epimorphism if the induced functor $\text{Mod} R$ (resp. $\text{Mod} R^{op}$),
i.e., \( Q_R \) (resp. \( RQ \)) is flat and \( Q \otimes_R Q \sim Q \) canonically (see, e.g., Silver [17], Lazard [11] and Popescu and Spîrcu [15] for details). A module \( X \) is called \( \tau \)-finitely generated if it contains a finitely generated dense submodule. A finitely generated module \( X \) is called \( \tau \)-finitely presented (resp. \( \tau \)-coherent) if for every epimorphism (resp. homomorphism) \( \pi : Y \to X \) with \( Y \) finitely generated, \( \text{Ker} \pi \) is \( \tau \)-finitely generated. A module \( X \) is called \( \tau \)-noetherian (resp. \( \tau \)-artinian) if it satisfies the ascending (resp. descending) chain condition on closed submodules. Finally, a ring \( R \) is called left (resp. right) \( \tau \)-noetherian if \( R \) (resp. \( R \)) is \( \tau \)-noetherian, left (resp. right) \( \tau \)-artinian if \( R \) (resp. \( R \)) is \( \tau \)-artinian, and left (resp. right) \( \tau \)-coherent if \( R \) (resp. \( R \)) is \( \tau \)-coherent.

1. \( \tau \)-absolutely pure and \( \tau \)-semicompact rings. In this section, we characterize rings \( R \) for which every finitely generated submodule of \( E(R) \) is torsionless.

Lemma 1.1 (Hoshino [7, Theorem A]). For a ring \( R \) the following are equivalent.

(a) \( \tau(X) = \text{Ker} \epsilon_X \) for every finitely presented \( X \in \text{Mod} \ R \).
(b) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(c) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(d) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(e) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(f) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(g) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(h) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(i) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).
(j) \( \text{Ext}^1(Y, R) \) is torsion for every finitely generated \( Y \in \text{Mod} \ R \).

Following [8], we call a ring \( R \) \( \tau \)-absolutely pure if it satisfies the equivalent conditions in Lemma 1.1. We call a homomorphism \( \pi : X \to Y \) a \( \tau \)-epimorphism if \( \text{Cok} \pi \) is torsion. Then we call a module \( X \) \( \tau \)-semicompact if for every inverse system of \( \tau \)-epimorphisms \( \{ \pi_{\lambda} : X \to Y_{\lambda} \}_{\lambda \in \Lambda} \) with each \( Y_{\lambda} \) torsionless, the induced homomorphism \( \lim_{\lambda} \pi_{\lambda} : X \to \lim_{\lambda} Y_{\lambda} \) is a \( \tau \)-epimorphism. Finally, we call a ring \( R \) left (resp. right) \( \tau \)-semicompact if \( R \) (resp. \( R \)) is \( \tau \)-semicompact.

Remarks. (1) The \( \tau \)-semicompactness is just the \( R \)-linear compactness, in the sense of Gómez Pardo [5], relative to Lambek torsion theory.

(2) Let \( \text{Mod} \ R/\tau \) denote the quotient category of \( \text{Mod} \ R \) over the full subcategory \( \text{Ker}(\text{Hom}_R(-, E(R))) \). Assume that the image of \( R \) in \( \text{Mod} \ R/\tau \) is linearly compact in the sense of Gómez Pardo [5]. Then \( R \) is left \( \tau \)-semicompact.

Theorem 1.2. For a ring \( R \) the following are equivalent.

(a) Every finitely generated submodule of \( E(R) \) is torsionless.
(b) \( \tau(X) = \text{Ker} \epsilon_X \) for every finitely generated \( X \in \text{Mod} \ R \).
(c) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(d) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(e) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(f) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(g) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(h) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(i) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).
(j) \( \text{Ext}^1(X, R) \) is torsion for every finitely generated \( X \in \text{Mod} \ R \).

Proof. (a) \( \iff \) (b). See Hoshino [7, Lemma 5].

(b) \( \iff \) (c). This is due essentially to Ohtake [14, Lemma 2.3]. Let \( 0 \to Y \to F \to X \to 0 \) be an exact sequence in \( \text{Mod} \ R \) with \( F \) finitely generated free
and let \( \pi: Y^* \to \text{Ext}_R^1(X,R) \) denote the canonical epimorphism. Let \( h \in Y^* \) and form a push-out diagram:

\[
\begin{array}{ccc}
0 & \to & Y & \to & F & \to & X & \to & 0 \\
& & h & \downarrow & \downarrow & \downarrow & & & \\
0 & \to & R & \to & Z & \to & X & \to & 0.
\end{array}
\]

Since \( Z \) is finitely generated, \( \text{Ker } \varepsilon_x \) is torsion. Thus \( \phi^{**} \circ \varepsilon_R = \varepsilon_Z \circ \phi \) is monic, so is \( \phi^{**} \). Hence \( (\text{Cok } \phi^*)^* \simeq \text{Ker } \phi^{**} = 0 \). Since \( \pi(h)R \) is an epimorphic image of \( \text{Cok } \phi^* \), \( (\pi(h)R)^* = 0 \) and thus \( \text{Ext}_R^1(X,R) \) is torsion.

(c) \( \Rightarrow \) (b). Let \( X \in \text{Mod } R \) be finitely generated. Let \( Y \) be a submodule of \( \text{Ker } \varepsilon_x \) and let \( j: Y \to X \) denote the inclusion. Then \( j^* = 0 \) and \( Y^* \) embeds in \( \text{Ext}_R^1(X/Y,R) \). Thus \( Y^* \) is torsion, so that \( Y^* = 0 \). Hence \( \text{Ker } \varepsilon_x \) is torsion and \( \varepsilon(x) = \text{Ker } \varepsilon_x \).

(c) \( \Leftrightarrow \) (d). This is easily deduced from [8, Lemma 2.7].

**Remark.** The equivalence (a) \( \Leftrightarrow \) (d) of Theorem 1.2 would yield a result of Gómez Pardo and Guil Asensio [6, Theorem 2.2].

**Corollary 1.3** (cf. Sumioka [20, Theorem 1]). Let \( R \) be left perfect. Then the following are equivalent.

(a) Every finitely generated submodule of \( \text{Ext}_R(X) \) is torsionless.

(b) \( R \) contains a faithful and injective left ideal.

**Proof.** (a) \( \Rightarrow \) (b). By Storrer [18] \( R \) contains an idempotent \( e \) with \( ReR \) a minimal dense right ideal. It is obvious that \( ReR \) is faithful. Since by Theorem 1.2 \( \text{Ext}_R^1(X,Re) \simeq \text{Ext}_R^1(X,R) \otimes ReR = 0 \) for every finitely generated \( X \in \text{Mod } R \), \( ReR \) is injective.

(b) \( \Rightarrow \) (a). Obvious.

**Corollary 1.4.** Let \( R \) be \( \tau \)-absolutely pure, left and right \( \tau \)-semicompact. Then both \( \text{Ker } \varepsilon_x \) and \( \text{Cok } \varepsilon_x \) are torsion for every finitely generated \( X \in \text{Mod } R \).

**Proof.** Let \( X \in \text{Mod } R \) be finitely generated. By Theorem 1.2 \( \text{Ker } \varepsilon_x \) is torsion. We know from the argument of Jans [9, Theorem 1.1] that \( \text{Cok } \varepsilon_x \simeq \text{Ext}_R^1(M,R) \) with \( M \in \text{Mod } R^{\text{op}} \) finitely generated. Thus again by Theorem 1.2 \( \text{Cok } \varepsilon_x \) is torsion.

**Remark.** Assume that \( R \) is a maximal left quotient ring of itself, i.e., \( E(R)/R \) is torsionfree. Then \( \text{Ext}_R^1(X,Y) = 0 \) for all torsion \( X \in \text{Mod } R \) and reflexive.
Corollary 1.5. Let $R$ be $\tau$-absolutely pure and left $\tau$-semicompact. Then every finitely generated $X \in \text{Mod } R$ is $\tau$-semicompact.

Proof. Let $X \in \text{Mod } R$ be finitely generated. Since every factor module of a $\tau$-semicompact module is $\tau$-semicompact, we may assume that $X$ is free. Then the argument of [8, Lemma 2.7] applies.

2. Flat epimorphic extension rings. Throughout this section, $Q$ stands for an extension ring of $R$.

The following lemmas seem to be known (cf. Silver [7], Lazard [11], Popescu and Spircu [15], Morita [13] and so on). However, for the benefit of the reader, we include proofs.

Lemma 2.1. The following are equivalent.

(1) The inclusion $R \to Q$ is a left flat epimorphism.
(2) $Q \otimes_R X = 0$ for every submodule $X$ of $Q/R$.

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Let $\pi : Q \otimes_R Q \to Q$ denote the multiplication map. Then $Q \text{Ker } \pi \simeq Q \otimes_R (Q/R) = 0$. Next, let $F_1 \to F_0 \to X \to 0$ be an exact sequence in $\text{Mod } R$ with each $F_i$ finitely generated free and put $Y = \text{Im}(F_1 \to F_0)$. We have a sequence of embeddings $\text{Tor}_1^R(Q,X) \subset \text{Tor}_1^R(Q/R,X) \subset (Q/R) \otimes_R Y$. Let us form a pull-back diagram:

$$
\begin{array}{ccc}
(Q/R) \otimes_R F_1 & \to & (Q/R) \otimes_R Y \\
\uparrow & & \uparrow \\
Z & \to & \text{Tor}_1^R(Q,X).
\end{array}
$$

Since $(Q/R) \otimes_R F_1$ is isomorphic to a finite direct sum of copies of $Q/R$, it follows by induction that $Q \otimes_R Z = 0$. Thus, since $Q \otimes_R Q \simeq Q$ canonically, $\text{Tor}_1^R(Q,X) \simeq Q \otimes_R \text{Tor}_1^R(Q,X) = 0$.

Lemma 2.2. The following are equivalent.

(1) $Q$ is a left quotient ring of $R$.
(2) (a) $Q \otimes_R (Q/R)$ is torsion.
(b) $Q \text{Tor}_1^R(Q,X)$ is torsion for every $X \in \text{Mod } R$.

Proof. Note that $\text{Hom}_R(Q \otimes_R (Q/R), E(Q)) \simeq \text{Hom}_R(Q/R, \text{Hom}_R(Q_R, E(Q)))$,
and that $\text{Hom}_Q(\text{Tor}^R(Q,X), E(Q)) \cong \text{Ext}_R^1(X, \text{Hom}_Q(QR, E(Q)))$ for every $X \in \text{Mod} R$.

(1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). It follows that $\text{Hom}_Q(QR, E(Q))$ is injective. Thus $E(Q)$ embeds in $\text{Hom}_Q(QR, E(Q))$. It then follows that $\text{Hom}_R(Q/R, E(Q)) = 0$.

The next lemma generalizes results of Cateforis [2, Proposition 2.2] and Masaike [12, Proposition 3] (cf. also Morita [13, Theorem 7.2]).

**Lemma 2.3.** The following are equivalent.

(1) The inclusion $R \to Q$ is a left flat epimorphism.

(2) (a) $Q$ is a left quotient ring of $R$.

(b) $Q \otimes_R X$ is torsionfree for every submodule $X$ of $R$.

**Proof.** (1) $\Rightarrow$ (2). By Lemma 2.2 (a) follows. It is obvious that (b) holds.

(2) $\Rightarrow$ (1). Let $Y$ be a submodule of $Q/R$. Since $R$ is torsion, so is $Q \otimes_R Y$. Next, let us form a pull-back diagram:

$$
\begin{array}{ccc}
0 & \to & R \\
\| & & \| \\
0 & \to & R \oplus X \to Y \to 0,
\end{array}
$$

where $j: R \to Q$ is an inclusion. Since $Q \otimes_R j$ is a split monomorphism, so is $Q \otimes_R Y$. Thus $Q \otimes_R X$ is torsionfree, so that $Q \otimes_R Y = 0$. By Lemma 2.1 the assertion follows.

**Lemma 2.4.** The following are equivalent.

(1) (a) $Q$ is a maximal left quotient ring of $R$.

(b) $E(Q)$ is an injective cogenerator in $\text{Mod} Q$.

(2) (a) $Q/R$ is torsion.

(b) $Q \otimes_R X = 0$ for every torsion $X \in \text{Mod} R$.

**Proof.** (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). By Lemma 2.1 the inclusion $R \to Q$ is a left flat epimorphism. Thus by Lemma 2.2 $Q$ is a left quotient ring of $R$. Next, let $X \in \text{Mod} Q$ be torsion. Then $R X$ is torsion and thus $Q \otimes Q \otimes_R X = 0$. Hence $E(Q)$ is an injective cogenerator in $\text{Mod} Q$, so that $Q$ is a maximal left quotient ring of $R$.

3. Flatness of the injective envelope. Throughout this section, $Q$ stands for a left quotient ring of $R$. 

Lemma 3.1. Let $R$ be left $\tau$-noetherian and let $X \in \text{Mod } R$ be flat. Then $Q \otimes_R X$ is torsionfree.

Proof. Let $I$ be a dense left ideal of $R$. By Faith [4, Proposition 3.1] $I$ contains a finitely generated subideal $J$ with $I/J$ torsion. Then $R/J$ is finitely presented torsion, so that $\text{Hom}_R(R/J, Q \otimes_R X) \simeq \text{Hom}_R(R/J, Q) \otimes_R X = 0$. Thus $R Q \otimes_R X$ is torsionfree, so is $Q \otimes_R X$.

Corollary 3.2. Let $R$ be left $\tau$-noetherian. Let $n \geq 1$ and let $X \in \text{Mod } R$ with weak dim $\Lambda X \leq n$. Then $\text{Tor}^R_n(Q, X) = 0$.

Proof. Let $\cdots \to F_1 \to F_0 \to X \to 0$ be an exact sequence in $\text{Mod } R$ with each $F_i$ free and put $Y = \text{Cok}(F_{n+1} \to F_n)$. Then $Y$ is flat and thus by Lemma 3.1 $Q \otimes_R Y$ is torsionfree. On the other hand, by Lemma 2.2 $Q \text{Tor}^R_n(Q, X)$ is torsion. It follows that $\text{Tor}^R_n(Q, X) = 0$.

Lemma 3.3. Let $X \in \text{Mod } Q$ with $Q \otimes_R X$ torsionfree. Then $Q \otimes_R X \simeq Q X$ canonically.

Proof. Let $\pi: Q \otimes_R X \to X$ denote the canonical epimorphism. Then $R \text{Ker } \pi \simeq_R (Q/R) \otimes_R X$ is torsion, so is $Q \text{Ker } \pi$. It follows that $\text{Ker } \pi = 0$.

Proposition 3.4. Let $R$ be left $\tau$-noetherian. Then every $X \in \text{Mod } Q$ with $R X$ flat is flat. In particular, $E(Q)$ is flat whenever $E(R)$ is.

Proof. Let $X \in \text{Mod } Q$ with $R X$ flat. Then by Lemmas 3.1 and 3.3 $Q \otimes_R X \simeq Q X$ canonically. Since both $\otimes_R Q$ and $\otimes_R X$ are exact, so is $\otimes_R X$.

Proposition 3.5. For a ring $R$ the following are equivalent.

(1) Arbitrary direct products of copies of $E(R)$ are flat.

(2) $R$ is $\tau$-absolutely pure and right $\tau$-coherent.

Proof. (1) $\Rightarrow$ (2). By Hoshino and Takashima [8, Lemma 1.4] $R$ is $\tau$-absolutely pure. Next, let $0 \to M \to F \to R$ be an exact sequence in $\text{Mod } R^{op}$ with $F$ finitely generated free. By Colby and Rutter [3, Theorem 1.3] $M$ contains a finitely generated submodule $N$ with $(M/N) \otimes_R E(R) \simeq 0$. It suffices to show that $M/N$ is torsion. For an $L \in \text{Mod } R^{op}$, there exists a natural homomorphism $	heta_L: L \otimes_R E(R) \to \text{Hom}_R(L^*, E(R))$ such that $\theta_L(x \otimes y)(z) = \alpha(x)y$ for $x \in L$, $y \in E(R)$ and $\alpha \in L^*$. Now, let $L$ be a cyclic submodule of $M/N$ and let $\pi: R \to L$ be epic in $\text{Mod } R^{op}$. Since $\theta_L \circ (\pi \otimes_R E(R))$
\( = \text{Hom}_R(\pi^*, E(R)) \cdot \theta_R \) is epic, so is \( \theta_L \). Note that \( L \otimes_R E(R) = 0 \). Thus \( \text{Hom}_R(L^*, E(R)) = 0 \) and hence \( L^* = 0 \). It follows that \( M/N \) is torsion.

(2) \( \Rightarrow \) (1). See Hoshino and Takashima [8, Proposition 1.6].

4. Quasi-Frobenius quotient rings. In this section, we provide a necessary and sufficient condition for an extension ring \( Q \) of \( R \) to be a quasi-Frobenius maximal two-sided quotient ring of \( R \).

**Lemma 4.1.** Let \( R \) be left \( \tau \)-noetherian and let \( Q \) be a maximal left quotient ring of \( R \). Assume that weak \( \dim R \leq 1 \). Then the inclusion \( R \rightarrow Q \) is a ring epimorphism.

Proof. We claim that \( (Q/R) \otimes_R Q = 0 \). Let \( I \) be a dense left ideal of \( R \). By Faith [4, Proposition 3.1] \( I \) contains a finitely generated subideal \( J \) with \( I/J \) torsion. Note that \( J \) is also a dense left ideal of \( R \). It follows that \( (Q/R)_R \) is an epimorphic image of the direct sum \( \oplus \text{Hom}_R(R/J, Q/R)_R \), where \( J \) runs over all finitely generated dense left ideals of \( R \). Let \( J \) be a finitely generated dense left ideal of \( R \). Since \( \text{Hom}_R(R/J, Q/R)_R \simeq \text{Ext}_R^1(R/J, Q/R) \), we have only to show that \( \text{Ext}_R^1(R/J, Q/R) = 0 \). For an \( X \in \text{Mod } R \), there exists a natural homomorphism

\[ \delta_X : X^* \otimes_R Q \rightarrow \text{Hom}_R(X, Q) \]

such that \( \delta_X(\alpha \otimes q)(x) = \alpha(x)q \) for \( \alpha \in X^*, q \in Q \) and \( x \in X \). As we remarked in [8], there exists an epimorphism \( \pi : X \rightarrow J \) with \( X \) finitely presented and \( \ker \pi \) torsion. Note that by Auslander [1, Proposition 7.1] \( \delta_X \) is monic. Since \( \pi^* \) is an isomorphism, \( \text{Hom}_R(\pi, Q) \circ \delta_X = \delta_X \circ (\pi^* \otimes_R Q) \) is monic, so is \( \delta_J \). Next, let \( j : J \rightarrow R \) denote the inclusion. Since \( \text{Hom}_R(j, Q) \) is an isomorphism, so is \( \text{Hom}_R(j, Q) \circ \delta_R = \delta_j \circ (j^* \otimes_R Q) \). Thus \( \delta_j \) is epic. Hence \( \delta_j \) is an isomorphism, so is \( j^* \otimes_R Q \). It follows that \( \text{Ext}_R^1(R/J, R) \otimes_R Q \simeq \text{Cok}(j^* \otimes_R Q) = 0 \).

In case \( Q = R \), the next theorem is due to Faith [4, Corollary 5.4].

**Theorem 4.2.** For an extension ring \( Q \) of \( R \) the following are equivalent.

1. \( Q \) is a quasi-Frobenius maximal two-sided quotient ring of \( R \).
2. (a) \( R \) is left \( \tau \)-noetherian.
   (b) \( RQ/R \) is torsion.
   (c) \( Q_R \) is injective.

Proof. (1) \( \Rightarrow \) (2). Obvious.
(2) \( \Rightarrow \) (1). For an \( X \in \text{Mod } R \), there exists a natural homomorphism
such that $\theta_x(q \otimes x)(x) = qx(x)$ for $q \in Q$, $x \in X$ and $x \in X^*$. Since $Q_R$ is injective, $\theta_x$ is an isomorphism for every finitely presented $X \in \text{Mod} \ R$. Let $I$ be a dense left ideal of $R$. By Faith [4, Proposition 3.1] $I$ contains a finitely generated subideal $J$ with $I/J$ torsion. Then $R/J$ is finitely presented torsion, so that $Q \otimes R(R/J) \simeq \text{Hom}_R((R/J)^*, Q) = 0$. Thus $Q \otimes R(R/I) = 0$. It follows that $Q \otimes R X = 0$ for every torsion $X \in \text{Mod} \ R$. Hence by Lemma 2.4, $Q$ is a maximal left quotient ring of $R$, and $E(QQ)$ is an injective cogenerator in $\text{Mod} \ Q$. Thus by Lemma 2.1 $Q_R$ is flat as well as injective, so that $E(R_R)$ is flat. Hence by Hoshino and Takashima [8, Proposition 1.7] and Masaike [12, Proposition 2] $Q$ is a right quotient ring of $R$. It follows that $Q$ is a right selfinjective maximal right quotient ring of $R$. On the other hand, since $R$ is left $\tau$-noetherian, so is $Q$. Thus $Q$ is left noetherian. Hence by Faith [4, Theorem 2.1] $Q$ is quasi-Frobenius.

Corollary 4.3. Let $R$ be left and right noetherian and let $Q$ be a maximal left quotient ring of $R$. Then the following are equivalent.

1) $Q$ is a quasi-Frobenius maximal two-sided quotient ring of $R$.

2) $RQ$ is flat and $\text{inj dim } RQ \leq 1$.

Proof. (1) $\Rightarrow$ (2). By Lemma 2.3 $RQ$ is flat. Also, $RQ$ is injective by Lambek [10, §5].

(2) $\Rightarrow$ (1). By Lemmas 4.1 and 2.2 $Q$ is a right quotient ring of $R$. Next, we claim that $RQ$ is injective. Since

$$\text{Tor}_2^R(E(R_R), X) \simeq \text{Hom}_R(\text{Ext}_R^2(X, R), E(R_R))$$

$$\simeq \text{Hom}_R(\text{Ext}_R^2(X, R), \text{Hom}_Q(RQ, E(QQ)))$$

$$\simeq \text{Hom}_Q(\text{Ext}_R^2(X, R) \otimes RQ, E(QQ))$$

$$\simeq \text{Hom}_Q(\text{Ext}_R^2(X, Q), E(QQ))$$

$$= 0$$

for every finitely generated $X \in \text{Mod} \ R$, we have weak dim $E(R_R) \leq 1$. Thus by Hoshino [7, Propositions $F$ and $C$] every finitely generated submodule of $E(R_R)$ is torsionless. Let $X \in \text{Mod} \ R$ be finitely generated. Since by Theorem 1.2 $X/\tau(X)$ is torsionless, there exists an exact sequence $0 \to X/\tau(X) \to F \to Y \to 0$ in $\text{Mod} \ R$ with $F$ free. Thus $\text{Ext}_R^1(X, Q) \simeq \text{Ext}_R^1(X/\tau(X), Q) \simeq \text{Ext}_R^2(Y, Q) = 0$. Hence $RQ$ is injective and by Theorem 4.2 the assertion follows.

Remark. Let $R$ be left noetherian and let $X \in \text{Mod} \ R$ be flat. Then $\text{Ext}_R^i(Y, R) \otimes X \simeq \text{Ext}_R^i(Y, X)$ for all $i \geq 0$ and finitely generated $Y \in \text{Mod} \ R$, so that $\text{inj dim } RX \leq \text{inj dim } R$. Thus, together with Lemma 2.3, Corollary 4.3 would
yield a result of Sato [16, Theorem].

5. Appendix. Throughout this section, \( Q \) stands for an extension ring of \( R \). We make some remarks on submodules of \( Q_R \).

The argument of Sumioka [19, Proposition 6] suggests the following lemma.

**Lemma 5.1.** The following are equivalent.

1. \( Q \) is a left quotient ring of \( R \).
2. (a) \( RQ/R \) is torsion.
   
   (b) \( R \cap I \neq 0 \) for every nonzero two-sided ideal \( I \) of \( Q \).

**Proof.** (1) \(\Rightarrow\) (2). Obvious.

(2) \(\Rightarrow\) (1). Put \( QE = \text{Hom}_R(RQ, E(R)) \). Then \( RE \simeq E(R) \) canonically, so that the composite of ring homomorphisms \( \text{End}(E(R)) \to \text{End}(QE) \to \text{End}(RE) \) is an isomorphism. Thus \( \text{End}(QE) \simeq \text{End}(RE) \) and hence \( \text{Biend}(QE) = \text{Biend}(RE) \). Let \( \phi: Q \to \text{Biend}(QE) \) denote the canonical ring homomorphism. Since \( RE \) is faithful, \( R \cap \ker \phi = 0 \) and thus \( \ker \phi = 0 \). Since \( \text{Biend}(RE) \) is a maximal left quotient ring of \( R \), the assertion follows.

**Lemma 5.2** (cf. Masaike [12, Proposition 2]). Assume that \( Q \) is a right quotient ring of \( R \). Let \( M \) be a submodule of \( Q_R \) containing \( R \) and put \( I = \{ a \in R | aM \subset R \} \). Then \( M \) is torsionless if and only if \( (R/R/I)^* = 0 \).

**Proof.** Let \( j: R_R \to M_R \) denote the inclusion. Then \( j \) is an essential monomorphism, so that \( \ker \varepsilon_M = 0 \) if and only if \( \ker j^* = 0 \). It suffices to show that \( \ker j^* \simeq (R/R/I)^* \). Identify \( (R,R/R)^* \) with \( R \). We claim that \( \text{Im} j^* \). It is obvious that \( I \subset \text{Im} j^* \). Conversely, let \( h \in M^* \). Since \( E(Q_R)_R \simeq E(R_R) \) is injective, \( h \) extends to some \( \phi: Q_R \to E(Q_R)_R \). It is easy to see that \( \phi \) is \( Q \)-linear. Thus \( h(1)x = \phi(1)x = \phi(x) = h(x) \in R \) for all \( x \in M \) and hence \( j^*(h) = h(1) \in I \).

For an \( M \in \text{Mod } R^{op} \), there exists a natural homomorphism

\[
\eta_M: M \to \text{Hom}_Q(\text{Hom}_R(M, Q), Q)
\]

such that \( \eta_M(x)(a) = a(x) \) for \( x \in M \) and \( a \in \text{Hom}_R(M, Q) \), and for an \( X \in \text{Mod } R \) there exists a natural homomorphism

\[
\xi_X: X^* \to \text{Hom}_Q(Q \otimes_R X, Q)
\]

such that \( \xi_X(a)(q \otimes x) = qa(x) \) for \( x \in X^* \), \( q \in Q \) and \( x \in X \). Also, for \( L, M \in \text{Mod } R^{op} \) there exists a natural homomorphism

\[
\delta_{L,M}: L \otimes_R M^* \to \text{Hom}_R(M, L)
\]
such that $\delta_{L,M}(x \otimes x)(y) = x \otimes y$ for $x \in L$, $x \in M^*$ and $y \in M$.

For each $M \in \text{Mod } R^{op}$, we have a commutative diagram:

$$
\begin{array}{c}
M \xrightarrow{\eta_M} \text{Hom}_Q(\text{Hom}_R(M,Q),Q) \\
\downarrow \epsilon_M \quad \downarrow \text{Hom}_Q(\delta_{Q,M},Q) \\
M^{**} \xrightarrow{\gamma_M} \text{Hom}_Q(Q \otimes_R M^*,Q)
\end{array}
$$

which yields the following lemma.

**Lemma 5.3.** Let $M \in \text{Mod } R^{op}$. Assume that both $\eta_M$ and $\text{Hom}_Q(\delta_{Q,M},Q)$ are monic. Then $M$ is torsionless.

Also, for each $M \in \text{Mod } R^{op}$, we have a commutative diagram with exact rows:

$$
\begin{array}{c}
R \otimes_R M^* \to Q \otimes_R M^* \to (Q/R) \otimes_R M^* \to 0 \\
\downarrow \delta_{R,M} \quad \downarrow \delta_{Q,M} \quad \downarrow \delta_{Q,R,M} \\
0 \to \text{Hom}_R(M,R) \to \text{Hom}_R(M,Q) \to \text{Hom}_R(M,Q/R).
\end{array}
$$

Note that, in case $M$ is finitely generated, $\text{Hom}_R(M,Q/R)$ embeds in a direct sum of copies of $RQ/R$. Thus Snake lemma yields the following two lemmas.

**Lemma 5.4.** Assume that $RQ/R$ is torsion. Then both $R\text{Ker} \delta_{Q,M}$ and $R\text{Cok} \delta_{Q,M}$ are torsion for every finitely generated $M \in \text{Mod } R^{op}$.

**Lemma 5.5.** Assume that the inclusion $R \to Q$ is a left flat epimorphism. Then $\delta_{Q,M} \approx Q \otimes_R \delta_{Q,M}$ is an isomorphism for every finitely generated $M \in \text{Mod } R^{op}$.

We are now in a position to formulate results of Masaike [12] as follows.

**Proposition 5.6** (Masaike [12]). For an extension ring $Q$ of $R$ the following hold.

1. If $Q$ is a left quotient ring of $R$, every finitely generated submodule of $Q_R$ is torsionless.
2. If the inclusion $R \to Q$ is a left flat epimorphism, every finitely generated submodule of $Q_R$ embeds in a free module.
3. Assume that $Q$ is a right quotient ring of $R$. Then $Q$ is a left quotient ring of $R$ if and only if every finitely generated submodule of $Q_R$ is torsionless.

Proof. (1) By Lemmas 5.3 and 5.4.
(2) By Lemma 5.5.
(3) By Lemmas 5.1 and 5.2.

References


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