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A NOTE ON AUSLANDER–REITEN QUIVERS FOR INTEGRAL GROUP RINGS

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1. Introduction

Let G be a finite group and $\mathcal{O}$ be a complete discrete valuation ring, with the maximal ideal (π) and residue field $k=\mathcal{O}/(\pi)$ of characteristic $p>0$. R will be used to denote either $\mathcal{O}$ or k. Let Θ be a connected component of the stable Auslander-Reiten quiver $\Gamma_S(RG)$ of the group algebra RG and set $V(\Theta)=\{vx(M)|M$ is an indecomposable RG-module in $\Theta\}$, where $vx(M)$ denotes the vertex of M. Due to Kawata ([4, Proposition 3.2]), we know that there is a minimal element Q in $V(\Theta)$ with respect to the partial order $\leq_G$ which is uniquely determined up to G-conjugation. We call Q a vertex of Θ.

Let $N=N_G(Q)$ and f be the Green correspondence with respect to (G,Q,N). Choose an indecomposable RG-module M_0 in Θ with Q as its vertex. Let Δ be the connected component of $\Gamma_S(RN)$ containing $fM_0=L_0$. In the case $R=k$, Kawata has shown the following theorem, which extends the Green correspondence, in his paper [4]:

There is a graph monomorphism from Θ to Δ which preserves edge-multiplicity and direction.

The purpose of this note is to ensure that the above result also holds for $\mathcal{O}G$-lattices (i.e., finitely generated $\mathcal{O}$-free $\mathcal{O}G$-modules). The important tools used here can be found in [4], indeed the whole argument in [4] is also valid for $\mathcal{O}G$-lattices with some modifications. In this note, we shall provide a slightly simple proof by examining the middle terms of Auslander-Reiten sequences (see Theorem 2.5 and Corollary 2.6 below). Our approach is valid for both $\mathcal{O}G$ and kG, and will make it clearer that Kawata's graph morphism is an extension of the Green correspondence. The graph morphism stated above is not always isomorphic. In Section 3, we shall give an example of $\mathcal{O}G$-lattices such that the graph morphism is actually not isomorphism on the component containing them.

The notation is almost standard. We shall work over the group ring RG. All the modules considered here are finitely generated free over R. We write $W|W'$

for RG-modules W and W', if W is a direct summand of W'. For an indecomposable non-projective RG-module M, we denote by $\mathscr{A}(M)$ the Auslander-Reiten (abbreviated to AR-) sequence terminating at M. Concerning some basic facts and terminologies used here, we refer to [1], [6] and [7], for example.

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2. The middle terms of AR-sequences

For later use, we shall exhibit some results on the AR-sequences for RG-modules, which are well-known or proved in [4] for kG-modules. We can easily see that they are also valid for $\mathcal{O}G$-lattices.

Lemma 2.1 ([4, Lemma 2.3]). *Let M be an indecomposable non-projective RG-module and H be a subgroup of G. Then the restricted exact sequence $\mathscr{A}(M)_H$ does not split if and only if $vx(M) \leq_G H$.*

Lemma 2.2 ([4, Lemma 2.4]). *Let H be a subgroup of G. Let M and L be indecomposable non-projective modules for G and H respectively. Assume that L is a dirct summand of ${L^G}_H$ with multiplicity one, and that M is a direct summand of L^G such that $L|M_H$. Then $\mathscr{A}(L)^G \simeq \mathscr{A}(M) \oplus \mathscr{E}$, where $\mathscr{E}$ is a split sequence.*

Let H and K be subgroups of G. By a direct computation, we can see that the Mackey decomposition theorem holds for short exact sequnces. Let φ: $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequnce of RH-modules. Then the exact sequnce ${\varphi^G}_K$ of RK-modules have the following form:

$$\bigoplus_{t \in H\backslash G/K} \{0 \to (A^t_{H^t \cap K})^K \xrightarrow{\alpha_t} (B^t_{H^t \cap K})^K \xrightarrow{\beta_t} (C^t_{H^t \cap K})^K \to 0\},$$

where α_t and β_t denote RK-homomorphisms ${\alpha^G}_K (= res^G_K \circ ind^G_H(\alpha))$ and ${\beta^G}_K$, restricted to the appropriate submodules, respectively. For short exact sequences, we shall also use the notation ${\varphi^G}_K \simeq \oplus_{t \in H\backslash G/K} (\varphi^t_{H^t \cap K})^K$. In particular, $\varphi | {\varphi^G}_H$ holds as RH-sequences.

Lemma 2.3 (see [4, Lemma 2.5]). *Let P be a non-trivial p-subgroup of G. Let L be an indecomposable non-projective module for $N_G(P)$. Assume that $P \leq_{N_G(P)} vx(L)$. The following hold.*

(1) *${\mathscr{A}(L)^G}_{N_G(P)} \simeq \mathscr{A}(L) \oplus \mathscr{E}$, where $\mathscr{E}$ is a P-split sequence.*

(2) *Assume further that $\mathscr{A}(L)^G \simeq \mathscr{A}(M) \oplus \mathscr{E}'$, where M is an indecomposable*

non-projective RG-module and $\mathscr{E}'$ *is a split sequence. Then* $\mathscr{A}(M)_{N_G(P)}\simeq\mathscr{A}(L)\oplus\mathscr{E}''$, *where* $\mathscr{E}''$ *is a P-split sequence.*

Proof. (1) For simplicity, put $N=N_G(P)$ and $\mathscr{A}=\mathscr{A}(L)$. By the Mackey decomposition, $\mathscr{A}^G{}_N\simeq\mathscr{A}\oplus\{\oplus_{t\in N\backslash G/N,t\notin N}(\mathscr{A}^t{}_{N^t\cap N})^N\}$. We shall show that $(\mathscr{A}^t{}_{N^t\cap N})^N$ is a P-split sequence for $t\notin N$. Again, by the Mackey decomposition, $(\mathscr{A}^t{}_{N^t\cap N})^N{}_P\simeq\oplus_{g\in(N^t\cap N)\backslash N/P}(\mathscr{A}^{tg}_{N^{tg}\cap N\cap P})^P$. Thus, for our purpose, it is enough to show that $\mathscr{A}^t_{N^t\cap N\cap P}$ splits for $t\notin N$. If this sequence does not split, we have that $P^t\leq_{N^t}vx(L^t)\leq_{N^t}(N^t\cap N\cap P)$, by the assumption and Lemma 2.1. So, $P=P^t$, but this contradicts the choices of t.

(2) By (1) and Krull-Schmidt theorem for the category of morphisms. ■

As we have mentioned in the introduction, every connected component of $\Gamma_S(RG)$ has a vertex. More precisely, the following holds.

Lemma 2.4 ([4, Lemma 3.1]). *Let* Ξ *be a connected subgraph of* $\Gamma_S(RG)$. *Take any* $Q\in V(\Xi)$ *with the smallest order among those p-subgroups in* $V(\Xi)$. *Then for any indecomposable RG-module* $M\in\Xi$, M_Q *has an indecomposable direct summand whose vertex is* Q.

Now we return to the situation in the introduction. Let Q be a vertex of Θ, put $N=N_G(Q)$. Let Λ be a subquiver of Δ consisting of $L_0=fM_0$ and all the RN-modules L in Δ with the property: There exist RN-modules $L_0,L_1,L_2,\cdots,L_m=L$ such that L_n and L_{n+1} are connected by an irreducible map for all n with $0\leq n\leq m-1$ and $Q\leq_G vx(L_n)$ for all n.

REMARK ([4, Lemma 4.1]). For any indecomposable RN-module L in Λ, $Q\leq vx(L)$ holds by Lemma 2.4.

We shall show that $\Theta\simeq\Lambda$ as graphs. Theorem 2.5 below is essential.

Let $\mathfrak{X}$ be the set of all p-subgroups of N whose orders are smaller than $|Q|$. Let L be an indecomposable RN-module in Λ, and M be an indecomposable RG-module in Θ. Assume that L and M satisfy the following two conditions:

(1) $L^G\simeq M\oplus W$, where W is a $\mathfrak{X}$-projective RG-module.

(2) $M_N\simeq L\oplus Z$, where Z_Q is a $\mathfrak{X}$-projective RQ-module.

Now we examine the relation of the middle terms of $\mathscr{A}(L)$ and $\mathscr{A}(M)$. Let Y be the set of all indecomposable direct summands of the middle term of $\mathscr{A}(L)$ whose vertices contain (a G-conjugation of) Q. Let X be the set of all indecomposable direct summands of the middle term of $\mathscr{A}(M)$. Then the modules in Y and X inherit the above conditions (1) and (2). More presisely, the following holds:

Theorem 2.5. *Use the above notations. For each $Y_i \in \boldsymbol{Y}$, $(Y_i)^G$ has a unique indecomposable direct summand, say $X_i = \Psi(Y_i)$, such that $Q \leq_G vx(X_i)$. The map Ψ gives a bijection from $\boldsymbol{Y}$ to $\boldsymbol{X}$ satisfying the following two conditions*:

(1′) $(Y_i)^G \simeq X_i \oplus U_i$, *where U_i is a $\mathfrak{X}$-projective RG-module.*

(2′) $(X_i)_N \simeq Y_i \oplus V_i$, *where $(V_i)_Q$ is a $\mathfrak{X}$-projective RQ-module.*

Moreover, $X_i \simeq X_j$ holds if and only if $Y_i \simeq Y_j$ holds, when $\Psi(Y_i) = X_i$ and $\Psi(Y_j) = X_j$.

Proof. Let Y_i be an element of $\boldsymbol{Y}$. First, we prove that $(Y_i)^G{}_N \simeq Y_i \oplus Y_i'$, where $(Y_i')_Q$ is $\mathfrak{X}$-projective. In particular, $Y_i | (Y_i)^G{}_N$ with multiplicity one by Lemma 2.4. By the conditions (1) and (2), $L^G{}_N \simeq L \oplus L'$, where $(L')_Q$ is $\mathfrak{X}$-projective. Let Y be the middle term of $\mathscr{A}(L)$. By Lemma 2.3, $Y^G{}_N \simeq Y \oplus Y'$, where Y' is the middle term of a Q-split sequence terminating at L'. Thus, $(Y_i')_Q | (L' \oplus \tau(L'))_Q$ and $(Y_i')_Q$ is $\mathfrak{X}$-projective, where τ denotes the Auslander-Reiten translation.

Next we prove that $(Y_i)^G$ has a unique indecomposable direct summand whose vertex contains Q. By Lemma 2.2, $\mathscr{A}(L)^G \simeq \mathscr{A}(M) \oplus \mathscr{E}$, where $\mathscr{E}$ is a split sequence termination at W. So, $Y^G \simeq X \oplus$(*$\mathfrak{X}$-projective RG-modules*), where X is the middle term of $\mathscr{A}(M)$. Let Y_t be an indecomposable direct summand of Y. If $(Y_t)^G$ and X have the same indecomposable direct summand, then $Q \leq_G vx(Y_t)$. So, if $Y_t \notin \boldsymbol{Y}$, $(Y_t)^G$ is $\mathfrak{X}$-projective. On the other hand, for $Y_i \in \boldsymbol{Y}$, $(Y_i)^G$ has a unique indecomposable direct summand, say X_i, satisfying $Y_i | (X_i)_N$, because $Y_i | (Y_i)^G{}_N$ with multiplicity one. Moreover, the condition $Y_i | (X_i)_N$ implies that $Q \leq_G vx(X_i)$ and $X_i \in \boldsymbol{X}$. Now we have to show the uniqueness of X_i. Let X_i' be an indecomposable summand of $(Y_i)^G$ such that $Q \leq_G vx(X_i')$. Because $X_i' | (Y_i)^G$, we have that $(X_i')_N | Y_i \oplus Y_i'$ and $(X_i')_Q | (Y_i \oplus Y_i')_Q$. We know that $(Y_i')_Q$ is $\mathfrak{X}$-projective, and that $(X_i')_Q$ and $(Y_i)_Q$ have indecomposable direct summands whose vertices are Q by Lemma 2.4. This implies that $Y_i | (X_i')_N$ and $X_i' \simeq X_i$.

Thus, for any $Y_i \in \boldsymbol{Y}$, we have that $(Y_i)^G \simeq X_i \oplus$(*$\mathfrak{X}$-projective RG-modules*), where $X_i \in \boldsymbol{X}$, and that $\oplus\Sigma(Y_i)^G \simeq X \oplus$(*$\mathfrak{X}$-projective RG-modules*), where the left-side sum runs over all $Y_i \in \boldsymbol{Y}$. Moreover, $(X_i)_N \simeq Y_i \oplus$(*some direct summands of* Y_i'). Hence, the correspondence Ψ: $Y_i \to X_i$ gives a bijective mapping from $\boldsymbol{Y}$ to $\boldsymbol{X}$ and we see that (1′) and (2′) hold. The last statement of the theorem is straightforward by (1′) and (2′). ■

REMARK FOR THEOREM 2.5. Assume that $L \in \Lambda$ and $M \in \Theta$ satisfy the conditions (1) and (2). Then the middle terms of $\mathscr{A}(\tau^{-1}(L))$ and $\mathscr{A}(\tau^{-1}(M))$ have also the properties which are satisfied by $\mathscr{A}(L)$ and $\mathscr{A}(M)$ in the above theorem.

Corollary 2.6 ([4, Theorem 4.6]). *For any RN-module $L \in \Lambda$, L^G has a unique indecomposable direct summand M such that $Q \leq_G vx(M)$. The correspondence $L \to M$ gives rise to a graph isomorphism from Λ to Θ, which preserves edge-multiplicity and direction. And the corresponding modules satisfy the conditions* (1) *and* (2).

Proof. First, we recall that $(L_0)^G$ has a unique indecomposable direct summand M_0 such that $Q \leq_G vx(M_0)$, and that L_0 and M_0 satisfy (1) and (2). By successive use of Theorem 2.5 and its remark, the proof will be done. ■

3. An Example

As we have seen in Corollary 2.6, there is a graph monomorphism from Θ to Δ. But this morphism is not always isomorphic (i.e., The case $\Lambda \subsetneq \Delta$ may occur). In this section, we shall provide an example of this type. (See Example 3.12.)

Throughout this seciton, we assume that $p=2$, $\mathcal{O}$ is of rank one (i.e., $(\pi)=(p)$) and has all the 3rd root of unity, and that P and V denote the cyclic group of order 2 and the Klein four group respectively, unless otherwise specified. Set $G=\mathfrak{A}_5 \times P$, $N=\mathfrak{A}_4 \times P$ and $Q=V\times P$, where $\mathfrak{A}_n$ is the alternating group of degree n. G and N have the common Sylow 2-subgroup Q and $N=N_G(Q)$.

Let M be an (not necessarily indecomposable) $\mathcal{O}G$-lattice. By abuse of the notations, we use the symbol $\Gamma_S(M)$ to denote the union of all the connected components of which contain some direct summand of M. (If M is indecomposable, $\Gamma_S(M)$ is just the connected component of $\Gamma_S(\mathcal{O}G)$ which contains M.) The map from the connected component Θ of $\Gamma_S(\mathcal{O}_P{}^G)$ to Δ of $\Gamma_S(\mathcal{O}_P{}^N)$, in the notations (3.6) and (3.7) below, is a desired one. (Example 3.12.) We proceed in several steps to achieve our purpose.

Step 1. In this step, p is an arbitrary prime and we do not assume $(\pi)=(p)$. Let G be a p-group and P be a non-trivial normal subgroup of G.

Following [9, §6], we construct the AR-sequence terminating at $\mathcal{O}_P{}^G \simeq \hat{P}\mathcal{O}G$, where $\hat{P}=\Sigma_{x\in P}x \in \mathcal{O}G$. Let $\overline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)$ be the sublattice of $End_{\mathcal{O}G}(\mathcal{O}_P{}^G)$ consisting of all homomorphisms which factor through some projective $\mathcal{O}G$-lattice. Put $\underline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)=End_{\mathcal{O}G}(\mathcal{O}_P{}^G)/\overline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)$. Since the $\mathcal{O}G$-map $\mathcal{O}G \to \hat{P}\mathcal{O}G \to 0$ $(1 \mapsto \hat{P})$ is a projective cover of $\hat{P}\mathcal{O}G$, we have that $End_{\mathcal{O}G}(\mathcal{O}_P{}^G) \simeq \mathcal{O}(G/P)$, $\overline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G) \simeq \{f \in End_{\mathcal{O}G}(\hat{P}\mathcal{O}G) | f(\hat{P}) \in |P|\hat{P}\mathcal{O}G\} \simeq |P|\mathcal{O}(G/P)$ and $\underline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G) \simeq (\mathcal{O}/|P|\mathcal{O})(G/P)$. $\underline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)(\simeq Ext^1_{\mathcal{O}G}(\Omega(\mathcal{O}_P{}^G),\mathcal{O}_P{}^G))$ has the simple socle. Put $\rho=|P|\pi^{-1}\hat{}_{G/P} \in End_{\mathcal{O}G}(\mathcal{O}_P{}^G) - \overline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)$. Then $\rho+\overline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)$ is a generator of the simple socle of $\underline{End}_{\mathcal{O}G}(\mathcal{O}_P{}^G)$.

Thus, we can obtain the AR-sequence $\mathscr{A}(\mathcal{O}_P{}^G)$ as a pull-back diagram of a projective cover of $\mathcal{O}_P{}^G$ along ρ, that is,

$$\begin{array}{ccccccccc} 0 & \to & \Omega(\mathcal{O}_P{}^G) & \to & M & \to & \mathcal{O}_P{}^G & \to & 0 \\ & & \| & & \downarrow & \text{P.B.} & \downarrow^{\rho} & & \\ 0 & \to & \Omega(\mathcal{O}_P{}^G) & \to & \mathcal{O}G & \to & \mathcal{O}_P{}^G & \to & 0 \end{array} \tag{3.1}$$

where the first row is $\mathscr{A}(\mathcal{O}_P{}^G)$ and the second one is a projective cover of $\mathcal{O}_P{}^G$.

For the middle term M of $\mathscr{A}(\mathcal{O}_P{}^G)$, the following holds.

Proposition 3.2.

(1) *M is indecomposable.*

(2) $vx(M)=\begin{cases} 1, & \textit{if}\ |G|=p\ \textit{and}\ (\pi)=(p) \\ G, & \textit{otherwise} \end{cases}$.

Proof. (1) Write $M=(|P|\pi^{-1}\Sigma_{a\in P\backslash G}a,\hat{P})\,\mathcal{O}G+\Sigma_{x\in P}(x-1,0)\,\mathcal{O}G\subset\mathcal{O}G\oplus\hat{P}\mathcal{O}G$. First, we prove that $\Omega(\mathcal{O}_P{}^G)=\{m\in M|m\hat{P}=0\}$. Put $X=\{m\in M|m\hat{P}=0\}$. It is clear that $\Omega(\mathcal{O}_P{}^G)\subset X$. Take an element $m=(|P|\pi^{-1}\Sigma_{a\in P\backslash G}a,\hat{P})\alpha+\beta$ of $X(\subset M)$, where $\alpha\in\mathcal{O}G$ and $\beta\in\Sigma_{x\in P}(x-1,0)\mathcal{O}G$. The equation $m\hat{P}=0$ implies that $|P|\hat{P}\alpha=0$, so, $\hat{P}\alpha=0$. Hence, we have that $m\in\Omega(\mathcal{O}_P{}^G)$ by the definition of M.

Next, we shall prove that M is indecomposable. Take any idempotent $f\in End_{\mathcal{O}G}(M)$ and fix it. Then f induces idempotents g and h of $End_{\mathcal{O}G}(X)$ and $End_{\mathcal{O}G}(\mathcal{O}_P{}^G)$ respectively, which satisfy the following commutative diagram with two $\mathscr{A}(\mathcal{O}_P{}^G)$'s as its rows;

$$\begin{array}{lccccccccl} (\varphi 1)\colon & 0 & \to & X & \to & M & \to & \mathcal{O}_P{}^G & \to & 0\ (\simeq\mathscr{A}(\mathcal{O}_P{}^G)) \\ (3.3) & & & {}^{g}\downarrow & & \downarrow^{f} & & \downarrow^{h} & & \\ (\varphi 2)\colon & 0 & \to & X & \to & M & \to & \mathcal{O}_P{}^G & \to & 0\ (\simeq\mathscr{A}(\mathcal{O}_P{}^G)). \end{array}$$

Then g and h are 0 or 1 by the indecomposability of X and $\mathcal{O}_P{}^G$. Note that neither the case $g=1$ and $h=0$ nor $g=0$ and $h=1$ happens; otherwise the sequence $(\varphi 1)$ or $(\varphi 2)$ splits. If $g=h=0$, then $f(M)\subset X$ and $f(M)=f^2(M)\subset f(X)=0$, so, $f=0$. If $g=h=1$, we have $f=1$ by the five lemma. Now, the proof of (1) is done.

(2) We prepare the following claim;

Let Q be a subgroup of G which contains P. If M is Q-projective, then $Q=G$.

Proof of the claim. We proceed by induction on $|G:Q|$, so we may assume $Q\lhd G$ and $|G:Q|=p$ or 1. To derive a contradiction, we assume that $Q\subsetneq G$. By [9, Proposition 4.10] and [3, Proposition 7.9 (ii)] for $\mathcal{O}G$-lattices (we can verify that the latter proposition holds for $\mathcal{O}G$-lattices by considering $\mathcal{O}$-length instead of k-dimension), we have

$$\mathscr{A}(\mathcal{O}_P{}^G)_Q\simeq\mathscr{A}(\mathcal{O}_P{}^Q)\oplus((p-1)\ \textit{non-zero split sequences}).$$

By the mackey decomposition, $M_Q|S^G{}_Q=\oplus_{g\in Q\backslash G}S^g$, where S is a Q-source of M. On the other hand, $(\mathcal{O}_P{}^G)_Q \simeq |G:Q|\mathcal{O}_P{}^Q$. But this contradicts that $\mathscr{A}(\mathcal{O}_P{}^G)_Q$ has non-zero split part. Now the proof of the claim is complete.

We return to the proof of (2). It is well-known that either $P \leq vx(M)$ or $vx(M) \lneq P$ occurs [2, (2.3) Lemma]. Thanks to the above claim, we know that the first case implies that $vx(M) = G$. To examine the second case, let's consider the diagram (3.1) modulo (π). Recall that $\hat{P}\mathcal{O}G \simeq \mathcal{O}_P{}^G$. If $|P|\pi^{-1}$ is not a unit of $\mathcal{O}$, the induced kG-map $\bar{\rho}: \hat{P}kG \to \hat{P}kG$ is the zero map. So $\bar{M} \simeq \bar{X} \oplus \hat{P}kG$ and $P = vx(\hat{P}kG) \leq vx(M)$. Thus $|P|\pi^{-1}$ must be a unit of $\mathcal{O}$. This fact yields that $|P| = p, (\pi) = (p)$ and $vx(M) = 1$. Moreover, we have $P = G$ by making use of our claim. Indeed, if $|G| = p$ and $(\pi) = (p)$, the map ρ in (3.1) is just an identity map and a projective cover of $\mathcal{O}_P{}^G$ is already $\mathscr{A}(\mathcal{O}_P{}^G)$. Now, the proof of (2) is done. ■

In the rest of this section, let $p = 2$ and $(\pi) = (2)$, P denote the cyclic group of order 2.

Step 2. In this step, set $G = V \times P$.

Let $\mathscr{A}(\mathcal{O}_P{}^G): 0 \to \Omega(\mathcal{O}_P{}^G) \to M \to \mathcal{O}_P{}^G \to 0$ be the AR-sequence terminating at $\mathcal{O}_P{}^G$. By Proposition 3.2, we know that M is indecomposable and $vx(M) = G$. Let Δ_0 be the connected component of $\Gamma_S(\mathcal{O}_P{}^G)$. Then $\Delta_0 \simeq ZA_\infty/(2)$ since $\mathcal{O}_P{}^G$ is periodic with period 2 (see [1, (2.31.6) and (2.31.11)], for example). And $\mathcal{O}_P{}^G$ lies at the end of Δ_0 by the indecomposability of M. Moreover, we have

Proposition 3.4. *Apart from $\mathcal{O}_P{}^G$ and $\Omega(\mathcal{O}_P{}^G)$, all the indecomposable $\mathcal{O}G$-lattices in Δ_0 have G as their vertices.*

Proof. By the shape of Δ_0, we know that $\mathscr{A}(M)$ has the form;

$$0 \to \Omega(M) \to \Omega(\mathcal{O}_P{}^G) \oplus S \to M \to 0,$$

where S is an indecomposable $\mathcal{O}G$-lattice.

First we shall prove that $vx(S) = G$. It is clear that $vx(S) \geq P$ by Lemma 2.4. It is well-known, or follows by applying Proposition 3.2 (1), (2) to $\mathcal{O}P$, that $\mathcal{O}P$ has three isomorphism classes of indecomposable $\mathcal{O}P$-lattices, that is, $\{\mathcal{O}_P, \Omega(\mathcal{O}_P), \mathcal{O}P\}$. If $vx(S) = P$, then the P-source of S must be $\mathcal{O}_P$ or $\Omega(\mathcal{O}_P)$, so, $S = \mathcal{O}_P{}^G$ or $S = \Omega(\mathcal{O}_P{}^G)$. But this is impossible. Hence, $vx(S) \gneq P$.

Put $t = (123) \times 1_P \in \mathfrak{A}_4 \times P$. Then, t acts on G by conjugation. So, t acts on the set of $\mathcal{O}G$-lattices. By the successive use of the uniqueness of AR-sequence, we have that $(\mathcal{O}_P{}^G)^t \simeq \mathcal{O}_P{}^G$, $M^t \simeq M$ and finally $S^t \simeq S$. Since $vx(S)$ is t-invariant and $vx(S) \gneq P$, we get $vx(S) = G$ as desired. For all the other indecomposable $\mathcal{O}G$-lattices in Δ_0, we can ensure that their vertices equal G by a way to similar to that for S. ■

From now on, we need the assumption that $\mathcal{O}$ has all the 3rd root of unity.

Step 3. In this step, set $G=\mathfrak{A}_4\times P$ and $Q=V\times P$.

Note that Q is a normal Sylow 2-subgroup of G and $\mathcal{O}_P{}^G\simeq\mathcal{O}\mathfrak{A}_4$ holds. We shall examine the connected components of $\Gamma_S(\mathcal{O}_P{}^G)$ by making use of the results of Step 2. $\mathcal{O}_P{}^G$ has just three isomorphism classes of primitive idempotents, say e_I,e_1 and e_2. $\mathcal{O}_P{}^G\simeq\hat{P}\mathcal{O}G=e_I\hat{P}\mathcal{O}G\oplus e_1\hat{P}\mathcal{O}G\oplus e_2\hat{P}\mathcal{O}G$ holds $\mathcal{O}G$-lattices.

For each primitive idempotent $e\in\{e_I,e_1,e_2\}$, the $\mathcal{O}G$-map $e\mathcal{O}G\rightarrow e\hat{P}\mathcal{O}G\rightarrow 0$ $(e\mapsto e\hat{P})$ is a projective cover of $e\hat{P}\mathcal{O}G$ and the $\mathcal{O}G$-map $e\hat{P}\mathcal{O}G\rightarrow e\hat{P}\mathcal{O}G$ $(e\mapsto e\hat{V})$ gives a generator of the simple socle of $\underline{End}_{\mathcal{O}G}(e\hat{P}\mathcal{O}G)$ modulo projectives. Then the following holds.

Proposition 3.5. *The connected component of* $\Gamma_S(e\hat{P}\mathcal{O}G)$, say Δ, *is isomorphic to that of* $\Gamma_S(\hat{P}\mathcal{O}Q)$. *In other words,* $\Delta\simeq ZA_\infty/(2)$. *Moreover, apart from* $e\hat{P}\mathcal{O}G$ *and* $\Omega(e\hat{P}\mathcal{O}G)$, *all the* $\mathcal{O}G$*-lattices in* Δ *has* Q *as their vertices.*

Proof. It is easy to see that $(e\hat{P}\mathcal{O}G)_Q\simeq\hat{P}\mathcal{O}Q$ and $(\hat{P}\mathcal{O}Q)^G\simeq\mathcal{O}_P{}^G$. So, $vx(e\hat{P}\mathcal{O}G)={}_GP$ and $e\hat{P}\mathcal{O}G$ has period 2. Moreover, $\mathscr{A}(e\hat{P}\mathcal{O}G)_Q\simeq\mathscr{A}(\hat{P}\mathcal{O}Q)$ holds, by [9, Proposition 4.10] and that Q is a normal Sylow 2-subgroup of G. Therefore, the middle term of $\mathscr{A}(e\hat{P}\mathcal{O}G)$, say L, is indecomposable, $vx(L)={}_GQ$ and $e\hat{P}\mathcal{O}G$ lies at the end of Δ.

Using this argument repeatedly, we can show that the restriction of the AR-sequence of each module in Δ to Q is still an AR-sequence, and consequently, we have a vertex preserving isomorphism $\Delta\simeq\Delta_0$ by restricting the modules in Δ to Q. Now the proof is complete by Proposition 3.4. ■

Step 4. In this step, set $G=\mathfrak{A}_5\times P, N=\mathfrak{A}_4\times P$ and $Q=V\times P$.

Note that Q is a common Sylow 2-subgroup of G and N, and $N=N_G(Q)$. For simplicity we put $Q_i=e_i\hat{P}\mathcal{O}N$ for $i=I,1,2$. In Proposition 3.5, we have determined the connected component $\Delta(\simeq ZA_\infty/(2))$ which contains Q_1 as the following;

$$(3.6)\qquad \Delta:\quad \begin{array}{ccccccccc} Q_1 & \leftarrow & L & \leftarrow & L_1 & \leftarrow & L_2 & \leftarrow & \cdots\cdots \\ & & & \nwarrow & & \nwarrow & & \nwarrow & \cdots \\ & & & & \Omega(Q_1) & \leftarrow & \Omega(L) & \leftarrow & \Omega(L_1)\leftarrow\cdots, \end{array}$$

where $vx(Q_1)=P, vx(L)=Q$ and $vx(L_i)=Q$ for $i=1,2,\cdots$.

Let M be the Green correspondent of L with respect to (G,Q,N) and Θ be the connected component which contains M. M has period 2, so $\Theta\simeq ZA_\infty/(2)$.

The rest of this section is devoted to proving that Θ has the following form;

$$\text{(3.7)}\qquad \Theta:\quad \begin{array}{ccccccc} M\leftarrow M_1 & \leftarrow & M_2 & \leftarrow & M_3 & \leftarrow & \cdots\cdots \\ \nwarrow & & \nwarrow & & \nwarrow & & \cdots \\ \Omega(M) & \leftarrow & \Omega(M_1) & \leftarrow & \Omega(M_2) & \leftarrow & \cdots, \end{array}$$

where M_i is the Green correspondent of L_i for $i=1,2,\cdots$, and all the indecomposable $\mathcal{O}G$-lattices in Θ have Q as their vertices.

Kawata's result ([5, Theorem]) guarantees that $\mathscr{A}(M_i)$ has the form just as in (3.7) for $i=1,2,\cdots$. (Kawata's theorem in [5] is valid for $\mathcal{O}G$-lattices.) So, for our purpose, it is enough to prove:

Proposition 3.8. *The middle term of $\mathscr{A}(M)$ is just M_1.*

To derive a contradiction, we assume to the contrary that the middle term of $\mathscr{A}(M)$ is not indecomposable. Let $\Omega(U)$ be the diret summand of it which is not M_1. By [5, Theorem] and the Green correspondence, $vx(U)\leq P$. If $vx(U)=1$, there is nothing to prove, so we may assume that $vx(U)=P$. The isomorphism classes of indecomposable $\mathcal{O}G$-lattices with vertex P are $\{P_I,P_1,P_2,P_3,\Omega(P_I),\Omega(P_1),\Omega(P_2),\Omega(P_3)\}$, where each P_i is projective as $\mathcal{O}\mathfrak{A}_5$-lattice and P acts trivially on it, and they have ranks 12, 8, 8 and 4 in turn for $i=I,1,2,3$. We shall eliminate the possibility that U might be isomorphic to any of them. Before doing so, we prepare the following lemma.

Lemma 3.9. *The following holds.*

(1) *There are exact sequences of the forms ;*

(*i*) $0\to\Omega(P_2)\to M\oplus X\to P_1\to 0$ *and*

(*ii*) $0\to\Omega(P_1)\to M\oplus X'\to P_2\to 0$,

where X and X' are some *P-projective $\mathcal{O}G$-lattices.* (X *and* X' *may be zero.*)

(2) $\mathscr{A}(Q_I)^G\simeq\mathscr{A}(P_I)\oplus\mathscr{A}(P_3)\oplus$ (*a split sequece*).

Proof. (1) The proofs for the sequences (i) and (ii) are given in entirely the same way. Here we refer to sequence (ii) only. Let's consider the diagram induced to $\mathcal{O}G$ from the following pull-back diagram of $\mathcal{O}N$-lattices:

$$\begin{array}{ccccccccc} O & \to & \Omega(Q_1) & \to & L & \to & Q_1 & \to & 0 \\ & & \| & & \downarrow & \text{P.B.} & \downarrow{\scriptstyle\sigma_1} & & \\ 0 & \to & \Omega(Q_1) & \to & P(Q_1) & \to & Q_1 & \to & 0, \end{array}$$

where the first row is $\mathscr{A}(Q_1)$, the second one is a projective cover of Q_1 and the map σ_1: $Q_1\to Q_1$ is given by $e_1\mapsto e_1\hat{V}$. Note that the induced diagram is also a pull-back and $(Q_1)^G=P_1\oplus P_2\oplus P_3$.

We shall examine σ_1^G: $(Q_1)^G \to (Q_1)^G$. Following [6, p.77 and p.185] (and keeping his notations), $\mathcal{O}G$-lattices P_1, P_2 and $Im(\sigma_1^G)$ has the following Loewy series;

$$\begin{array}{cccccc} & 2_1 & & 2_2 & & \\ & I & & I & & 2_2 \\ P_1: & 2_2 & , \ P_2: & 2_1 & , \ Im(\sigma_1^G): & I \\ & I & & I & & 2_1 \\ & 2_1 & & 2_2 & & . \end{array}$$

Thus, we may regard P_1 as an injective hull of $Im(\sigma_1^G)$ as $\mathcal{O}\mathfrak{A}_5$-lattices. Let $\sigma_1^G|_{P'_2}$: $P'_2 \to Im(\sigma_1^G)$ be a projective cover of $Im(\sigma_1^G)$ as $\mathcal{O}\mathfrak{A}_5$-lattices. By composing suitable isomorphisms to σ_1^G, we may assume $P'_2 = P_2$ by which the diagram is still pull-back. Therefore, we can conclude that σ_1^G: $(Q_1)^G \to (Q_1)^G$ is the sum of a projective cover $\sigma_1^G|_{P_2}$: $P_2 \to Im(\sigma_1^G)(\subset P_1)$ as $\mathcal{O}\mathfrak{A}_5$-lattices and two zero maps $P_1 \to 0$, $P_3 \to 0$. On the other hand, the projective cover of Q_1 is induced to a direct sum of three projective covers of P_1, P_2 and P_3. Now, we have that

$$\mathscr{A}(Q_1)^G \simeq (0 \to \Omega(P_1) \to M \oplus X' \to P_2 \to 0) \oplus (\textit{two split sequences})$$

and X' is a P-projective $\mathcal{O}G$-lattice, since $X'|L^G$.

(2) Next, we shall induce the pull-back diagram of $\mathscr{A}(Q_I)$ to $\mathcal{O}G$. Let σ_I be the $\mathcal{O}N$-map $Q_I \to Q_I$ $(e_I \mapsto e_I \hat{V})$. Note that $(Q_I)^G = P_I \oplus P_3 \oplus P_3$ and $Im(\sigma_1^G) \simeq I \oplus P_3$, where I is the simple socle of P_I. By the same argument as in (1), we have that σ_I^G: $(Q_I)^G \to (Q_I)^G$ is the sum of a projective cover κ: $P_I \to I(\subset P_I)$ as $\mathcal{O}\mathfrak{A}_5$-lattices, identity map $P_3 \to P_3$ and zero map $P_3 \to 0$, and that the projective cover of Q_I is induced to a direct sum of three projective covers of P_I, P_3 and P_3. Since κ is (left-)annihilated by any non-automorphism in $End_{\mathcal{O}G}(P_I)$, $\bar{\kappa} = \kappa + \overline{End}_{\mathcal{O}G}(P_I) \in Soc(\underline{End}_{\mathcal{O}G}(P_I))$ and we have (2). ($\mathscr{A}(P_3)$ is just a projective cover of P_3.) ∎

Now, we return to the proof of Proposition 3.8. By the Brauer's third main theorem and [7, Corollary 3.11 on p.325], U belongs to the principal block of G, so, we have that $U \not\approx P_3$. If P_I is connected to M, then Q_I and L are connected by the Green correspondence and Lemma 3.9(2). But this does not happen since Q_I and Q_1 belong to the different components (see Proposition 3.5). So, $U \not\approx P_I$.

Next we shall prove $U \not\approx P_1$. We assume by way of contradiction that $\mathscr{A}(P_1)$ is of the form;

$$0 \to \Omega(P_1) \to M \oplus \Omega(Y) \to P_1 \to 0, \tag{3.10}$$

where Y is an indecomposable $\mathcal{O}G$-lattice (possibly $Y=0$). We shall compare two sequences (3.10) and (3.9) (1)(i). We need to consider the following two cases.

Case 1. Let $Y=0$.

Then we have that $rank_{\mathcal{O}}M=16$ and $X=0$ by Lemma 3.9 (1)(i). For simplicity, put $H=\mathfrak{A}_5$. Recall that $\Omega(P_i)_H \simeq (P_i)_H$ holds for $i=1,2$. Since the restrictions of (3.10) and (3.9)(1)(i) to $\mathcal{O}H$ split, we have that $M_H \simeq (P_1)_H \oplus (P_1)_H$ and $M_H \simeq (P_1)_H \oplus (P_2)_H$. This is a contradiction.

Case 2. Let $Y \neq 0$.

Then $X \neq 0$ and $M \neq 0$. The P-projectivity of X implies that X is indecomposable and $rank_{\mathcal{O}}X=8$, since the ranks of projective (resp. P-projective) indecomposable modules which may occur are 24 or 16 (resp. 12 or 8). So, we have $rank_{\mathcal{O}}M=8$ and $rank_{\mathcal{O}}\Omega(Y)=8$. Moreover,

Lemma 3.11.

(1) $rank_{\mathcal{O}}\Omega(M)= 8$.

(2) $rank_{\mathcal{O}}Y=8$.

Proof. First we note that $rank_{\mathcal{O}}\Omega(X)=8$ since $X \simeq P_1, P_2, \Omega(P_1)$, or $\Omega(P_2)$.

(1) The tensor product of the sequence (3.9)(1)(i) with $\Omega(\mathcal{O})$ is a direct sum of $0 \to P_2 \to \Omega(M) \oplus \Omega(X) \oplus (projective) \to \Omega(P_1) \to 0$ and a split sequence. The argument over the ranks tells us that the above $(projective)=0$ and $rank_{\mathcal{O}}\Omega(M)=8$.

(2) This follows immediately from $\mathscr{A}(\Omega(P_1))$ and (1). ■

Now let the tree class of Θ be $\cdots - Y_2 - Y_1 - Y - P_1 - M - M_1 - M_2 - \cdots$.

Let $\mathscr{A}(Y)$ be $0 \to \Omega(Y) \to P_1 \oplus \Omega(Y_1) \oplus (projective) \to Y \to 0$. Since the ranks of $Y, P_1, \Omega(Y)$ and $\Omega(P_1)$ are 8, we have that (a) $(projective)=0$, (b) $\Omega(Y_1) \neq 0$ and its rank is 8, and from $\mathscr{A}(\Omega(Y))$, (c) $rank_{\mathcal{O}}(Y_1)=8$. Similarly, for $\mathscr{A}(Y_1)$, we have that (a_1) its middle term has no projective modules, (b_1) $\Omega(Y_2) \neq 0$ and its rank is 8, and (c_1) $rank_{\mathcal{O}}(Y_2)=8$, since the ranks of $Y_1, Y, \Omega(Y_1)$ and $\Omega(Y)$ are 8. This inductive argument can be continued for $Y_i\,(i=2,3,\cdots)$. But this contradicts that Θ has tree class A_∞.

In both cases, (3.10) gives a contradiction. So we have that $U \not\simeq P_1$. In the same way, we have that $U \not\simeq P_2$, using the sequence (ii) in Lemma 3.9(1).

Finally, it is easy to see that $U \not\simeq \Omega(P_i)$ $(i=I,1,2,3)$ by virtue of the above argument for $P_i (i=I,1,2,3)$. Hence, such a U does not exist. Now, the proof of Proposition 3.8 is done and we are ready to exhibit the example that we have mentioned;

EXAMPLE 3.12. With the above notations. Put $M=M_0$ and $L=L_0$. Let Λ be the subquiver of Δ obtained by removing Q_1 and $\Omega(Q_1)$ from Δ. Then $\Lambda \simeq \Theta$ holds by Corollary 2.6. That is, Kawata's morphism from $\Theta(3.7)$ to $\Delta(3.6)$ is not isomorphic.

REMARK. In the case $R=k$, an example similar to ours has already given by Okuyama in [8].

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