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Osaka J. Math.
32 (1995), 475–482

NOTES ON TRIVIAL SOURCE MODULES

Dedicated to Professor S. Endo on his 60th birthday

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(Received July 8, 1993)

1. Introduction

Let G be a finite group and k an algebraically closed field of characteristic p. An indecomposable kG-module with a vertex Q is said to be *a weight module* if its Green correspondent with respect to $(G,Q,N_G(Q))$ is simple. Let B be a block of kG. Alperin [1] conjectured that the number of the weight modules belonging to B equals that of the simple modules in B. If this is the case and a defect group of B a TI set, then it can be shown under some additional assumption that the socles of weight modules are simple, which in turn determine the isomorphism classes of the weight modules ; this holds if G is a simple group with a cyclic Sylow p-subgroup. This rather surprising property has been known to hold for finite groups of Lie type of characteristic p. However little is known about general properties of weight modules. In the final section we shall study solvable groups that have only simple weight modules.

Throughout this paper G denotes a finite group and k an algebraically closed field of prime characteristic p. For a kG-module M, hd(M), soc(M) and P(M) denote the head, socle and projective cover of M respectively. If N is a kG-module, $N|M$ indicates that N is isomorphic to a direct summand of M, and (N,M) denotes the multiplicity of N as a summand of M. We fix a block B of kG and let D be its defect group. IRR(B) denotes a full set of non-isomorphic simple modules in $B, l(B)$ its cardinality and WM(B) a full set of non-isomorphic weight modules belonging to B.Let f be the Green correspondence with respect (G,D,H), where $H=N_G(D)$. If WM$(B|D)$ denotes the subset of WM(B) consisting of the weight modules with vertices D and b the Brauer correspondent of B in kH, then f induces a bijection between WM$(B|D)$ and IRR(b).

The author thanks the referee for improving the proof of Proposition 4 below.

2. Weight modules over blocks with TI defect groups

To begin with, we quote the following as a preliminary lemma.

Lemma 1 (Robinson [8]). *Let T be a subgroup of G. Let M (resp. N) be* *a simple kG (resp. kT)-module. Then we have* $(\mathrm{P}(M),N^G)=(\mathrm{P}(N),M_{|H})$.

Throughout this section D is assumed to be a non-trivial TI subgroup of G, i.e., $D\cap D^x=1$ if $x\in G\backslash H$. Let $\mathrm{IRR}(B)=\{M_1,\cdots,M_r\}$, $\mathrm{IRR}(b)=\{W_1,\cdots,W_e\}$ and $n_i=\dim_k W_i$. We set $\mathrm{WM}(B|D)=\{V_i=f^{-1}(W_i);\ 1\le i\le e\}$. Note that $\mathrm{WM}(B)=\mathrm{WM}(B|D)$. In fact, let $V\in\mathrm{WM}(B)$ and $Q=\mathrm{vx}(V)$. We may assume that $D\supset Q$. If $D>Q$, then $N_D(Q)>Q$. On the other hand, since D is a TI set, it follows that $H\supset N_G(Q)$ and hence $N_D(Q)$ is normal in $N_G(Q)$. So, $N_G(Q)/Q$ fails to have a block of defect zero. This is a contradiction, since $f(V)$ is simple and projective as an $N_G(Q)/Q$-module.

Lemma 2. $M_{i|H}=f(M_i)\oplus N_i$, *where* $N_{i|D}$ *is projective and* $f(M_i)_{|D}$ *has no projective summand.*

Proof. If L is an indecomposable component of N_i with vertex P, then P lies in $\mathfrak{Y}(D,H)$, where

$$\mathfrak{Y}(D,H)=\{Q;Q\subset D^x\cap H,x\in G\backslash H\}.$$

By the Mackey decomposition theorem we have

$$(L\otimes_P kH)_{|D}=\oplus\sum_{y\in P\backslash H/D}(L\otimes_P y)\otimes_{P^y\cap D}kD.$$

There is $x\in G\backslash H$ such that $P\subset D^x\cap H$. Hence for any $y\in H$, we have

$$P^y\cap D\subset D^{xy}\cap D\cap H=1,\ \text{as } xy\in G\backslash H.$$

Therefore $(L\otimes_P kH)_{|D}$ is projective. Since $L|L\otimes_P kH$, $L_{|D}$ is also projective.

We next show that $f(M_i)_{|D}$ is projective-free. Actually, this is a general fact. Note that $f(M_i)$ belongs to b and b has the normal defect group D. So, it suffices to show that if L is a non-projective indecomposable b-module, then $L_{|D}$ is projective-free. But since L is D-projective, this is a routine work, using Mackey decomposition.

Lemma 3. $\mathrm{Hom}_{kG}(M_i,V_j)\simeq\mathrm{Hom}_{kH}(f(M_i),W_j)$ *for all* i, j.

Proof. There is an isomorphism

$$\mathrm{Hom}_{KG}(M_i,V_j)/\mathrm{Tr}_{\mathfrak{X}}^G(M_i,V_j)\simeq\mathrm{Hom}_{kH}(f(M_i),W_j)/\mathrm{Tr}_{\mathfrak{X}}^H(f(M_i),W_j),$$

where $\mathfrak{X}=\mathfrak{X}(D,H)=\{Q;Q\subset D^x\cap D,\ x\in G\backslash H\}$. However, since D is a TI set, we have $\mathfrak{X}=\{1\}$. And if M and V are non-projective indecomposable and if one of them is simple, then $\mathrm{Tr}_1^G(M,V)=0$, whence the result follows.

Proposition 4. *Let ε be the block idempotent of B. Then we have*

$$(k_D)^G\varepsilon \simeq \oplus \sum_{i=1}^{e} n_i V_i \oplus \oplus \sum_{i=1}^{r} a_i \mathrm{P}(M_i), \text{ with } a_i=(kD, M_{i|D}).$$

Proof. Let

$$k[H/D]=\sum_{i=1}^{m} n_i W_i$$

be an indecomposable decomposition. Note that no W_j belongs to b if $j \geq e+1$. Since D is a TI set, we have

$$W_i^G = f^{-1}(W_i) \oplus (\text{projectives}).$$

Moreover we know by Green's theorem that $V_j=f^{-1}(W_j)$ does not belongs to B if $j \geq e+1$. Thus

$$(k_D)^G=(k_D^H)^G=k[H/D]^G=\oplus \sum_{i=1}^{e} n_i V_i \oplus \oplus \sum_{j=e+1}^{m} n_j V_j \oplus (\text{projectives}),$$

whence we have

$$(k_D)^G\varepsilon = \oplus \sum_{i=1}^{e} n_i V_i \oplus \oplus \sum_{i=1}^{r} a_i \mathrm{P}(M_i), \text{ with } a_i \geq 0$$

and by Lemma 1, $a_i=(kD, M_{i|D})$ for $i=1,2,\cdots,r$.

Theorem 5. *Assume that D is a* TI *set and that* $\mathrm{hd}(f(M_i))$ *is simple for all i. Then we have the following*:

(1) $l(B) \geq l(b)$;

(2) *the equality sign in the above holds if and only if* $\mathrm{soc}(V_i)$ *is simple for all i ($1 \leq i \leq e$), in which case we have that*

$$\mathrm{soc}(V_i) \simeq \mathrm{soc}(V_j) \text{ if and only if } V_i \simeq V_j.$$

Proof. From the assumption we may set $\mathrm{hd}(f(M_i))=W_{\tau(i)}$ ($1 \leq i \leq r$, $1 \leq \tau(i) \leq e$). By lemma 3 we find easily that

(i) $M_i | \mathrm{soc}(V_{\tau(i)})$ with multiplicity one.

(ii) If $M_i | \mathrm{soc}(V_j)$, then $j=\tau(i)$.

Now, the second assertion yields that the map

$$\tau\colon \{1,2,\cdots,r\} \to \{1,2,\cdots,e\}$$

is a surjection. In fact, for an arbitrary V_j, take M_i such that $M_i|\mathrm{soc}(V_j)$. Then $j=\tau(i)$. Thus τ is surjective. In particular, we have that $r\geq e$.

To show the second part of the theorem, suppose that $\mathrm{soc}(V_j)$ is is simple for all j.Then $M_i=\mathrm{soc}(V_{\tau(i)})$ and hence τ is a bijection. Therefore we have $r=e$. If, convesely, $r=e$, then τ is a bijection. This implies by (ii) above that $\mathrm{soc}(V_i)$ must be simple and $V_i\simeq V_j$ if and only if $\mathrm{soc}(V_i)\simeq\mathrm{soc}(V_j)$.

REMARK 1. If the Alperin conjecture is true, we always have $l(B)=l(b)$ when D is a TI set.

3. Weight modules for the symmetric group S_p

In this section we assume that $G=S_p$ is the symmetric group on p letters. If D is a Sylow p-subgroup of G, then D has order p and $C_G(D)=D, H/D\simeq(Z/(p))^*$, the group of units of $Z/(p)$. In particular, it follows that $b=kH$ is the block of kH. Let us wtite

$$\mathrm{IRR}(b)=\{W_0,\cdots,W_{p-2}\},\ \text{where } \dim_k W_i=1\ (0\leq i\leq p-2).$$

If B denotes the principal block of G, then B is a unique block of kG of positive defect and $l(B)=p$-1. The decomposition matrix of B is known. It can be displayed as follows, see James [5].

	φ_0	φ_1	φ_2		φ_{p-2}
$\chi_0=(p)=1_G$	1				
$\chi_1=(p-1,1)$	1	1			**0**
$\chi_2=(p-2,1^2)$		1	1		
$\vdots$			$\ddots$	$\ddots$	
$\chi_{p-2}=(2,1^{p-2})$	**0**			1	1
$\chi_{p-1}=(1^p)$					1

Since $\chi_i=\varphi_{i-1}+\varphi_i$ and $\deg\chi_i={}_{p-1}C_i$, we find via induction that $\deg\varphi_i={}_{p-2}C_i$ $(0\leq i\leq p-2)$. So we can label the simple modules is B such that

$$\mathrm{IRR}(B)=\{M_0,\cdots,M_{p-2}\},\ \text{with } m_i=\dim_k M_i={}_{p-2}C_i.$$

Here we note the following facts on binomial coefficients ${}_nC_i$.

Lemma 6.

(1) $$m_i={}_{p-2}C_i=\begin{cases} i+1 \bmod p, & \text{if } i \text{ is even;} \\ p-i-1 \bmod p, & \text{if } i \text{ is odd.} \end{cases}$$

(2) *Suppose that* $n \geq 4$. *If* $2 \leq i \leq n\text{-}2$, *then* ${}_nC_i \geq n+2$.

Now, since H/D is abelian, every principal indecomposable module over kH has dimension p, and thus every non-projective indecomposable module has dimension smaller than p. In particular if follows that $\dim_k f(M_i) < p$. By Lemma 2, we can write

$$M_{i|D} = f(M_i)_{|D} \oplus a_i kD.$$

For $i=0,1,p-3$ or $p-2$, we have that $m_i < p$ and so $a_i = 0$. This is true for all i, provided $p \leq 5$. Suppose $p > 5$. If $2 \leq i \leq p-4$, then $m_i \geq p$ by Lemma 6(2) and hence $a_i > 0$. This, together with Lemma 6(1) yields that $\dim_k f(M_i) = i+1$ or $p-i-1$ according as whether i is even or odd $(2 \leq i \leq p-4)$. Thus we have:

$$a_i = \begin{cases} (m_i - i - 1)/p, & \text{if } i \text{ is even;} \\ (m_i - (p-i-1))/p, & \text{if } i \text{ is odd.} \end{cases}$$

Now we have the following result by Lemma 1 and Proposition 4.

Proposition 7. *Let* $\mathrm{WM}(B) = \{V_0, \cdots, V_{p-2}\}$, *where* $V_i = f^{-1}(W_i)$, *and let* $\{U_j;\ 1 \leq j \leq q\}$ *be the set of simple kG-modules belonging to the blocks of defect zero. Then we have*

$$(k_D)^G \simeq \oplus \sum_{i=0}^{p-2} V_i \oplus \oplus \sum_{i=2}^{p-4} a_i \mathrm{P}(M_i) \oplus \oplus \sum_{i=1}^{q} (\dim_k U_i/p) U_i.$$

4. Socles of weight modules

In view of Theorem 5, it seems to be natural to consider the following situation:

(♯) Every weight module belonging to B has a simple socle, and for U, $V \in \mathrm{WM}(B)$, we have

$$\mathrm{soc}(U) \simeq \mathrm{soc}(V) \quad \text{if and only if} \quad U \simeq V.$$

We first remark that

Proposition 8. *If G is a simple group with a cyclic Sylow p-subgroup, the condition* (♯) *holds for every block B.*

In fact we know that a Sylow p-subgroup is a TI set (Blau[2]) and that $l(B) = l(b)$, hence the result follows from Theorem 5.

On the other hand, we have the following, as is shown on pp.370–371 in

Alperin [1].

Proposition 9 (Alperin). *Let G be a finite group of Lie type of characteristic p. Then the condition (♯) holds for every block B.*

Before proceeding let us recall that *a simple module is a weight module if and only if it has trivial source* (Okuyama [7]).

Now, for the rest of this paper we assume that G is solvable. In this case the Alperin conjecture has been proved by Okuyama.

DEFINITION. A solvable group G is said to be p'-supersolvable if all of its chief composition factors of order prime to p are cyclic.

Proposition 10. *If G is p'-supersolvable, every simple module has trivial source. Hence* $\mathrm{WM}(B)=\mathrm{IRR}(B)$ *for every block B.*

Proof. Let G be a counter-example of minimum order and let V be a simple kG-module with source not isomorphic to k. Let K be a maximal abelian normal p'-subgroup of G and W a simple summand of $V_{|K}$. By Fong's reduction and the minimality of G, W must be G-invariant. So W is faithful as K-module and hence K must be central. If $O_p(G/K)=1$, G/K has a cyclic normal p'-subgroup, say M/K. Then M is abelian, contradicting the choice of K.Thus $O_p(G/K)>1$, which implies that $O_p(G)>1$, since K is central. This is a contradiciton.
The second statement is clear since the number of weight modules belonging to B equals $l(B)$.

Now we give a definition:

DEFINITION. A finite group is said to be a CR1-group if all of its characteristic abelian subgroup are cyclic.

We say that the group G involves a group T provided there are subgroups $L \triangleright M$ of G such that $L/M \simeq T$. For a prime number q, let us denote a Sylow q-subgroup of G by G_q.

Theorem 11. *Let G be a solvable group and suppose that G_q involves no non-adelian CR1-group for each prime q different from p. Then the conclusion of Proposition 10 holds.*

Proof. We shall show that every simple kG-module has trivial source by the induction on the order of G.We may assume $O_p(G)=1$. Let K be the Fitting subgroup of G, so we have that $C_G(K) \subset K$. If K is cyclic, Aut(K) is abelian and so is G/K. Thus G is supersolvable and the result follows from Proposition 10. If K is non-cyclic, our assumption implies that G has a non-cylic abelian normal q-subgroup, say L, for some prime q. Let V be a simple kG-module and W a simple summand of $V_{|L}$. If the inertial group of W is proper, the result follows by induction. If W is G-invariant, $N=\mathrm{Ker}(W)$ is a non-trivial normal subgroup of G.Then we get the result by applying the inductive hypothesis to G/N.

REMARK 2. The CR1-q-groups are classified (Gorenstein [3], Chap. 5). In particular, a non-abelian CR1-q-group contains D_3 or Q_3 if $q=2$, while it contains $M(q)$ if q is odd, where

$$M(q)=\langle x,y,z;\ x^q=y^q=z^q=1,\ [x,z]=[y,z]=1,\ [x,y]=z\rangle,$$

which has order q^3 and exponent q.

REMARK 3. One may show that the following q-group Q involves no non-abelian CR1-q-group:

$$Q=\langle x,y\colon\ x^{q^a}=y^{q^b}=1,\ x^y=x^{1+q^{a-1}}\rangle,$$

where $a\geq 2$, $b\geq 1$, and $a\geq 3$ if $q=2$.

In fact every proper subgroup of Q is abelian (cf. Huppert [4] III, Aufgaben 22). So it suffices to show that Q has no factor group isomorphic to D_3, Q_3 or $M(q)$, which will be easily done.

REMARK 4. Let $G=\langle\sigma\rangle$ be the semidirect product, where σ is an automorphism of the quaternion group Q_3 of order 3. Then kG has a simple module whose source is not trivial, where k is of characteristic 3. On the other hand, if G is the symmetric group S_4, every simple kG-module has trivial source, k being the same as above. In both groups the Sylow 2-subgroups are CR1-groups.

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