Sturm, K-T.
Osaka J. Math.
32 (1995), 275–312

# ANALYSIS ON LOCAL DIRICHLET SPACES - II. UPPER GAUSSIAN ESTIMATES FOR THE FUNDAMENTAL SOLUTIONS OF PARABOLIC EQUATIONS

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(Received August 6, 1993)

## 0. Introduction

For every $t \in \boldsymbol{R}$, let $\mathscr{E}_t$ be a regular, local Dirichlet form with common domain $\mathscr{F} \subset L^2(X,m)$ on a locally compact, separable Hausdorff space $X$. We study the behaviour of (global as well as local) solutions of the parabolic equation

$$L_t u = \tfrac{\partial}{\partial t} u \quad \text{on } \boldsymbol{R} \times X. \tag{0.1}$$

The (not necessarily selfadjoint) operators $L_t$ on $L^2(X,m)$ are supposed to be associated with the (not necessarily symmetric) Dirichlet forms $\mathscr{E}_t$ on $L^2(X,m)$ according to

$$-(L_t u, v) = \mathscr{E}_t(u,v).$$

To simplify things, let us first of all consider the case where all the $\mathscr{E}_t$'s are symmetric and strongly local. In this case, the only assumption in the whole paper which is imposed on the forms $\mathscr{E}_t$ is the uniform parabolicity condition

$$k \cdot \mathscr{E}(u,u) \leq \mathscr{E}_t(u,u) \leq K \cdot \mathscr{E}(u,u). \tag{0.2}$$

Here $\mathscr{E}$ is a fixed symmetric and strongly local, regular Dirichlet form. In terms of $\mathscr{E}$ we define the intrinsic distance $\rho$ on $X$ which is *assumed* to reproduce the original topology on $X$. The main result of the first part of this paper is the following integrated upper Gaussian estimate.

**Theorem 0.1.** *The transition operators $T_t^s$, $s<t$, associated with* (0.1) *can be estimated as follows*

$$(T_t^s 1_A, 1_B) \leq \sqrt{m(A)} \cdot \sqrt{m(B)} \cdot \exp\left(-\frac{\rho^2(A,B)}{4K(t-s)}\right) \cdot \exp(-k\lambda(t-s)) \tag{0.3}$$

for all $s<t$ *and all subsets* $A,B\subset X$ *of finite measure.*

Here $\lambda\geq 0$ denotes the bottom of the spectrum of the selfadjoint operator $-L$. Note that if there exists a fundamental solution $p(t,.,s,.)$ for equation (0.1) then the LHS of (0.3) is just $\int_A\int_B p(t,y,s,x)m(dx)m(dy)$.

In order to obtain pointwise estimates we have to impose further conditions. These are assumptions on the Dirichlet form $\mathscr{E}$. Namely, in the second part of this paper we assume that a doubling property and a scale invariant Sobolev inequality hold true on the state space $X$. Under these assumptions we carry out the Moser iteration for local sub- or supersolutions to obtain pointwise (upper and lower) estimates in term of $L^p$-means over suitable parabolic cylinders. We combine these subsolution estimates with the integrated Gaussian estimate (0.3) in order to obtain pointwise estimates for the fundamental solution $p(t,.,s,.)$ of (0.1).

**Theorem 0.2.** *There exist constants* $C$ *and* $N$ *such that*

$$p(t,y,s,x)\leq C\cdot m^{-1/2}(B_{\sqrt{t-s}}(x))\cdot m^{-1/2}(B_{\sqrt{t-s}}(y))\cdot \\ \cdot\exp\left(-\frac{\rho^2(x,y)}{4K(t-s)}\right)\cdot\left(1+\frac{\rho(x,y)^2}{K(t-s)}\right)^{N/2} \tag{0.4}$$

*uniformly for all points* $(s,x)$ *and* $(t,y)\in\boldsymbol{R}\times X$ *with* $s<t$.

Note that for every $\varepsilon>0$ the RHS of (0.4) can be estimated by

$$p(t,y,s,x)\leq C'\cdot m^{-1/2}(B_{\sqrt{t-s}}(x))\cdot m^{-1/2}(B_{\sqrt{t-s}}(y))\cdot\exp\left(-(1-\varepsilon)\frac{\rho^2(x,y)}{4K(t-s)}\right)$$

with a constant $C'=C'(\varepsilon)$.

For parabolic divergence form operators on $\boldsymbol{R}^N$ this type of estimate is due to E.B. Davies [6] improving previous results by D.G. Aronson [1]. For Laplace-Beltrami operators on Riemannian manifolds it is due to P. Li and S.T. Yau [23] (whose result was improved by E.B. Davies, L. Saloff-Coste, N. Varopoulos and many others). Finally, for Hörmander type and general subelliptic operators on $\boldsymbol{R}^N$ this Gaussian estimate is due to D. Jerison and A. Sanchez-Calle [18] and to S. Kusuoka and D.W. Stroock [19].

In the particular time-independent case $L_t\equiv L$, (0.4) is just an estimate for the heat kernel for $L$. From this heat kernel estimate one easily deduces the following Green function estimate:

$$g(x,y)\leq C\cdot\int_{\rho^2(x,y)}^{\infty} m^{-1}(B_{\sqrt{t}}(x))dt$$

which in this generality is due to M.Biroli and U.Mosco [2,3] (extending previous results by many other authors).

## 1. Time-dependent Dirichlet spaces and parabolic equations

### 1.1. The initial Dirichlet space

#### A) The Hilbert spaces $\mathcal{H}, \mathcal{F}, \mathcal{F}^*$

The basic object for the sequel is a fixed regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}=\mathcal{F}(X)$ on a real Hilbert space $\mathcal{H}=L^2(X,m)$ with norm $\|u\|=(\int_X u^2\,dm)^{1/2}$. $\mathcal{F}$ is again a real Hilbert space with norm $\|u\|_{\mathcal{F}}:=\sqrt{\mathcal{E}(u,u)+\|u\|^2}$. We identify $\mathcal{H}$ with its own dual; the dual of $\mathcal{F}$ is denoted by $\mathcal{F}^*$. Thus we have

$$\mathcal{F} \subset \mathcal{H} \subset \mathcal{F}^*$$

with continuous and dense embeddings. We shall use the same notation $(.,.)$ for the inner product in $\mathcal{H}$ and for the pairing between $\mathcal{F}^*$ and $\mathcal{F}$.

#### B) The Dirichlet form $\mathcal{E}$

The underlying topological space $X$ is a locally compact separable Hausdorff space and $m$ is a positive Radon measure with $\operatorname{supp}[m]=X$. The initial Dirichlet form $\mathcal{E}$ is always assumed to be *symmetric* (i.e. $\mathcal{E}(u,v)=\mathcal{E}(v,u)$) and *strongly local* (i.e. $\mathcal{E}(u,v)=0$ whenever $u\in\mathcal{F}$ is constant on a neighborhood of the support of $v\in\mathcal{F}$ or, in other words, $\mathcal{E}$ has no killing measure and no jumping measure). The selfadjoint operator associated with the initial form $\mathcal{E}$ is denoted by $L$.

#### C) The energy measure $\Gamma$

Any such form can be written as

$$\mathcal{E}(u,v)=\int_X d\Gamma(u,v)$$

where $\Gamma$ is a positive semidefinite, symmetric bilinear form on $\mathcal{F}$ with values in the signed Radon measures on $X$ (the so-called energy measure). It can be defined by the formulae

$$\begin{aligned}\int_X \phi\, d\Gamma(u,u) &= \mathcal{E}(u,\phi u)-\frac{1}{2}\mathcal{E}(u^2,\phi)\\ &=\lim_{t\to 0}\frac{1}{2t}\int_X\int_X \phi(x)\cdot[u(x)-u(y)]^2 T_t(x,dy)m(dx)\end{aligned}$$

for every $u \in \mathscr{F}(X) \cap L^\infty(X,m)$ and every $\phi \in \mathscr{F}(X) \cap \mathscr{C}_0(X)$. Since $\mathscr{E}$ is assumed to be strongly local, the energy measure $\Gamma$ is local and satisfies the Leibniz rule as well as the chain rule, cf. [13],[21],[36]. As usual we extend the quadratic forms $u \mapsto \mathscr{E}(u,u)$ and $u \mapsto \Gamma(u,u)$ to the whole spaces $L^2(X,m)$ resp. $L^2_{\text{loc}}(X,m)$ in such a way that $\mathscr{F}(X) = \{u \in L^2(X,m) : \mathscr{E}(u,u) < \infty\}$ and $\mathscr{F}_{\text{loc}}(X) = \{u \in L^2_{\text{loc}}(X,m) : \Gamma(u,u)$ is a Radon measure$\}$.

### D) The intrinsic metric $\rho$

The energy measure $\Gamma$ defines in an intrinsic way a pseudo metric $\rho$ on $X$ by

$$\rho(x,y) = \sup\{u(x) - u(y) : u \in \mathscr{F}_{\text{loc}}(X) \cap \mathscr{C}(X), d\Gamma(u,u) \leq dm \text{ on } X\}. \tag{1.1}$$

The condition $d\Gamma(u,u) \leq dm$ in (1.1) means that the energy measure $\Gamma(u,u)$ is absolutely continuous w.r.t. the reference measure $m$ with Radon-Nikodym derivative $\frac{d}{dm}\Gamma(u,u) \leq 1$. The density $\frac{d}{dm}\Gamma(u,u)(z)$ should be interpreted as the square of the (length of the) gradient of $u$ at $z \in X$. In general, $\rho$ may be degenerate (i.e. $\rho(x,y) = \infty$ or $\rho(x,y) = 0$ for some $x \neq y$). This (pseudo) metric will be discussed again at the end of chapter 1.

### E) Examples

The main examples which we have in mind are:

- $L$ is the Laplace-Beltrami operator on a Riemannian manifold and $m$ is the Riemannian volume; in this case, $\rho$ is just the Riemannian distance.
- $L$ is a uniformly elliptic operator with a nonnegative weight $\phi$ on $\boldsymbol{R}^N$, i.e. $\mathscr{E}(u,v) = \Sigma_{i,j=1}^N \int a_{ij} \cdot \frac{\partial}{\partial x_i} u \cdot \frac{\partial}{\partial x_j} v \cdot \phi dx$ and $(u,v) = \int uv\phi dx$ with $(a_{ij})$ symmetric and uniformly elliptic and $\phi$ as well as $\phi^{-1} \in L^1_{loc}(\boldsymbol{R}^N, dx)$; in this case, $\rho$ is equivalent to the Euclidean distance (cf. [3],[24],[37a],[37c]).
- $L$ is a subelliptic operator on $\boldsymbol{R}^N$, i.e. $\mathscr{E}(u,v) = \Sigma_{i,j=1}^N \int a_{ij} \cdot \frac{\partial}{\partial x_i} u \cdot \frac{\partial}{\partial x_j} v \, dx$ and $(u,v) = \int uv \, dx$ with $(a_{ij})$ symmetric and elliptic and such that $\mathscr{E}(u,u) \geq \delta \cdot \|u\|^2_{H^\varepsilon} - \|u\|^2$ for some $\delta, \varepsilon > 0$; in this case, $\rho$ is equal to the metric used e.g. by Fefferman/Phong [11], Fefferman/Sanchez-Calle [12], Jerison/Sanchez-Calle [18], Nagel/Stein/ Wainger [27]; it can be locally estimated by the Euclidean distance $|.|$ as follows

$$\frac{1}{C} \cdot |x-y| \leq \rho(x,y) \leq C \cdot |x-y|^\varepsilon.$$

Further examples will be discussed in [37a] and [37c].

## 1.2. The Dirichlet forms $\mathscr{E}_t$

In the sequel we will study the behaviour of solutions of a parabolic equation

$L_t u = \frac{\partial}{\partial t} u$ on (an open subset of) $\boldsymbol{R} \times X$. Here $\{L_t\}_{t \in \boldsymbol{R}}$ is a uniformly parabolic operator in the following sense.

### A) One-parameter family of Dirichlet forms

We assume that for every $t \in \boldsymbol{R}$ we are given a regular, local Dirichlet form $\mathscr{E}_t$ with domain $\mathscr{D}(\mathscr{E}_t) \equiv \mathscr{F}$. We do *not* require that $\mathscr{E}_t$ is symmetric or strongly local.

The negative semidefinite, closed and densely defined operator on $\mathscr{H} = L^2(X,m)$ associated with the Dirichlet form $\mathscr{E}_t$ is denoted by $L_t$ and $\hat{L}_t$ denotes its adjoint operator which is associated with the adjoint form $\hat{\mathscr{E}}_t : (u,v) \mapsto \mathscr{E}_t(v,u)$. That is, for all $u,v \in \mathscr{F}$

$$-(L_t u, v) = \mathscr{E}_t(u,v) = \hat{\mathscr{E}}_t(v,u) = -(\hat{L}_t v, u).$$

As usual we can decompose $\mathscr{E}_t$ into its symmetric part $\tilde{\mathscr{E}}_t = \frac{1}{2}(\mathscr{E}_t + \hat{\mathscr{E}}_t)$ and its antisymmetric part $\check{\mathscr{E}}_t = \frac{1}{2}(\mathscr{E}_t - \hat{\mathscr{E}}_t)$ such that $\mathscr{E}_t = \tilde{\mathscr{E}}_t + \check{\mathscr{E}}_t$. The symmetric part $\tilde{\mathscr{E}}_t$ in turn can be decomposed into its diffusion part $\tilde{\mathscr{E}}_t^{diff}$ and its killing part $\tilde{\mathscr{E}}_t^{kill}$.

### B) The uniform parabolicity condition

The one-parameter family $\{\mathscr{E}_t\}_{t \in \boldsymbol{R}}$ of these Dirichlet forms is assumed to be *uniformly parabolic* with respect to the initial Dirichlet form $\mathscr{E}$ in the following sense:

**Assumption (UP).** *There exist constants $K,k \in ]0,\infty[$ and $\gamma \in [0,\infty[$ such that*

$$-\mathscr{E}_t(u,v) \leq K \cdot \mathscr{E}^{\#}(u,v) - k \cdot \mathscr{E}(\sqrt{uv}, \sqrt{uv}) + \gamma \cdot (u,v) \tag{1.2}$$

*for all $t \in \boldsymbol{R}$ and all $u,v \in \mathscr{F}$ with $uv \geq 0$ and $\sqrt{uv} \in \mathscr{F}$ where*

$$\mathscr{E}^{\#}(u,v) = \mathscr{E}(\sqrt{uv}, \sqrt{uv}) - \mathscr{E}(u,v).$$

Moreover, we assume that for all $u,v \in \mathscr{F}$ the map $t \mapsto \mathscr{E}_t(u,v)$ is measurable and that the sector condition holds uniformly in $t$, that is, there exists a constant $C \in ]0,\infty[$ such that

$$\mathscr{E}_t(u,v) \leq C \cdot \|u\|_{\mathscr{F}} \cdot \|v\|_{\mathscr{F}} \tag{1.3}$$

for all $t \in \boldsymbol{R}$ and all $u,v \in \mathscr{F}$.

Note that in the case $\mathscr{E}_t \equiv \mathscr{E}$, condition (1.2) is always satisfied with $K = k = 1$ and $\gamma = 0$. In the general case, condition (1.2) for $(\mathscr{E}_t)_t$ is equivalent to condition (1.2) for $(\hat{\mathscr{E}}_t)_t$ (with the same constants). Analogous statements hold for condition (1.3).

Also note that according to the following Lemma 1.1 for symmetric, strongly local Dirichlet forms $\mathscr{E}_t$ condition (1.2) is equivalent to the condition

$$k \cdot \mathscr{E}(u,u) - \gamma \cdot (u,u) \leq \mathscr{E}_t(u,u) \leq K \cdot \mathscr{E}(u,u) \tag{1.4}$$

which in turn implies the uniform sector condition (1.3) with $C \equiv K$.

REMARK. The quantity $\mathscr{E}^*(u,v)$ appearing in the condition (UP) has a particular nice representation in the case $u=e^{-\psi}w$ and $v=e^{\psi}w$ with $w,\psi \in \mathscr{F}(X)\cap L^\infty(X)$. Namely,

$$\mathscr{E}^*(e^{-\psi}w,e^{\psi}w)=\int_X w^2 d\Gamma(\psi,\psi).$$

In order to see that, write

$$\mathscr{E}^*(e^{-\psi}w,e^{\psi}w)=\mathscr{E}(w,w)-\mathscr{E}(e^{-\psi}w,e^{\psi}w)=\int d\Gamma(w,w)-\int d\Gamma(e^{-\psi}w,e^{\psi}w)$$

$$=\int d\Gamma(w,w)-\left[\int d\Gamma(w,w)+\int e^{-\psi}w d\Gamma(w,e^{\psi})+\int e^{\psi}w d\Gamma(e^{-\psi},w)+\int w^2 d\Gamma(e^{-\psi},e^{\psi})\right]$$

$$=\int_X w^2 d\Gamma(\psi,\psi).$$

Similarly, one sees that for $u,\phi \in \mathscr{F}(X)\cap L^\infty(X)$

$$\mathscr{E}^*(u,u\phi^2)=\int_X u^2 d\Gamma(\phi,\phi).$$

From these formulae one concludes by approximation that

$$\mathscr{E}^*(u,v)\geq 0$$

for all $u,v\in\mathscr{F}$ with $uv\geq 0$ and $\sqrt{uv}\in\mathscr{F}$. Finally, note that of course $\mathscr{E}^*(u,u)=0$ for all $u\in\mathscr{F}$.

**Lemma 1.1.**

*i) Condition* (1.2) *is equivalent to each of the following conditions*:

$$-\mathscr{E}_t(u,u\phi^2)\leq K\cdot\int_X u^2 d\Gamma(\phi,\phi)-k\cdot\mathscr{E}(u\phi,u\phi)+\gamma\cdot(u\phi,u\phi) \tag{1.5}$$

*for all* $t\in \boldsymbol{R}$ *and all* $u,\phi\in\mathscr{F}(X)\cap L^\infty(X)$ *or*

$$-\mathscr{E}_t(e^{-\psi}w,e^{\psi}w)\leq K\cdot\int_X w^2 d\Gamma(\psi,\psi)-k\cdot\mathscr{E}(w,w)+\gamma\cdot(w,w) \tag{1.6}$$

*for all* $t\in\boldsymbol{R}$ *and all* $w,\psi\in\mathscr{F}(X)\cap L^\infty(X)$.

*ii) Condition* (1.2) *implies*

$$\mathscr{E}_t(u,u) \geq k \cdot \mathscr{E}(u,u) - \gamma \cdot (u,u) \tag{1.7}$$

*and*

$$\tilde{\mathscr{E}}_t^{diff}(u,u) \leq K \cdot \mathscr{E}(u,u) \tag{1.8}$$

(*whereas condition* (1.3) *implies* $\mathscr{E}_t(u,u) \leq C \cdot (\mathscr{E}(u,u) + (u,u)))$ *for all* $t \in \boldsymbol{R}$ *and all* $u \in \mathscr{F}$. *Here* $\tilde{\mathscr{E}}_t^{diff}$ *denotes the diffusion part of* $\mathscr{E}_t$ (*cf. sect.* 1.2.*A*).

*iii) Let* $\mathscr{E}_t$ *be symmetric. Then condition* (1.2) *is equivalent to the pair of conditions* (1.7) *and* (1.8). *Moreover, condition* (1.3) *is equivalent to the condition*

$$\mathscr{E}_t(u,u) \leq C \cdot (\mathscr{E}(u,u) + (u,u)) \tag{1.9}$$

*for all* $t \in \boldsymbol{R}$ *and all* $u \in \mathscr{F}$.

Proof. i) The implications (1.2) $\Rightarrow$ (1.5) and (1.2) $\Rightarrow$ (1.6) are obvious from the preceding Remark and the fact that under the stated assumptions all functions under consideration lie in $\mathscr{F}$. For the implication (1.6) $\Rightarrow$ (1.5), approximate $\mathscr{E}_t(u,u\phi^2)$ by $\mathscr{E}_t(u,u(\phi^2+\varepsilon))$ and replace $u$ by $e^{-\psi}w$ and $\phi^2+\varepsilon$ by $e^{2\psi}$. For(1.5) $\Rightarrow$ (1.2), approximate $\mathscr{E}_t(u,v)$ by $\mathscr{E}_t(u+\varepsilon\cdot v,v)$ and replace the $v$ in the second place by $(u+\varepsilon v)\phi^2$.

ii) (1.7) is obvious since $\mathscr{E}^{\#}(u,u)=0$. To see (1.8), use the fact that (1.2) (or, more obviously, (1.5)) for $\mathscr{E}_t$ implies

$$-\tilde{\mathscr{E}}_t(e^{-\psi}w, e^{\psi}w) \leq K \cdot \int w^2 d\Gamma(\psi,\psi) + \gamma \cdot (w,w). \tag{1.10}$$

Now (using the symmetry of $\tilde{\mathscr{E}}_t$) write the LHS of (1.10) as

$$\int w^2 d\Gamma_t(\psi,\psi) - \mathscr{E}_t(w,w) = -\tilde{\mathscr{E}}_t(e^{-\psi}w, e^{\psi}w). \tag{1.11}$$

Here $\Gamma_t$ is the energy measure associated with the strongly local, symmetric Dirichlet form $\tilde{\mathscr{E}}_t^{diff}$. In (1.10) and (1.11) replace $\psi$ by $n\cdot\psi$ and $w$ by $\frac{1}{n}w$ with $n \to \infty$. This yields

$$\int w^2 d\Gamma_t(\psi,\psi) \leq K \cdot \int w^2 d\Gamma(\psi,\psi)$$

and thus $\tilde{\mathscr{E}}_t^{diff}(\psi,\psi) \leq K \cdot \mathscr{E}(\psi,\psi)$.

iii) If $\mathscr{E}_t$ is symmetric, then

$$-\mathscr{E}_t(e^{-\psi}w, e^{\psi}w) = \int w^2 d\Gamma_t(\psi,\psi) - \mathscr{E}_t(w,w).$$

Using (1.7) and (1.8), the RHS can be estimated from above by $K \cdot \int w^2 d\Gamma(\psi,\psi) -$

$k\cdot\mathscr{E}(w,w)+\gamma\cdot\|w\|^2$. This proves (1.6). In order to see that (1.9) implies (1.3) it suffices to note that for symmetric $\mathscr{E}_t$ the Cauchy-Schwarz inequality yields $\mathscr{E}_t(u,v)\le\mathscr{E}_t(u,u)^{1/2}\cdot\mathscr{E}_t(v,v)^{1/2}$. □

REMARKS. i) Let us mention that all estimates in the sequel only depend on the constants $K,k,\gamma$ in (1.2) and not on the constant $C$ in (1.3). We recall that if the $\mathscr{E}_t$'s are symmetric and strongly local, then (1.2) implies (1.3).

ii) If the family $(\mathscr{E}_t)_t$ satisfies (1.2) with constants $K=K_0>0$, $k=0$ and $\gamma=\gamma_0\ge 0$ then for any $\delta>0$ the family $(\mathscr{E}_t+\delta\cdot\mathscr{E})_t$ satisfies (1.2) with constants $K=K_0+\delta$, $k=\delta>0$ and $\gamma=\gamma_0$.

iii) If the family $(\mathscr{E}_t)_t$ satisfies (1.2) with constants $K,k$ and $\gamma=\gamma_0\ge 0$ then the family $(\mathscr{E}_t+\gamma_0\cdot(.,.))_t$ satisfies (1.2) with the same constants $K,k$ and with $\gamma=0$. Thus we can and will restrict ourselves in the sequel to the case

$$\boxed{\gamma=0.}$$

Before looking for too complicated examples of $\mathscr{E}_t$'s satisfying (1.2), we should mention that the main results of this paper are new even in the case $\mathscr{E}_t\equiv\mathscr{E}$ (for all $t\in\boldsymbol{R}$). Besides this (time-independent and symmetric) example there are two types of examples which we want to discuss now.

### C) Symmetric examples

*Let for every $t\in\boldsymbol{R}$ the Dirichlet form $\mathscr{E}_t$ be (just like $\mathscr{E}$ itself) symmetric and strongly local and the family $\{\mathscr{E}_t\}_{t\in\boldsymbol{R}}$ satisfy*

$$k\cdot\mathscr{E}(u,u)\le\mathscr{E}_t(u,u)\le K\cdot\mathscr{E}(u,u) \tag{1.12}$$

*for all $t\in\boldsymbol{R}$ and $u\in\mathscr{F}$ with constants $0<k\le K<\infty$.*

In the situations considered at the end of section 1.1, one might think of $\mathscr{E}$ having smooth coefficients $a_{ij}$ (or even $a_{ij}=\delta_{ij}$) and $\mathscr{E}_t$ being of the same type with measurable coefficients. The classical situation is that $L$ is the Laplace operator on $\boldsymbol{R}^N$ and $\{L_t\}_t$ is a second order, uniformly parabolic differential operator in divergence form (cf. [10] including references to the pioneering contributions of Nash, De Giorgi, Moser, Aronson, Davies).

This can be generalized by choosing $L$ to be the Laplace-Beltrami operator on a Riemannian manifold and $\{L_t\}_t$ to be a suitable "uniformly parabolic" operator on that manifold (cf. [33] for the elliptic case).

A similar situation occurs when $L$ is a Hörmander type operator (with $\mathscr{C}^\infty$-coefficents) and $\{L_t\}_t$ is derived from $L$ by means of bounded measurable coefficeints (cf. [34] for the elliptic case).

The previous framework can slightly be enlarged by merely assuming that for every $t\in\boldsymbol{R}$ the Dirichlet form $\mathscr{E}_t$ is symmetric and local, i.e.

$$\mathscr{E}_t(u,v)=\mathscr{E}_t^{diff}(u,v)+\int_X uv\,d\mu_t$$

where $\mathscr{E}_t^{diff}$ is of the previously considered type (i.e. symmetric and strongly local and satisfying $k\cdot\mathscr{E}(u,u)\le\mathscr{E}_t^{diff}(u,u)\le K\cdot\mathscr{E}(u,u)$) and $\mu_t$ is a nonnegative Radon measure on $X$ ("killing measure"). We emphasize that the uniform parabolicity condition (UP) gives no condition on $\mu_t$. The only restriction arises from the uniform sector condition (1.3) (and from the assumption $\mathscr{D}(\mathscr{E}_t)\equiv\mathscr{F}$).

### D) A nonsymmetric example

We will give an example of a local Dirichlet form whose antisymmetric part is controlled by the symmetric part in such a way that (1.2) holds true. To be specific, let $L$ be the Laplace operator on $L^2(\boldsymbol{R}^N,dx)$ (which implies that $\rho$ is the Euclidean distance) and let

$$\mathscr{E}_t(u,v)=\sum_{i,j=1}^N\int a_{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}v\,dx+\sum_{i=1}^N\int b_i\frac{\partial}{\partial x_i}uv\,dx+\sum_{i=1}^N\int \hat{b}_iu\frac{\partial}{\partial x_i}v\,dx+\int cuv\,dx$$

with time-dependent coefficents $a=(a_{ij})$, $b=(b_i)$, $\hat{b}=(\hat{b}_i)$, $c\in L^\infty(\boldsymbol{R}\times\boldsymbol{R}^N,dtdx)$ satisfying

(i) $k_0\cdot|\xi|^2\le\Sigma_{i,j}a_{ij}\xi_i\xi_j\le K_0\cdot|\xi|^2$ for all $\xi\in\boldsymbol{R}^N$;

(ii) $|\breve{a}|:=(\Sigma_{i,j}\breve{a}_{ij}^2)^{1/2}\le A$ (where $\breve{a}_{ij}=\frac{1}{2}(a_{ij}-a_{ji})$);

(iii) $|b|+|\hat{b}|\le 2B$;

(iv) $-c\le C$

uniformly on $\boldsymbol{R}\times\boldsymbol{R}^N$ with constants $k_0$, $K_0$, $A$, $B$ and $C$. In order that $\mathscr{E}_t$ is Markovian we also have to assume that $c-\operatorname{div}b\ge 0$ and $c-\operatorname{div}\hat{b}\ge 0$. Then for a.e. $t\in\boldsymbol{R}$, the form $\mathscr{E}_t$ on $C_0^\infty(\boldsymbol{R}^N)$ is closable with closure being a regular Dirichlet form on $\mathscr{F}=H^1(\boldsymbol{R}^N)$ satisfying (uniformly in $t$) the "sector condition" (1.3) ([24], II.2.d). Actually, also (1.2) is satisfied since

$$\begin{aligned}
&-\mathscr{E}_t(e^{-\psi}w,e^{\psi}w)\\
&=\sum\int w^2a_{ij}\frac{\partial}{\partial x_i}\psi\frac{\partial}{\partial x_j}\psi\,dx-\sum\int a_{ij}\frac{\partial}{\partial x_i}w\frac{\partial}{\partial x_j}w\,dx\\
&\quad-2\sum\int w\breve{a}_{ij}\frac{\partial}{\partial x_i}\psi\frac{\partial}{\partial x_j}w\,dx+\sum\int w^2b_i\frac{\partial}{\partial x_i}\psi dx-\sum\int wb_i\frac{\partial}{\partial x_i}w\,dx\\
&\quad-\sum\int w^2\hat{b}_i\frac{\partial}{\partial x_i}\psi\,dx-\sum\int w\hat{b}_i\frac{\partial}{\partial x_i}w\,dx-\int w^2c\,dx\\
&\le K_0\cdot\int w^2|\nabla\psi|^2dx-k_0\cdot\int|\nabla w|^2dx+C\cdot\int w^2dx
\end{aligned}$$

$$+2A\cdot\int w\cdot|\nabla w|\cdot|\nabla\psi|dx+2B\cdot\int w^2|\nabla\psi|dx+2B\cdot\int w|\nabla w|dx$$

$$\leq K\cdot\int w^2|\nabla\psi|^2dx-k\cdot\int|\nabla w|^2dx+\gamma\cdot\int w^2dx$$

with

$$K=K_0+\frac{A}{\delta}+\frac{B}{\varepsilon},\quad k_0-\delta\cdot A-\frac{B}{\varepsilon},\quad \gamma=C+2\varepsilon\cdot B$$

where $\delta$ and $\varepsilon$ are arbitrary parameters with values in $]0,\infty[$.

Note that the constants only depend on bounds of the coefficients and not of their derivatives. If one admits also conditions on $\mathrm{div}(b+\hat{b})$ then one may replace (iii) and (iv) by

(iii') $|b-\hat{b}|\leq 2B'$

(iv') $-c+\frac{1}{2}\mathrm{div}(b+\hat{b})\leq C'$

in order to get $-\mathscr{E}_t(e^{-\psi}w,e^{\psi}w)\leq K'\cdot\int w^2|\nabla\psi|^2dx-k'\cdot\int|\nabla w|^2dx+\gamma'\cdot\int w^2dx$ with

$$K'=K_0+\frac{A}{\delta}+\frac{B'}{\varepsilon},\quad k'=k_0-\delta\cdot A,\quad \gamma'=C'+\varepsilon\cdot B'$$

where $\delta$ and $\varepsilon$ are arbitrary parameters with values in $]0,\infty[$.

### 1.3. Time-dependent Dirichlet-spaces

Let $I=]\sigma,\tau[\subset \boldsymbol{R}$ be an open interval.

#### A) Function spaces

We will be concerned with the following Banach spaces (cf. [22],[32]):

- $\mathscr{C}(I\to\mathscr{H})$
  being the set of continuous and bounded functions of the form $u:I\to\mathscr{H}$, $t\mapsto u_t=u(t,.)$ equipped with the norm $\sup_{t\in I}\|u_t\|=\sup_{t\in I}[\int_X u^2(t,x)m(dx)]^{1/2}$. What one has in mind is that actually $u:(t,x)\to\boldsymbol{R}$, $(t,x)\mapsto u(t,x)$ is a function of space and time which is regarded as a one-parameter family $(u_t)_{t\in I}$ of functions $u_t$ depending only on space. Note that in this paper $u_t$ *always* denotes the function $x\mapsto u(t,x)$ and *never* the time derivative of $u$. The latter is always denoted by $\frac{\partial}{\partial t}u$.
- $L^2(I\to\mathscr{F})$
  being the Hilbert space of functions $u:I\to\mathscr{F}$ with norm $(\int_I\|u_t\|^2_{\mathscr{F}}dt)^{1/2}$.
- $H^1(I\to\mathscr{F}^*)$

the Sobolev space of functions $u \in L^2(I \to \mathscr{F}^*)$ with distributional time derivative $\frac{\partial}{\partial t}u \in L^2(I \to \mathscr{F}^*)$ equipped with the norm $(\int_I \|u_t\|^2_{\mathscr{F}^*} + \|\frac{\partial}{\partial t}u_t\|^2_{\mathscr{F}^*}dt)^{1/2}$.

- $\mathscr{F}(I \times X) := L^2(I \to \mathscr{F}) \cap H^1(I \to \mathscr{F}^*)$
  being a Hilbert space with norm $\|u\|_{\mathscr{F}(I \times X)} = (\int_I \|u_t\|^2_{\mathscr{F}} + \|\frac{\partial}{\partial t}u_t\|^2_{\mathscr{F}^*}dt)^{1/2}$. We mention the following important result from [32, Lemma 10.4]:

$$L^2(I \to \mathscr{F}) \cap H^1(I \to \mathscr{F}^*) \subset C(\bar{I} \to \mathscr{H}).$$

*It implies that functions* $u \in \mathscr{F}(I \times X)$ can be extended onto $\bar{I} \times X$ in the sense that $t \mapsto u_t$ is continuous on the closed interval $\bar{I}$ as a map with values in $\mathscr{H} = L^2(X,m)$.

**B) The time-dependent Dirichlet form**

On $\mathscr{F}(I \times X)$ we define the *time-dependent Dirichlet form* $\mathscr{E}_I$ associated with the parabolic operator $L_t - \frac{\partial}{\partial t}$ by

$$\mathscr{E}_I(u,v) = \int_I \mathscr{E}_t(u_t, v_t)dt + \int_I (\tfrac{\partial}{\partial t}u_t, v_t)dt.$$

The time-dependent Dirichlet form associated with the coparabolic operator $L_t + \frac{\partial}{\partial t}$ is given by $\int_I \mathscr{E}_t(u_t, v_t)dt - \int_I(\frac{\partial}{\partial t}u_t, v_t)$. Its domain is still $\mathscr{F}(I \times X)$.

We will also be concerned with the (co-) parabolic operators $\hat{L}_t \pm \frac{\partial}{\partial t}$ where for any $t \in I$ the elliptic operator $\hat{L}_t$ is just the adjoint operator of $L_t$. Of particular importance is the coparabolic operator $\hat{L}_t + \frac{\partial}{\partial t}$. Note that (at least formally) $(L_t - \frac{\partial}{\partial t})^\wedge = \hat{L}_t + \frac{\partial}{\partial t}$. The time-dependent Dirichlet form associated with this operator is given by

$$\widehat{\mathscr{E}}_I(u,v) = \int_I \mathscr{E}_t(v_t, u_t)dt - \int_I (\tfrac{\partial}{\partial t}u_t, v_t)dt.$$

Its domain is again $\mathscr{F}(I \times X)$.

## 1.4. Parabolic equations

**A) The notion of solution**

DEFINITION. i) A function $u$ is called a (global) solution of the parabolic equation

$$L_t u = \tfrac{\partial}{\partial t}u \quad \text{on } I \times X \tag{1.13}$$

iff $u \in \mathscr{F}(I \times X)$ and $\mathscr{E}_I(u,\phi) = 0$ for all $\phi \in \mathscr{F}(I \times X)$, that is,

$$\mathscr{E}_I(u,\phi) = \int_I \mathscr{E}_t(u_t, \phi_t)dt + \int_I (\tfrac{\partial}{\partial t}u_t, \phi_t)dt = 0. \tag{1.14}$$

It is called a subsolution of (1.13) if instead of the equality (1.14) the inequality $\mathscr{E}_I(u,\phi)\geq 0$ holds true for all nonnegative $\phi\in\mathscr{F}(I\times X)$.

ii) Given a function $f\in\mathscr{H}$, the function $u$ is called a (global) solution of the initial value problem

$$\begin{aligned} L_t u &= \tfrac{\partial}{\partial t}u \quad \text{on } (I\times X) \\ u_\sigma &= f \quad\ \ \text{on } X \end{aligned} \tag{1.15}$$

iff $u$ solves (1.13) and $u_t \xrightarrow{t\to\sigma} f$ in $\mathscr{H}$.

We recall that every solution $u$ of (1.13) extends to $u\in\mathscr{C}([\sigma,\tau]\to\mathscr{H})$. Hence, the condition $u_\sigma=f$ in (1.15) is well-defined and equivalent to the condition $u_t \xrightarrow{t\to\sigma} f$ in $\mathscr{H}$.

REMARKS. i) The set $\mathscr{F}(I\times X)$ of test functions in (1.14) can equivalently replaced by the (larger) set $L^2(I\to\mathscr{F})$ as well as by the (smaller) set $\{\phi\in\mathscr{F}(I\times X)$ with $\phi_\sigma\equiv\phi_\tau\equiv 0\}$.

ii) Integration by parts yields that condition (1.14) can equivalently be replaced by

$$\int_I \mathscr{E}_t(u_t,\phi_t)dt - \int_I (u_t,\tfrac{\partial}{\partial t}\phi_t)dt = -(u_\tau,\phi_\tau)+(u_\sigma,\phi_\sigma) \tag{1.16}$$

for all $\phi\in\mathscr{F}(I\times X)$ or by $\int_I\mathscr{E}_t(u_t,\phi_t)dt-\int_I(u_t,\frac{\partial}{\partial t}\phi_t)dt=0$ for all $\phi\in\mathscr{F}(I\times X)$ with $\phi_\sigma\equiv\phi_\tau\equiv 0$. Using (1.16) instead of (1.14) one can obviously weaken the a priori assumption $u\in\mathscr{F}(I\times X)$ since in this formulation no time derivative of $u$ is used. We will come back to this point in Proposition 1.3.

iii) Reversing the time direction, the above formalism can be used to treat solutions of the coparabolic equation $L_t u=-\frac{\partial}{\partial t}u$ on $I\times X$ and solutions of the corresponding terminal value problem $L_t u=-\frac{\partial}{\partial t}u$ on $I\times X$, $u_\tau=f$ on $X$.

iv) The general set-up can slightly be modified in order to treat also boundary value problems of the form

$$\begin{aligned} L_t u &= \tfrac{\partial}{\partial t}u \quad \text{on } I\times G \\ u &= 0 \qquad \text{on } I\times\partial G \\ u &= f \qquad \text{on } \{\sigma\}\times G \end{aligned} \tag{1.17}$$

on a cylinder $I\times G$ with an open set $G\subset X$. For this purpose, we replace $\mathscr{F}=\mathscr{F}(X)$ by

$$\mathscr{F}(G)=\{u\in\mathscr{F}:\tilde{u}=0 \text{ q.e. } \text{ on } X\backslash G\}$$

where $\tilde{u}$ denotes a quasi-continuous version of $u$. With $\mathscr{F}(G)$ in the place of $\mathscr{F}(X)$ we define $\mathscr{F}(I\times G)$ and we say that $u$ is a solution of the boundary value problem (1.17) iff $u\in\mathscr{F}(I\times G)$ and $\mathscr{E}_I(u,\phi)=0$ for all $\phi\in\mathscr{F}(I\times G)$. Actually, however, this is no extension of our general set-up since $(\mathscr{E},\mathscr{F}(G))$ and $(\mathscr{E}_t,\mathscr{F}(G))$ are again Dirichlet spaces of the previously considered type.

## B) Existence and uniqueness

Taking $\phi=u$ in the definition (1.14) and in (1.16), we immediately obtain that every solution $u$ of (1.13) satisfies the following basic inequality

$$\|u_t\|^2=\|u_s\|^2-2\int_s^t \mathscr{E}_r(u_r,u_r)dr \tag{1.18}$$

for all $\sigma\leq s\leq t\leq\tau$. This identity implies that for every solution $u$ of (1.13) the function

$$t\mapsto\|u_t\| \quad \text{is continuous and decreasing on } \bar{I}. \tag{1.19}$$

The fact that $\|u\|$ decays is sometimes called *integrated maximum principle.* In particular, we get $\|u_t\|\leq\|f\|$ for every solution $u$ of the initial value problem (1.15). This of course implies *uniqueness* of the solution of (1.15).

More sophisticated is the proof of the existence of a solution of (1.15). From [22] (Chap. III, Thm 4.1 and Rem. 4.3) we quote

**Proposition 1.2.** *For every $f\in\mathscr{H}$ there exists a unique solution $u\in\mathscr{F}(I\times X)$ of the initial value problem* (1.15).

We recall that every solution of the parabolic equation (1.13) is by definition in $\mathscr{F}(I\times X)$ and every function in $\mathscr{F}(I\times X)$ is already in $\mathscr{C}(\bar{I}\to\mathscr{H})$.

Now we give another definition of solution of (1.13) where a priori no condition on the time derivative $\frac{\partial}{\partial t}u$ is imposed. The function space under consideration is $L^2(I\to\mathscr{F})\cap\mathscr{C}(\bar{I}\to\mathscr{H})$ which is a proper superset of $\mathscr{F}(I\times X)$. The remarkable fact is that any function $u\in L^2(I\to\mathscr{F})\cap\mathscr{C}(\bar{I}\to\mathscr{H})$ which solves the equation $L_t u=\frac{\partial}{\partial t}u$ on $I\times X$ (in the sense of the following Proposition 1.3) already lies in $\mathscr{F}(I\times X)$ which in particular means that $\frac{\partial}{\partial t}u\in L^2(I\to\mathscr{F}^*)$.

**Proposition 1.3.** *A function $u$ is a solution of the parabolic equation* (1.13) *if and only if*

$$u\in L^2(I\to\mathscr{F})\cap\mathscr{C}(\bar{I}\to\mathscr{H})$$

*and*

$$\int_\sigma^T \mathscr{E}_t(u_t,\phi_t)dt-\int_\sigma^T (u_t,\tfrac{\partial}{\partial t}\phi_t)dt=-(u_T,\phi_T)+(u_\sigma,\phi_\sigma) \tag{1.20}$$

*for all* $T\in]\sigma,\tau[$ *and all* $\phi\in\mathscr{F}(]\sigma,T[\times X)$.

Proof. The "only if"-part is obvious. Therefore, let $u\in L^2(I\to\mathscr{F})\cap\mathscr{C}(\bar{I}\to\mathscr{H})$ satisfy (1.20) and let $w=u-v$ where $v\in\mathscr{F}(I\times X)$ is the solution of the initial value problem on $L_t v=\frac{\partial}{\partial t}v$ on $I\times X$, $v_\sigma=u_\sigma$ on $X$. Then

$$\int_\sigma^T \mathscr{E}_t(w_t,\phi_t)dt-\int_\sigma^T(w_t,\tfrac{\partial}{\partial t}\phi_t)dt=-(w_T,\phi_T)+(w_\sigma,\phi_\sigma) \tag{1.21}$$

for all $T$ and $\phi$ as above. Now choose $\phi\in\mathscr{F}(I\times X)$ to be the solution of the terminal value problem $\hat{L}_t\phi=-\frac{\partial}{\partial t}\phi$ on $]\phi,T[\times X$, $\phi_T=w_T$ on $X$ (which exists according to a suitable modification of Proposition 1.2). Then the LHS of (1.21) vanishes since $\hat{L}_t\phi=-\frac{\partial}{\partial t}\phi$. Thus $(w_T,\phi_T)=(w_\sigma,\phi_\sigma)$. But $w_T=\phi_T$ and $w_\sigma=0$. Hence, $w_T=0$. Since $T\in]\sigma,\tau[$ was arbitrary this yields the claim. □

With a slightly modified argument one actually can relax the a priori assumption $u\in L^2(I\to\mathscr{F})\cap\mathscr{C}(\bar{I}\to\mathscr{H})$ and replace it by $u\in L^2(I\to\mathscr{F})\cap L^\infty(\bar{I}\to\mathscr{H})$, cf. [22], [20].

### C) Contraction properties

From Proposition 1.2 we deduce that for every $t\geq\sigma$ there exists a uniquely determined operator $T_t^\sigma:\mathscr{H}\to\mathscr{H}$ with the property that for every $f\in\mathscr{H}$ the unique solution $u\in\mathscr{F}(I\times X)$ of (1.15) is given by $u:t\mapsto T_t^\sigma f$. The operator $T_t^\sigma$ is called *transition operator* from $\sigma$ to $t$ associated with the parabolic operator $L_t-\frac{\partial}{\partial t}$.

The family $(T_t^s)_{s\leq t}$ satisfies

- $t\mapsto T_t^s$ is strongly continuous on $[s,\infty[$;
- $\|T_t^s\|_{2,2}\leq 1$ whenever $s\leq t$;
- $T_t^r=T_t^s\circ T_s^r$ whenever $r\leq s\leq t$.

Here and in the sequel, $\|\,.\,\|_{p,p}$ denotes the norm of an operator from $L^p(X,m)$ into itself ($p\in[1,\infty]$). In particular, $\|\,.\,\|_{2,2}$ denotes the operator norm on $\mathscr{H}$.

**Lemma 1.4.** *For all* $f\in\mathscr{H}$, *all* $s\leq t$ *and all* $f\in\mathscr{H}$: *if* $0\leq f\leq 1$ *then* $0\leq T_t^s f\leq 1$.

Proof. For $u\in\mathscr{F}(I\times X)$ put $u^\#=(u\vee 0)\wedge 1$. Then (according to the Markov property of the Dirichlet forms $\mathscr{E}$ and $\mathscr{E}_t$) $u^\#\in\mathscr{F}(I\times X)$ and $\mathscr{E}_t(u_t,u_t-u_t^\#)\geq 0$ for all $t\in 1$ ([28],[24]). Moreover, $(u_t,\frac{\partial}{\partial t}(u_t-u_t^\#))=0$ if $u_t\in[0,1]$ and $=\frac{1}{2}\frac{\partial}{\partial t}\|u_t\|^2$ if $u_t\notin[0,1]$. Hence, if $u$ is a solution of (1.13) then $(u_t,\frac{\partial}{\partial t}(u_t-u_t^\#))\leq 0$ for all $t\in I$ and thus

$$(u_t,u_t-u_t^\#)-(u_s,u_s-u_s^\#)=-\int_s^t\mathscr{E}_r(u_r,u_r-u_r^\#)dr+\int_s^t(u_r,\tfrac{\partial}{\partial r}(u_r-u_r^\#))dr\leq 0.$$

This implies

$$(u_t - u_t^{\#}, u_t - u_t^{\#}) \le 2(u_t, u_t - u_t^{\#}) \le (u_s, u_s - u_s^{\#}).$$

But if we now assume that $0 \le u_s \le 1$ then $u_t - u_t^{\#} = 0$, that is $0 \le u_t \le 1$. This proves the claim since for any $f \in \mathscr{H}$ the function $u:(t,x) \mapsto T_t^s f(x)$ is a solution of (1.13). □

REMARK. Reversing the time direction we deduce from Proposition 1.2 that for every $f \in \mathscr{H}$ there exists a unique solution $u \in \mathscr{F}(I \times X)$ of the terminal value problem $L_t u = -\frac{\partial}{\partial t} u$ on $I \times X$, $u_\tau = f$ on $X$. This solution is given by $u: t \mapsto S_t^\tau f$ where the family $(S_t^s)_{s \ge t}$ satisfies properties analogous to that of $(T_t^s)_{s \le t}$. $S_t^s$ is called *transition operator* from $s$ to $t$ associated with the coparabolic operator $L_t + \frac{\partial}{\partial t}$. Similarly, we define the transition operators $\hat{T}_t^s$ and $\hat{S}_t^s$ associated with the operators $\hat{L}_t \mp \frac{\partial}{\partial t}$.

**Lemma 1.5.** *For all $f, g \in \mathscr{H}$, all $s \le t$*

$$(f, T_t^s g) = (\hat{S}_s^t f, g).$$

*That is, $(T_t^s)^\wedge = \hat{S}_s^t$ where $\hat{S}_s^t$ denotes the transition operator from $t$ to $s$ associated with the coparabolic operator $\hat{L}_t + \frac{\partial}{\partial t}$.*

Proof. Let $u_r = T_r^s f$ and $v_r = \hat{S}_r^t g$ with $f, g \in \mathscr{H}$ and $s \le r \le t$. Then

$$(u_t, v_t) - (u_s, v_s) = -\int_s^t \mathscr{E}_r(u_r, v_r) dr + \int_s^t (u_r, \tfrac{\partial}{\partial t} v_r) dr = 0$$

where the first equality comes from the fact that $L_t u = \frac{\partial}{\partial t} u$ and the second equality from $\hat{L}_t v = -\frac{\partial}{\partial t} v$. Together with the fact that $T_s^s = Id = \hat{S}_t^t$ this already yields

$$(T_t^s f, g) = (u_t, v_t) = (u_s, v_s) = (f, \hat{S}_s^t g).$$

**Proposition 1.6.** *For all $p \in [1, \infty]$ and all $s \le t$, the operator $T_t^s$ extends to a contraction operator on $L^p(X,m)$, that is,*

$$\|T_t^s\|_{p,p} \le 1.$$

Proof. The Markov property (Lemma1.4) implies that $T_t^s$ extends to a contraction operator on $L^\infty(X,m)$. The same argument applies to $\hat{S}_s^t$. According to Lemma 1.5 the latter implies that $T_t^s$ extends to a contraction operator on $L^1(X,m)$. The rest follows by interpolation. □

REMARK. The extension of $T_t^s$ to $L^p(X,m)$ is unique by density for $1 \le p < \infty$ and it is unique for $p = \infty$ if one imposes the extra condition of weak* continuity which will be done henceforth.

### 1.5. Integrated Gaussian estimates ("The Method of Davies")

#### A) Integrated maximum principle

The basic step in order to obtain Gaussian estimates for $T_t^s$ consists in an estimate for the norm of the operator $f \to e^{-\psi} \cdot T_t^s(e^{\psi} \cdot f)$ on $\mathscr{H}$. In the "classical" context of uniformly elliptic operators on Riemannian manifolds, this type of estimate was used e.g. by Gaffney [14], Davies [9] and Grigor'yan [15]. The latter called it (weighted) integrated maximum principle. We recall that $\lambda = \inf\{\frac{\mathscr{E}(u,u)}{\|u\|^2} : u \in \mathscr{F}, u \neq 0\}$ denotes the bottom of the selfadjoint operator $-L$. Note that always $\lambda \geq 0$.

**Lemma 1.7.** *Let $\psi \in \mathscr{F} \cap L^\infty(X,m)$ with $d\Gamma(\psi,\psi) \leq \gamma^2\, dm$ and let $u$ be a solution of the parabolic equation $L_t u = \frac{\partial}{\partial t} u$ on $I \times X$. Then for all $t \geq s \geq \sigma$*

$$\|e^{\psi} u_t\| \leq e^{(K\gamma^2 - k \cdot \lambda)(t-s)} \cdot \|e^{\psi} u_s\|.$$

Proof.

$$\begin{aligned}
\|e^{\psi} u_t\|^2 - \|e^{\psi} u_s\|^2 &= 2\int_s^t \left(\frac{\partial}{\partial r} u_r, e^{2\psi} u_r\right) dr = -2\int_s^t \mathscr{E}_r(u_r, e^{2\psi} u_r)\, dr \\
&\leq 2K \int_s^t e^{2\psi} u_r^2\, d\Gamma(\psi,\psi) - 2k \int_s^t \mathscr{E}(e^{\psi} u_r, e^{\psi} u_r)\, dr \\
&\leq 2(K\gamma^2 - k\lambda) \cdot \int_s^t \|e^{\psi} u_r\|^2\, dr.
\end{aligned}$$

From this inequality, the claim follows by Gronwall's Lemma. □

#### B) Integrated Gaussian estimate

For subsets $A$ and $B$ of $X$ we define

$$\tilde{\rho}(A,B) = \sup\{\psi(A,B) : \psi \in \mathscr{F}_{loc}(X) \cap \mathscr{C}(X),\ d\Gamma(\psi,\psi) \leq dm\}$$

where $\psi(A,B) := \inf\{\psi(x) - \psi(y) : x \in A, y \in B\}$. Note that always $\tilde{\rho}(A,B) \leq \rho(A,B) := \inf\{\rho(x,y) : x \in A, y \in B\}$ and in the next section we shall see that equality holds true under a weak assumption on $(X,\rho)$.

**Theorem 1.8.** *For any measurable subsets $A, B \subset X$ of finite measure and any $s < t$*

$$(T_t^s 1_A, 1_B) \leq \sqrt{m(A)} \cdot \sqrt{m(B)} \cdot \exp\left(-\frac{\tilde{\rho}^2(A,B)}{4K(t-s)}\right) \cdot \exp(-k\lambda(t-s)).$$

Proof. For $\gamma>0$ and $\psi\in\mathscr{F}_{loc}(X)\cap L^\infty(X)\cap\mathscr{C}(X)$ with $d\Gamma(\psi,\psi)\le dm$ the preceding integrated maximum principle yields

$$\begin{aligned}(T_t^s 1_A, 1_B) &= (e^{\gamma\psi}T_t^s 1_A, e^{-\gamma\psi}1_B)\\ &\le e^{(K\gamma^2-k\lambda)(t-s)}\cdot\|e^{\gamma\psi}1_A\|\cdot\|e^{-\gamma\psi}1_B\|\\ &\le e^{(K\gamma^2-k\lambda)(t-s)}\cdot e^{-\gamma\psi(A,B)}\cdot\sqrt{m(A)}\cdot\sqrt{m(B)}.\end{aligned}$$

Taking the supremum over all such $\psi$ and choosing $\gamma=\frac{\tilde\rho(A,B)}{2K(t-s)}$ yields the claim. $\square$

### C) The metric $\rho$ and Assumption (A)

Here and henceforth we make the

**Assumption (A).** *The topology induced by $\rho$ is equivalent to the original topology on $X$.*

This assumption in particular implies that $\rho$ is non-degenerate and that for any $y\in X$ the function $x\mapsto\rho(x,y)$ is continuous on $X$.

It is discussed in more details in the paper [36]. Note that it does not necessarily imply that all balls $B_r(x)=\{y\in X:\ \rho(x,y)<r\}$ are relatively compact in $X$. The latter is true if and only if the metric space $(X,\rho)$ is *complete.*

In [36] we proved that under (A) for every $y\in X$ the distance function $\rho_y: x\mapsto\rho(x,y)$ on $X$ satisfies $\rho_y\in\mathscr{F}_{loc}(X)\cap\mathscr{C}(X)$ and

$$d\Gamma(\rho_y,\rho_y)\le dm. \tag{1.22}$$

Hence, the distance function $\rho_x$ can be used to construct cut-off functions on intrinsic balls $B_r(x)$ of the form

$$\rho_{x,r}: y\mapsto(r-\rho(x,y))_+. \tag{1.23}$$

Obviously, $\rho_{x,r}\in\mathscr{F}_{loc}(X)\cap\mathscr{C}(X)$ and $d\Gamma(\rho_{x,r},\rho_{x,r})\le dm$. Moreover, $\rho_{x,r}\in\mathscr{F}(X)\cap\mathscr{C}_0(X)$ if (and only if) $B_r(x)$ is relatively compact.

In the sequel we shall need the following generalization.

**Lemma 1.9.** *For every relatively compact set $Y\subset X$* the distance function *$\rho_Y: x\mapsto\rho(x,Y):=\inf\{\rho(x,y):\ y\in Y\}$ satisfies $\rho_Y\in\mathscr{F}_{loc}(X)\cap\mathscr{C}(X)$ and*

$$d\Gamma(\rho_Y,\rho_Y)\le dm. \tag{1.24}$$

Proof. Note that for any $x\in X$ the function $y\mapsto\rho(x,y)$ is continuous. Hence, $\rho(x,Y)=\rho(x,\bar Y)$. Now let $Y$ be compact and fix $\varepsilon>0$. There exists a finite number of points $\{y_i\}_{i=1,\cdots,n}$ such that $Y\subset\bigcup_{i=1}^n B_\varepsilon(y_i)$. Let $\psi_i=\rho(.,y_i)-\varepsilon$ and $\Psi=\inf_{i=1,\cdots,n}\psi_i$. Then $\psi_i\leq 0$ in $B_\varepsilon(y_i)$ and $\psi_i\in\mathscr{F}_{loc}(X)\cap\mathscr{C}(X)$ with $d\Gamma(\psi_i,\psi_i)\leq dm$ ([36], Lemma 1.1) . Hence, $\Psi\leq 0$ in $Y$ and $\Psi\in\mathscr{F}_{loc}(X)\cap\mathscr{C}(X)$ with $d\Gamma(\Psi,\Psi)\leq dm$. The latter implies $\Psi(x)\leq\rho(x,y)$ for all $x\in X$ and $y\in Y$. That is, $\Psi\leq\rho_Y$. On the other hand,

$$\rho_Y=\inf\{\rho(.,y):\ y\in Y\}\leq\inf\{\rho(.,y_i):\ i=1,\cdots,n\}=\Psi+\varepsilon.$$

Following the argumentation in [36, proof of Lemma 1'] one concludes that $\rho_Y\in\mathscr{F}_{loc}(X)\cap\mathscr{C}(X)$ with $d\Gamma(\psi,\psi)\leq dm$.

REMARK. If in addition to Assumption (A) we assume that $(X,\rho)$ is complete then the assertion of Lemma 1.9 holds true for *all subsets* $Y\subset X$. In order to see that, let $x_0\in X$ and consider $\rho_Y$ in $B_1(x_0)$. For that purpose, let $r=\rho(x_0,Y)$ and $Y_0=Y\cap B_{r+2}(x_0)$. Then $Y_0$ is relatively compact and $\rho_Y=\rho_{Y_0}$ on $B_1(x_0)$.

**Lemma 1.10.** *For all compact sets* $K,L\subset X$

$$\rho(K,L)=\tilde\rho(K,L).$$

Proof. We recall the trivial inequality $\tilde\rho(K,L)\leq\rho(K,L)$. From Lemma 1.9 we conclude that $\psi_0=\rho(.,L)$ belongs to the class $\Psi$ of functions which were used to define $\tilde\rho$, hence, $\tilde\rho(K,L)=\sup_{\psi\in\Psi}\psi(K,L)\geq\psi_0(K,L)$. But $\psi_0(K,L)=\inf_{x\in K}\psi_0(x)-\sup_{y\in L}\psi_0(y)=\inf_{x\in K}\rho(x,L)-0=\rho(K,L)$. □

**Corollary 1.11.** *For any measurable subsets* $A$, $B\subset X$ *of finite measure and any* $s<t$

$$(T_t^s 1_A,1_B)\leq\sqrt{m(A)}\cdot\sqrt{m(B)}\cdot\exp\left(-\frac{\rho^2(A,B)}{4K(t-s)}\right)\cdot\exp(-k\lambda(t-s)).$$

Proof. Let $K$ and $L$ be compact subsets of $A$ and $B$, respectively. Then according to Theorem 1.8 together with Lemma 1.10

$$\begin{aligned}(T_t^s 1_K,1_L)&\leq\sqrt{m(K)}\cdot\sqrt{m(L)}\cdot\exp\left(-\frac{\rho^2(K,L)}{4K(t-s)}\right)\cdot\exp(-k\lambda(t-s))\\&\leq\sqrt{m(A)}\cdot\sqrt{m(B)}\cdot\exp\left(-\frac{\rho^2(A,B)}{4K(t-s)}\right)\cdot\exp(-k\lambda(t-s)).\end{aligned}$$

Now let $K_n\uparrow A$ and $L_n\uparrow B$ such that $1_{K_n}\to 1_A$ and $1_{L_n}\to 1_B$ in $\mathscr{H}$. Then $(T_t^s 1_{K_n},1_{L_n})\to(T_t^s 1_A,1_B)$ and thus

$$(T_t^s 1_A,1_B)\leq\sqrt{m(A)}\cdot\sqrt{m(B)}\cdot\exp\left(-\frac{\rho^2(A,B)}{4K(t-s)}\right)\cdot\exp(-k\lambda(t-s)).$$ □

REMARKS. i) If there exists a fundamental solution $p(t,y,s,x)$ for $L_t - \frac{\partial}{\partial t}$ then for any $s<t$ and any measurable subsets $A$, $B$ of $X$

$$\int_A\int_B p(t,y,s,x)m(dx)m(dy) \le \sqrt{m(A)}\cdot\sqrt{m(B)}\cdot\exp\left(-\frac{\rho^2(A,B)}{4K(t-s)}\right)\cdot\exp(-k\lambda(t-s)). \tag{1.25}$$

The existence of a fundamental solution is guaranteed under certain assumptions which will be discussed in the next chapter. (But it is also satisfied in much more general situations.)

ii) In the next chapter we will combine (1.25) with subsolution estimates of the form

$$p(t,y,s,x) \le C\cdot\frac{1}{m(A)}\cdot\frac{1}{m(B)}\cdot(T_t^s 1_A, 1_B)$$

in order to obtain pointwise Gaussian estimates of the form

$$p(t,y,s,x) \le C'\frac{1}{\sqrt{m(A)}}\cdot\frac{1}{\sqrt{m(B)}}\cdot\exp\left(-\frac{\rho^2(x,y)}{4K(t-s)}\right)\cdot\exp(-k\lambda(t-s))$$

where $A=B_r(x)$, $B=B_r(y)$ and $r=\sqrt{t-s}$.

## 2. Sobolev inequality and pointwise Gaussian estimates

Our aim is to derive certain regularity and smoothness properties for local solutions of the equation $L_t u = \frac{\partial}{\partial t}u$ on an open set $Q \subset \boldsymbol{R}\times X$. To this end, we introduce certain assumptions which will be discussed in the following sections.

### 2.1. The Assumptions on $\mathscr{E}$

We always assume that Assumption (A) holds true on $X$.

In order to derive pointwise estimates for the density of the transition operator $T_t^s$ we assume from now on that in addition a *doubling property* and a scale invariant *Sobolev inequality* holds true on $X$ or at least on sufficiently many open sets $Y \subset X$. These assumptions will be formulated in terms of the initial Dirichlet form $\mathscr{E}$ on $L^2(X,m)$.

### A) The doubling property

**Assumption (B).** *There exists a constant* $N=N(Y)$ *such that*

$$m(B_{2r}(x)) \le 2^N\cdot m(B_r(x)) \tag{2.1}$$

("*doubling property*") *whenever* $B_{2r}(x) \subset Y$ *and all these balls* $B_{2r}(x) \subset Y$ *are relatively compact in* $X$.

We say that Assumpion (B) holds true *locally on* $X$ if it is satisfied for every relatively compact open set $Y \subset X$ or, equivalently, if every point $y \in X$ has a neighborhood $Y \subset X$ which satisfies (B). It is easy to see that (under (A)) Assumption (B) holds true locally on $X$ if and only if there exist an upper semi-continuous function $N: X \to [0,\infty[$ and a lower semi-continuous function $R: X \to ]0,\infty]$ with the property that for all $x \in X$ and all $r < R(x)$ the balls $B_{2r}(x)$ are relatively compact and

$$m(B_{2r}(x)) \leq 2^{N(x)} \cdot m(B_r(x)) \tag{2.2}$$

If these functions $N$ and $R$ can be chosen to be constants, then we say that the doubling property holds true *locally uniformly on* $X$.

Note that (2.1) implies

$$m(B_{r'}(x)) \leq (2r'/r)^N \cdot m(B_r(x)) \tag{2.3}$$

for all $x \in Y$ and $r' > r > 0$ with $B_{r'}(x) \subset Y$ and

$$m(B_{r'}(x')) \leq (4r'/r)^N \cdot m(B_r(x)) \tag{2.4}$$

for all $x, x' \in Y$ and $r, r' \in ]0,\infty[$ with $B_r(x) \subset B_{r'}(x')$ and $B_{2r}(x) \subset Y$.

The number $N$ plays the role of the dimension of the space X. Note, however, that it may be a fractional number and that it may vary on $X$. Let us mention that without restriction this number $N$ in (2.1) (and the function $N$ in (2.2)) can and will(!) always be chosen to satisfy

$$N > 2.$$

Assumption (B) implies that the metric space $(Y, \rho)$ is a homogeneous space in the sense of R. Coifman and G. Weiss [5] which in turn implies that several covering properties hold true.

### B) The Sobolev inequality

**Assumption (C).** *There exist constants* $C_S = C_S(Y)$ *and* $N = N(Y) > 2$ *such that for all* $B_r(x) \subset\subset Y$

$$\left(\int_{B_r(x)} |u|^{\frac{2N}{N-2}} dm\right)^{\frac{N-2}{N}} \leq C_S \cdot \frac{r^2}{m(B_r(x))^{2/N}} \int_{B_r(x)} d\Gamma(u,u) + r^{-2} \cdot u^2 \, dm \tag{2.5}$$

*for all* $u \in \mathscr{F}(X) \cap \mathscr{C}_0(B_r(x))$.

If Assumptions (B) and (C) both hold true on some set $Y \subset X$ then without restriction we always assume that the constants $N = N(Y)$ in both Assumptions coincide (otherwise take the maximum of both).

Using the cut-off functions $\rho_{x,r}$ from section 1.5.C one can deduce from (2.5) the following alternative form of the Sobolev inequality

$$\left(\int_{B_{(1-\delta)r}(x)} |u|^{\frac{2N}{N-2}} dm\right)^{\frac{N-2}{N}} \leq 3C_S \cdot \frac{r^2}{m(B_r(x))^{2/N}} \int_{B_r(x)} d\Gamma(u,u) + (\delta r)^{-2} \cdot u^2 \, dm \qquad (2.6)$$

for all $B_r(x) \subset \subset Y$, all $\delta \in ]0,1[$ and all $u \in \mathscr{F}_{loc}(X)$.

Indeed, with $\delta \in ]0,1[$ and $\psi = (\frac{1}{\delta r}\rho_{x,r}) \wedge 1$ we get from (2.5)

$$\left(\int_{B_{(1-\delta)r}(x)} |u|^{\frac{2N}{N-2}} dm\right)^{\frac{N-2}{N}} \leq \left(\int_{B_r(x)} |u \cdot \psi|^{\frac{2N}{N-2}} dm\right)^{\frac{N-2}{N}}$$

$$\leq C_S \cdot \frac{r^2}{m(B_r(x))^{2/N}} \int_{B_r(x)} d\Gamma(u \cdot \psi, u \cdot \psi) + r^{-2} \cdot u^2 \cdot \psi^2 \, dm$$

$$\leq C_S \cdot \frac{r^2}{m(B_r(x))^{2/N}} \int_{B_r(x)} 2d\Gamma(u,u) + \left(\frac{2}{\delta^2} + 1\right) r^{-2} \cdot u^2 \, dm.$$

Of course, (2.6) (e.g. with $\delta = 1/2$) also implies (2.5) for all balls $B_r(x)$ satisfying $B_{2r}(x) \subset Y$ (with a new constant $C_S$ being $12\times$ the original constant $C_S$). Note that in the formulation (2.6) the functions $u$ are not required to vanish on the boundary of $B_r(x)$.

We say that Assumption (C)(or, in other words, a scale invariant Sobolev inequality) holds true *locally* on $X$ if every point $y \in X$ has an open neighborhood $Y \subset X$ on which Assumption (C) is satisfied. Using the formulation (2.6) one can show that Assumption (C) holds true locally on $X$ if and only if it is satisfied for every relatively compact open set $Y \subset X$.

We say that a scale invariant Sobolev inequality holds true *uniformly locally* on $X$ if there exist constants $C_S$, $R$ and $N>2$ such that property (2.5) is satisfied with the same constants $C_S$ and $N$ for all $B_r(x) \subset X$ with $r<R$.

### C) Examples

Let us first of all discuss these assumptions for the examples from section 1.1.E.

- Let $L$ be the Laplace-Beltrami operator on a smooth Riemannian manifold $X$ and let $m$ be the Riemannian volume. Then Assumptions (B) and (C) are always satisfied locally on $X$. They are both satisfied locally uniformly on $X$ if the Ricci curvature on $X$ is bounded from below and they are both satisfied globally on $X$ if the Ricci curvature on $X$ is nonnegative, cf. [33]
- Let $L$ be a uniformly elliptic operator on $\boldsymbol{R}^N$ with a nonnegative weight $\phi$ belonging to the Muckenhaupt class $A_2$. Then Assumptions (B) and (C) are both satisfied globally on $X$, cf. [3].

- Let $L$ be a subelliptic operator on $\boldsymbol{R}^N$. Then Assumptions (B) and (C) are both satisfied globally on $X$, cf. [3].

Besides these three (classes of ) examples there exist a huge amount of further examples satisfying both assumptions (B) and (C) according to the following two basic facts which are proven in [37]:

*i) If a doubling property and a scale invariant Poincaré inequality hold true on an open set $Y \subset X$, then also a scale invariant Sobolev inequality holds true on $Y$.*

*ii) A doubling property and a scale invariant Poincaré inequality hold true simultaneously on an open set $Y \subset X$ if and only if a scale invariant Harnack inequality for the parabolic operator $L - \frac{\partial}{\partial t}$ on $\boldsymbol{R} \times Y$ holds true.*

Here we say that a scale invariant Poincaré inequality holds true on $Y$ if there exists a constant $P = P(Y)$ such that for all $B_r(x) \subset \subset Y$

$$\int_{B_{r/2}(x)} |u - u_{x,r}|^2 dm \leq P \cdot r^2 \int_{B_r(x)} d\Gamma(u,u) \tag{2.7}$$

for all $u \in \mathscr{F}(X)$ where $u_{x,r} = \frac{1}{m(B_{r/2}(x))} \int_{B_{r/2}(x)} u \, dm$.

### 2.2. The assumptions on $\mathscr{E}_t$

We always assume that the uniform parabolicity condition (UP) holds true with constants $0 < k \leq K < \infty$ and $\gamma = 0$. In addition, we assume that the following *strong uniform parabolicity condition* holds true on $X$ or at least on sufficiently many open sets $Y \subset X$.

**Assumption (SUP).** *There exists a constant $\kappa = \kappa(Y) \geq 1$ such that*

$$-\frac{p-1}{2} \cdot \mathscr{E}_t(u, u^{p-1}\phi^2) \leq \kappa \cdot \int u^p d\Gamma(\phi,\phi) - \frac{1}{\kappa} \cdot \left(1 - \frac{1}{p}\right)^2 \cdot \int \phi^2 d\Gamma(u^{p/2}, u^{p/2}) \tag{2.8}$$

*for all $p \in \boldsymbol{R}$, all nonnegative $u \in \mathscr{F}_{loc}(Y)$ with $u + u^{-1} \in L^\infty(Y)$ and all $\phi \in \mathscr{F}_{comp}(Y) \cap L^\infty(Y)$.*

Here $\mathscr{F}_{comp}(Y)$ denotes the set of all $u \in \mathscr{F}(X)$ which vanish $m$-a.e. outside some compact subset of $Y$. For a given $p \in \boldsymbol{R}$, we say that the condition ($\mathrm{SUP}_p$) holds true on $Y$ if there exist a constant $\kappa = \kappa(Y,p)$ such that (2.8) is satisfied for all $u$ and $\phi$ as above.

REMARKS. i) If the condition ($\mathrm{SUP}_2$) holds true on $Y = X$ (with a constant $\kappa$), then also the condition (UP) from section 1.2.B holds true (with constants $K = 1 + 2\kappa$ and $k = \frac{1}{1+2\kappa}$).

If conversely (UP) holds true (with constants $K$ and $k$) then also ($SUP_2$) holds true (with $\kappa = \sup\{K, 1/k\}$).

ii) Assume that all the $\mathscr{E}_t$'s are symmetric. Then ($SUP_2$) (with $\phi \equiv 1$) implies $\mathscr{E}_t(u,u) \geq \frac{1}{2\kappa} \cdot \mathscr{E}(u,u)$ and ($SUP_0$) (with $\phi \equiv u$) implies $\mathscr{E}_t(u,u) \leq 2\kappa \cdot \mathscr{E}(u,u)$. Moreover, ($SUP_p$) (with $u \equiv 1$) for $p \to -\infty$ implies that $\mathscr{E}_t$ has no killing (i.e. it is strongly local).

On the other hand, the condition $k \cdot \mathscr{E}(u,u) \leq \mathscr{E}_t(u,u) \leq K \cdot \mathscr{E}(u,u)$ together with the fact that $\mathscr{E}_t$ is strongly local implies (SUP) on $X$ with $\kappa = \sup\{K, 1/k\}$.

iii) To derive upper Gaussian estimates it actually suffices to assume that ($SUP_p$) holds true for all $p \geq 2$ with a constant $\kappa = \kappa(Y)$ not depending on $p$. For other estimates (like Harnack's inequality), however, it is appropriate to assume ($SUP_p$) for all $p$.

In order to give examples of Dirichlet forms $\mathscr{E}_t$ satisfying (SUP) we only have to recall the above Remark ii). *For any $t \in \boldsymbol{R}$, let $\mathscr{E}_t$ be a symmetric, strongly local Dirichlet form satisfying*

$$k \cdot \mathscr{E}(u,u) \leq \mathscr{E}_t(u,u) \leq K \cdot \mathscr{E}(u,u). \tag{2.9}$$

*Then condition (SUP) is satisfied on $X$ with $\kappa = \sup\{K, 1/k\}$.*

A nonsymmetric example for (SUP) is again given by the Dirichlet forms

$$\mathscr{E}_t(u,v) = \sum_{i,j=1}^{N} \int a_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} v \, dx$$

of section 1.2.D with a not necessarily symmetric, time-dependent diffusion matrix $a = (a_{ij}) \in L^\infty(\boldsymbol{R} \times \boldsymbol{R}^N, dtdx)$ (and with vanishing low order coefficients $b$, $\hat{b}$, $c$). Let $k_0 \cdot |\xi|^2 \leq \Sigma_{i,j} a_{ij} \xi_i \xi_j \leq K_0 \cdot |\xi|^2$ for all $\xi \in \boldsymbol{R}^N$ and let $|\check{a}| := (\Sigma_{i,j} \check{a}_{ij}^2)^{1/2} \leq A$ with constants $0 < k_0 \leq K_0 < \infty$ and $0 \leq A < \infty$. Then obviously

$$\begin{aligned}
&-\frac{p-1}{2} \mathscr{E}_t(u, u^{p-1}\phi^2) \\
&= -2(1-1/p)^2 \int \phi^2 \sum a_{ij} \frac{\partial}{\partial x_i} u^{p/2} \frac{\partial}{\partial x_j} u^{p/2} dx \\
&\quad - 2(1-1/p) \int \phi u^{p/2} \sum a_{ij} \frac{\partial}{\partial x_i} u^{p/2} \frac{\partial}{\partial x_j} \phi dx \\
&\leq -1(1-1/p)^2 \int \phi^2 \sum a_{ij} \frac{\partial}{\partial x_i} u^{p/2} \frac{\partial}{\partial x_j} u^{p/2} dx + \int u^p \sum a_{ij} \frac{\partial}{\partial x_i} \phi \frac{\partial}{\partial x_j} \phi dx \\
&\quad - 2(1-1/p) \int \phi u^{p/2} \sum \check{a}_{ij} \frac{\partial}{\partial x_i} u^{p/2} \frac{\partial}{\partial x_j} \phi dx
\end{aligned}$$

$$\leq -k_0(1-1/p)^2\int \phi^2|\nabla u^{p/2}|^2 dx + K_0\int u^p|\nabla\phi|^2 dx$$

$$+2A|1-1/p|\int \phi u^{p/2}|\nabla u^{p/2}|\cdot|\nabla\phi| dx$$

$$\leq -\frac{1}{\kappa}(1-1/p)^2\int \phi^2|\nabla u^{p/2}|^2 dx + \kappa\cdot\int u^p|\nabla\phi|^2\, dx$$

with $\kappa = \sup\{K_0 + 2\frac{A^2}{k_0}, \frac{2}{k_0}\}$.

In order to include low order terms (arising from non-vanishing coefficients $b$, $\hat{b}$, $c$) one has to admit an additional zero order term $\pm(p\text{-}1)\alpha\cdot\int u^p\phi^2\, dm$ on the RHS of (2.8). This, however, can be reduced to the previously considered situation if one replaces $\mathscr{E}_t$ by $\mathscr{E}_t \pm \alpha\cdot(.,.)$. The following estimates will be proven for local sub- and supersolutions of the equation $(L_t \pm \alpha)u = \frac{\partial}{\partial t}u$.

### 2.3. Sub- and supersolution estimates

In this section we study $L^p$-mean value properties for nonnegative local solutions of the parabolic equation $(L_t + \alpha)u = \frac{\partial}{\partial t}u$ on certain subsets $Q \subset \mathbf{R}\times X$. Some of these properties also hold true for local sub- or supersolutions.

#### A) Local solutions

Let $G$ be an open subset of $X$, let $I$ be the interval $]\sigma,\tau[\, \subset \mathbf{R}$ and let $Q$ be the *parabolic cylinder* $I\times G$. Denote the measure $dt\otimes dm$ on $\mathbf{R}\times X$ by $d\underline{m}$. We define $\mathscr{F}_{loc}(Q)$ to be the set of all $\underline{m}$-measurable functions on $Q$ such that for every relatively compact, open set $G' \subset\subset G$ and every open interval $I' \subset\subset I$ there exists a function $u' \in \mathscr{F}(I\times X)$ with $u = u'$ on $I'\times G'$. We say that a function $u$ belongs to $\mathscr{F}_\square(I\times G)$ if $u \in \mathscr{F}(I\times X)$ and if for a.e. $t\in I$ the function $u_t$ has compact support in $G$. Note that a function $u\in\mathscr{F}_\square(I\times G)$ only has to vanish on the lateral boundary $I\times\partial G$ but neither on the upper boundary $\{\tau\}\times G$ nor on the lower boundary $\{\sigma\}\times G$.

DEFINITION. Let $\alpha\in\mathbf{R}$. We say that $u$ is a *local subsolution* (resp. *local supersolution*) of the equation

$$(L_t + \alpha)u = \tfrac{\partial}{\partial t}u \quad \text{on } Q$$

if $u\in\mathscr{F}_{loc}(Q)$ and

$$\mathscr{E}_J(u,\phi) - \alpha\int_{J\times X} u\cdot\phi\, d\underline{m} \leq 0 \tag{2.10}$$

(or $\mathscr{E}_J(u,\phi)-\alpha\int_{J\times X}u\cdot\phi\,d\underline{m}\geq 0$, resp.) for all $J\subset\subset I$ and all nonnegative $\phi\in\mathscr{F}_\square(Q)$. The function $u$ is called a *local solution* if it is a local subsolution and a local supersolution. In this case, (2.10) holds true with "$\equiv$" for all $\phi\in\mathscr{F}_\square(Q)$.

In order to be precise one could call our solutions *local weak solutions.* Note that if for a.e. $t\in\boldsymbol{R}$ the function $u(t,.)$ is locally in the domain of the operator $L_t$ then (2.10) is satisfied if and only if the functions $L_tu(t,.)$ satisfy $L_tu(t,x)+\alpha\cdot u(t,x)\geq\frac{\partial u}{\partial t}(t,x)$ for $\underline{m}$-a.e. $(t,x)\in Q$.

### B) $L^p$-estimates

The main result of this section is the fact that any nonnegative local solution $u$ of the equation $L_tu=\frac{\partial}{\partial t}u$ in a parabolic cylinder $Q$ can be estimated from above (and also from below) by its $L^p$-mean $(\int u^p)^{1/p}$ with arbitrary $p>0$ (or $p<0$, resp.).

This result partly holds true also for nonnegative local sub- or supersolutions of the equation $L_tu=\frac{\partial}{\partial t}u$ and even for nonnegative local sub- or supersolutions of the equation $(L_t+\alpha)u=\frac{\partial}{\partial t}u$. Note that if we consider *subsolutions* then we may restrict ourselves to the case $\alpha\geq 0$ (since for every $\alpha<0$ all nonnegative local subsolutions of the equation $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ are also local subsolutions of the equation $L_tu=\frac{\partial}{\partial t}u$) and considering *supersolutions* we similarly may restrict oureselves to $\alpha\leq 0$.

In the sequel, $Q$ will always chosen to be a parabolic ball. Given a parabolic ball $Q=Q_r(s,x)=]s-r^2,s+r^2[\times B_r(x)$ and a parameter $\gamma\in]0,1[$ we simplify notation and use the following abbreviations: $Q^-(\gamma)=]s-\gamma\cdot r^2,s[\times B_{\gamma\cdot r}(x)$, $Q^+(\gamma)=]s,s+\gamma\cdot r^2[\times B_{\gamma\cdot r}(x)$ and $Q(\gamma)=]s-\gamma\cdot r^2,s+\gamma\cdot r^2[\times B_{\gamma\cdot r}(x)$.

**Theorem 2.1.** *Assume that (B), (C) and (SUP) hold true on $Y=B_R(y)\subset X$ with constants $N=N(Y)$, $C_S=C_S(Y)$ and $\kappa=\kappa(Y)$. Then there exists a constant $C_1=C_1(N)$ such that for all $p\in\boldsymbol{R}\backslash\{0,1\}$, all $\alpha\in\boldsymbol{R}$, all balls $B_r(x)\subset B_R(y)$, all $s\in\boldsymbol{R}$, all $\delta\in]0,1[$ and with*

$$C=C_1\cdot\frac{C_S^{N/2}\cdot(1+|\alpha p|r^2)^{1+N/2}\cdot\kappa^{1+N}}{\delta^{2+N}\cdot r^2\cdot m(B_r(x))}$$

*we have the following estimates:*

$$\sup_{Q^-(1-\delta)}u^p\leq C\cdot\left(\tfrac{p}{p-1}\right)^{1+N}\cdot\int_{Q^-(1)}u^p\,d\underline{m}\tag{2.11.a}$$

*whenever $p>1$ and $u$ is a nonnegative subsolution of the equation $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ on $Q^-(1)$;*

$$\sup_{Q(1-\delta)}u^p\leq C\cdot\int_{Q(1)}u^p\,d\underline{m}\tag{2.11.b}$$

*whenever* $p \neq 0$ *and* $u$ *is a nonnegative solution* (!) *of the equation* $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ *on* $Q(1)$;

$$\sup_{Q^-(1-\delta)} u^p \leq C \cdot \int_{Q^-(1)} u^p \, d\underline{m} \tag{2.11.c}$$

*whenever* $p<0$ *and* $u$ *is a nonnegative supersolution of the equation* $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ *on* $Q^-(1)$.

Note that (2.11.c) is actually a pointwise lower estimate for $u$ on $Q^-(1-\delta)$.

The proof of the Theorem depends essentially on the following Lemma which yields the steps for the iteration.

**Lemma 2.2.** *Under the above assumaptions there exists a constant* $C_2=C_2(N)$ *such that for all* $p \in \boldsymbol{R}\setminus\{0,1\}$, *all* $\alpha \in \boldsymbol{R}$, *all balls* $B_r(x) \subset B_R(y)$, *all* $s \in \boldsymbol{R}$, *all* $\delta \in ]0,1[$ *and with*

$$C=C_2 \cdot C_S \cdot \frac{R^2}{m(B_R(y))^{2/N}} \cdot (1+(p\alpha)^{\pm}R^2)^{1+2/N} \cdot \left(1+\frac{\kappa}{|1-p^{-1}|}\right)^{1+2/N} \cdot \left(\frac{1}{\delta r}\right)^{2+4/N}$$

(*where* $(p\alpha)^{\pm}=p\cdot(\alpha \vee 0)$ *if* $p>1$ *and* $=|p|\cdot(-\alpha \vee 0)$ *if* $p<1$) *we have the following estimates*:

$$\int_{Q^-(1-\delta)} (u^p)^{1+2/N} d\underline{m} \leq C \cdot \left(\int_{Q^-(1)} u^p d\underline{m}\right)^{1+2/N} \tag{2.12.a}$$

*whenever* $p>1$ *and* $u$ *is a nonnegative subsolution of the equation* $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ *on* $Q^-(1)$;

$$\int_{Q^+(1-\delta)} (u^p)^{1+2/N} d\underline{m} \leq C \cdot \left(\int_{Q^+(1)} u^p d\underline{m}\right)^{1+2/N} \tag{2.12.b}$$

*whenever* $0<p<1$ *and* $u$ *is a nonnegative supersolution of the equation* $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ *on* $Q^+(1)$;

$$\int_{Q^-(1-\delta)} (u^p)^{1+2/N} d\underline{m} \leq C \cdot \left(\int_{Q^-(1)} u^p d\underline{m}\right)^{1+2/N} \tag{2.12.c}$$

*whenever* $p<0$ *and* $u$ *is a nonnegative supersolution of the equation* $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ *on* $Q^-(1)$.

Proof. We follow the classical proof by J. Moser [26, Lemma 1]. We begin with an inequality which holds true for all $v \in \mathscr{F}_{loc}(I \times B_r(x))$. From the Sobolev inequality (2.6) we deduce that for a.e. $t \in I$

$$\left(\int_{B[1-\delta]} |v_t|^{\frac{2N}{N-2}} dm\right)^{\frac{N-2}{N}}$$

$$\leq 3\cdot C_S \cdot \frac{(1-\frac{\delta}{2})^2 r^2}{m(B_{(1-\frac{\delta}{2})r}(x))^{2/N}} \int_{B[1-\frac{\delta}{2}]} d\Gamma(v_t,v_t) + (\frac{2}{\delta r})^2 v_t^2 dm$$

$$\leq 48\cdot C_S \frac{R^2}{m(B_R(y))^{2/N}} \int_{B[1-\frac{\delta}{2}]} d\Gamma(v_t,v_t) + \frac{1}{\delta^2 r^2} v_t^2 dm \tag{2.13}$$

where $B[\gamma]=B_{\gamma\cdot r}(x)$ and $v_t=v(t,.)$. Following [25, Lemma 2] we derive from (2.13) that

$$\int_{J\times B[1-\delta]} |v|^{2(1+2/N)} dm \leq 48\cdot C_S \cdot \frac{R^2}{m(B_R(y))^{2/N}}$$

$$\cdot\left(\int_{J\times B[1-\frac{\delta}{2}]} d\Gamma(v_t,v_t)dt + \frac{1}{\delta^2 r^2} v^2 dm\right)\cdot\left(\sup_{t\in J}\int_{B[1-\frac{\delta}{2}]} v_t^2 dm\right)^{2/N} \tag{2.14}$$

for any interval $J\subset\subset I$ and any ball $B_r(x)\subset B_R(y)$.

Now we fix a number $p\in \mathbf{R}\setminus\{0,1\}$ and in the case $p>1$ (resp. $p<1$) we assume that $u$ is a local subsolution (or supersolution, resp.) of the equation $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ on $Q^{\mp}(1)$. That is,

$$(p-1)\cdot\left(-\int_J \mathscr{E}_t(u_t,\phi_t)dt + \alpha\int_{J\times X}(u_t,\phi_t)dt - \int_{J\times X}\left(\frac{\partial u_t}{\partial t},\phi_t\right)dt\right)\geq 0 \tag{2.15}$$

for all $J\subset\subset I$ and all $\phi\in\mathscr{F}_{\square}(Q^{\mp}(1))$. We choose $\phi=u^{p-1}\cdot\psi^2\cdot\eta^2$ where $\psi(t,y)=\psi(y)=(\frac{2}{\delta\cdot r}\cdot\rho_{x,(1-\frac{\delta}{2})r}(y))\wedge 1$ and $\eta(t,y)=\eta(t)$. The choice of $\eta$ depends on $p$. If $\frac{1}{p}<1$ we take $\eta(t)=1$ for $t>s-(1-\frac{\delta}{2})r^2$, $\eta(t)=0$ for $t<s-r^2$ and linearly interpolated in the remaining interval. If $\frac{1}{p}<1$ we take $\eta(t)=1$ for $t<s+(1-\frac{\delta}{2})r^2$, $\eta(t)=0$ for $t>s+r^2$ and linearly interpolated in the remaining interval.

In general, however, this function $\phi$ will not be in $\mathscr{F}_{\square}(Q^{\mp}(1))$. We have to approximate it. For instance, in the definition of $\phi$ we can replace $u$ by $u_n=(u\wedge n)\vee\frac{1}{n}$ (with $n\in\mathbf{N}$). Also in order to apply (SUP), we have to approximate $u$ by $u_n\in L^{\infty}(X,m)\cap L^{-\infty}(X,m)$. In the limit $n\to\infty$ we actually obtain that (2.15) holds true with the above choice of $\phi=u^{p-1}\cdot\psi^2\cdot\eta^2$ and that we may apply the strong uniform parabolicity condition (SUP). For details we refer to the proof of Theorem 1 in [36]. Thus, we deduce from (2.15)

$$\frac{2}{\kappa}\left(1-\frac{1}{p}\right)^2\int_J\int_X \psi^2 d\Gamma(u_t^{p/2},u_t^{p/2})\eta_t^2 dt + \frac{p-1}{p}\int_J (\tfrac{\partial}{\partial t}u_t^p,\psi^2)\eta_t^2 dt$$

$$\leq 2\kappa \int_J \int_X u_t^p d\Gamma(\psi,\psi)\eta_t^2 dt + \alpha(p-1)\int_J \int_X u_t^p \psi^2 dm\eta_t^2 dt$$

and following the argumentation of [26, section 4] we conclude from the latter that

$$\sup_{t\in I} \int_{B[1-\frac{\delta}{2}]} u_t^p dm + \frac{\varepsilon}{\kappa}\iint_{Q^{\mp}(1-\frac{\delta}{2})} d\Gamma(u_t^{p/2},u_t^{p/2})dt$$
$$\leq 9\left(\frac{\kappa}{\varepsilon}\frac{1}{\delta^2 r^2}+\frac{1}{\delta^2 r^2}+\frac{(p\alpha)^\pm}{4}\right)\cdot \int_{Q^{\mp}(1)} u^p d\underline{m} \tag{2.16}$$

where $\varepsilon = \frac{1}{2}|1-p^{-1}|$ and (depending on whether $\frac{1}{p}<1$ or $\frac{1}{p}>1$) $I$ is either the interval $]s-(1-\frac{\delta}{2})r^2,s[$ or the interval $]s,s+(1-\frac{\delta}{2})r^2[$, $Q^{\mp}(1)$ is either $Q^-(1)$ or $Q^+(1)$ and $(p\alpha)^\pm$ is either $p\cdot(\alpha\vee 0)$ or $|p|\cdot(-\alpha\vee 0)$. Finally, (2.14) (applied to $v=u^{p/2}$) and (2.16) together imply

$$\int_{Q^{\mp}(1-\delta)} (u^p)^{1+2/N} d\underline{m}$$
$$\leq 48\cdot C_S\cdot \frac{R^2}{m(B_R(y))^{2/N}}\cdot\left[9\cdot\frac{\kappa}{\varepsilon}\left(\frac{\kappa}{\varepsilon}\frac{1}{\varepsilon\delta^2 r^2}+\frac{1}{\delta r^2}+\frac{(p\alpha)^\pm}{4}\right)+\frac{1}{\delta^2 r^2}\right]\cdot$$
$$\cdot\left[9\left(\frac{\kappa}{\varepsilon}\frac{1}{\varepsilon\delta^2 r^2}+\frac{1}{\delta r^2}+\frac{(p\alpha)^\pm}{4}\right)\right]^{2/N}\cdot\left[\int_{Q^{\mp}(1)} u^p d\underline{m}\right]^{1+2/N}$$
$$\leq C_3(N)\cdot C_S\cdot\frac{R^2}{m(B_R(y))^{2/N}}\cdot\left[\frac{\kappa}{\varepsilon}+(p\alpha)^\pm r^2+1\right]^{1+2/N}\cdot\left[\frac{\kappa}{\varepsilon}+1\right]$$
$$\cdot\left(\frac{1}{\delta^2 r^2}\right)^{1+2/N}\cdot\left[\int_{Q^{\mp}(1)} u^p d\underline{m}\right]^{1+2/N}$$
$$\leq C_4(N)\cdot C_S\cdot\frac{R^2}{m(B_R(y))^{2/N}}\cdot(1+(p\alpha)^\pm R^2)^{1+2/N}\cdot\left(1+\frac{\kappa}{|1-p^{-1}|}\right)^{2+2/N}$$
$$\cdot\left(\frac{1}{\delta r}\right)^{2+4/N}\cdot\left[\int_{Q^{\mp}(1)} u^p d\underline{m}\right]^{1+2/N}. \tag{2.17}$$

□

**Proof of Theorem 2.1.** a), c): Now we are in a position to carry out the Moser iteration. In the case $\frac{1}{p}<1$ we choose $\rho_\nu = 1-\delta+\delta\cdot 2^{-\nu}$ and $p_\nu = p\cdot(1+\frac{2}{N})^\nu$.

Note that the constant $c_1$ in [26] depends linearly on $c_{11}^{n/2}$ (with $n$ being the dimension which comes in via the Sobolev inequality). Hence, in our case the iteration of (2.12.c) resp. (2.12.a) yields

$$\sup_{Q^{\pm}(1-\delta)} u^p \leq C_5(N)\cdot C_S^{N/2}\cdot \frac{R^2}{m(B_R(y))}\cdot(1+(p\alpha)^{\pm}R^2)^{1+N/2}\cdot\kappa^{1+N}\cdot$$

$$\cdot\left(1+\frac{1}{|1-p^{-1}|}\right)^{1+N}\cdot\left(\frac{1}{\delta r}\right)^{N+2}\cdot\int_{Q^{\pm}(1)} u^p d\underline{m}. \tag{2.18}$$

b) The case $\frac{1}{p}>1$ is a little bit more delicate. We have to make a finite number of iterations with values $p_\nu\in\,]0,1]$ (here using estimate (2.12.b)) and then as before an infinite number of iterations with values $p_\nu\in\,]1,\infty]$ (now using estimate (2.12.a)). Hence, we have to restrict ourselves to *solutions* of the equation $(L_t+\alpha)u=\frac{\partial}{\partial t}u$ and we have to consider them on cylinders of the form $Q(\gamma)$ (instead of $Q^{\pm}(\gamma)$). Note that the estimates in Lemma 2.1 remain true if we replace all cylinders $Q^{\pm}(\gamma)$ by $Q(\gamma)$.

The iteration on $Q(\gamma)$ now yields for any $p\neq 0$

$$\sup_{Q(1-\delta)} u^p \leq C_6(N)\cdot C_S^{N/2}\cdot\frac{R^N}{m(B_R(y))}\cdot(1+|p\alpha|\cdot R^2)^{1+N/2}\cdot\kappa^{1+N}\cdot\left(\frac{1}{\delta r}\right)^{N+2}\cdot\int_{Q^{\pm}(1)} u^p d\underline{m}. \tag{2.19}$$

Note that in the case $\alpha=0$ the constant on the RHS of (2.19)(and of (2.11.b)) does *not* depend on $p$. For $p\in\,]0,2[$ this can be seen by the argument of [26, p.739] (with $\frac{1}{2}$ instead of "$\mu$") and for $p\in \boldsymbol{R}\setminus[0,2]$ it follows from (2.18). □

### 2.4. Estimates for the fundamental solution

Our goal is to derive upper estimates of Gaussian type for the fundamental solution $p(s,x,t,y)$ of the parabolic operator $L_t-\frac{\partial}{\partial t}$ on $I\times X$. The idea is to combine the integrated Gaussian estimates from section 1.5 (which hold true globally on $X$) with the subsolution estimates from the previous section (which should hold true in suitable neighborhoods of $x$ and $y$). To this end, we assume that the (global) assumptions (A) and (UP) hold true on $X$ and that the assumptions (B),(C) and (SUP) hold true locally on $X$ (or at least in suitable neighborhoods of given points $x$ and $y$).

### A) The fundamental solution

**Proposition 2.3.** *There exists a measurable function* $p:\boldsymbol{R}\times X\times\boldsymbol{R}\times X\to[0,\infty[$ *with the following properties:*

*(i) for every $t>s$, m-a.e. $x,y\in X$ and every $f\in L^1(X,m)+L^\infty(X,m)$*

$$T_t^s f(y)=\int_X p(t,y,s,z)f(z)m(dz) \tag{2.20.a}$$

*and*

$$\hat{S}_s^t f(y)=\int_X p(t,z,s,x)f(z)m(dz); \tag{2.20.b}$$

*(ii) for every $s<\sigma<\tau$ and m-a.e. $x\in X$ the function*

$$u\colon (t,y)\mapsto p(t,y,s,x) \tag{2.21.a}$$

*is a global solution of the equation $L_t u=\frac{\partial}{\partial t}u$ on $]\sigma,\tau[\times X$ and for every $t>\tau>\sigma$ and m-a.e. $y\in X$ the function*

$$u\colon (s,x)\mapsto p(t,y,s,x) \tag{2.21.b}$$

*is a global solution of the equation $\hat{L}_t u=-\frac{\partial}{\partial t}u$ on $]\sigma,\tau[\times X$;*

*(iii) for every $s<r<t$ and m-a.e. $x,y\in X$*

$$p(t,y,s,x)=\int_X p(t,y,r,z)p(r,z,s,x)m(dz); \tag{2.22}$$

*(iv) for every $s<t$*

$$\int_X\int_X p(t,y,s,x)^2 m(dx)m(dy)\le e^{-2k\lambda(t-s)}\le 1; \tag{2.23}$$

*(v) $p$ is locally bounded on the set $\{(s,x,t,y)\colon s<t\}$.*

Proof. (i),(v): We define $p(t,y,s,x)$ as the density of the transition operator $T_t^s$. Fix $(s',x',t',y')\in I\times X\times I\times X$ with $s'<t'$ and a neighborhood $W$ of $(s',x',t',y')$ in $I\times X\times I\times X$. Without restriction $W=Q_r(s',x')\times Q_r(t',y')$ with some $r<\frac{1}{4}\sqrt{t'-s'}$. For $s\in]s'-r^2,s'+r^2[$ and $f\in L^1(X,m)\cap L^\infty(X,m)$ we consider the function

$$u:(t,y)\mapsto T_t^s f(y)$$

on $Q_{2r}(t',y')$. By definition of the operators $(T_t^s)_t$ (cf. section 1.4.C), it is a solution of the equation $L_t u=\frac{\partial}{\partial t}u$ on $Q_{2r}(t',y')$ and according to Theorem 2.1 it therefore satisfies

$$\begin{aligned}\sup_{Q_r(t',y')} u &\le C\cdot\underline{\int}_{Q_{2r}(t',y')} u\,dm\\ &\le C\cdot\int_{t'-4r^2}^{t'+4r^2}\int_X T_t^s f(y)m(dy)dt\le C\cdot 8r^2\cdot\int_X f(y)m(dy)\end{aligned}$$

where the latter inequality follows from the contracion property of $T_t^s$ on $L^1(X,m)$ (which in turn is a consequence of the Markov property of $\hat{\mathscr{E}}_t$, see Proposition 1.6). From the Theorem of Dunford-Pettis it now follows that $T_t^s$ is an integral operator with a density $p(t,.,s,.)$ satisfying

$$\sup_{x\in X}\sup_{y\in B_r(y')} p(t,y,s,x)\leq C\cdot 8r^2$$

whenever $|t-t'|$ and $|s-s'|<r^2$. The claim for $f\in L^1(X,m)$ now follows by density and for $f\in L^\infty(X,m))$ by weak* continuity. Both together imply the assertion for $f\in L^1(X,m)+L^\infty(X,m)$.

An analogous argument applies to the operator $\hat{S}_s^t$ which also turns out to be an integral operator with a locally bounded density, say $q(s,.,t,.)$. According to Lemma 1.5, we can choose $p$ and $q$ such that $p(t,y,s,x)=q(s,x,t,y)$ for all $t,y,s,x$.

(iii) is an obvious consequence of the operator identity $T_t^s=T_t^r\circ T_r^s$ (cf. section 1.4.C).

(iv) follows from the fact that $\int_X\int_X p(t,y,s,x)^2 m(dx)m(dy)=\|T_t^s\|^2$ and that $\|T_t^s\|\leq e^{-k\lambda(t-s)}\leq 1$ (cf. section 1.4.C).

(ii) From (iv) we deduce that $\int_X\int_X p(\sigma,y,s,x)^2 m(dx)m(dy)\leq 1$, hence, that for $m$-a.e. $x\in X$

$$\int_X p(\sigma,y,s,x)^2 m(dy)<\infty,$$

or, in other words, that the function $f\colon y\mapsto p(\sigma,y,s,x)$ satisfies $f\in L^2(X,m)$. Therefore, there exists a unique solution of the initial value problem $L_t u=\frac{\partial}{\partial t}u$ on $]\sigma,\tau[\times X$, $u=f$ on $\{\sigma\}\times X$. By definition of $T_\psi^\sigma$, this solution is given by

$$u(t,y)=T_t^\sigma f(y)=\int p(t,y,\sigma,z)f(z)m(dz).$$

Using the particular choice of $f$ and property (iii) we get

$$u(t,y)=\int p(t,y,\sigma,z)p(\sigma,z,s,x)m(dz)=p(t,y,s,x)$$

which proves that $(t,y)\mapsto p(t,y,s,x)$ is a solution of $L_t u=\frac{\partial}{\partial t}u$ on $]\sigma,\tau[\times X$. An analogous reasoning yields the assertion for the formally adjoint operator $\hat{L}_t=\frac{\partial}{\partial t}$.

□

**B) Gaussian estimates**

We still assume that the (global) assumptions (A) and (UP) hold true (with constants $0<k\leq K<\infty$) and that assumptions (B), (C) and (SUP) hold true locally on $X$. The latter means that for any points $y_i\in X$ $(i=1,2)$ there exist numbers $R_i>0$ such that (B), (C) and (SUP) hold true in $B_{R_i}(y_i)$ with constants $N_i$, $C_{S,i}$ and $\kappa_i$.

Instead of that , it would be sufficient to suppose that the estimate (2.11.a) with $p=2$ holds true for all nonnegative local solutions in $B_{R_i}(y_i)$.

**Theorem 2.4.** *Under the above assumptions, the following estimate holds true for all points $(t_1,y_1)$ and $(t_2,y_2)\in I\times X$ with $t_1<t_2$*

$$p(t_2,y_2,t_1,y_1)\leq\sqrt{A_1A_2}\cdot\exp\left(-\frac{\rho^2(y_1,y_2)}{4K(t_2-t_1)}\right)\cdot\left(1+\frac{\rho(y_1,y_2)^2}{K(t_2-t_1)}\right)^{N/2}$$

$$\cdot\exp(-k\lambda(t_2-t_1))\cdot(1+k\lambda(t_2-t_1))^{1+N/2}$$

*where $N=\frac{N_1+N_2}{2}$, $A_i=C_3(N_i)\cdot C_{S,i}^{N_i/2}\cdot\kappa^{1+N_i}\cdot m^{-1}(B_{\sqrt{\tau_i}}(y_i))$, $\tau_i=\inf\{K(t_2-t_1),R_i^2\}$ and $C_3(N_i)$ is a constant depending only on $N_i$ $(i=1,2)$.*

Proof. Let us fix radii $r_1$ and $r_2\in]0,\sqrt{K(t_2-t_1)}[$.

For $\psi\in\mathscr{F}_{loc}\cap\mathscr{C}_b(X)$ with $d\Gamma(\psi,\psi)\leq dm$ on $X$ and $f\in L^2(X,m)$, $f\geq 0$, we consider the function $v_t=e^{k\lambda(t-s)}\cdot T_t^s(e^{-\beta\psi}\cdot f)$ (where $\beta,s,t\in\boldsymbol{R}$, $s<t$). Then according to Lemma 1.7, $\|e^{\beta\psi}\cdot v_t\|\leq e^{\beta^2\cdot K\cdot(t-s)}\cdot\|f\|$ and $v:(t,y)\mapsto v_t(y)$ is a solution of the equation $(L_t+k\lambda)v=\frac{\partial}{\partial t}v$ on $]s,\infty[\times X$. Applying the subsolution estimate (2.11.a) to $v$ on the cylinder $Q_2=Q_{r_2}^-(t_2,y_2)$ yields

$$\begin{aligned}v^2(t_2,y_2)&\leq C_1(N_2)\cdot C_{S,2}^{N_2/2}\cdot\kappa_2^{1+N_2}\cdot(1+k\lambda r_2^2)^{1+N_2/2}\cdot\frac{1}{r_2^2\cdot m(B_{r_2}(y_2))}\int_{Q_2}v^2\underline{dm}\\
&\leq C_1(N_2)\cdot C_{S,2}^{N_2/2}\cdot\kappa_2^{1+N_2}\cdot(1+k\lambda r_2^2)^{1+N_2/2}\cdot e^{-2\beta\psi(y_2)}\cdot e^{2|\beta|\cdot r_2}\\
&\quad\cdot\frac{1}{r_2^2\cdot m(B_{r_2}(y_2))}\int_{Q_2}e^{2\beta\psi}\cdot v^2\underline{dm}\\
&\leq C_1(N_2)\cdot C_{S,2}^{N_2/2}\cdot\kappa_2^{1+N_2}\cdot(1+k\lambda r_2^2)^{1+N_2/2}\cdot e^{-2\beta\psi(y_2)}\cdot e^{2|\beta|\cdot r_2}\\
&\quad\cdot e^{2\beta^2K(t_2-s)}\cdot\frac{1}{m(B_{r_2}(y_2))}\cdot\|f\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}v^2(t_2,y_2)&=\left[e^{k\lambda(t_2-s)}\cdot\int_X p(t_2,y_2,s,x)\cdot e^{-\beta\psi(x)}\cdot f(x)m(dx)\right]^2\\
&\geq e^{2k\lambda(t_2-s)}\cdot e^{-2\beta\psi(y_1)}\cdot e^{-2|\beta|\cdot r_1}\cdot\left[\int_{Br_1(y_1)}p(t_2,y_2,s,x)\cdot f(x)m(dx)\right]^2.\end{aligned}$$

Taking the supremum over all $f \in L^2(B_{r_1}(y_1),m)$ with $\|f\|=1$ we obtain

$$\int_{Br_1(y_1)} p(t_2,y_2,s,x)^2 m(dx)$$
$$\leq C_1(N_2)\cdot C_{S,2}^{N_2/2}\cdot \kappa_2^{1+N_2}\cdot(1+k\lambda r_2^2)^{1+N_2/2}\cdot e^{2\beta(\psi(y_1)-\psi(y_2))}\cdot e^{2|\beta|\cdot(r_1+r_2)}$$
$$\cdot e^{2(\beta^2K-k\lambda)\cdot(t_2-s)}\cdot\frac{1}{m(B_{r_2}(y_2))}. \tag{2.24}$$

Now we use the fact that the function $w:(s,x)\mapsto e^{k\lambda(t_2-s)}\cdot p(t_2,y_2,s,x)$ is a solution of the time reversed equation $(L_t+k\lambda)w=-\frac{\partial}{\partial t}w$ on the cylinder $Q_1=Q^+_{r_1}(t_1,y_1)$ and therefore satisfies the subsolution estimates of Theorem 2.1 with the same constants but with reversed time direction. In particular,

$$e^{2k\lambda(t_2-t_1)}\cdot p^2(t_2,y_2,t_1,y_1)$$
$$\leq C_1(N_1)\cdot C_{S,1}^{N_1/2}\cdot\kappa_1^{1+N_1}\cdot(1+k\lambda r_1^2)^{1+N_1/2}\cdot\frac{1}{r_1^2\cdot m(B_{r_1}(y_1))}$$
$$\cdot\int_{Q_1} e^{2k\lambda(t_2-s)}\cdot p^2(t_2,y_2,s,x)\cdot d\underline{m}(s,x). \tag{2.25}$$

Together with (2.24) we get

$$p(t_2,y_2,t_1,y_1)\leq\sqrt{A'_1A'_2}\cdot e^{\beta(\psi(y_1)-\psi(y_2))}\cdot e^{2|\beta|(r_1+r_2)}\cdot e^{\beta^2K(t_2-t_1)}\cdot e^{-k\lambda(t_2-t_1)} \tag{2.26}$$

where

$$A'_i=C_1(N_i)\cdot C_{S,i}^{N_i/2}\cdot\kappa_i^{1+N_i}\cdot(1+k\lambda\cdot r_i^2)^{1+N_i/2}\cdot m^{-1}(B_{r_i}(y_i)).$$

We now choose

$$r_i=\inf\{\sqrt{K\tau},\frac{K\tau}{\rho},R_i\}\quad(i=1,2),$$

$\beta=\frac{\rho}{2K\cdot\tau}$ and $\psi\in\mathscr{F}_{loc}\cap\mathscr{C}_b(X)$ with $\psi(y_1)-\psi(y_2)$ arbitrarily close to $-\rho$ where $\rho=\rho(y_1,y_2)$ and $\tau=t_2-t_1$. Then(2.26) states that

$$p(t_2,y_2,t_1,y_1)\leq\sqrt{A''_1A''_2}\cdot\exp\left(-\frac{\rho^2(y_1,y_2)}{4K(t_2-t_1)}\right)\cdot\exp(-k\lambda(t_2-t_1))$$

where

$$A_i''=C_1(N_i)\cdot C_{S,i}^{N_i/2}\cdot\kappa_i^{1+N_i}\cdot(1+k\lambda\cdot r_i^2)^{1+N_i/2}\cdot m^{-1}(B_{r_i}(y_i))$$
$$\leq C_3(N_i)\cdot C_{S,i}^{N_i/2}\cdot\kappa_i^{1+N_i}\cdot(1+k\lambda\cdot\tau)^{1+N_i/2}\cdot\left(1+\frac{\rho^2(y_1,y_2)}{K\tau}\right)^{N_i/2}$$
$$\cdot m^{-1}(B_{\sqrt{\tau_i}}(y_i)).$$

and $\tau_i=\inf\{K\tau,R_i^2\}$ $(i=1,2)$. □

The Gaussian estimates for the fundamental solutions have a particular nice form if the assumptions (B), (C) and (SUP) hold true globally on $X$.

**Corollary 2.5.** *Under the above assumptions,there exists a constant $C_4(N)$ only depending on $N$ such that the following estimate holds true uniformly for all points $(t_1,y_1)$ and $(t_2,y_2)\in I\times X$ with $t_1<t_2$*

$$p(t_2,y_2,t_1,y_1)\leq C_4(N_2)\cdot C_S^{N/2}\cdot\kappa^{1+N}\cdot m^{-1/2}(B_{\sqrt{K(t_2-t_1)}}(y_2))\cdot m^{-1/2}(B_{\sqrt{K(t_2-t_1)}}(y_2))$$
$$\cdot\exp\left(-\frac{\rho^2(y_1,y_2)}{4K(t_2-t_1)}\right)\cdot\left(1+\frac{\rho(y_1,y_2)^2}{K(t_2-t_1)}\right)^{N/2}$$

Note that if (SUP) hold true globally, then the constants $k$, $K$ can be chosen as functions of $\kappa$ as follows: $K=1+2\kappa$ and $k=(1+2\kappa)^{-1}$.

Also note that if assumption (B) holds true globally, then all balls $B_r(x)$ are relatively compact and the volume of (concentric balls of) $X$ grows at most polynomially. According to [36], Theorem 5, this implies that the spectral bound $\lambda=0$.

## C) The time-independent, symmetric case

In the time-independent, symmetric case things are much easier. Without restriction, we may assume that $\mathscr{E}_t\equiv\mathscr{E}$. Then of course $k=K=\kappa=1$.

The fundamental solution $p(t,y,s,x)$ satisfies

$$p(t,y,s,x)=p(t-s,y,0,x)=p(t,-s,x,0,y)$$

and instead of the latter we simply write $p(t,-s,x,y)$ and call it heat kernel. Since the map $t\mapsto T_{t-s}^0$ is analytic we also obtain estimates for the time derivative of the heat kernel.

Let us assume that assumptions (A),(B) and (C) hold true locally on $X$. That is, for any points $y_i \in X$ $(i=1,2)$ there exist numbers $R_i>0$ such that (A), (B), (C) hold true in $B_{R_i}(y_i)$ with constants $N_i$ and $C_{S,i}$.

**Theorem 2.6.** *Under the above assumptions, the following estimate holds true for all $j \in N \cup \{0\}$, for all points $y_1, y_2 \in X$ and for all $t>0$*

$$(\tfrac{\partial}{\partial t})^j p(t,y_1,y_2) \le \sqrt{A_1 A_2} \cdot t^{-j} \cdot \exp\left(-\frac{\rho^2(y_1,y_2)}{4t}\right) \cdot \left(1+\frac{\rho^2(y_1,y_2)^2}{t}\right)^{N/2+j} \cdot$$

$$\cdot \exp(-\lambda t) \cdot (1+\lambda t)^{1+N/2+j}$$

*where $N=\frac{N_1+N_2}{2}$, $A_i = C_3(N_i) \cdot C_4(j) \cdot C_{S,i}^{N_i/2} \cdot m^{-1}(B_{\sqrt{\tau_i}}(y_i))$, $\tau_i = \inf\{t, R_i^2\}$, $C_3(N_i)$ is a constant depending only on $N_i$ $(i=1,2)$, and $C_4(j)$ is a constant depending only on $j$.*

Proof. In the case $j=0$, the result is already contained in the previous Theorem. For the canonical Dirichlet form on a Riemannian manifold the result is obtained by L. Saloff-Coste [33]. The general case is proven along the same lines using the $L^2$-analyticity of the map $t \mapsto T^0_{t-s}$ (see the proof of Theorem 6.3 in [33]). □

**Corollary 2.7.** *If the assumptions (A), (B) and (C) hold true globally on X, then the following estimate holds true for all $j \in N \cup \{0\}$ and uniformly for all points $x,y \in X$ and all $t>0$*

$$(\tfrac{\partial}{\partial t})^j p(t,x,y) \le C \cdot t^{-j} \cdot m^{-1/2}(B_{\sqrt{t}}(x)) \cdot m^{-1/2}(B_{\sqrt{t}}(y))$$

$$\cdot \exp\left(-\frac{\rho^2(x,y)}{4t}\right) \cdot \left(1+\frac{\rho(x,y)^2}{t}\right)^{N/2+j}.$$

*with $C = C_4(j) \cdot C_5(N) \cdot C_S^{N/2}$ where $C^4(j)$ is a constant only depending on j and $C_5(N)$ only depends on N.*

REMARK. i) The polynomial correction term $(1+\frac{\rho(x,y)^2}{t})^{N/2+j}$ in the above estimate can of course be absorbed by the Gaussian term $\exp(-\frac{\rho^2(x,y)}{4t})$ if we replace the number 4 by some larger one. That is, *for every $\varepsilon>0$ there exists a constant $C = C_6(\varepsilon, N, j) \cdot C_S^{N/2}$ such that*

$$(\tfrac{\partial}{\partial t})^j p(t,x,y) \le C \cdot t^{-j} \cdot m^{-1/2}(B_{\sqrt{t}}(x)) \cdot m^{-1/2}(B_{\sqrt{t}}(y)) \cdot \exp\left(-\frac{\rho^2(x,y)}{(4+\varepsilon)t}\right)$$

*for all $j \in N \cup \{0\}$, all $x,y \in X$ and all $t>0$.*

ii) From the doubling property it follows that $m(B_{\sqrt{t}}(x)) \le m(B_{\sqrt{t}}(y)) \cdot 2^N \cdot$

$(1+\frac{\rho(x,y)}{\sqrt{t}})^N$. Hence, *for every* $\varepsilon>0$ *there exists a constant* $C=C_7(\varepsilon,N,j)\cdot C_S^{N/2}$ *such that*

$$(\tfrac{\partial}{\partial t})^j p(t,x,y)\leq C\cdot t^{-j}\cdot m^{-1}(B_{\sqrt{t}}(x))\cdot\exp\left(-\frac{\rho^2(x,y)}{(4+\varepsilon)t}\right) \tag{2.27}$$

*for all* $j\in N\cup\{0\}$, *all* $x,y\in X$ *and all* $t>0$.

Following the argumentation in [23] one easily deduces from a heat kernel estimate like (2.27) an estimate for the Green function $g(x,y)=\int_0^\infty p(t,x,y)dt$.

**Corollary 2.8.** *If the assumptions* (*A*), (*B*) *and* (*C*) *hold true globally on X*, then

$$g(x,y)\leq C\cdot\int_{\rho(x,y)}^{\infty}\frac{r\,dr}{m(B_r(x))} \tag{2.28}$$

*for all* $x,y\in X$ *with* $C=C_8(N)\cdot C_S^{N/2}$ *where* $C_8(N)$ *is a constant depending only on N.*

This type of Green function estimate was already obtained in full generality by M. Biroli and U. Mosco [2,3]. It is well-known in more concrete situations like in Riemannian geometry or in the theory of subelliptic operators. From the upper bound (2.28) for the Green function we easily derive a necessary and sufficient criterion for recurrence. In Riemannian geometry, this citerion was established by N. Varopoulos [38]. See also [37a].

**Corollary 2.9.** *Let* $(\mathscr{E},\mathscr{F})$ *be irreducible and let the assumptions* (*A*), (*B*) *and* (*C*) *hold true globally on X. Moreover, fix an arbitrary point* $x\in X$. *Then* $(\mathscr{E},\mathscr{F})$ *is recurrent if and only if*

$$\int_1^\infty\frac{r\,dr}{m(B_r(x))}=\infty. \tag{2.29}$$

Proof. In [36] we already proved that (2.29) is sufficient for recurrence (which requires neither assumption (B) nor (C)). From Corollary 2.8 we now see that (2.29) is also necessary for recurrence. Namely, if (2.29) is not satisfied for some $x\in X$, then first of all we deduce that $X$ can not have finite volume, in particular, $X\backslash B_1(x)$ can not be a $m$-zero set. Moreover, it implies that according to (2.28) the Green function $g(x,y)$ is finite for all $y\in X\backslash B_1(x)$. This however, implies that $(\mathscr{E},\mathscr{F})$ is not recurrent ([36], Theorem 3). □

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