Takakuwa, S.
Osaka J. Math.
32 (1995), 171–184

BEHAVIOR OF MINIMIZING SEQUENCES FOR THE YAMABE PROBLEM

SHOICHIRO TAKAKUWA

(Received May 10, 1993)

1. Introduction

In 1960, Yamabe [14] presented the following problem.

The Yamabe problem. *Given a compact Riemannian manifold* (M,g) *of dimension* $n\,(\geq 3)$, *find a conformally equivalent metric with constant scalar curvature.*

He reduced this problem of finding a smooth function u together with a constant λ satisfying the nonlinear eigenvalue problem

$$\text{(Y)} \qquad -\kappa\Delta_g u+Ru=\lambda u^{N-1}, \qquad u>0 \qquad \text{in } M,$$

$$\kappa=\frac{4(n-1)}{n-2}, \qquad N=\frac{2n}{n-2},$$

where Δ_g denotes the negative definite Laplacian and R is the scalar curvature of g. He attempted to solve equation (Y) by finding a positive extremal of the functional

$$\text{(1.1)} \qquad Q(u)=\int_M(\kappa|\nabla u|^2+Ru^2)dV/(\int_M |u|^N\,dV)^{2/N}.$$

He claimed that for any M quation (Y) has a solution which attains the minimum

$$\text{(1.2)} \qquad \lambda(M)=\inf\{Q(u)|u\in C^\infty(M),\ u\not\equiv 0\}.$$

This constant $\lambda(M)$ is a conformal invariant, which is often called the *Yamabe invariant.* In 1968, however, Trudinger [13] discovered an error in Yamabe's proof and showed that Yamabe's proof works in case $\lambda(M)$ is bounded above by a small constant. In 1976, Aubin [1] extended Trudinger's result. He proved that if M satisfies the inequality

$$\text{(1.3)} \qquad \lambda(M)<\Lambda:=\lambda(S^n)=n(n-1)\,\mathrm{vol}(S^n)^{2/n},$$

then the minimum $\lambda(M)$ is attained by a positive smooth function on M. This result turned the focus of the proof to the question whether M satisfies inequality (1.3). Aubin also proved that if $n \geq 6$ and M is not locally conformally flat, then inequality (1.3) holds. In 1984, Schoen [10] proved in the remaining case that inequality (1.3) holds unless M is conformal to the standard sphere and solved the Yamabe problem in the affirmative. In the proof of the above results [1], [10], [13], [14] the special minimizing sequence of approximate solutions for (Y) is used as a basic tool.

The purpose of this paper is to study the Yamabe problem as a problem in analysis. We shall describe the behavior of any minimizing sequence completely under no assumption on $\lambda(M)$. To prove our main theorem we apply to the minimizing problem of the functional Q several techniques in real analysis combined with the theory of partial differential equations. Through our main theorem we are able to understand the condition (1.3) in analytic standpoint. Our proof is independent of Schoen's result in [10].

We denote by $H^1(M)$ the Sobolev space of order one. Take a minimizing sequence $\{u_j\} \subset H^1(M)$, that is, $Q(u_j)$ tends to $\lambda(M)$ as $j \to \infty$. We may assume that u_j is non-negative almost everywhere and $\|u_j\|_N = 1$, where $\|\cdot\|_N$ denotes L^N norm on M. Indeed, if $\{u_j\}$ is a minimizing sequence, then so is $\{|u_j|/\|u_j\|_N\}$ (see [14].) We esily see that

$$\begin{aligned} \|u_j\|^2_{H^1(M)} &= \int_M |\nabla u_j|^2 dV + \int_M |u_j|^2 dV, \\ &\leq \frac{1}{\kappa} Q(u_j) + C\|u_j\|^2_N. \end{aligned}$$

This guarantees that any minimizing sequence is bounded in $H^1(M)$. If $\{u_j\}$ has a subsequence $\{u_k\}$ converging to some u in $L^N(M)$, then u is a positive smooth function satisfying (Y) with $\lambda = \lambda(M) = Q(u)$ (see [2], [7].) As pointed out in [2], [13], however, $\{u_j\}$ is not always compact in $L^N(M)$. The main result of this paper states how the minimizing sequence $\{u_j\}$ behaves in case its compactness fails.

Main Theorem. *Suppose that (M,g) is a compact connected Riemannian manifold and that $n = \dim M \geq 3$. Let $\{u_j\} \subset H^1(M)$ be a minimizing sequence for Q with $u_j \geq 0$ a.e. on M and $\|u_j\|_N = 1$. Then, either $\{u_j\}$ has a convergent subsequence in $H^1(M)$, or there exist*

(i) *a subsequence $\{k\} \subset \{j\}$,*

(ii) *a point $a \in M$,*

(iii) *a sequence $\{r_k\}$ of $\boldsymbol{R}_+$ with $r_k \to 0$ as $k \to \infty$,*

(iv) *a sequence $\{a_k\}$ of M with $a_k \to a$ as $k \to \infty$,* *satisfying the followings*:

(1) *The sequence* u_k *converges to* 0 *in* $H^1_{loc}(M-\{a\})$,

(2) *The measure* $u_k^N\, dV$ *converges to Dirac measure* δ_a *weakly in the sense of probability measures on* M,

(3) *The renormalized sequence* $\tilde{u}_k(x)=r_k^{(n-2)/2}u_k(\exp_{a_k}(r_k x))$ *converges to the function*

$$v(x)=c(n)\left(\frac{\rho}{\rho^2+|x|^2}\right)^{(n-2)/2} \qquad (\rho>0),$$

in $H^1_{loc}(\boldsymbol{R}^n)$. *Here*, $\exp_{a_k}$ *denotes the exponentil map of* M *at* a_k *and* $c(n)=2^{(n-2)/2}\mathrm{vol}(S^n)^{-(n-2)/2n}$.

(4) *The Yamabe invariant* $\lambda(M)$ *equals to* Λ.

A similar phenomenon to our theorem has been observed in various nonlinear problems under the name of bubble theorem or concentration compactness theorem (for example, see [3], [5], [9]). Our proof is inspired by the work of Brezis-Coron [3] for H-systems. Struwe [11] proved an analogous convergence theorem for a Yamabe-type equation in a bounded domain of $\boldsymbol{R}^n$. Our proof differs from his in idea. Also, we do not use the general method of concentration compactness due to P.L. Lions [8], [9]. We only need a notion of the concentration function intoduced in [8]. In the above theorem, we give a more precise conclusion than both of theirs.

In Section 2 we prove the local convergence theorem, which ensures the strong convergence of sequences under the smallness condition on their L^N norm. In Section 3 we present two propositions on the blowing up of sequences by using the result in Section 2. In both Section 2 and 3 we discuss a more general class of sequences which satisfy th assumption of the Palais-Smale condition. In the final section we give the proof of Main Theorem and an example of non-compact minimizing sequences which blows up at a given point.

Acknowledgement. The author would like to thank Professor O. Kobayashi for his valuable advice.

2. Local Convergence Theorem

In this section we prove the following.

Theorem 2.1. *Let* Ω *be a domain in* M. *Suppose that sequences* $\{u_j\}\subset H^1(\Omega)$ *and* $\{\lambda_j\}\subset \boldsymbol{R}$ *satisfy*

(1) $u_j \to u$ *weakly in* $H^1(\Omega)$,

(2) $\lambda_j \to \lambda \in \boldsymbol{R}$,

(3) $\kappa\Delta_g u_j - Ru_j + \lambda_j |u_j|^{N-2} u_j \to 0$ *in* $H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

Then, there exists a positive constant ε *depending only on* n, λ *such that if* $\{u_j\}$ *satisfies*

$$\int_\Omega |u_j|^N dV \leq \varepsilon \qquad \text{for any } j, \tag{2.1}$$

then $u_j \to u$ *in* $H^1_{\text{loc}}(\Omega)$.

To prove this theorem, we need the following lemma.

Lemma 2.2. *If a sequence* $\{u_j\}$ *converges to* u *weakly in* $H^1(\Omega)$ *and* $u \in L^\infty_{\text{loc}}(\Omega)$, *then*

$$|u_j|^{N-2} u_j - |u|^{N-2} u - |u_j - u|^{N-2}(u_j - u) \to 0 \quad \text{in } H^{-1}(\Omega')$$

for each $\Omega' \Subset \Omega$.

Proof. Set $u_j = u + v_j$. By Rellich theorem, $v_j \to 0$ strongly in $L^2(\Omega)$. Using the mean value theorem

$$|u_j|^{N-2}u_j - |u|^{N-2}u = \int_0^1 \frac{d}{dt}\left(|u + tv_j|^{N-2}(u + tv_j)\right) dt,$$

$$= (N-1)\int_0^1 |u + tv_j|^{N-2} v_j dt.$$

Thus, we obtain

$$|u_j|^{N-2}u_j - |u|^{N-2}u - |v_j|^{N-2}v_j = (N-1)\int_0^1 (u + tv_j|^{N-2} - |tv_j|^{N-2}) v_j dt.$$

We notice that for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) \geq 0$ so that

$$\| u + tv_j|^{N-2} - |tv_j|^{N-2}| \leq \varepsilon |tv_j|^{N-2} + C(\varepsilon)|u|^{N-2},$$

holds. Then, we obtain

$$\| u_j|^{N-2}u_j - |u|^{N-2}u - |v_j|^{N-2}v_j| \leq \varepsilon |v_j|^{N-1} + (N-1)C(\varepsilon)|u|^{N-2}|v_j|.$$

We fix any $\Omega' \Subset \Omega$. For any $\zeta \in H_0^1(\Omega')$ with $\|\zeta\|_{H_0^1(\Omega')} \leq 1$,

$$\left|\int_{\Omega'}(|u_j|^{N-2}u_j-|u|^{N-2}u-|v_j|^{N-2}v_j)\zeta\,dV\right|$$

$$\leq\varepsilon\int_{\Omega'}|v_j|^{N-1}|\zeta|\,dV+(N-1)C(\varepsilon)\int_{\Omega'}|u|^{N-2}|v_j||\zeta|eV,$$

$$\leq\varepsilon\|v_j\|_{L^N(\Omega)}^{N-1}\|\zeta\|_{L^N(\Omega')}+(N-1)C(\varepsilon)\|u\|_{L^\infty(\Omega')}^{N-2}\|v_j\|_{L^2(\Omega')}\|\zeta\|_{L^2(\Omega')},$$

$$\leq C_1\varepsilon+C_2(\varepsilon)\|v_j\|_{L^2(\Omega)}.$$

Since $v_j\to 0$ in $L^2(\Omega)$, we get

$$C_2(\varepsilon)\|v_j\|_{L^2(\Omega)}<\varepsilon,\qquad\text{for sufficiently large } j.$$

Then, we obtain

$$\|\,|u_j|^{N-2}u_j-|u|^{N-2}u-|v_j|^{N-2}v_j\|_{H^{-1}(\Omega')}<(1+C_1)\varepsilon,$$

for any j large enough. This completes the proof. □

Proof of Theorem 1.1. By Rellich theorem, $\{u_j\}$ converges to some u strongly in $L^2(\Omega)$. The limit u satisfies

$$\kappa\Delta_g u-Ru+\lambda|u|^{N-2}u=0\in H^{-1}(\Omega).$$

From the regularity result of Trudinger [13], it follows that $u\in L^\infty_{\mathrm{loc}}(\Omega)$.

We set $u_j=v_j+u$. The sequence $\{v_j\}\subset H^1(\Omega)$ satisfies

$$\begin{aligned} v_j\to 0 &\quad \text{weakly in } H^1(\Omega)\\ &\quad \text{strongly in } L^2(\Omega),\\ v_j(x)\to 0 &\quad \text{a.e. on } \Omega,\end{aligned}$$

Using the theorem of Brezis-Lieb [4], we have

$$\int_\Omega |u_j|^N dV=\int_\Omega |u|^N dV+\int_\Omega |v_j|^N dV+o(1).$$

Applying Lemma 1.2, for each $\Omega'\Subset\Omega$,

$$|u_j|^{N-2}u_j-|u|^{N-2}u-|v_j|^{N-2}v_j\to 0,\quad\text{in } H^{-1}(\Omega').$$

Then, we have

$$\kappa\Delta v_j + \lambda_j |v_j|^{N-2} v_j \to 0, \quad \text{in } H^{-1}(\Omega').$$

Since $\{v_j\}$ is bounded in $H^1(\Omega)$, $(\lambda_j - \lambda)|v_j|^{N-2}v_j \to 0$ in $H^{-1}(\Omega)$. Therefore, we obtain

$$\kappa\Delta_g v_j + \lambda |v_j|^{N-2} v_j \to 0 \quad \text{in } H^{-1}(\Omega') \quad \text{for any } \Omega' \Subset \Omega. \tag{2.2}$$

For any $\zeta \in C_0^1(\Omega)$, we have

$$\kappa\int_\Omega \nabla v_j \cdot \nabla(\zeta^2 v_j) dV = \lambda \int_\Omega \zeta^2 |v_j|^N dV + o(1),$$

$$\kappa\int_\Omega |\nabla(\zeta v_j)|^2 dV = \lambda \int_\Omega (\zeta v_j)^2 |v_j|^{N-2} dV + o(1). \tag{2.3}$$

Using Hölder inequality,

$$\kappa\int_\Omega |\nabla(\zeta v_j)|^2 dV \le \lambda^+ \Big(\int_\Omega |v_j|^N dV\Big)^{2/n} \Big(\int_\Omega |\zeta v_j|^N dV\Big)^{2/N} + o(1),$$

$$\le \lambda^+ \Big(\int_\Omega |u_j|^N dV\Big)^{2/n} \Big(\int_\Omega |\zeta v_j|^N dV\Big)^{2/N} + o(1),$$

where $\lambda^+ = \max\{\lambda, 0\}$. The sharp Sobolev inequality of Aubin [2] then states that for any $\sigma > 0$ there exists a constant $A(\sigma) \ge 0$ such that

$$\Big(\int_\Omega |\zeta v_j|^N dV\Big)^{2/n} \le \frac{\kappa(1+\sigma)}{\Lambda} \int_\Omega |\nabla(\zeta v_j)|^2 dV + A(\sigma) \int_\Omega (\zeta v_j)^2 dV,$$

where $\Lambda = n(n-1)\ \mathrm{vol}(S^n)^{2/n}$. Substituting this inequality, we have

$$\begin{aligned} \int_\Omega |\nabla(\zeta v_j)|^2 dV &\le \frac{\lambda + (1+\sigma)}{\Lambda} \Big(\int_\Omega |u_j|^N dV\Big)^{2/n} \int_\Omega |\nabla(\zeta v_j)|^2 dV \\ &\quad + \frac{\lambda + A(\sigma)}{\kappa} \|u_j\|_N^{N-2} \int_\Omega (\zeta v_j)^2 dV + o(1). \end{aligned} \tag{2.4}$$

We take $\varepsilon > 0$ so that $\lambda^+ \varepsilon^{2/n} < \Lambda$. Then, by choosing σ sufficiently small so that

$$1 - \frac{\lambda^+}{\Lambda}(1+\sigma)\varepsilon^{2/n} > 0.$$

holds, we obtain

$$\left(1-\frac{\lambda^{+}}{\Lambda}(1+\sigma)\varepsilon^{2/n}\right)\int_{\Omega}|\nabla(\zeta v_j)|^2\,dV=o(1).$$

This completes the proof. □

REMARK 2.3.

(1) If $\lambda\leq 0$, from (2.4) it follows directly that

$$\int_{\Omega}|\nabla(\zeta v_j)|^2\,dV=o(1).$$

Therefore, the conclusion of Theorem 2.1 lways holds in case $\lambda\leq 0$.

(2) Consider the case $\lambda>0$. As in the proof of Theorem 1.1, the constant ε in general has a bound $\varepsilon<(\Lambda/\lambda)^{n/2}$. We can take ε arbitrarily close to the constant $(\Lambda/\lambda)^{n/2}$.

3. Blowing up Phenomenon

We first show the following result.

Proposition 3.1. *Let $\{u_j\}$ be a sequence of $H^1(M)$ and $\{\lambda_j\}$ be a sequence of* $\boldsymbol{R}$. *Suppose that*

(i) $u_j\geq 0$ *a.e. on M and* $\|u_j\|_N=1$,

(ii) $u_j\to u$ *weakly in* $H^1(M)$,

(iii) $\lambda_j\to\lambda$ *for some* $\lambda\in\boldsymbol{R}$,

(iv) $\kappa\Delta_g u_j-Ru_j+\lambda_j u_j^{N-1}\to 0,\qquad$ *in* $H^{-1}(M)=(H^1(M))^*$.

Then, there exist a subsequence $\{k\}\subset\{j\}$ and a (possibly empty) finite subset $\mathscr{S}=\{a_0,\cdots,a_m\}$ of M such that

(1) u_k *converges to u in* $H^1_{\mathrm{loc}}(M-\mathscr{S})$.

(2) *For each $i=0,\cdots,m$, there exists a constant $\alpha_i>0$ such that*

$$u_k^N\,dV\to u^N\,dV+\sum_{i=0}^{m}\alpha_i\delta_{a_i},\tag{3.1}$$

weakly in the sense of probability measures on M.

Proof. We take $\varepsilon=\varepsilon(n,\lambda)$ as in Theorem 1.1. We define the set $\mathscr{S}$ by

$$\mathscr{S}=\bigcap_{r>0}\{x\in M\,|\,\liminf_{j\to\infty}\int_{B(x,r)}|u_j|^N\,dV\geq\varepsilon\}.\tag{3.2}$$

We first show that $\mathscr{S}$ is closed. Let $\{x_k\}$ be a sequence in $\mathscr{S}$ converging to x in M and let r be an arbitrary positive number. Then, for any $r'<r$, $B(x_k, r') \subset B(x, r)$ holds for sufficiently large k. Thus,

$$\liminf_{j\to\infty}\int_{B(x,r)}|u_j|^N dV \geq \liminf_{j\to\infty}\int_{B(x_k,r')}|u_j|^N dV \geq \varepsilon.$$

By letting $r'\to r$, we get $x\in\mathscr{S}$. This shows that $\mathscr{S}$ is closed in M, and hence is compact.

Next, we show that $\mathscr{S}$ is at most a finite set. For any $r>0$, we take a maximal family $\{B(x_1, r),\cdots,B(x_I, r)\}$ of $I=I(r)$ disjoint geodesic blls of radius r with center $x_i\in\mathscr{S}$. By maximality $\mathscr{S}$ is covered by $B(x_1, 2r),\cdots,B(x_I, 2r)$. Since $x_i\in\mathscr{S}$, for any $\delta>0$ and each i

$$\int_{B(x_i,r)} u_j^N dV \geq \varepsilon(1-\delta),$$

holds if j is sufficiently large. Summing up, we get

$$(3.3)\qquad I\leq\frac{1}{\varepsilon(1-\delta)}\sum_{i=1}^{I}\int_{B(x_i,r)}u_j^N dV\leq\frac{1}{\varepsilon(1-\delta)}\int_M u_j^N dV\leq\frac{1}{\varepsilon(1-\delta)}.$$

This shows $\mathscr{H}^0(\mathscr{S})\leq 1/(\varepsilon(1-\delta))<\infty$ where $\mathscr{H}^0$ denotes the 0-dimensional Hausdorff measure on M. Since the 0-dimensional Hausdorff measure coincides with the counting measure, $\mathscr{S}$ is at most finite.

We show that a subsequence of $\{u_j\}$ converges strongly in $H^1_{\mathrm{loc}}(M-\mathscr{S})$. If y in $M-\mathscr{S}$, then there exist $r>0$ and infinitely many j such that the inequality

$$\int_{B(y,r)} u_j^N dV \leq \varepsilon$$

holds. By Theorem 2.1 we show that such u_j converges to u strongly in $H^1(B(y, r/2))$. By a diagonal subsequence argument, a subsequence $\{u_k\}$ of $\{u_j\}$ converges strongly in $H^1(\Omega)$ for each $\Omega\Subset M-\mathscr{S}$.

For each j, we define the (signed) Radon measure μ_j on M by

$$\mu_j(A)=\int_A (u_j^N-u^N)\,dV \qquad \text{for } A\subset M.$$

By Fatou's lemma

$$\|\mu_j\| \le \int_M (u_j^N + u^N)\, dV \le 1 + \liminf_{j\to\infty} \int_M u_j^N\, dV = 2,$$

where $\|\mu_k\|$ is the total variation of μ_j. Then a subsequence $\{\mu_k\}$ converges to some non-negative Radon measure μ weakly. Since M is compact, $u_k^N\, dV \to u^N\, dV + \mu$ weakly in the sense of probability measures on M. Since $\{u_k\}$ converges to u in $H^1_{\rm loc}(M-\mathscr{S})$, the support of the measure μ is contined in the finite set $\mathscr{S}$. Thus, we have $\mu = \sum_{i=0}^{m} \alpha_i \delta_{a_i}$ for some $\alpha_i \ge 0$ $(i=1,\cdots,m)$ where we set $\mathscr{S} = \{a_0, \cdots, a_m\}$.

We show each α_i is positive. Fix any $a_i \in \mathscr{S}$. For arbitrarily small $r>0$, we take $\eta_r \in C_0^\infty(M)$ satisfying $0 \le \eta_r \le 1$ in M and

$$\eta_r(x) = \begin{cases} 1 & \text{if } x \in B(a_i, r), \\ 0 & \text{if } x \notin B(a_i, 2r), \end{cases}$$

By the definition of $\mathscr{S}$ we have

$$\varepsilon \le \liminf_{k\to\infty} \int_{B(a_i,r)} u_k^N\, dV \le \lim_{k\to\infty} \int_M \eta_r u_k^N\, dV \le \alpha_i + \int_{B(a_i,2r)} u^N\, dV, \tag{3.4}$$

Letting r tend to 0, we obtain $\alpha_i \ge \varepsilon 0$. This complete the proof. □

Corollary 3.2. *Suppose that sequences $\{u_j\}$, $\{\lambda_j\}$ are as in Proposition* 3.1.

(1) *If $\lambda \le 0$, then $\mathscr{S}$ is empty.*

(2) *If $\lambda \le \Lambda$ and $\mathscr{S}$ is not empty, then $\mathscr{S} = \{a_0\}$ for some $a_0 \in M$, $u \equiv 0$ and the probability measure $u_k^N\, dV$ converges to Dirac measure δ_{a_0} weakly.*

(3) *If $\lambda < p^{2/n}\Lambda$ for some integer $p \ge 2$, then $\mathscr{S}$ consists of at most $p-1$ points.*

Proof. (1) From Remark 2.3 (1), if $\lambda \le 0$, then $\{u_k\}$ converges to u strongly in $H^1(\Omega)$ for each domain $\Omega \subset M$. Thus, the set $\mathscr{S}$ is empty.

(2) From Remark 2.3 (2), we can chooe $\varepsilon = 1-\delta$ for sufficiently small $\delta > 0$. By (3.3), (3.4) we have

$$\mathscr{H}^0(\mathscr{S}) \le \frac{1}{(1-\delta)^2} < 2, \qquad \varepsilon = 1-\delta \le \alpha_0 \le 1.$$

This leads to $\mathscr{S} = \{a_0\}$, $\alpha_0 = 1$ and $u \equiv 0$. (3) From Remark 2.3 (2), we may taken $\varepsilon > 1/p$. Passing to the limit in (3.1), we have

$$1=\lim_{k\to\infty}\int_M u_k^N\,dV=\int_M u^N\,dV+\sum_i \alpha_i.$$

Since $\alpha_i \geq \varepsilon > 1/p$ holds for each i, we have $\mathscr{H}^0(\mathscr{S})<p(1-\|u\|_N^N)\leq p$. □

We next consider the behavior of $\{u_j\}$ near the *blowing up set* $\mathscr{S}$.

Proposition 3.3. *Let $\{u_j\}$, $\{\lambda_j\}$ be as in Proposition 3.1. Suppose a is an arbitrary point in $\mathscr{S}$. Then, there exist*

(i) *a subsequence $\{k\}\subset\{j\}$,*

(ii) *a sequence r of R with $r_k\to 0$ as $k\to\infty$,*

(iii) *a sequence a of M with $a_k\to 0$ as $k\to\infty$*

such that

(1) *the renormalized sequence $\tilde{u}_k(x)=r_k^{(n-2)/2}u_k\,(\exp_{a_k}(r_kx))$ converges to some function v in $H^1_{\mathrm{loc}}(\boldsymbol{R}^n)$,*

(2) *the limit v i a positive smooth function on $\boldsymbol{R}^n$ and satisfies*

$$\begin{cases} -\kappa\Delta v=\lambda v^{N-1} \text{ in } \boldsymbol{R}^n, \\ \displaystyle\int_{\boldsymbol{R}^n}|\nabla v|^2\,dx<\infty, \quad \int_{\boldsymbol{R}^n} v^N\,dx<\infty. \end{cases} \tag{3.5}$$

Proof. We take a normal coordinate neighborhood W of a and a normal coordinate system x of M centered at a. Through this coordinate we identify W as a neighborhood of the origin 0 in $\boldsymbol{R}^n$. So we note that the metric g satisfies $g_{\alpha\beta}=\delta_{\alpha\beta}+O(|x|^2)$ in the x-coordinate. Let $B(x,r)$ be the open ball with center x and radius r and let $B(r)=B(0,r)$. We note that for any integrable function f on W,

$$(1-C_1R^2)\int_{B(R)}|f|\,dV\leq\int_{B(R)}|f|\,dx\leq(1+C_1R^2)\int_{B(R)}|f|\,dV \tag{3.6}$$

holds. We choose $R>0$ small enough so that $\mathscr{S}\cap B(2R)=\{0\}$.

As in [8], we introduce the concentration function

$$Q_j(t)=\sup_{y\in B(R)}\int_{B(y,t)}u_j^N\,dx \qquad \text{for } 0\leq t\leq R.$$

Each function Q_j is continuous and non-decreasing in t, and $Q_j(0)=0$. We fix an arbitrary small $\delta>0$. By the definition of $\mathscr{S}$ and (3.6)

$$Q_j(R)\geq\int_{B(R)}u_j^N\,dx\geq\varepsilon(1-\delta)$$

holds for sufficiently large j. Here, the constant ε is taken as in Theorem 2.1. By continuity of Q_j, there exist $0<r_j<R$ and $a_j\in\overline{B(R)}$ such that

$$Q_j(r_j)\int_{B(a_j,r_j)} u_j^N dx=\varepsilon(1-2\delta).$$

Since the origin is a unique point in $\mathscr{S}\cap B(2R)$, we obtain

$$r_j\to 0 \quad\text{and}\quad a_j\to 0 \quad\text{as}\quad j\to\infty.$$

We set $U(j)=B(a_j/r_j, 2R/r_j)\subset \boldsymbol{R}^n$ and

$$\tilde{u}_j(x)=r_j^{(n-2)/2}u_j(a_j+r_jx).$$

Since a_j lies in $B(R/2)$ for sufficiently large j, we have $B(R/r_j)\subset U(j)$ which leads to $U(j)\to \boldsymbol{R}^n$ as $j\to\infty$. Then, we have

$$\int_{U(j)}|\nabla\tilde{u}_j|^2dx\le C_2,\qquad \int_{U(j)}\tilde{u}_j^N dx\le 1+C_1R^2. \tag{3.7}$$

From the definition of a_j and r_j, $\tilde{u}_j$ satisfies

$$\int_{B(1)}\tilde{u}_j^N dx=\varepsilon(1-2\delta). \tag{3.8}$$

From the assumption we have

$$\kappa\Delta_j\tilde{u}_j-R(a_j+r_j\cdot)r_j^2\tilde{u}_j+\lambda_j\tilde{u}_j^{N-1}\to 0 \quad\text{in } H^{-1}(\boldsymbol{R}^n)$$

where Δ_j is the Laplacian with respect to the metric $g_j=g(a_j+r_j\cdot)$. Since the metric g is Euclidean up to second order,

$$(\Delta_j-\Delta)\tilde{u}_j\to 0,\qquad \text{in } H^{-1}_{\text{loc}}(\boldsymbol{R}^n).$$

Then we get

$$\kappa\Delta\tilde{u}_j+\lambda_j\tilde{u}_j^{N-1}\to 0,\qquad \text{in } H^{-1}_{\text{loc}}(\boldsymbol{R}^n). \tag{3.9}$$

Using the diagonal subsequence argument, we can take a subsequence $\{k\}\subset\{j\}$ so that

$$\tilde{u}_k\to v \qquad \text{weakly in } H^1(\Omega) \text{ for each domain } \Omega\Subset\boldsymbol{R}^n,$$

$$\tilde{u}_k(x)\to v(x) \qquad \text{a.e. on } \boldsymbol{R}^n,$$

for some $v \in H^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ with $v \geq 0$ a.e. By (3.7) we have

$$\int_{\boldsymbol{R}^n} |\nabla v|^2 dx < \infty, \qquad \int_{\boldsymbol{R}^n} v^N dx < \infty.$$

Passing to the limit in (2.9), we see that v is a weak solution of

$$\kappa \Delta v + \lambda v^{N-1} = 0 \quad \text{in } \boldsymbol{R}^n. \tag{3.10}$$

Using the regularity theorem of Trudinger [13], $v \in C^\infty(\boldsymbol{R}^n)$ and v satisfies (3.10) in the classical sense. By the maximum principle, we see that v is either positive everywhere or identically zero.

We prove that $\{\tilde{u}_k\}$ converges in $H^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. Fix any $z \in \boldsymbol{R}^n$. By the definition of a_j, r_j, we have

$$\int_{B(z,1)} \tilde{u}_j^N dx \leq \int_{B(1)} \tilde{u}_j^N dx \leq \varepsilon(1-2\delta) < \varepsilon. \tag{3.11}$$

By Theorem 2.1, $\tilde{u}_j$ converges to v strongly in $H^1(B(z, 1/2))$. Also, we obtain $v \not\equiv 0$, that is, v is positive everywhere. This completes the proof. □

Remark 3.4. In the proof of Proposition 3.3 the fact $U(j) \to \boldsymbol{R}^n$ implies that $|a_k|/r_k \to \infty$ as $k \to \infty$.

4. Proof of Main Theorem

We now give the proof of Main Theorem.

Proof of Main Theorem. We note that Q is a C^2 functional and for any $u \in H^1(M)$ with $\|u\|_N = 1$, Fréchet derivative $Q'(u)$ is given by

$$Q'(u) = -2(\kappa \Delta_g u - Ru + Q(u)|u|^{N-2}u) \in H^{-1}(M).$$

Hence, if $\{u_j\}$ is a minimizing sequence with $u_j \geq 0$ and $\|u\|_N = 1$, then

$$\lambda_j = Q(u_j) \to \lambda(M),$$

$$\kappa \Delta_g u_j - Ru_j + \lambda_j u_j^{N-1} \to 0 \quad \text{in } H^{-1}(M).$$

From the result of Aubin [1], the inequality $\lambda(M) \leq \Lambda$ holds for any M. By Corollary 3.2, we may assume $\lambda(M) > 0$ and $\mathscr{S} = \{a_0\}$. Otherwise $\{u_j\}$ has a strongly convergent subsequence in $H^1(M)$. Then there exist $\{r_k\}$, $\{a_k\}$ and v as stated in Proposition 3.3. The positive smooth function v satisfies (3.5) with $\lambda = \lambda(M)$. By the theorem of Gidas-Ni-Nirenberg [6], positive solutions of (3.5) are completely described. Thus we get

$$v(x)=\left(\frac{4n(n-1)}{\lambda(M)}\right)^{(n-2)/4}\left(\frac{\rho}{\rho^2+|x-b|^2}\right)^{(n-2)/2}, \tag{4.1}$$

for some $\rho>0$ and $b\in \boldsymbol{R}^n$. Passing to the limit in (3.11), we have

$$\int_{B(z,1)} v^N dx \le \int_{B(1)} v^N dx \qquad \text{for any } z\in \boldsymbol{R}^n.$$

This implies that v is radially symmetric. Then, we have $b=0$ in (4.1). By the result of Talenti [12] on Sobolev inequality in $\boldsymbol{R}^n$, such v satisfies

$$\left(\int_{\boldsymbol{R}^n} v^N dx\right)^{2/n}=\frac{\kappa}{\Lambda}\int_{\boldsymbol{R}^n} |\nabla v|^2 dx.$$

Thus we have

$$\int_{\boldsymbol{R}^n} v^N dx=\left(\frac{\Lambda}{\lambda(M)}\right)^{n/2}.$$

Using Fatou's lemma, we have

$$1\le\left(\frac{\Lambda}{\lambda(M)}\right)^{n/2}=\int_{\boldsymbol{R}^n} v^N dx\le \liminf_{j\to\infty}\int_{U(j)} \tilde{u}_j^N dx\le \alpha_0\le 1.$$

Therefore we obtain $\lambda(M)=\Lambda$. This completes the proof. □

From Main Theorem we easily observe that any minimizing sequence is compact in the strong topology of $H^1(M)$ if (M, g) *satisfies* $\lambda(M)<\Lambda$. Finally we remark that the blowing up phenomenon in Main Theorem may occur at each point of M if it occurs at one point.

Proposition 4.1. *Let M be a compact Riemannian manifold with $\lambda(M)=\Lambda$. For each $p\in M$, there exists a minimizing sequence $\{u_j\}\subset C^\infty(M)$ satisfying*

(1) $u_j\ge 0$ *and* $\|u_j\|_N=1$,

(2) $u_j^N\, dV\to\delta_p$ *weakly in the sense of probability measures on M.*

Proof. We take a radial cutoff function $\eta\in C_0^\infty(\boldsymbol{R}^n)$ satisfying

$$\eta(x)=\begin{cases}1 & \text{if } |x|\le 1,\\ 0 & \text{if } |x|\ge 2,\end{cases}$$

$$0\le\eta\le 1, \qquad |\nabla\eta|=|\partial\eta/\partial r|\le 2. \tag{4.2}$$

We take normal coordinates of M centered at p. For small $\varepsilon>0$ and $t>0$ we define

$$u_{\epsilon,t}(x)=\eta\left(\frac{x}{\varepsilon}\right)\left(\frac{t}{t^2+|x|^2}\right)^{(n-2)/2}. \tag{4.3}$$

The calculation [1], [7, Lemma 3.4] gives

$$\Lambda\leq Q(u_{\epsilon,t})\leq\Lambda(1+C_1\varepsilon)(1+C_2t).$$

Taking sequences $\varepsilon_j\to 0$ and $t_j\to 0$ as $j\to\infty$, we obtain the sequence $u_j(x)=u_{\epsilon_j,t_j}(x)/\|u_{\epsilon_j,t_j}\|_N$ having the desired properties. □

References

[1] T. Aubin: *Équations différentielles non linéares et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269–296.

[2] ————: Nonlinear Analysis on Manifolds: Monge-Ampère Equations, Springer-Verlag, New York, 1982.

[3] H. Brezis and J.M. Coron: *Convergence of solutions of H-systems or how to blow bubbles*, Archive Rat. Mech. Anal. **89** (1985), 21–56.

[4] H. Brezis and E.H. Lieb: *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[5] S.K. Donaldson: *An application of gauve theory to 4-dimensional tpology*, J. Diff. Geom. **18** (1983), 279–315.

[6] B. Gidas, W.-M. Ni and L. Nirenberg: *Symmety and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[7] J.M. Lee and T.M. Parker: *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37–91.

[8] P.L. Lions: *The concentration-compactness principle in the calculus of variations: the locally compact case, Part I*, Ann. Inst. H. Poincarè 1 (1984), 109–145; *Part. II*, Ann. Inst. H. Poincarè **1** (1984), 223–283.

[9] ————: *The concentration-compactness principle in the calculus of variations: the limit case, Part I*, Revista. Mat. Ibero. **1(1)** (1985), 145–201; *Part II*, Revista. Mat. Ibero. **1(2)** (1985), 45–121.

[10] R. Schoen: *Conformal deformation of a Riemannian metric o constant scalar curvature*, J. Diff. Geom. **20** (4984), 479–495.

[11] M. Struwe: *A global compactness results for elliptic boundary value problems in volving limiting nonlinearities*, Math. Z. **187** (1984), 511–517.

[12] G. Talenti: *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

[13] N.S. Trudinger: *Remarks converning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 265–274.

[14] H. Yamabe: *On a deformation of Riemannian structures on compact mnifolds*, Osaka Math. J. **12** (1960), 21–37.

Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1–1, Hachioji-shi, Tokyo 192–03,
Japan