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ENDOMORPHISMS OF HOMOGENEOUS SPACES OF LIE GROUPS

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If H is a closed subgroup of a topological group G it is well-known that there is a bijection

$$\mathrm{Map}_G(G/H, G/H) \rightleftarrows (G/H)^H$$

which is actually a homeomorphism when the mapping space is equipped with compact-open topology. Homeomorphisms correspond to the subspaces

$$\mathrm{Homeo}_G(G/H) \rightleftarrows NH/H.$$

Our main purpose is to prove

Theorem. *If G is a Lie group and H is a closed subgroup then NH/H is open in $(G/H)^H$.*

In [tD, Ch. IV.1] Tammo tom Dieck defines a universal additive invariant $U(G)$ of pointed finite G-CW-complexes for arbitrary topological groups G and computes it for compact Lie groups. As a corollary we obtain that his result is valid for arbitrary Lie groups, too.

Corollary. *$U(G)$ is a free abelian group on elements $u(G/H^+)$ where H runs through a complete set of conjugacy classes of closed subgroups H in G for any Lie group G.*

The condition

$$(O) \quad NH/H \text{ is open in } (G/H)^H$$

was introduced in a study with Wolfgang Lück [LL] in order to define the equivariant Lefschetz class of a G-endomorphism $f\colon X \to X$ of a finite G-CW-complex.

The inverse images of the subspaces $NH/H \subset (G/H)^H \subset G/H$ are

$$NH=\{g\in G|g^{-1}Hg=H\} \text{ and } SH=\{g\in G|g^{-1}Hg\subset H\}$$

and we claim that NH is open in SH, when G is a Lie group. It is well-known that $NH=SH$ when H is compact. As H is closed in G, both NH and SH are always closed in G, so that for Lie groups G the fixed point space splits as a topological sum

$$(G/H)^H=NH/H+(G/H)^{>H}.$$

The Lie theory we need can be found e.g. in the books Helgason [H, Ch.II] or Kawakubo [K, Ch.3].

Reduction to a discrete subgroup

Let G be a Lie group and H be a closed subgroup. We first claim that it suffices to prove the Theorem for all Lie groups G in the case where H is discrete. Indeed, let H_0 denote the unit component of H. Then H_0 is a closed and open subgroup of H and $H/H_0=\pi_0(H)$. If $g^{-1}Hg\subset H$ then $g^{-1}H_0g\subset H$ is a connected set which contains e, whence $g^{-1}H_0g\subset H_0$. Then it holds for the Lie algebras that $L(g^{-1}H_0g)\subset L(H_0)$, but as they have the same dimension they must coincide. By connectedness $g^{-1}H_0g=H_0$ and $NH\subset SH\subset N(H_0)$. We can therefore assume that $G=N(H_0)$, i.e. that H_0 is normal in G. Then the normalizer of the discrete subgroup $\pi_0(H)=H/H_0$ of G/H_0 is $N\pi_0(H)=NH/H_0$, $S\pi_0(H)=SH/H_0$ and it clearly suffices to prove the claim for the subgroup $\pi_0(H)$ of G/H_0.

Lie algebra of the centralizer

Let G be a Lie group and let H be a discrete closed subgroup of G. The centralizer

$$ZH=\{g\in G|ghg^{-1}=h \text{ for } h\in H\}$$

is a closed subgroup of G and is normal in NH for any closed H. When H is moreover discrete, then it holds $(NH)_0=(ZH)_0$: each $g\in(NH)_0$ can be connected to e by a path g_t in NH. The corresponding conjugations $c_{g_t}:H\to H$ give a homotopy from c_g to $c_e=id_H$. As H is discrete, the homotopy is constant and therefore $c_g=id_H$, i.e. $g\in ZH$. We conclude that $LNH=LZH$.

Recall that the adjoint representation of G in LG is defined by attaching to an element $g\in G$ the differential $Ad(g)$: $LG\to LG$ of the conjugation $c_g:G\to G$.

Lemma. $LZH=\{X\in LG\,|\,X=Ad(h)X \text{ for } h\in H\}$.

Proof. A closed subgroup H of a Lie group G is itself a Lie group with Lie algebra

$$LH=\{X\in LG \mid exp(tX)\in H \text{ for } t\in \mathbf{R}\},$$

see [H, Theorem II 2.3] or [K, Theorem 3.36]. Hence

$$\begin{aligned} LZH &= \{X\in LG \mid exp(tX)\in ZH \text{ for } t\in\mathbf{R}\} \\ &= \{X\in LG \mid c_h(exp(tX))=exp(tX) \text{ for } h\in H, t\in\mathbf{R}\} \end{aligned}$$

as $Ad(h)$ is the differential of c_h, the last set equals to

$$\begin{aligned} &= \{X\in LG \mid exp(tX)=exp(tAd(h)X) \text{ for } h\in H, t\in\mathbf{R}\} \\ &= \{X\in LG \mid X=Ad(h)X \text{ for } h\in H\} \end{aligned}$$

as *exp* is a diffeomorphism near the origin. This proves the Lemma.

Proof of the Theorem

Let G be a Lie group and let H be a closed discrete subgroup. The quotient space G/H is then a smooth manifold and the projection $\pi\colon G\to G/H$ is a smooth covering projection. Then the diagrams

$$\begin{array}{ccc} G & \xrightarrow{c_h} & G \\ \pi\downarrow & & \downarrow\pi \\ G/H & \xrightarrow{l_h} & G/H \end{array} \qquad\qquad \begin{array}{ccc} LG & \xrightarrow{Ad(h)} & LG \\ exp\downarrow & & \downarrow exp \\ G & \xrightarrow{c_h} & G \end{array}$$

commute for each $h\in H$, where $l_g\colon G/H\to G/H$ is left translation by g and $c_g\colon G\to G$ is conjugation by g. As the exponential map *exp* is a local diffeomorphism at the origin 0, $exp(0)=e$ and similarly π is a local diffeomorphism at the unit element e and $\pi(e)=eH$, the composite πexp is a diffeomorphism of a small enough open disk $U_h\subset LG$ onto its image $V_h\subset G/H$. V_h is an open neighborhood of eH and the diagram

$$\begin{array}{ccc} U_h & \xrightarrow{A(d)(h)} & Ad(h)U_h \\ \pi exp\downarrow & & \downarrow \pi exp \\ V_h & \xrightarrow{l_h} & hV_h \end{array}$$

commutes. In paricular $U_h^{Ad(h)}$ is diffeomorphic to V_h^h.

By the Lemma $LZH=\{X\in LG \mid X=Ad(h)X \text{ for } h\in H\}$. As the spaces in question are finite-dimensional vector spaces, we can choose a finite set $h_1, h_2, \cdots, h_n\in H$

such that $LZH=\{X\in LG \mid X=Ad(h_i)X \text{ for } i=1,\cdots,n\}$. Let $U=\bigcap_{i=1}^n U_{h_i}$ and $V=\bigcap_{i=1}^n V_{h_i}$. The map πexp restricts to a diffeomorphism $U\to V$, which induces a diffeomorphism

$$U\cap LZH=\bigcap_{i=1}^n U_{h_i}^{Ad(h_i)} \overset{\sim}{\to} \bigcap_{i=1}^n V_{h_i}^{h_i}=V^{\{h_1,\cdots,h_n\}}.$$

Choose U and consequently V is so small that $\pi exp(U\cap LZH)=V\cap(ZH/H)$ holds. Then the neighborhood V of the point $eH=G/H$ satisfies

$$V\cap(ZH/H)=V\cap(G/H)^{\{h_1,\cdots,h_n\}}.$$

But clearly we have $ZH/H\subset NH/H\subset(G/H)^H\subset(G/H)^{\{h_1,\cdots,h_n\}}$ so in fact equality

$$V\cap(NH/H)=V\cap(G/H)^H$$

holds. Hence $NH/H\subset(G/H)^H$ is open at the point $eH\in NH/H$. Using the left action of NH/H we see that NH/H is open in $(G/H)^H$. This proves the Theorem.

Proof of the Corollary

Recall tom Dieck's definition of the universal additive invariant of a topological group G. An additive invariant consists of an pair (B,b), B an abelian group and b an assignement which associates to each pointed finite G-CW-complex X an element $b(X)\in B$ such that $b(X)=b(Y)$ if X and Y are pointed G-homotopy equivalent and that the condition

$$b(X)=b(A)+b(X/A)$$

holds when A is a pointed subcomplex of X. An additive invariant (U,u) is universal if every other additive invariant factors through it uniquely. A universal additive invariant is uniquely determined by the usual Grothendieck construction, and it is denoted by $(U(G),u)$.

It follows by an easy argument that $U(G)$ is always generated by the classes $u(G/H^+)$ [tD, Proposition IV.1.8]. Although not explicitly stated, the proof that there are no relations between the classes $u(G/H^+)$ uses implicitly the fact that G is a compact Lie group since it is based on the Euler characteristics $\chi(X^H/NH)$, which are guaranteed to exist if G is a compact Lie group since then X^H/NH is a compact ENR but not otherwise (cf. the example given below.)

Let G be a Lie group. Using our Theorem we can alternatively proceed as follows. As noted in [LL, p. 495], the condition (O) implies that a G-CW-complex structure on X induces a relative NH/H-CW-complex structure on the pair $(X^H,X^{>H})$. The quotient $(X^H/NH,X^{>H}/NH)$ is then an ordinary relative CW-complex (possibly non-Hausdorff) whose cells correspond to the G-cells of X of type G/H. If $n(X,H,i)$ is the number of such i-cells, it follows that the numbers

$n(X,H)=\sum_{i\geqq 0}n(X,H,i)$ are G-homotopy invariants of X as

$$n(X,H)=\chi(X^H/NH, X^{>H}/NH).$$

Then $(\mathbf{Z}, n(X,H))$ is an additive invariant such that

$$n(G/H^+,H)=1, n(G/K^+,H)=0 \text{ for } K \text{ not conjugate to } H.$$

This proves the Corollary.

Example

We conclude with an example taken from Fuchsian groups. Let $G=PSL(2,\mathbf{R})$ considered as the group of Möbius transformations

$$g(z)=\frac{az+b}{cz+d},\ ad-bc=1,\ a,b,c,d\in\mathbf{R}$$

of the complex plane. Let H be the discrete subgroup of translations

$$h(z)=z+n,\quad n\in\mathbf{Z}.$$

Then it is easy to check that the normalizer NH of H in G consists of translations

$$n(z)=z+b,\quad b\in\mathbf{R},$$

whereas SH equals the affine transformations

$$s(z)=mz+b,\quad m=1,2,\cdots,b\in\mathbf{R}.$$

In particular NH/H is a circle S^1 and $SH/H=\mathbf{N}\times S^1$.

Taking $X=G/H$ gives an example of a finite G-CW-complex (a zero-cell) with X^H/NH a countable discrete set and therefore of infinite Euler characteristic.

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