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# THE CHERN CHARACTER OF THE SYMMETRIC SPACE $SU(2n)/SO(2n)$

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## 0. Introduction

For $n \geq 1$, let $t$: $SU(n) \to SU(n)$ be the map defined by $t(x)=\bar{x}$ for $x \in SU(n)$, where $\bar{x}$ is the complex conjugate of a unitary matrix $x$. The natural inclusion $\boldsymbol{R} \subset \mathbf{C}$ yields a monomorphism $i_1$: $SO(n) \to SU(n)$ of topological groups. Clearly $i_1(SO(n))=\{x \in SU(n) \mid t(x)=x\}$. So the quotient space $SU(n)/i_1(SO(n))$, which we abbreviate to $SU(n)/SO(n)$, forms a compact symmetric space. It is denoted by $AI$ (of rank $n-1$) in É. Cartan's notation. In this paper we compute its Chern character

$$ch\colon K^*(SU(2n)/SO(2n)) \to H^{**}(SU(2n)/SO(2n);\, \boldsymbol{Q}),$$

while that of $SU(2n+1)/SO(2n+1)$ has been described in [8].

## 1. $K$-rings

In this section we collect some results on $K$-theory of related spaces needed in the sequel.

Let $G$ be a compact Lie group. Then the complex representation ring $R(G)$ forms a $\lambda$-ring. For each integer $k \geq 0$, let $\lambda^k$: $R(G) \to R(G)$ be the $k$-th exterior power operation. The following is a known result: see [4, Chapter 13] or [9, Chapter 4].

**Proposition 1.** *For $n \geq 2$, put $\lambda_1=[\boldsymbol{C}^n] \in R(SU(n))$ and let $\lambda_k=\lambda^k(\lambda_1)$. Then*

$$R(SU(n))=\boldsymbol{Z}[\lambda_1,\lambda_2,\cdots,\lambda_{n-1}],$$

*where $\lambda_0=1$ and $\lambda_n=1$.*

**Proposition 2.** *$t^*$: $R(SU(n)) \to R(SU(n))$ satisfies*

$$t^*(\lambda_k)=\lambda_{n-k} \quad \textit{for} \quad k=1,\cdots,n-1.$$

Proof. This follows from [9, Example 2.9 and Proposition 2.12].

Let $T'$ be the group of "diagonal" matrices consisting of $n$ $2\times 2$ diagonal boxes

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}, \qquad (\theta_i \in \boldsymbol{R}).$$

Then $T'$ is a maximal torus of $SO(2n)$. Let $i'\colon T' \to SO(2n)$ be the inclusion. There are classes $\eta_1,\cdots,\eta_n$ of 1-dimensional $T'$-$\boldsymbol{C}$-modules such that

$$R(T') = \boldsymbol{Z}[\eta_1,\eta_1^{-1},\cdots,\eta_n,\eta_n^{-1}]/(\eta_1\eta_1^{-1}-1,\cdots,\eta_n\eta_n^{-1}-1) \tag{1.1}$$

(see [9, Chapter 3, §3] and also (2.1)).

The following is a known result: see [4, Chapter 13] or [9, Chapter 4].

**Proposition 3.** *For $n\geq 1$, put $\mu_1=[\boldsymbol{R}^{2n}\otimes_{\boldsymbol{R}}\boldsymbol{C}]\in R(SO(2n))$ and let $\mu_k=\lambda^k(\mu_1)$. There are two representations $\mu_n^+,\mu_n^-$ of $SO(2n)$ of dimension $\binom{2n}{n}/2$ (where $\binom{a}{b}$ denotes the binomial coefficient) such that*

$$\mu_n=\mu_n^+ + \mu_n^-, \qquad i'^*(\mu_n^+ - \mu_n^-)=\prod_{i=1}^{n}(\eta_i-\eta_i^{-1})$$

*and*

$$R(SO(2n))=\boldsymbol{Z}[\mu_1,\mu_2,\cdots,\mu_{n-1},\mu_n^+,\mu_n^-]/(r_n),$$

*where $\mu_{2n-k}=\mu_k$ for $k=1,\cdots,n-1$ and*

$$r_n=(\mu_n^+ + \mu_{n-2}+\cdots)(\mu_n^- + \mu_{n-2}+\cdots)-(\mu_{n-1}+\mu_{n-3}+\cdots)^2.$$

For $n\geq 2$ the universal covering group of $SO(2n)$ is the spinor group $Spin(2n)$. Let $p\colon Spin(2n)\to SO(2n)$ be the covering map. For simplicity we write $\mu_i$ for $p^*(\mu_i)$. Then

$$R(Spin(2n))=\boldsymbol{Z}[\mu_1,\mu_2,\cdots,\mu_{n-2},\Delta_{2n}^+,\Delta_{2n}^-],$$

where $\Delta_{2n}^+$, $\Delta_{2n}^-$ are the half-spin representations, each of dimension $2^{n-1}$, and

$$\begin{aligned} p^*(\mu_{n-1}) &= \Delta_{2n}^+\Delta_{2n}^- - \mu_{n-3} - \mu_{n-5} - \cdots, \\ p^*(\mu_n^+) &= \Delta_{2n}^+\Delta_{2n}^+ - \mu_{n-2} - \mu_{n-4} - \cdots, \\ p^*(\mu_n^-) &= \Delta_{2n}^-\Delta_{2n}^- - \mu_{n-2} - \mu_{n-4} - \cdots \end{aligned} \tag{1.2}$$

(see [4, Chapter 13] or [9, Chapter 4]).

**Proposition 4.** *$i_1^*$: $R(SU(2n)) \to R(SO(2n))$ satisfies*

$$i_1^*(\lambda_k) = \mu_k = i_1^*(\lambda_{2n-k}) \quad \textit{for } k = 1, \cdots, n-1$$
$$i_1^*(\lambda_n) = \mu_n^+ + \mu_n^- .$$

Proof. Since $i_1^*(\lambda_1) = \mu_1$, this follows from Proposition 1 and 3.

Let $G$ be a compact Lie group and $H$ a closed subgroup of $G$. Then the inclusion $i$: $H \to G$ induces a fibre sequence

$$G \xrightarrow{\pi} G/H \xrightarrow{j} BH \xrightarrow{Bi} BG.$$

There are two constructions of elements of $K^*(G/H)$ (see [6, p. 624]). First, a unitary representation $\mu$: $H \to U(n)$ induces a map $B\mu$: $BH \to BU(n)$, which determines an $n$-dimensional complex vector bundle over $BH$ and hence an element $\alpha(\mu) \in K(BH) = K^0(BH)$. This correspondence extends to a $\lambda$-ring homomorphism $\alpha$: $R(H) \to K^0(BH)$. Let $\varepsilon$: $R(G) \to \boldsymbol{Z}$ be the augmentation and $I(G)$ its kernel. Then the composite $j^*\alpha$: $R(H) \to K^0(G/H)$ factors through the projection

$$R(H) \to R(H)/(I(G)) = R(H) \otimes_{R(G)} \boldsymbol{Z},$$

where $(I(G))$ is the ideal in $R(H)$ generated by the $i^*$-image of $I(G)$, and $\boldsymbol{Z}$ is a $R(G)$-module by pulling back along $\varepsilon$. We write $\alpha(\tilde{\mu}) \in \tilde{K}^0(G/H)$ for the image of $\mu - n \in I(H)$ under $j^*\alpha$: $I(H) \to \tilde{K}^0(G/H)$. Secondly, suppose that two representations $\lambda, \lambda'$: $G \to U(n)$ (of the same dimension) agree on $H$. Then there is a map $f$: $G/H \to U(n)$ defined by

$$f(xH) = \lambda(x)\lambda'(x)^{-1} \quad \text{for } xH \in G/H.$$

The composition of $f$ with the canonical injection $\iota_n$: $U(n) \to U$ (where $U$ is the stable unitary group) defines a (base point preserving) homotopy class $\beta(\lambda - \lambda')$ in $[G/H, U] = \tilde{K}^{-1}(G/H)$. When $H$ is the trivial group and $\lambda'$ is the trivial representation of dimension $n$, we have its absolute version $\beta(\lambda) \in \tilde{K}^{-1}(G)$.

Since the composite

$$R(G) \xrightarrow{\alpha} K(BG) \xrightarrow{ch} H^*(BG; \boldsymbol{Q})$$

maps $I(G)$ into $H^+(BG; \boldsymbol{Q}) = \Sigma_{q>0} H^q(BG; \boldsymbol{Q})$, the composite $ch\,\alpha$:

$R(H)\to H^*(BH;Q)$ induces a homomorphism

$$R(H)/(I(G))\to H^*(BH;\,Q)/(H^+(BG\,;\,Q)),$$

where $(H^+(BG;\,Q))$ is the ideal in $H^*(BH;Q)$ generated by the $Bi^*$-image of $H^+(BG;\,Q)$. Since $(Bi)j$ is null-homotopic, $j^*\colon H^*(BH;\,Q)\to H^*(G/H;\,Q)$ induces a homomorphism

$$H^*(BH;\,Q)/(H^+(BG;\,Q)\to H^*(G/H;\,Q).$$

Thus there is a commutative diagram

$$\begin{array}{ccc} R(H)/(I(G)) & \overset{ch\,\alpha}{\to} & H^*(BH;\,Q)/(H^+(BG;\,Q)) \\ j^*\alpha\downarrow & & \downarrow j^* \\ K^0(G/H) & \overset{ch}{\to} & H^*(G/H;\,Q) \end{array} \tag{1.3}$$

**Lemma 5.** *With the above notation,*

$$R(SO(2n))/(I(SU(2n)))=Z[\mu_n^+]/((\mu_n^+-\binom{2n}{n}/2)^2),$$

*where the relation* $\mu_n^- -\binom{2n}{n}/2=-(\mu_n^+ -\binom{2n}{n}/2)$ *holds.*

Proof. By Proposition 4,

$$(I(SU(2n)))=(\mu_1-\binom{2n}{1},\cdots,\mu_{n-1}-\binom{2n}{n-1},\mu_n^+ +\mu_n^- -\binom{2n}{n}).$$

So the lemma follows from Proposition 3.

The $K$-theory of $SU(n)/SO(n)$ was determined by H. Minami [6].

**Theorem 6** ([6, Proposition 8.2]). *With the above notation, as a* $Z/(2)$*-graded algebra over* $Z$, $\tilde{K}^*(SU(2n)/SO(2n))$ *is an exterior algebra* $\Lambda_Z(\beta(\lambda_1-\lambda_{2n-1}),\cdots,\beta(\lambda_{n-1}-\lambda_{n+1}))\otimes\Lambda_Z(\alpha(\mu_n^+))$ *generated by elements* $\beta(\lambda_k-\lambda_{2n-k})$ *of degree* $-1$ and $\alpha(\mu_n^+)$ *of degree* 0.

## 2. Cohomology rings

In this section we collect some results on the cohomology of related spaces needed in the sequel. We refer the reader to [5] for a more complete exposition.

A compact connected Lie group $G$ has a maximal torus $T$. If $\{\alpha_1,\cdots,\alpha_n\}$(where $n=\dim T$) is a simple system of roots of $G$ with respect to $T$, then

$$H^*(BT;\ \boldsymbol{Q})=\boldsymbol{Q}[\alpha_1,\cdots,\alpha_n],$$

where $\alpha_i\in H^2(BT;\ \boldsymbol{Q})$. Suppose further that $G$ is simply connected. Let $\omega_1,\cdots,\omega_n$ be the fundamental weights determined by $\{\alpha_1,\cdots,\alpha_n\}$, i.e.,

$$2(\omega_i,\alpha_j)/(\alpha_j,\alpha_j)=\delta_{ij}\quad(1\leq i,j\leq n).$$

Then

$$H^*(BT;\ \boldsymbol{Z})=\boldsymbol{Z}[\omega_1,\cdots,\omega_n],$$

where $\omega_i\in H^2(BT;\ \boldsymbol{Z})$. Each $\omega_i$ is expressed as a $\boldsymbol{Q}$-linear combination of $\alpha_i$'s.

Let $T$ be the group of diagonal matrices having $2n$ entries $e^{i\theta_i}$ with $e^{i\theta_1}\cdots e^{i\theta_{2n}}=1$. Then $T$ is a maximal torus of $SU(2n)$, and $i_1(T')\subset T$. If $\{\alpha_1,\cdots,\alpha_{2n-1}\}$ is a simple system of roots of $SU(2n)$ with respect to $T$, then

$$H^*(BT;\ \boldsymbol{Q})=\boldsymbol{Q}[\alpha_1,\cdots,\alpha_{2n-1}].$$

**Proposition 7.** *$Bt^*$: $H^*(BT;\ \boldsymbol{Q})\to H^*(BT;\ \boldsymbol{Q})$ satisfies*

$$Bt^*(\alpha_i)=\alpha_{2n-i}\quad \textit{for } i=1,\cdots,2n-1.$$

Proof. This is a restatement of Proposition 2, since $\lambda_i$ is just the irreducible representation determined by $\alpha_i$, that is, $\lambda_i$ admits a highest weight $\omega_i$ below.

Let $\omega_1,\cdots,\omega_{2n-1}$ be the fundamental weights determined by $\{\alpha_1,\cdots,\alpha_{2n-1}\}$. Then

$$H^*(BT;\ \boldsymbol{Z})=\boldsymbol{Z}[\omega_1,\cdots,\omega_{2n-1}].$$

The following is also a restatement of Proposition 2.

**Corollary 8.** *$Bt^*$: $H^*(BT;\ \boldsymbol{Z})\to H^*(BT;\ \boldsymbol{Z})$ satisfies*

$$Bt^*(\omega_i)=\omega_{2n-i}\quad \textit{for } i=1,\cdots,2n-1.$$

Let $R_i$ denote the reflection relative to $\alpha_i$. If we put

$$t_1=\omega_1$$

$$t_i=R_{i-1}(t_{i-1})==-\omega_{i-1}+\omega_i \text{ for } i=2,\cdots,2n-1,$$

$$t_{2n}=R_{2n-1}(t_{2n-1})=-\omega_{2n-1},$$

then

$$H^*(BT;\ \boldsymbol{Z})=\boldsymbol{Z}[t_1,\cdots,t_{2n}]/(t_1+\cdots+t_{2n}).$$

In fact, $\{t_i|i=1,\cdots,2n\}$ is the set of weights of $\lambda_1$: $SU(2n)\to U(2n)$.

**Corollary 9.** *$Bt^*$: $H^*(BT;\ \boldsymbol{Z})\to H^*(BT;\ \boldsymbol{Z})$ satisfies*

$$Bt^*(t_i)=-t_{2n+1-i} \quad \textit{for } i=1,\cdots,2n.$$

$\tilde{T}'=p^{-1}(T')$ is a maximal torus of $Spin(2n)$. Since $Bp^*$: $H^2(BT';\ \boldsymbol{Q})\to H^2(B\tilde{T}';\ \boldsymbol{Q})$ is an isomorphism, we may identify them. If $\{\alpha'_1,\cdots,\alpha'_n\}$ is a simple system of roots of $Spin(2n)$ with respect to $\tilde{T}'$, then

$$H^*(BT';\ \boldsymbol{Q})=H^*(B\tilde{T}';\ \boldsymbol{Q})=\boldsymbol{Q}[\alpha'_1,\cdots,\alpha'_n].$$

**Proposition 10.** *$Bi_1{}^*$: $H^*(BT;\ \boldsymbol{Q})\to H^*(BT';\ \boldsymbol{Q})$ satisfies*

$$Bi_1{}^*(\alpha_i)=\alpha'_i=Bi_1{}^*(\alpha_{2n-1}) \quad \textit{for } i=1,\cdots,n-1$$

$$Bi_1{}^*(\alpha_n)=-\alpha'_{n-1}+\alpha'_n.$$

Proof. This is a restatement of Proposition 4, since $\mu_i$ is the irreducible representation determined by $\alpha'_i$ for $i=1,\cdots,n-2$ and $\Delta^+_{2n},\Delta^-_{2n}$, are the irreducible representations determined by $\alpha'_n,\alpha'_{n-1}$ respectively (see (1.2) and the next corollary).

Let $\omega'_1,\cdots,\omega'_n$ be the fundamental weights determined by $\{\alpha'_1,\cdots,\alpha'_n\}$. Then

$$H^*(B\tilde{T}';\ \boldsymbol{Z})=\boldsymbol{Z}[\omega'_1,\cdots,\omega'_n].$$

For simplicitly we write $i_1$ for the composite $i_1p$: $\tilde{T}'\to T$.

**Corollary 11.** *$Bi_1{}^*$: $H^*(BT;\ \boldsymbol{Z})\to H^*(B\tilde{T}';\ \boldsymbol{Z})$ satisfies*

$$Bi_1{}^*(\omega_i)=\omega'_i=Bi_1{}^*(\omega_{2n-i}) \quad \textit{for } i=1,\cdots,n-2$$

$$Bi_1{}^*(\omega_{n-1})=\omega'_{n-1}+\omega'_n=Bi_1{}^*(\omega_{n+1})$$

$$Bi_1{}^*(\omega_n)=2\omega'_n.$$

Proof. This is also a restatement of Proposition 4. We only prove that Proposition 10 implies this corollary. (The converse follows similarly.) We will use implicitly the expression of fundamental weights in terms of simple roots given in [1]. First, for $i=1,\cdots,n-2$,

$$Bi_1{}^*(\omega_i)=Bi_1{}^*(\frac{1}{2n}[\sum_{j=1}^{i-1} j(2n-i)\alpha_j+\sum_{j=1}^{2n-1} i(2n-j)\alpha_j])$$

$$=\frac{1}{2n}[\sum_{j=1}^{i-1} j(2n-i)Bi_1{}^*(\alpha_j)+\sum_{j=1}^{n-1} i(2n-j)Bi_1{}^*(\alpha_j)$$

$$+inBi_1{}^*(\alpha_n)+\sum_{k=1}^{n-1} ikBi_1{}^*(\alpha_{2n-k})]$$

$$=\frac{1}{2n}[\sum_{j=1}^{i-1} j(2n-i)\alpha'_j+\sum_{j=i}^{n-1} i(2n-j)\alpha'_j$$

$$+in(-\alpha'_{n-1}+\alpha'_n)+\sum_{k=1}^{n-1} ik\alpha'_k]$$

$$=\frac{1}{2n}[\sum_{j=1}^{i-1} j(2n-i)\alpha'_j+\sum_{j=i}^{n-2} i(2n-j)\alpha'_j+i(n+1)\alpha'_{n-1}$$

$$-in\alpha'_{n-1}+in\alpha'_n+\sum_{k=1}^{i-1} ik\alpha'_k+\sum_{k=i}^{n-2} ik\alpha'_k+i(n-1)\alpha'_{n-1}]$$

$$=\frac{1}{2n}[\sum_{j=1}^{i-1} (j(2n-i)+ij)\alpha'_j+\sum_{j=i}^{n-2} (i(2n-j)+ij)\alpha'_j$$

$$+(i(n+1)-in+i(n-1))\alpha'_{n-1}+in\alpha'_n]$$

$$=\frac{1}{2n}[2n\sum_{j=1}^{i-1} j\alpha'_j+2in\sum_{j=i}^{n-2}\alpha'_j+in\alpha'_{n-1}+in\alpha'_n]$$

$$=\sum_{j=1}^{i-1} j\alpha'_j+i\sum_{j=i}^{n-2}\alpha'_j+\frac{1}{2}i\alpha'_{n-1}+\frac{1}{2}i\alpha'_n$$

$$=\omega'_i.$$

Next

$$Bi_1{}^*(\omega_{n-1})=Bi_1{}^*(\frac{1}{2n}[\sum_{j=1}^{n-2} j(n+1)\alpha_j+\sum_{j=n-1}^{2n-1} (n-1)(2n-j)\alpha_j])$$

$$=\frac{1}{2n}[\sum_{j=1}^{n-2} j(n+1)Bi_1{}^*(\alpha_j)+(n-1)(n+1)Bi_1{}^*(\alpha_{n-1})$$

$$+(n-1)nBi_1{}^*(\alpha_n)+\sum_{k=1}^{n-1}(n-1)kBi_1{}^*(\alpha_{2n-k})]$$

$$=\frac{1}{2n}[\sum_{j=1}^{n-2} j(n+1)\alpha'_j+(n-1)(n+1)\alpha'_{n-1}$$

$$+(n-1)n(-\alpha'_{n-1}+\alpha'_n)$$

$$+\sum_{k=1}^{n-2}(n-1)k\alpha'_k+(n-1)^2\alpha'_{n-1}]$$

$$=\frac{1}{2n}[\sum_{j=1}^{n-2}(j(n+1)+(n-1)j)\alpha'_j$$

$$+(n-1)((n+1)-n+(n-1))\alpha'_{n-1}$$

$$+(n-1)n\alpha'_n]$$

$$=\frac{1}{2n}[2n\sum_{j=1}^{n-2} j\alpha'_j+(n-1)n\alpha'_{n-1}+(n-1)n\alpha'_n]$$

$$=\sum_{j=1}^{n-2} j\alpha'_j+\frac{1}{2}(n-1)\alpha'_{n-1}+\frac{1}{2}(n-1)\alpha'_n$$

$$=\frac{1}{2}\sum_{j=1}^{n-2} j\alpha'_j+\frac{1}{4}n\alpha'_{n-1}+\frac{1}{4}(n-2)\alpha'_n$$

$$+\frac{1}{2}\sum_{j=1}^{n-2} j\alpha'_j+\frac{1}{4}(n-2)\alpha'_{n-1}+\frac{1}{4}n\alpha'_n$$

$$=\omega'_{n-1}+\omega'_n.$$

Finally

$$Bi_1{}^*(\omega_n)=Bi_1{}^*(\frac{1}{2n}[\sum_{j=1}^{n-1} jn\alpha_j+\sum_{j=n}^{2n-1} n(2n-j)\alpha_j])$$

$$=\frac{1}{2n}[n\sum_{j=1}^{n-1} njBi_1{}^*(\alpha_j)+n^2Bi_1{}^*(\alpha_n)+n\sum_{k=1}^{n-1} kBi_1{}^*(\alpha_{2n-k})]$$

$$=\frac{1}{2}\sum_{j=1}^{n-1} j\alpha'_j+\frac{1}{2}n(-\alpha'_{n-1}+\alpha'_n)+\frac{1}{2}\sum_{k=1}^{n-1} k\alpha'_k$$

$$=\sum_{j=1}^{n-2} j\alpha'_j+\frac{1}{2}(n-1)\alpha'_{n-1}-\frac{1}{2}n\alpha'_{n-1}$$

$$+\frac{1}{2}n\alpha'_n+\frac{1}{2}(n-1)\alpha'_{n-1}$$

$$=\sum_{j=1}^{n-2} j\alpha'_j+\frac{1}{2}(n-2)\alpha'_{n-1}+\frac{1}{2}n\alpha'_n$$

$$=2\omega'_n.$$

If we put

$$t'_1=\omega'_1,$$

$$t'_i=R'_{i-1}(t'_{i-1})=-\omega'_{i-1}+\omega'_i \text{ for } i=2,\cdots,n-2$$

$$t'_{n-1}=R'_{n-2}(t'_{n-2})=-\omega'_{n-2}+\omega'_{n-1}+\omega'_n,$$

$$t'_n=R'_{n-1}(t'_{n-1})=-\omega'_{n-1}+\omega'_n,$$

then

$$H^*(BT';\ \boldsymbol{Z})=\boldsymbol{Z}[t'_1,\cdots,t'_n]$$

and the element $\eta_i$ of (1.1) can be chosen so that

(2.1) $$(ch\ \alpha)(\eta_i)=\exp(t'_i)=\sum_{j\geq 0}\frac{t'^j_i}{j!}\in H^*(BT';\ \boldsymbol{Q}).$$

In fact, $\{\pm t'_i | i=1,\cdots,n\}$ is the set of weights of $\mu_1\colon SO(2n)\to U(2n)$.

**Corollary 12.** *$Bi_1{}^*\colon H^*(BT;\ \boldsymbol{Z})\to H^*(BT';\ \boldsymbol{Z})$ satisfies*

$$Bi_1{}^*(t_i)=t'_i, \quad \textit{for } i=1,\cdots,n$$

$$Bi_1{}^*(t_{2n+1-i})=-t'_i, \quad \textit{for } i=1,\cdots,n.$$

For a compact connected Lie group $G$ with maximal torus $T$, the Weyl group $W(G)=N_G(T)/T$ acts on $T$ and hence on $H^*(BT;\ \boldsymbol{Q})$. Let

$H^*(BT; \boldsymbol{Q})^{W(G)}$ be the subalgebra of the invariants of $W(G)$ in $H^*(BT; \boldsymbol{Q})$. According to the result of Borel [2], under the homomorphism $H^*(BG; \boldsymbol{Q}) \to H^*(BT; \boldsymbol{Q})$ induced by the inclusion $T \to G$, $H^*(BG; G)$ is mapped isomorphically onto $H^*(BT; \boldsymbol{Q})^{W(G)}$, so we may identify them. Furthermore, if $G$ has no torsion, this statement holds over the integers $\boldsymbol{Z}$ as well.

Let $\sigma_i(\ )$ denote the $i$-th elementary symmetric function. If $G=SU(2n)$, then it has no torsion and $W(SU(2n))$ acts on $H^2(BT; \boldsymbol{Z})$ as the full permutation group on $\{t_1, \cdots, t_{2n}\}$. Thus

$$H^*(BSU(2n); \boldsymbol{Z}) = \boldsymbol{Z}[c_2, c_3, \cdots, c_{2n}],$$

where $c_i = \sigma_i(t_1, \cdots, t_{2n}) \in H^{2i}(BSU(2n); \boldsymbol{Z})$. If $G=SO(2n)$, then it has 2-torsion and $W(SO(2n))$ acts on $H^2(BT'; \boldsymbol{Q})$ as the group generated by permutations of the $t_i'$ and transformations $(t_1', \cdots, t_n') \mapsto (\varepsilon_1 t_1', \cdots, \varepsilon_n t_n')$ with $\varepsilon_i = \pm 1$ and $\varepsilon_1 \cdots \varepsilon_n = 1$. Thus

$$H^*(BSO(2n); \boldsymbol{Q}) = \boldsymbol{Q}[p_1, p_2, \cdots, p_{n-1}, \chi] \tag{2.2}$$

where $p_i = \sigma_i(t_1'^2, \cdots, t_n'^2) \in H^{4i}(BSO(2n); \boldsymbol{Z}), \chi = \sigma_n(t_1', \cdots, t_n') \in H^{2n}(BSO(2n); \boldsymbol{Z})$ and the relation $\chi^2 = p_n$ holds. Since

$$\begin{aligned}\sum_{j=0}^{2n} Bi_1^*(c_j) &= \prod_{i=1}^{2n} (1 + Bi_1^*(t_i)) \\ &= \sum_{i=1}^{n} (1 - t_i'^2) \qquad \text{by Corollary 12} \\ &= \sum_{i=0}^{n} (-1)^i p_i,\end{aligned}$$

we have

$$\begin{aligned} Bi_1^*(c_{2i}) &= (-1)^i p_i \qquad && \textit{for } i = 1, \cdots, n-1 \\ Bi_1^*(c_{2i+1}) &= 0 && \textit{for } i = 1, \cdots, n-1 \\ Bi_1^*(c_{2n}) &= (-1)^n \chi^2. \end{aligned} \tag{2.3}$$

**Lemma 13.** *With the above notation,*

$$H^*(BSO(2n); \boldsymbol{Q})/(H^+(BSU(2n); \boldsymbol{Q})) = \Lambda_{\boldsymbol{Q}}(\chi).$$

Proof. By (2.3),

$$(H^+(BSU(2n);\ \boldsymbol{Q}))=(p_1,\cdots,p_{n-1},\chi^2).$$

So the lemma follows from (2.2).

Consider the principal $SU(2n)$-bundle

$$SU(2n)\overset{\pi_1}{\to}SU(2n)/SO(2n)\overset{j_1}{\to}BSO(2n) \tag{2.4}$$

whose classifying map is $Bi_1$. As a Hopf algebra over $\boldsymbol{Z}$,

$$H^*(SU(2n);\ \boldsymbol{Z})=\Lambda_{\boldsymbol{Z}}(x_3,x_5,\cdots,x_{4n-1}),$$

where $x_{2i+1}\in H^{2i+1}(SU(2n);\ \boldsymbol{Z})$ is primitive and transgresses to $c_{2i}\in H^{2i}(BSU(2n);\ \boldsymbol{Z})$ in the Serre spectral sequence of the universal $SU(2n)$-bundle. By naturality and (2.2), the transgression $H^*(SU(2n);\ \boldsymbol{Q})\to H^{*+1}(BSO(2n);\ \boldsymbol{Q})$ sends $x_{4i-1}$ to $p_i$ for $i=1,\cdots,n$ and $x_{4i+1}$ to 0 for $i=1,\cdots,n-1$. Therefore, there exist elements

$$e_{4i+1}\in H^{4i+1}(SU(2n)/SO(2n);\ \boldsymbol{Q})\quad (i=1,\cdots,n-1),$$

$$e_{2n}\in H^{2n}(SU(2n)/SO(2n);\ \boldsymbol{Q})$$

such that $\pi_1{}^*(e_{4i+1})=x_{4i+1}$, $j_1{}^*(\chi)=e_{2n}$ and

$$H^*(SU(2n)/SO(2n);\ \boldsymbol{Q})=\Lambda_{\boldsymbol{Q}}(e_5,e_9,\cdots,e_{4n-3})\otimes\Lambda_{\boldsymbol{Q}}(e_{2n}).$$

Then, by Lemma 13, the right vertical homomorphism

$$j_1{}^*\colon H^*(BSO(2n);\ \boldsymbol{Q})/(H^+(BSU(2n);\ \boldsymbol{Q}))\to H^*(SU(2n)/SO(2n);\ \boldsymbol{Q})$$

of (1.3) coincides with the injection

$$\Lambda_{\boldsymbol{Q}}(\chi)\to\Lambda_{\boldsymbol{Q}}(e_5,e_9,\cdots,e_{4n-3})\otimes\Lambda_{\boldsymbol{Q}}(e_{2n})$$

sending $\chi$ to $e_{2n}$.

In view of consequences of the Poincaré duality theorem, we can choose elements $e_{4i+1}\in H^{4i+1}(SU(2n);\ \boldsymbol{Z})$ $(i=1,\cdots,n-1)$ and $e_{2n}\in H^{2n}(SU(2n)/SO(2n);\ \boldsymbol{Z})$ so that

$$H^*(SU(2n)/SO(2n);\ \boldsymbol{Z})/\mathrm{Tor}=\Lambda_{\boldsymbol{Z}}(e_5,e_9,\cdots,e_{4n-3})\otimes\Lambda_{\boldsymbol{Z}}(e_{2n}).$$

For a field $\boldsymbol{k}$ of characteristic $\neq 2$,

$$H^*(BSO(2n);\ \boldsymbol{k})=\boldsymbol{k}[p_1,p_2,\cdots,p_{n-1},\chi],$$

where $p_i \in H^{4i}(BSO(2n); \boldsymbol{k})$ and $\chi \in H^{2n}(BSO(2n); \boldsymbol{k})$. According to [5, Volume I, Chapter 3, Theorem 6.7 (2)], from the same equalities as in (2.3) for the cohomology with coefficients in $\boldsymbol{k}$, it follows that

$$H^*(SU(2n)/SO(2n); \boldsymbol{k}) = \Lambda_{\boldsymbol{k}}(e_5, e_9, \cdots, e_{4n-3}) \otimes \Delta_{\boldsymbol{k}}(e_{2n})$$

where $e_i \in H^i(SU(2n)/SO(2n); \boldsymbol{k})$, $\Delta_{\boldsymbol{k}}$ denotes a graded ring over $\boldsymbol{k}$ with a simple system of generators, $\pi_1^*(e_{4i+1}) = x_{4i+1}$ for $i = 1, \cdots, n-1$ and $j_1^*(\chi) = e_{2n}$.

For a field $\boldsymbol{k}$ of characteristic 2,

$$H^*(BSO(2n); \boldsymbol{k}) = \boldsymbol{k}[w_2, w_3, \cdots, w_{2n}]$$

where $w_i \in H^i(BSO(2n); \boldsymbol{k})$ and the action of the mod 2 Bockstein operator $Sq^1$ on it is given by

$$Sq^1(w_{2i}) = w_{2i+1} \qquad \text{for } i = 1, \cdots, n$$

$$Sq^1(w_{2i+1}) = 0 \qquad \text{for } i = 1, \cdots, n-1.$$

According to [5, Volume I, Chapter 3, Theorem 6.7 (3)], since $Bi_1^*(c_i) = w_i^2$, it follows that

$$H^*(SU(2n)/SO(2n); \boldsymbol{k}) = \Lambda_{\boldsymbol{k}}(e_2, e_3, \cdots, e_{2n-1}, e_{2n}),$$

where $e_i \in H^i(SU(2n)/SO(2n); \boldsymbol{k})$ and $j_1^*(w_i) = e_i$ for $i = 2, \cdots, 2n$.

Recall from [5, Volume I, Chapter 3, Theorem 5.17] that there exists a unique element $\chi \in H^{2n}(BSO(2n); \boldsymbol{Z})$ such that $Bi'^*(\chi) = t'_1 t'_2 \cdots t'_n$ and $\chi \equiv w_{2n} \pmod 2$.

**Proposition 14.** $\pi_1^*$: $(SU(2n)/SO(2n); \boldsymbol{Z}) \to H^*(SU(2n); \boldsymbol{Z})$ *satisfies*

$$\pi_1^*(e_{4i+1}) = 2x_{4i+1} \textit{ for } i = 1, \cdots, n-1$$

*and* $j_1^*$: $H^*(BSO(2n); \boldsymbol{Z}) \to H^*(SU(2n)/SO(2n); \boldsymbol{Z})$ *satisfies*

$$j_1^*(\chi) = e_{2n}.$$

Proof. Consider the Serre spectral sequence $\{E_r, d_r\}$ for the integral cohomology of the bundle (2.4). Then the above facts imply the following: for $i = 1, \cdots, n$, each $x_{4i-1} \in H^{4i-1}(SU(2n); \boldsymbol{Z})$ transgresses to a generator of a certain summand $\boldsymbol{Z}$ in $H^{4i}(BSO(2n); \boldsymbol{Z})$; for $i = 1, \cdots, n-1$, each $x_{4i+1} \in H^{4i+1}(SU(2n); \boldsymbol{Z})$ transgresses to a generator of a certain summand $\boldsymbol{Z}/(2)$ in $H^{4i+2}(BSO(2n); \boldsymbol{Z})$ and $2x_{4i+1} \in H^{4i+1}(SU(2n); \boldsymbol{Z})$ survives to $E_\infty$. This proves the first statement.

By Lemma 13 and the characterization of the generator $e_{2n}\in H^{2n}(SU(2n)/SO(2n);\ \boldsymbol{Z})$, we may set

$$j_1^*(e_{2n})=m\chi \text{ for some nonzero } m\in \boldsymbol{Z}$$

in $H^{2n}(BSO(2n);\ \boldsymbol{Z})$. Examining the Serre spectral sequence for the mod $p$ cohomology of the bundle (2.4), we see that $m\not\equiv 0 \pmod p$ for every prime $p$. Hence $m=1$ up to sign. This proves the second statement, and completes the proof.

## 3. Proof of main result

In this section we deduce our main result.
Let $\phi\colon \boldsymbol{N}\times\boldsymbol{N}\times\boldsymbol{N}\to\boldsymbol{Z}$ be the function defined by

$$\phi(n,k,q)=\sum_{i=1}^{k}(-1)^{i-1}\binom{n}{k-i}i^{q-1}.$$

It is known that

$$K^*(SU(2n))=\Lambda_{\boldsymbol{Z}}(\beta(\lambda_1),\beta(\lambda_2),\cdots,\beta(\lambda_{2n-1})).$$

Then the following is shown in [7,§2].

**Proposition 15.** *$ch\colon K^*(SU(2n))\to H^{**}(SU(2n);\ \boldsymbol{Q})$ is given by*

$$ch(\beta(\lambda_k))=\sum_{i=1}^{2n-1}\frac{1}{i!}\phi(2n,k,i+1)x_{2i+1}$$

*for $k=1,\cdots,2n-1$.*

Now our main result is

**Theorem 16.** *$ch\colon K^*(SU(2n)/SO(2n))\to H^{**}(SU(2n)/SO(2n);\ \boldsymbol{Q})$ is given by*

$$ch(\beta(\lambda_k-\lambda_{2n-k}))=\sum_{i=1}^{n-1}\frac{1}{(2i)!}\phi(2n,k,2i+1)e_{4i+1}$$

*for $k=1,\cdots,n-1$ and*

$$ch(\alpha(\widetilde{\mu_n^+}))=2^{n-1}e_{2n}.$$

Proof. Let $\xi_1\colon SU(2n)/SO(2n)\to SU(2n)$ be the map defined by

$$\xi_1(xSO(2n)) = xt(x)^{-1} = x{}^t x \quad \text{for } x \in SU(2n),$$

where ${}^t x$ is the transposed matrix of $x$. By [3], if $x \in H^*(SU(2n); \boldsymbol{Q})$ is primitive, then $\pi_1{}^*\xi_1{}^*(x) = x - t^*(x)$. It follows from Corollary 9 that $t^*$: $H^*(BSU(2n); \boldsymbol{Z}) \to H^*(BSU(2n); \boldsymbol{Z})$ satisfies

$$t^*(c_{i+1}) = (-1)^{i+1} c_{i+1} \quad \text{for } i = 1, \cdots, 2n-1.$$

Therefore, $t^*$: $H^*(SU(2n); \boldsymbol{Z}) \to H^*(SU(2n); \boldsymbol{Z})$ satisfies

$$t^*(x_{2i+1}) = (-1)^{i+1} x_{i+1} \quad \text{for } i = 1, \cdots, 2n-1.$$

Hence, for $i = 1, \cdots, 2n-1$,

$$\pi_1{}^*\xi_1{}^*(x_{2i+1}) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 2x_{2i+1} & \text{if } i \text{ is even.} \end{cases}$$

By the first statement of Proposition 14, for $i = 1, \cdots, 2n-1$,

$$\xi_1{}^*(x_{2i+1}) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ e_{i+1} & \text{if } i \text{ is even.} \end{cases}$$

On the other hand, since $\lambda_k$; $SU(2n) \to U(\binom{2n}{k})$ is a homomorphism of topological groups and $t^*(\lambda_k) = \lambda_{2n-k}$ by Proposition 2, a map representing the homotopy class $\xi_1{}^*(\beta(\lambda_k))$ is given by

$$\begin{aligned} xSO(2n) \mapsto \iota_{\binom{2n}{k}} \lambda_k(xt(x)^{-1}) &\\ = \iota_{\binom{2n}{k}} (\lambda_k(x)\lambda_k(t(x))^{-1}) &\\ = \iota_{\binom{2n}{k}} (\lambda_k(x)\lambda_{2n-k}(x)^{-1}), & \end{aligned}$$

which represents the homotopy class $\beta(\lambda_k - \lambda_{2n-k})$. Hence $\xi_1{}^*(\beta(\lambda_k)) = \beta(\lambda_k - \lambda_{2n-k})$ for $k = 1, \cdots, n-1$. Then

$$\begin{aligned} ch(\beta(\lambda_k - \lambda_{2n-k})) &= ch(\xi_1{}^*(\beta(\lambda_k))) \\ &= \xi_1{}^*(ch(\beta(\lambda_k))) \\ &= \xi_1{}^*\Big(\sum_{i=1}^{2n-1} \frac{1}{i!} \phi(2n, k, i+1) x_{2i+1}\Big) \end{aligned}$$

by Proposition 15

$$= \sum_{i=1}^{2n-1} \frac{1}{i!} \phi(2n, k, i+1) \xi_1{}^*(x_{2i+1}).$$

Substituting 0 (resp. $e_{2i+1}$) for $\xi_1{}^*(x_{2i+1})$ if $i$ is odd (resp. even), we obtain the first equality.

In order to prove the second equality, we compute the image of $\mu_n^+ - \mu_n^-$ under the upper horizontal homomorphism

$$R(SO(2n))/(I(SU(2n))) \xrightarrow{ch\,\alpha} H^*(BSO(2n); \boldsymbol{Q})/(H^+(BSU(2n); \boldsymbol{Q}))$$

of (1.3). To do this, since $Bi'^*$: $H^*(BSO(2n); \boldsymbol{Q}) \to H^*(BT'; \boldsymbol{Q})$ is injective, it suffices to determine the image of $i'^*(\mu_n^+ - \mu_n^-) = \Pi_{i=1}^n (\eta_i - \eta_i{}^{-1})$ (see Proposition 3) under

$$R(T')/(I(SU(2n))) \xrightarrow{ch\,\alpha} H^*(BT'; \boldsymbol{Q})/(H^+(BSU(2n); \boldsymbol{Q})).$$

Then

$$\begin{aligned}
(ch\ \alpha)(\prod_{i=1}^n (\eta_i - \eta_i{}^{-1})) &= \prod_{i=1}^n (\exp(t_i') - \exp(-t_i')) \quad \text{by (2.1)} \\
&= \prod_{i=1}^n (\sum_{j\geq 0} \frac{t_i'^j}{j!} - \sum_{j\geq 0} \frac{(-t_i')^j}{j!}) \\
&= \prod_{i=1}^n (\sum_{j\geq 0} \frac{(1-(-1)^j)t_i'^j}{j!}) \\
&= \prod_{i=1}^n (\sum_{j\geq 0} \frac{2t_i'^{2j+1}}{(2j+1)!}) \\
&= \prod_{i=1}^n (2t_i' \sum_{j\geq 0} \frac{t_i'^{2j}}{(2j+1)!}) \\
&= 2^n \chi \prod_{i=1}^n (\sum_{j\geq 0} \frac{t_i'^{2j}}{(2j+1)!}) \quad \text{since } \chi = \prod_{i=1}^n t_i' \\
&\equiv 2^n \chi \text{ modulo } (p_1, p_2, \cdots, p_n)
\end{aligned}$$

in $H^*(BT'; \boldsymbol{Q})$. Therefore, by the second statement of Proposition 14,

$$(j_1{}^* ch\ \alpha)(\mu_n^+ - \mu_n^-) = 2^n e_{2n}$$

in $H^*(SU(2n)/SO(2n); \boldsymbol{Q})$. By the commutativity of (1.3) and the definitions of $\alpha(\widetilde{\mu_n^+})$ and $\alpha(\widetilde{\mu_n^-})$,

$$ch(\alpha(\widetilde{\mu_n^+}))-ch(\alpha(\widetilde{\mu_n^-}))=2^n e_{2n}.$$

Since $\alpha(\widetilde{\mu_n^-})=-\alpha(\widetilde{\mu_n^+})\in\tilde{K}^0(SU(2n)/SO(2n))$ by Lemma 5, the left hand side is equal to $2ch(\alpha(\widetilde{\mu_n^+}))$. Thus we obtain the second equality, and the proof is completed.

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