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CONTINUITY OF DIRECTIONAL ENTROPY

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1. Introduction

J. Milnor [3] has introduced the notion of directional entropy in cellular automata. Cellular automata can be briefly described as a dynamical system where K consists of finite alphabets and S is a continuous map on a compact space $K^{\mathbf{Z}^n}$ to itself which commutes with the translations of the lattice $\mathbf{Z}^n$. Since the space $K^{\mathbf{Z}^n}$ is compact, S is uniformly continuous. Hence it is not difficult to show that S is a block map (a finite code) [1]. (S is said to have a finite memory.) Since the directional entropy is defined in all directions, it can be considered as a generalization of the entropy of non cocompact subgroups. J. Milnor asked if the directional entropy is continuous.

When $n=1$, we denote by T the shift action on $K^{\mathbf{Z}}$. Let μ be a measure invariant under T. We assume that S preseves the measure μ. $\{T,S\}$ generate a $\mathbf{Z}\times\mathbf{N}$ action, which can be extended to a $\mathbf{Z}\times\mathbf{Z}=\mathbf{Z}^2$ action.

Let $(X,T,\mathscr{F},\mu)$ be an ergodic dynamical system of finite entropy. By Krieger's theorem, this system is isomorphic to a product space of finite alphabets with the shift. Without confusion, we will denote this symbolic system by $(X,T,\mathscr{F},\mu)$. Let S be a measure preserving invertable map of X generated by a block map. Hence $\{T,S\}$ generate a $\mathbf{Z}^2$-action on X. In this setting, Sinai [4] has shown the following: We assume $h(T^kS^l)<\infty$ for all $(k,l)\in\mathbf{Z}^2$. (Clearly this forces the entropy of the $\mathbf{Z}^2$-action to be zero). If $\{q_i/p_i\}$ converges to an irrational τ, then $\frac{1}{\sqrt{q_i^2+p_i^2}}h_{p_i,q_i}$ converges, where h_{p_i,q_i} denotes $h(T^{p^i}S^{q^i})$. Main tool of this proof is to express the directional entropy in a rational direction as an integral.(**)

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(**) In [4], he expressed the limit in the form of an integral. In private communications, we have agreed that in order to be able to write the limit as an integral, the proof needs further assumptions on T and S. (see Remark 2.1)

In this paper we will show that the directional entropy in the direction v_θ where $\tan\theta=\tau$ can be represented as an integral in the form of [4]. And we show that the limit is always less than or equal to the integral. Hence directional entropy is upper semi-continuous. This partly answers Milnor's question. It is not yet clear if the directional entropy is in fact continuous even in the case of cellular automata.

We may mention a couple of results in this direction. In a topological setting of Cellular Automata, D. Lind and J. Smillie constructed an example in their unpublished work (see [2]) whose topological directional entropy function is upper semi-continuous on rationals. J.-P. Thouvenot [5], simultaneously B. Weiss, constructed an example by cutting and stacking method such that if S is not continuous (that is, S is of infinite memory), then $\{\frac{1}{\sqrt{q_i^2+p_i^2}}h_{p_i,q_i}\}$ does not necessarily converge. Hence the directional entropy is far from being continuous in a more general setting.

We may sketch the Thouvenot's example briefly [5]. We first construct a $\boldsymbol{Z}^2$-action $(Y,\boldsymbol{Z}^2,\mathscr{G},v)$ which has the following condition.

(1) $h_{1,0}>0$

(2) $h_{p,q}=0$ for all (p,q) which is not a scalar multiple of $(1,0)$.

He uses the above system to construct a sequence $\{(X_i,\boldsymbol{Z}^2,\mathscr{F}_i,\mu)\}$ of $\boldsymbol{Z}^2$-actions such that

$$(3)\quad \frac{1}{\sqrt{p_{2i}^2+q_{2i}^2}}h_{p_{2i},q_{2i}}>1$$

$$(4)\quad h_{p,q}=0 \qquad \text{if } (p,q)\neq(p_{2i},q_{2i}).$$

Let $(X,\boldsymbol{Z}^2,\mu)=\prod_{i=1}^{\infty}(X_i,\boldsymbol{Z}^2,\mathscr{F}_i,\mu_i)$. We have

$$(5)\quad \frac{1}{\sqrt{q_i^2+p_i^2}}h_{p_i,q_i}>1 \qquad \text{if } i \text{ is even}$$

$$(6)\quad \frac{1}{\sqrt{q_i^2+p_i^2}}h_{p_i,q_i}=0 \qquad \text{if } i \text{ is odd}$$

Hence it is clear that $\{\frac{1}{\sqrt{q_i^2+p_i^2}}h_{p_i,q_i}\}$ does not converge.

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2. Preliminaries

Let $(X,T,\mathscr{F},\mu)$ be an ergodic dynamical system where X is a space of doubly infinite sequences of finitely many alphabets. Let T be a shift in X and S be a homomorphism of the compact space given by the rule f, i.e. $(Sx)_n=f(x_{n-s},x_{n-s+1},\cdots,x_0,\cdots,x_{n+s})$. We call S a block map of size 2s. First we embed the lattice $\boldsymbol{Z}^2$ into the 2-dimensional real vector space $\boldsymbol{R}^2$. Let P be a partition according to the alphabets. We denote the partition $T^iS^j(P)$ by $P_{i,j}$. The directional entropy $h(\vec{v})$ in the direction of $\vec{v}$ is defined as follows:

$$h(\vec{v})=\sup_B \overline{\lim_{t\to\infty}} \frac{1}{t} H(\bigvee_{(i,j)\in B+[0,t]v} P_{i,j}).$$

where $B+[0,t]\vec{v}=\{(i,j)\in\boldsymbol{Z}^2;$ there exists $(k,l)\in B$ such that $(i,j)\text{-}(k,l)=\alpha\vec{v}$ for some $\alpha\in[0,t]\}$. Supremum is taken over all bounded subsets of $\boldsymbol{Z}^2$.

Given a vector $v\in\boldsymbol{R}^2$, which is not a scalar multiple of $(1,0)$, we define $w=\cot\theta$ where θ is the angle between the vector $\vec{v}$ and x-axis. We note that $(w,1)$ is a scalar multiple of $\vec{v}$. We call a vector a rational vector if the corresponding w is rational. Otherwise, we call the vector an irrational vector.

It is not difficult to check that

$$h(\vec{v})=\lim_{n\to\infty} \overline{\lim_{t\to\infty}} \frac{1}{t} H(\bigvee_{j=0}^{[ty]} \bigvee_{-n+jw+a<i<n+jw+a} P_{i,j}) \text{ for any } a\in[0,1].$$

Clearly $h(\vec{v})$ is homogeneous and $h(T^pS^q)=h(\vec{v})$ where $\vec{v}=(p,q)$. (see [2])

We will follow the notations in [4]. The following left and right entropies have been introduced in [4].

$$H_r(a,\vec{v})=H(\bigvee_{i\geq a+w} P_{i,1}|\bigvee_{j\leq 0}\bigvee_{i\geq a+jw} P_{i,j})$$

$$H_l(a,\vec{v})=H(\bigvee_{i\leq a+w} P_{i,1}|\bigvee_{j\leq 0}\bigvee_{i\leq a+jw} P_{i,j})$$

(Note: $H(\bigvee_{i\geq a+w} P_{i,1} | \bigvee_{j\leq 0} \bigvee_{i\geq a+jw} P_{i,j}) =$

$$\lim_{m\to\infty} \lim_{n\to\infty} H(\bigvee_{a+w\leq a+w+m} P_{i,1} | \bigvee_{j\leq 0} \bigvee_{a+jw\leq i\leq a+jw+n} P_{i,j})$$

We define

$$H(a,\vec{v}) = H_r(a,\vec{v}) + H_l(a,\vec{v}).$$

We note that H_r, H_l and H are defined independent of the size of the vector.

Since S is a block map of size $2s$, we have

$$H_r(a,\vec{v}) = \begin{cases} H(\bigvee_{a+w\leq i<a+s} P_{i,1} | \bigvee_{j\leq 0} \bigvee_{i\geq a+jw} P_{i,j}) & \text{if } w\leq s \\ 0 & \text{if } w>s \end{cases}$$

$$H_l(a,\vec{v}) = \begin{cases} H_l \bigvee_{a-s\leq i\leq a+w} P_{i,1} | \bigvee_{j\leq 0} \bigvee_{i\leq a+jw} P_{i,j}) & \text{if } w\geq -s \\ 0 & \text{if } w<-s \end{cases}$$

We need following observations.

(I) Given a vector $\vec{v}$, $H_r(a,\vec{v})$ and $H_l(a,\vec{v})$ are bounded functions of $a\in[0,1]$. Hence $H(a,\vec{v})$ is a bounded function.

(II) For each fixed vector $\vec{v}$, $H_r(a,\vec{v})$ and $H_l(a,\vec{v})$ are periodic functions of a with period 1. Hence so is H $(a,\vec{v})$.

(III) We fix a_0 and $\vec{v}_0=(x_0,y_0)$ and assume that a_0+jw_0 is not an integer for any $j\leq 1$. (We note that given $\vec{v}_0$, hence w_0, there exist only countably many a's in $[0,1]$ such that $a+jw$ is an integer for some $j\leq 1$.) If a and $\vec{v}$ are sufficiently close to a_0 and $\vec{v}_0$ respectively, then

$$\bigvee_{a+w\leq i<a+s} P_{i,1} = \bigvee_{a_0+w_0\leq i<a_0+s} P_{i,1}$$

(IV) Let (x_k,y_k) be a sequence of vectors such that $\frac{y_k}{x_k}\to\frac{y_0}{x_0}$. We have $w_k\to w_0$. If $\frac{y_k}{x_k}\searrow\frac{y_0}{x_0}$, then

$$\lim_{k\to\infty} \bigvee_{j\le 0} \bigvee_{i\ge a+jw_k} P_{i,j} = \bigvee_{j\le 0} \bigvee_{i\ge a+jw_o} P_{i,j} \quad \text{and}$$

$$\lim_{k\to\infty} \bigvee_{j\le 0} \bigvee_{i\le a+jw_k} P_{i,j} \ge \bigvee_{j\le 0} \bigvee_{i\le a+jw_o} P_{i,j}$$

whenever $a+jw_0$ is not an integer for any $j\le 1$. Hence by (III) and the continuity of conditional entropy, we have $\lim_{k\to\infty} H_r(a,\vec{v_k})=H_r(a,\vec{v_o})$ for all except countably many a's. And we have $\lim_{k\to\infty} H_l(a,\vec{v_k})\le H_l(a,\vec{v_o})$ for all except countably many a's. Hence we have $\lim_{k\to\infty} H(a,\vec{v_k})\le H(a,\vec{v_o})$ for the set of a's of full measure.

Likewise if $\frac{y_k}{x_k} \nearrow \frac{y_0}{x_0}$, then we have

$$\lim_{k\to\infty} H_l(a,\vec{v_k})=H_l(a,\vec{v_o}) \text{ and } \lim_{k\to\infty} H_r(a,\vec{v_k})\le H_r(a,\vec{v_o}).$$

(V) If $\vec{v_k}$'s are sufficiently close to $\vec{v_o}$, then the sequence $\{H(a,\vec{v_k})\}$ is uniformly bounded for all $a\in[0,1]$ and all $\vec{v_k}$'s.

REMARK 2.1. We note that $\lim_{k\to\infty} \bigvee_{j\le 0} \bigvee_{i\le a+jw_k} P_{i,j}$ is $\bigvee_{j\le 0} \bigvee_{i\le a+jw_0} P_{i,j}$ plus some kind of "tail field". If this "tail field" does not affect the entropy, then the continuity of directional entropy follows from the result in the next section.

3. Directional entropy of an irrational vector

We fix an irrational vector $\vec{v}$. Without loss of generality we assume $\frac{y}{x}>0$. In this section, we write $H_r(a)$ for $H_r(a,\vec{v})$. Let $\varepsilon>0$ be given. Let $m_r(a,\varepsilon)$ be an integer such that if $m\ge m_r(a,\varepsilon)$, then

$$H_r^m(a)=H\Big(\bigvee_{a+w\le i<a+s} P_{i,1} \,\Big|\, \bigvee_{j\le 0} \bigvee_{a+jw\le i\le a+jw+m} P_{i,j}\Big)\le H_r(a)+\varepsilon.$$

We let $n_r(a,m,\varepsilon)$ where $m\ge m_r(a,\varepsilon)$ be an integer such that if $n\ge n_r(a,m,\varepsilon)$, then

$$|H(\bigvee_{a+w\le i<a+s} P_{i,1} | \bigvee_{-n<j\le 0} \bigvee_{a+jw\le i\le a+jw+m} P_{i,j}) - H_r^m(a)| < \varepsilon.$$

We denote by $H_r^{m,n}(a)$

$$H(\bigvee_{a+w\le i<a+s} P_{i,1} | \bigvee_{-n<j\le 0} \bigvee_{a+jw\le i\le a+jw+m} P_{i,j}).$$

Hence if $n\ge n_r$, then

$$|H_r^{m_o,n}(a) - H_r(a)| < 2\varepsilon.$$

We choose $m_r(\varepsilon)$ sufficiently large so that

$$\nu\{a\in[0,1]\colon |H_r^m(a) - H_r(a)| < \varepsilon\} > 1-\varepsilon \text{ for } m\ge m_r(\varepsilon) \qquad \cdots\cdots(3.1)$$

where ν denotes the Lebesgue measure on $[0,1]$.
Given $m_r(\varepsilon)$, we can choose $n_r(\varepsilon)$ sufficiently large so that

$$\nu\{a\in[0,1]\colon |H_r^{m_r,n}(a) - H_r^{m_r}(a)| < \varepsilon\} > 1-\varepsilon \text{ for } n\ge n_r(\varepsilon). \qquad \cdots\cdots(3.2)$$

Likewise we can choose $m_l(\varepsilon)$ so that it satisfies the condition corresponding to (3.1).

Recall that s denotes the size of the block map. We let $m_0(\varepsilon) = \max\{m_r(\varepsilon), m_l(\varepsilon)\}$. We choose $n_o(m_o,\varepsilon)$ sufficiently large so that the following conditions are satisfied.

(i) The set

$$E_r^{m_o,n}(\varepsilon) = \left\{a\in[0,1]\colon |H_r^{m_o,n}(a) - H_r^{m_o}(a)| < \varepsilon,\ |H_r^{m_o+s,n}(a) - H_r^{m_o+s}(a)| < \varepsilon\right\}$$

has measure bigger than $1-\varepsilon$ for all $n\ge n_o$.

(ii) The set

$$E_l^{m_o,n}(\varepsilon) = \left\{a\in[0,1]\colon |H_l^{m_o,n}(a) - H_l^{m_o}(a)| < \varepsilon,\ |H_l^{m_o+s,n}(a) - H_l^{m_o+s}(a)| < \varepsilon\right\}$$

has measure bigger than $1-\varepsilon$ for all $n\ge n_o$.

Remark 3.1. We note that $H_r(a) \le H_r^{m+s,n}(a) \le H_r^{m,n}(a)$.

REMARK 3.2. If $n \geq n_o$, then each of the sets $F_r^{m,n} = \{a \in [0,1]: |H_r^{m,n}(a) - H_r(a)| < 2\varepsilon\}$ and $F_l^{m,n} = \{a \in [0,1]: |H_l^{m,n}(a) - H_l(a)| < 2\varepsilon\}$ has measure bigger than $1-2\varepsilon$. Also each of the sets $F_r^{m+s,n} = \{a \in [0,1]: |H_r^{m+s,n}(a) - H_r(a)| < 2\varepsilon\}$ and $F_n^{m+s,n} = \{a \in [0,1]: |H_l^{m+s,n}(a) - H_l(a)| < 2\varepsilon\}$ has measure bigger than $1-2\varepsilon$.

REMARK 3.3. We note that $H_r^{m,n}(a)$ and $H_l^{m,n}(a)$ are uniformly bounded in all a, m and $n \geq 1$.

REMARK 3.4. If $a \in E_r^{m_o,n}$, then $|H_r^{m,n}(a) - H_r^{m+s,n}(a)| < 3\varepsilon$ for all $n \geq n_o$. Also for a given $n \geq n_o$ and any $k \geq n$, if $a \in E_r^{m_o,n}$, then we have

$$|H_r^{m,n}(a) - H_r^{m,k}(a)| < 4\varepsilon.$$

We let

$$H^{m,n}(a) = H_l^{m,n}(a) + H_l^{m,n}(a)$$

$$F^{m,n} = F_r^{m,n} \cap F_l^{m,n}$$

Hence, given $\varepsilon > 0$, there exist m and n such that

$$\left| \int_0^1 H(a)da - \int_0^1 H^{m,n}(a)da \right| \leq$$

$$\int_{F^{m,n}} |H(a) - H^{m,n}(a)|\, da + \int_{(F^{m,n})^c} |H(a) - H^{m,n}(a)|\, da$$

$$< 4\varepsilon + 4\varepsilon L \qquad \cdots\cdots(3.3)$$

where L denotes the uniform bound of $H^{m,n}(a)$ for all a, m and $n \geq 1$.

Theorem 3.1. *Let $\vec{v} = (x,y)$ be an irrational vector. Then we have*

$$\frac{1}{y} h(\vec{v}) = \int_0^1 H(a,\vec{v})da.$$

Proof. Let $\varepsilon > 0$ be given. Let $[\tau]$ denote the greatest integer $\leq \tau$. We choose $m_0(\frac{\varepsilon}{10})$ and $n_0(m_0, \frac{\varepsilon}{10})$ so that if $n \geq n_0$, then the subset E of $[0,1]$ satisfying

(i) $\left| H_r^{m_0,n}(a) - H_r(a) \right| < \frac{\varepsilon}{5}$ and $\left| H_l^{m_0,n}(a) - H_l(a) \right| < \frac{\varepsilon}{5}$

(ii) $\left|H_r^{m_0+s,n}(a)-H_r(a)\right|<\frac{\varepsilon}{5}$ and $\left|H_l^{m_0+s,n}(a)-H_l(a)\right|<\frac{\varepsilon}{5}$

has measure bigger than $1-\frac{\varepsilon}{2}$.

We choose an even positive integer m such that

(m.1) $\frac{m}{2}>>s$,

(m.2) $m>>m_o(\frac{\varepsilon}{10})$,

(m.3) $|h(\vec{v})-\lim_{t\to\infty}\frac{1}{t}H(\bigvee_{j=0}^{[ty]}\bigvee_{\frac{k}{2}+jw+a\leq i\leq\frac{k}{2}+jw+a}P_{i,j})|<\varepsilon$ for all $k\geq m$.

Given this m, we choose an integer $n\geq n_o(m,\frac{\varepsilon}{10})$. We choose t_o such that

(t.1) $\frac{2n}{[t_oy]}\cdot L<\varepsilon$,

(t.2) $|h(\vec{v})-\frac{1}{t}H(\bigvee_{j=0}^{[ty]}\bigvee_{-\frac{m}{2}+jw+a\leq i\leq\frac{m}{2}+jw+a}P_{i,j})|<2\varepsilon$ for all $t\geq t_o$,

(t.3) If $t\geq t_o$, then the set

$$B=\{a\in[0,1]:\ |\frac{1}{[ty]}\sum_{j=0}^{[ty]}H^{m,n}(a+jw)-\int_0^1 H^{m,n}(a)da|<\varepsilon\}$$

has measure bigger than $1-\varepsilon$,

(t.4) If $t\geq t_o$, then

$G=\{a\in[0,1]:\ \frac{1}{[ty]}\sum_{j=0}^{[ty]}\chi_E(a+jw)>1-\varepsilon\}$ has measure bigger than $1-\varepsilon$,

(t.5) $\frac{1}{[t_oy]}H(\bigvee_{a\leq i\leq a+m}P_{i,o})<\varepsilon$ for all $a\in[0,1]$.

Given m, we fix $t \geq t_o$. We let $u=[ty]$ and $m'=\frac{m}{2}$. We choose $a_o \in B \cap G$. We denote $kw+a_o$ (mod 1) by a_k and $[kw+a_o]$ by w_k. It is sufficient to show that

$$\left|\frac{1}{u} H\left(\bigvee_{j=0}^{u} \bigvee_{-m'+jw+a_0 \leq i \leq m'+jw+a_0} P_{i,j}\right)-\frac{1}{u}\sum_{k=1}^{u} H^{m,n}(a_{k-1})\right|$$

is less than $\varepsilon \cdot \beta$ for some number β. By our conditions (t.2),(t.3) and (3.3),

$$\frac{1}{u} H\left(\bigvee_{j=0}^{u} \bigvee_{-m'+jw+a_0 \leq i \leq m'+jw+a_0} P_{i,j}\right)$$

$$=\frac{1}{u}\sum_{k=0}^{u} H\left(\bigvee_{-m'+kw+a_0 \leq i \leq m'+kw+a_0} P_{i,k} \Big| \bigvee_{0 \leq j<k} \bigvee_{-m'+jw+a_0 \leq i \leq m'+(j-1)w+a_0} P_{i,j}\right)$$

Since $H(\cdots|\cdots)$ is invariant under the shift T, the above formula is equal to

$$\frac{1}{u}\sum_{k=0}^{u} H\left(\bigvee_{kw+a_0 \leq i \leq m+kw+a_0} P_{i,k} \Big| \bigvee_{0 \leq j<k} \bigvee_{jw+a_0 \leq i \leq m+jw+a_0} P_{i,j}\right)$$

$$=\frac{1}{u}\sum_{k=0}^{u} H\left(\bigvee_{a_k+w \leq i \leq a_k+w+m} P_{i,k} \Big| \bigvee_{0 \leq j<k} \bigvee_{jw+a_0-w_{k-1} \leq i \leq jw+a_o-w_{k-1}+m} P_{i,j}\right)$$

$$=\frac{1}{u}\sum_{k=1}^{u} H\left(\bigvee_{a_{k-1}+w \leq i \leq a_{k-1}+w+m} P_{i,1} \Big| \right.$$

$$\left. \bigvee_{0 \leq j<k} \bigvee_{jw+a_0-w_{k-1} \leq i \leq jw+a_o-w_{k-1}+m} P_{i,j-k+1}\right)$$

$$+\frac{1}{u} H\left(\bigvee_{a_0 \leq i \leq a_0+m} P_{i,0}\right)$$

Let $b_{k,j}=a_{k-1}-jw$ for $k=1,\cdots,u$ and $j=0,\cdots,k-1$. Note that $b_{k,o}=a_{k-1}$. We have

$$\frac{1}{u}\sum_{k=1}^{u} H\Big(\bigvee_{a_{k-1}+w\le i\le a_{k-1}+w+m} P_{i,1} \,\Big|\, \bigvee_{0\le j<k}\ \bigvee_{jw+a_0-w_{k-1}\le i\le jw+a_0-w_{k-1}+m} P_{i,j-k+1}\Big)$$

$$=\sum_{k=1}^{u} H\Big(\bigvee_{a_{k-1}+w\le i\le a_{k-1}+w+m} P_{i,1} \,\Big|\, \bigvee_{0\le j<k}\ \bigvee_{b_{k,j}\le i\le b_{k,j}+m} P_{i,-j}\Big)$$

$$=\sum_{k=1}^{u} H\Big(\bigvee_{a_{k-1}+w\le i\le a_{k-1}+s} P_{i,1} \,\Big|\, \bigvee_{0\le j<k}\ \bigvee_{b_{k,j}\le i\le b_{k,j}+m} P_{i,-j}\Big)+$$

$$H\Big(\bigvee_{a_{k-1}+m-s\le i\le a_{k-1}+m+w} P_{i,1} \,\Big|\Big(\bigvee_{0\le j<k}\ \bigvee_{b_{k,j}\le i\le b_{k,j}+m} P_{i,-j}\Big)$$

$$\bigvee\Big(\bigvee_{a_{k+1}+w\le i\le a_{k-1}+s} P_{i,1}\Big)\Big)$$

We note that the first term in the summand is equal to $H_r^{m,k}(a_{k-1})$. Let us consider the second term.

$$H\Big(\bigvee_{a_{k-1}+m-s\le i\le a_{k-1}+m+w} P_{i,1} \,\Big|\Big(\bigvee_{0\le j\le k}\ \bigvee_{b_{k,j}\le i\le b_{k,j}+m} P_{i,-j}\Big)$$

$$\bigvee\Big(\bigvee_{a_{k+1}+w\le i\le a_{k-1}+s} P_{i,1}\Big)\Big)$$

$$=H\Big(\bigvee_{a_{k-1}-s\le i\le a_{k-1}+w} P_{i,1} \,\Big|\Big(\bigvee_{0\le j<k}\ \bigvee_{b_{k,j}-m\le i\le b_{k,j}} P_{i,-j}\Big)$$

$$\bigvee\Big(\bigvee_{a_{k-1}+w-m\le i\le a_{k-1}-m+s} P_{i,1}\Big)\Big) \tag{3.4}$$

Clearly (3.4) is bigger than

$$H\Big(\bigvee_{a_{k-1}-s\le i\le a_{k-1}+w} P_{i,1} \,\Big|\Big(\bigvee_{0\le j\le k}\ \bigvee_{b_{k,j}-m\le i\le b_{k,j}} P_{i,j}\Big) \bigvee\Big(\bigvee_{b_{k,o}-m-s\le i\le b_{k,o}-m} P_{i,o}\Big)\Big)$$

which is greater than

$$H\Big(\bigvee_{a_{k-1}-s\le i\le a_{k-1}+w} P_{i,1} \,\Big|\, \bigvee_{0\le j\le k}\ \bigvee_{b_{k,j}-m-s\le i\le b_{k,j}} P_{i,-j}\Big)=H_l^{m+s,k}(a_{k-1}).$$

Also (3.4) is smaller than

$$H(\bigvee_{a_{k-1}-s\leq i\leq a_{k-1}+w} P_{i,1}|\bigvee_{0\leq j\leq k}\bigvee_{b_{k,j}-m\leq i\leq b_{k,j}} P_{i,-j})=H_l^{m,k}(a_{k-1})$$

Now, we have that

$$\frac{1}{[ty]}H(\bigvee_{j=o}^{[ty]}\bigvee_{-m'+jw+a_o\leq i\leq m'+jw+a_o} P_{i,-j})$$

is bigger than

$$\frac{1}{u}(H(\bigvee_{a_o\leq i\leq a_o+m} P_{i,o})+\sum_{k=1}^{u}(H_r^{m,k}(a_{k-1})+H_l^{m+s,k}(a_{k-1})))$$

and smaller than

$$\frac{1}{u}(H(\bigvee_{a_o\leq i\leq a_o+m} P_{i,o})+\sum_{k=1}^{u}(H_r^{m,k}(a_{k-1})+H_l^{m,k}(a_{k-1})))$$

$$=\frac{1}{u}(H(\bigvee_{a_o\leq i\leq a_o+m} P_{i,o})+\sum_{k=1}^{u}H^{m,k}(a_{k-1})).$$

Since we have

$$\sum_{k-1}^{u}(H_r^{m,k}(a_{k-1})+H_l^{m+s,k}(a_{k-1}))\geq\sum_{k=1}^{u}(H_r^{m+s,k}(a_{k-1})+H_l^{m+s,k}(a_{k-1}))$$

$$=\sum_{k-1}^{u}H^{m+s,k}(a_{k-1}),$$

We have that

$$|\frac{1}{u}H(\bigvee_{j=o}^{u}\bigvee_{-m'+jw+a_o\leq i\leq m'+jw+a_o} P_{i,j}-\frac{1}{u}\sum_{k=1}^{u}H^{m,n}(a_{k-1})| \qquad \cdots\cdots(3.5)$$

is between

$$|\frac{1}{u}H(\bigvee_{a_o\leq i\leq a_o+m} P_{i,o})-\frac{1}{u}\sum_{k=1}^{u}(H^{m,n}(a_{k-1})-H^{m+s,k}(a_{k-1}))| \qquad \cdots\cdots(3.6)$$

and

$$\left|\frac{1}{u}H\Big(\bigvee_{a_o\le i\le a_o+m} P_{i,o}\Big)-\frac{1}{u}\sum_{k=1}^{u}(H^{m,n}(a_{k-1})-H^{m,k}(a_{k-1}))\right| \qquad \cdots\cdots(3.7)$$

We will show that (3.7) is small.
By our choice of u, satisfying (t.1), (t.3), (t.4), and (t.5) we have

$$\left|\frac{1}{u}H\Big(\bigvee_{a_o\le i\le a_o+m} P_{i,o}\Big)-\frac{1}{u}\sum_{k=1}^{u}(H^{m,n}(a_{k-1})-H^{m,k}(a_{k-1}))\right|$$

$$\le \varepsilon+\frac{2n}{u}L+\frac{1}{u}\sum_{k=n}^{u}|H^{m,n}(a_{k-1})-H^{m,k}(a_{k-1})|$$

$$<2\varepsilon+\frac{1}{u}\Big(\sum_{\{k:a_{k-1}\in E\}}|H^{m,n}(a_{k-1})-H^{m,k}(a_{k-1})|+$$

$$\sum_{\{k:a_{k-1}\in E^c\}}|H^{m,n}(a_{k-1})-H^{m,k}(a_{k-1})|\Big)$$

$$<2\varepsilon+\frac{1}{u}\Big(\sum_{\{k:a_{k-1}\in E\}}\frac{\varepsilon}{5}+\frac{1}{u}\sum_{\{k:a_{k-1}\in E^c\}}|H^{m,n}(a_{k-1})-H^{m,k}(a_{k-1})|$$

$$<2\varepsilon+\frac{1}{u}\,\frac{\varepsilon}{5}\,(1-\varepsilon)u+\frac{2\varepsilon u}{u}\,2L$$

$$=2\varepsilon+\frac{\varepsilon(1-\varepsilon)}{5}+4\varepsilon L.$$

Recall that if $a\in E$, then we have $|H^{m,n}(a)-H^{m+s,k}(a)|<\varepsilon$ for all $k\ge n$. Hence we have that (3.7) is also smaller than $\varepsilon(2+(1-\varepsilon)+4L)\le \varepsilon(3+4L)$. Likewise, we can show that (3.6) is small. This completes the proof of the Theorem.

4. Continuity

We need the following two theorems from [4].

Theorem 4.1. *Let $\vec{v}=(p,q)(q\ne 0)$ be a given vector where p and q are relatively prime.*

$$\frac{1}{q}h(\vec{v})=\frac{1}{q}\sum_{k=0}^{q-1}H\Big(\frac{p}{q}k,\vec{v}\Big)$$

Corollary 4.2. *Under the condition of Theorem* 4.1, *we have*

$$\frac{1}{q}h(\vec{v}) = \frac{1}{q}\sum_{k=0}^{q-1} H\left(\frac{k}{q},\vec{v}\right) = \int_0^1 H(a,\vec{v})da.$$

REMARK 4.1. The first equality comes from observation (II) in §2. The second equality comes from the following observation;

$$H_r(a,\vec{v}) = H_r(b,\vec{v}) \text{ if } a,b \in \left(\frac{k}{q},\frac{k+1}{q}\right], \ k=0,\cdots,q-1.$$

$$H_l(a,\vec{v}) = H_l(b,\vec{v}) \text{ if } a,b \in \left[\frac{k}{q},\frac{k+1}{q}\right), \ k=0,\cdots,q-1.$$

That is, $H_r(a,\vec{v})$ $\left(H_l(a,\vec{v})\right)$ is a left (right) continuous step function.

Corollary 4.3. $h(T^pS^q)<\infty$ *for all* $(p,q)\in \boldsymbol{Z}^2$.

REMARK 4.2. If $\vec{v}$ is a scalar multiple of (0,1), then we note that $H_r(a,\vec{v})(H_l(a,\vec{v}))$ is a constant function on [0,1].

Theorem 4.4. *Given a vector* $\vec{v}=(x,y)(y\neq 0)$, *let* $\{\vec{v}_k=(p_k,q_k)\}$ be a sequence of vectors such that $\frac{q_k}{p_k}\to\frac{y}{x}$.

$$\lim_{k\to\infty}\frac{1}{q_k}h(\vec{v}_k) = \lim_{k\to\infty}\int_0^1 H(a,\vec{v}_k)da \leq \int_0^1 H(a,\vec{v})da$$

By Corollary 4.2 and the observation (IV) in §2, the above theorem is clear.

Corollary 4.5. *If* $\vec{v}\neq(1,0)$, *then the directional entropy function is upper semi-continuous at* $\vec{v}$.

REMARK 4.3. It is not hard to see that $yH(a,\vec{v})$ is uniformly bounded for all unit vectors. Hence $h(\vec{v})$ is uniformly bounded for all unit vectors.

Theorem 4.6. *Dirctional entropy function is continuous at* $\vec{v}=(1,0)$.

Proof. Althogh this is proved in [2], we will prove it here to illustrate the ideas in a general setting. We let $\vec{w}=(x_0,y_0)$ where x_0

and y_o satisfy $y=-\frac{1}{s}(x-1)$. By the homogeneity of the directional entropy, it is sufficient to prove that $h(\vec{w})\to h(\vec{v})$ as $x_o\to 1$. We may assume $x_o>0$ and $d=\frac{y_o}{x_o}<<\frac{1}{s}$.

We note that

$$h(\vec{v})=\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}H(\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}\le j\le\frac{m}{2}}P_{i,j})$$

and

$$h(\vec{w})=\lim_{m\to\infty}\lim_{t\to\infty}\frac{1}{t}H(\bigvee_{0\le i\le tx}\ \bigvee_{-\frac{m}{2}+di\le j\le\frac{m}{2}+di}P_{i,j})$$

Since S is a finite code of length $2s$, it is easy to see that for each m,

$$\bigvee_{0\le i\le nx_o}\ \bigvee_{-\frac{m}{2}+di\le j\le\frac{m}{2}+di}P_{i,j}\subset\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}\le j\le\frac{m}{2}}P_{i,j}.$$

Hence we have $h(\vec{w})\le h(\vec{v})$.

We show that

$$\frac{1}{n}H(\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}\le j\le\frac{m}{2}}P_{i,j})-\frac{1}{n}H(\bigvee_{0\le i\le nx_o}\ \bigvee_{-\frac{m}{2}+di\le j\le\frac{m}{2}+di}P_{i,j})$$

is small for large n and large m, *as* $x_o\to 1$.
Let d_n denote the largest integer $<nd+1$. We assume n to be sufficiently large so that $n(1-x_o)>s$. We note that

$$H(\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}\le j\le\frac{m}{2}}P_{i,j})=H(\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}+d_n\le j\le\frac{m}{2}+d_n}P_{i,j})$$

and

$$\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}+d_n\le j\le\frac{m}{2}+d_n}P_{i,j}\subset Q^1\vee Q^2 \text{ where}$$

$$Q^1=\bigvee_{0\le i\le n}\ \bigvee_{-\frac{m}{2}+di\le j\le\frac{m}{2}+di}P_{i,j} \text{ and}$$

$$Q^2=\bigvee_{0\le i\le n}\ \bigvee_{\frac{m}{2}+di<j<\frac{m}{2}+d_n}P_{i,j}.$$

Now, we compute $\frac{1}{n}H(Q^2|Q^1)$ and $\frac{1}{n}H(Q^1)$ as $n \to \infty$ and $m \to \infty$.

$$\frac{1}{n}H(Q^2|Q^1) = \frac{1}{n}H\Big(\bigvee_{0\le i\le n}\ \bigvee_{\frac{m}{2}+di<j\le\frac{m}{2}+d_n} P_{i,j}|Q^1\Big)$$

$$= \frac{1}{n}H\Big(\bigvee_{\frac{m}{2}<j\le\frac{m}{2}+d_n}\ \bigvee_{0\le i<\frac{1}{d}(j-\frac{m}{2})} P_{i,j}|Q^1\Big)$$

$$= \frac{1}{n}\sum_{j=\frac{m}{2}+1}^{\frac{m}{2}+d_n} H\Big(\bigvee_{0\le i<\frac{1}{d}(j-\frac{m}{2})} P_{i,j}\Big|\bigvee_{\frac{m}{2}+1\le k<j}\ \bigvee_{0\le i<\frac{1}{d}(k-\frac{m}{2})} P_{i,j} \vee Q^1\Big)$$

$$\le \frac{1}{n}\sum_{j=\frac{m}{2}+1}^{\frac{m}{2}+d_n} H\Big(\bigvee_{0\le i<s} P_{i,j}\Big|\bigvee_{\frac{m}{2}+1\le k<j-1}\ \bigvee_{0\le i<s(k-\frac{m}{2})} P_{i,j}\Big) + H\Big(\bigvee_{n-s<i\le n} P_{i,\frac{m}{2}+d_n-1}\Big)$$

$$+ H\Big(\bigvee_{n-s<i\le n} P_{i,\frac{m}{2}+d_n}\Big)\Big)$$

$$\le \frac{1}{n}\sum_{j=\frac{m}{2}+1}^{\frac{m}{2}+d_n} H\Big(\bigvee_{0\le i<s} P_{i,j}\Big) + 2H\Big(\bigvee_{n-s<i\le n} P_{i,o}\Big)\Big)$$

$$\le \frac{1}{n}(d_n+2)H\Big(\bigvee_{0\le i<s} P_{i,o}\Big) \to dH\Big(\bigvee_{0\le i<s} P_{i,o}\Big) \text{ as } n\to\infty.$$

We also note that

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}H(Q^1) = \frac{1}{x_o}h(\vec{w}).$$

Hence we have

$$h(\vec{v}) \le \frac{1}{x_o}h(\vec{w}) + dH\Big(\bigvee_{0\le i<s} P_{i,o}\Big).$$

Since the directional entropy is uniformly bounded, this completes our proof.

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