Burns, J.M. Clancy, M.J.
Osaka J. Math.
31 (1994), 533–559

POLAR SETS AS NONDEGENERATE CRITICAL SUBMANIFOLDS IN SYMMETRIC SPACES

JOHN M. BURNS and MICHAEL J. CLANCY

(Received June 7, 1993)

1. Introduction

In recent times a new approach has been taken to the study of compact symmetric spaces. This approach, begun by B-Y. Chen and T. Nagano [4], involves the study of pairs $(M^+(p), M^-(p))$ of totally geodesic submanifolds associated with closed geodesics. The submanifold $M^+(p)$, called the polar set through p, is the orbit through p of the isotropy subgroup. The space $M^-(p)$ intersects $M^+(p)$ orthogonally at p and, when M is irreducible, is usually a local product of two irreducible symmetric spaces.

The purpose of this paper is to exhibit the close connection between these pairs (together with their generalizations through the method of Borel-De Siebenthal [1]) and the Morse-Bott theory of isotropy-invariant functions on M. If $M=G/K$, we consider conjugation-invariant functions on G_1 which we pull back to (K_1-invariant functions on) M by means of the quadratic representation. In this way, we reduce our study to that of class-functions at the group level, and calculations may be restricted to a maximal torus in the group. If H is a vertex of the fundamental simplex and if $p=\exp H$ is a critical point of a class-function on G_1, then the eigenspaces of the Hessian coincide with the factoring obtained from the Borel-De Siebenthal splitting at p. Thus, the question of nondegeneracy and the calculation of the index is reduced to finding the eigenvalues corresponding to each factor. This can be done easily. To construct suitable class-functions we consider the real parts of the characters of irreducible representations. Some care must be taken with the choice of representation so that we do not obtain degenerate critical submanifolds. For example, in the case of the groups E_6 and G_2, the character of the adjoint representation has some M^+'s as degenerate critical submanifolds. The correct choice of representation usually seems to be one having lowest degree and, generically, the critical submanifolds (all of which are nondegenerate) are either M^+'s or are of the form $K_1.p$, where p is the exponential of a vertex of the fundamental simplex. The explicit calculations are given for the groups F_4 and E_6 where we have taken the lowest dimensional (non-trivial) irreducible representations of these groups. In the case of E_6 there are two additional (nondegenerate) critical submanifolds.

Similar functions have appeared in the literature for various subclasses of symmetric spaces, namely: T. Frankel's function for the classical groups [6], S. Ramanujam's for symmetric spaces of classical type [10] and M. Takeuchi's for the broader class of symmetric R-spaces [11]. Our methods give a unified approach to the subject and have the advantage of allowing one to construct functions for the entire class of compact symmetric spaces.

The authors wish to thank the referee for providing Remark (ii) following Lemma 3.5, Remark (ii) following Theorem 4.1 and for suggestions which led to improvements and simplifications in the presentation of this work. We would also like to thank E. Buffet for some helpful conversations.

2. Preliminaries

A (connected) Riemannian manifold M is called a (Riemannian) symmetric space if for each $x \in M$ there exists an isometry s_x of M satisfying:

1. x is an isolated fixed point of s_x and
2. s_x is involutive (i.e. $s_x \circ s_x =$ the identity map on M).

For each $x \in M$, s_x is uniquely determined by reflection in geodesics through x and is called the *symmetry* at x.

Denote by G the closure (in the compact open topology) of the group of isometries of M generated by $\{s_x\colon x \in M\}$. Then G is a Lie group which acts transitively on M. If we fix $o \in M$, then we can write $M = G/K$ where $K := \{g \in G\colon g.o = o\}$. Henceforth, we will assume that M, and therefore G, is compact.

If $\mathcal{G}$ denotes the Lie algebra of G, then we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & End(\mathcal{G}) \arrow[d, "e"] \\ G \arrow[r, "\mathrm{Ad}"] & GL(\mathcal{G}) \end{tikzcd} \qquad (1)$$

where $(\mathrm{Ad}g)X := (d/dt)g(\exp tX)g^{-1}|_{t=0}$, $(\mathrm{ad}X)Y := [X, Y]$, and *exp* and *e* are the exponential maps. Consequently, if $<\ ,\ >$ is a bi-invariant Riemannian metric on G, then

$$<(\mathrm{ad}X)Y, Z> = -<Y, (\mathrm{ad}X)Z> \tag{2}$$

for every $X, Y, Z \in \mathscr{G}$. That is, for each $X \in \mathscr{G}$ the endomorphism $\mathrm{ad}X$ is skew-symmetric with respect to $<\ ,\ >$.

The map σ: $G \to G$: $g \mapsto s_o g s_o^{-1} = s_o g s_o$ is an involutive automorphism of G and, therefore,its derivative $\sigma_* = \mathrm{Ad}s_o$ is an involutive automorphism of $\mathscr{G}$, which we may assume preserves $<\ ,\ >$. Thus, $\sigma_* \in GL(\mathscr{G})$ which satisfies:

1. $\sigma_*^2 =$ the identify map on $\mathscr{G}$
2. $\sigma_*[X, Y] = [\sigma_* X, \sigma_* Y]$ $\forall X, Y \in \mathscr{G}$ and
3. $<\sigma_* X, \sigma_* Y> = <X, Y>$ $\forall X, Y \in \mathscr{G}$.

Accordingly, we obtain an orthogonal decomposition $\mathscr{G} = \mathscr{M} \oplus \mathscr{K}$ where $\mathscr{M}$ and $\mathscr{K}$ are the -1 and $+1$ eigenspaces of σ_*, respectively. $\mathscr{K}$ is the Lie algebra of K and $\mathscr{M}$ may be identified with $T_o M$, the tangent space to M at o. It follows that

$$[\mathscr{M}, \mathscr{M}] \subseteq \mathscr{K},\ [\mathscr{K}, \mathscr{K}] \subseteq \mathscr{K},\ [\mathscr{M}, \mathscr{K}] \subseteq \mathscr{M} \tag{3}$$

If $\mathscr{A}$ is an abelian subalgebra of $\mathscr{G}$ (i.e. $[\mathscr{A}, \mathscr{A}] = 0$) then the Jacobi identity implies that $\{\mathrm{ad}H\colon H \in \mathscr{A}\}$ is a commuting family of (skew-symmetric) endomorphisms of $\mathscr{G}$. Hence, $\{(\mathrm{ad}H)^2\colon H \in \mathscr{A}\}$ is a commuting family of symmetric endomorphisms of $\mathscr{G}$ and so can be simultaneously orthogonally diagonalised. In particular, the eigenvalues of $(\mathrm{ad}H)^2$ are all ≤ 0.

From now on we will assume $\mathscr{A}$ is an abelian subalgebra of $\mathscr{G}$ such that $\mathscr{A} \subset \mathscr{M}$ and is maximal with respect to this property. The torus $A = \{\exp H.o\colon H \in \mathscr{A}\}$ for such an $\mathscr{A}$ is called a maximal torus of M. Now, for each $H \in \mathscr{A} \subset \mathscr{M}$ it follows from (3) that $(\mathrm{ad}H)^2$: $\mathscr{M} \to \mathscr{M}$ and $\mathscr{K} \to \mathscr{K}$. Accordingly, we can decompose $\mathscr{G}$ into an orthogonal sum of (simultaneous) eigenspaces of $\{(\mathrm{ad}H)^2\colon H \in \mathscr{A}\}$ as follows:

$$\mathscr{G} = \mathscr{A} \oplus \sum_{\alpha \in \Re_M} \mathscr{M}_\alpha \oplus \mathscr{K}_0 \oplus \sum_{\alpha \in \Re_M} \mathscr{K}_\alpha \tag{4}$$

where $\mathscr{K}_0 := \{Y \in \mathscr{K}\colon [Y, \mathscr{A}] = 0\}$,

$$\mathrm{ad}H\colon \mathscr{M}_\alpha \to \mathscr{K}_\alpha \text{ satisfies } (\mathrm{ad}H)^2|_{\mathscr{M}_\alpha} = -\alpha^2(H)id_{\mathscr{M}_\alpha}$$

and

$$\mathrm{ad}H\colon \mathscr{K}_\alpha \to \mathscr{M}_\alpha \text{ satisfies } (\mathrm{ad}H)^2|_{\mathscr{K}_\alpha} = -\alpha^2(H)id_{\mathscr{K}_\alpha}$$

The non-zero linear functions $\alpha\colon\mathscr{A}\to\mathbf{R}$ so defined up to a multiple of ± 1 are called the *roots* of M and will be denoted by $\Re_M$. For each $\alpha\in\Re_M$ let $\mu_\alpha := \dim\mathscr{M}_\alpha = \dim\mathscr{K}_\alpha$, which is called the multiplicity of the root α. Thus, for each root α there exist orthonormal bases $X_1^\alpha,\cdots,X_{\mu_\alpha}^\alpha$ for $\mathscr{M}_\alpha$ and $Y_1^\alpha,\cdots,Y_{\mu_\alpha}^\alpha$ for $\mathscr{K}_\alpha$ such that $\forall H\in\mathscr{A}$ and $1\le i\le\mu_\alpha$ we have

$$(\mathrm{ad}H)X_i^\alpha = \alpha(H)Y_i^\alpha \quad \text{and} \quad (\mathrm{ad}H)Y_i^\alpha = -\alpha(H)X_i^\alpha \tag{5}$$

DEFINITION 2.1. Vectors $X^\alpha\in\mathscr{M}_\alpha$ and $Y^\alpha\in\mathscr{K}_\alpha$ which satisfy (5) $\forall H\in\mathscr{A}$ will be called conjugate (relative to $\mathscr{A}$).

Any conjugate vectors X^α, Y^α span a two dimensional subspace of $\mathscr{G}$ which is invariant under $\mathrm{ad}\mathscr{A}$. Hence, $\forall H\in\mathscr{A}$, $\mathrm{ad}H$ and $e^{\mathrm{ad}H}$ restricted to $\mathbf{R}X^\alpha\oplus\mathbf{R}Y^\alpha$ have the following matrix representations with respect to X^α, Y^α:

$$\mathrm{ad}H = \begin{pmatrix} 0 & -\alpha(H) \\ \alpha(H) & 0 \end{pmatrix}, \; e^{\mathrm{ad}H} = \begin{pmatrix} \cos\alpha(H) & -\sin\alpha(H) \\ \sin\alpha(H) & \cos\alpha(H) \end{pmatrix} \tag{6}$$

Thus,

$$\left.\begin{aligned} e^{\mathrm{ad}H}X^\alpha &= \cos\alpha(H)X^\alpha + \sin\alpha(H)Y^\alpha \\ e^{\mathrm{ad}H}Y^\alpha &= -\sin\alpha(H)X^\alpha + \cos\alpha(H)Y^\alpha \end{aligned}\right\} \tag{7}$$

We now describe the tangent bundle to M on a maximal torus. For any $p\in M$ we obtain a surjective linear map:

$$p_*\colon\mathscr{G}\to T_pM\colon\; X\mapsto X_p := (d/dt)\exp tX.p|_{t=0} \tag{8}$$

In this way, $\forall X\in\mathscr{G}$ we obtain a C^∞ vector field on M (again denoted by X) called the field on M *induced* by X. We remark that at o this map: $\mathscr{G}=\mathscr{M}\oplus\mathscr{K}\to T_oM$ has kernel $\mathscr{K}$ and, after a suitable choice of metric $<\ ,\ >$ on G, we may assume it maps $\mathscr{M}$ isometrically onto T_oM. Now, fix a maximal torus $A := \{\exp H.o : H\in\mathscr{A}\}$ in M.

Proposition 2.2. *Let $X^\alpha\in\mathscr{M}_\alpha$ and $Y^\alpha\in\mathscr{K}_\alpha$ be conjugate relative to $\mathscr{A}$, then $\forall\ H,H'\in\mathscr{A}$ we have*:

1. $H'_{\exp H.o} = (\exp H)_* H'_o$

2. $X^{\alpha}_{\exp H.o} = \cos\alpha(H)(\exp H)_* X^{\alpha}_o$
3. $Y^{\alpha}_{\exp H.o} = \sin\alpha(H)(\exp H)_* X^{\alpha}_o$

Proposition 2.3. *Let $A = \{\exp H.o{:}H \in \mathscr{A}\}$ be a maximal torus in M and let $h \in \exp\mathscr{A} \subset G$.*

1. *If $H^1,\cdots,H^r$ is an orthonormal basis for $\mathscr{A}$, then $H^1_{h.o},\cdots,H^r_{h.0}$ is an orthonormal basis for $T_{h.o}A$.*
2. *$\{X^{\alpha}_{h.o},\ Y^{\alpha}_{h.o}{:}\ X^{\alpha} \in \mathscr{M}_{\alpha},\ Y^{\alpha} \in \mathscr{K}_{\alpha}$ and $\alpha \in \Re_M\}$ spans the normal space to A at $h.o$.*

3. K_1-invariant Morse functions on M

Let K_1 be the identity component of the isotropy subgroup K at o. Throughout this section f: $M \to \boldsymbol{R}$ will be a K_1-invariant function on M. We recall ([7] ChV, §6) that if A is a maximal torus in M, then $\forall x \in M\ \exists k \in K_1$ such that $kx \in A$. Therefore, f is completely determined by its values on A.

Proposition 3.1. *For every $p \in A$ $\operatorname{grad} f_p \in T_pA$.*

Proof. Since $\{X^{\alpha}_p,\ Y^{\alpha}_p{:}\ X^{\alpha} \in \mathscr{M}_{\alpha},\ Y^{\alpha} \in \mathscr{K}_{\alpha}$ and $\alpha \in \Re_M\}$ spans the normal space to A at p, it is sufficient to show that $\langle \operatorname{grad} f_p, Y^{\alpha}_p\rangle = \langle \operatorname{grad} f_p, X^{\alpha}_p\rangle = 0$.

1. $\langle \operatorname{grad} f_p, Y^{\alpha}_p\rangle = 0$:
   Observe that $\forall Y^{\alpha} \in \mathscr{K}_{\alpha}$ and $\forall t \in \mathbf{R}$ $\exp tY^{\alpha} \in K_1$ so that
   $$\begin{aligned}\langle \operatorname{grad} f_p, Y^{\alpha}_p\rangle &= Y^{\alpha}_p f = (d/dt) f(\exp tY^{\alpha}.p)|_{t=0}\\ &= (d/dt) f(p)|_{t=0} = 0.\end{aligned}$$
2. $\langle \operatorname{grad} f_p, X^{\alpha}_p\rangle = 0$:
   Choose $Y^{\alpha} \in \mathscr{K}_{\alpha}$ so that it is conjugate to X^{α} (relative to $\mathscr{A}$). If $h.o \in A$ is such that $h = \exp H$ where $H \in \mathscr{A}$ satisfies $\alpha(H) \notin \mathbf{Z}\pi$, then we have:
   $$\begin{aligned}\langle \operatorname{grad} f_{h.o}, X^{\alpha}_{h.o}\rangle &= \langle \operatorname{grad} f_{h.o}, \cos\alpha(H) h_* X^{\alpha}_o\rangle\\ &= \cot\alpha(H) \langle \operatorname{grad} f_{h.o}, Y^{\alpha}_{h.o}\rangle\\ &= 0.\end{aligned}$$
   That is, $\langle \operatorname{grad} f_{\exp H.o}, X^{\alpha}_{\exp H.o}\rangle = 0$ $\forall H \in \mathscr{A}$ which satisfies $\alpha(H) \notin \mathbf{Z}\pi$. Therefore, by continuity the innerproduct is zero $\forall H \in \mathscr{A}$.

The Hessian: If $p \in \mathrm{M}$ is a critical point of f, then the *Hessian* of f at p (denoted Hf_p) is the symmetric bilinear form on T_pM defined as follows: given $X, Y \in T_pM$ extend them to C^∞ vector fields on M (again denoted by X and Y) and define:

$$Hf_p(X, Y) = X_p(Yf)$$

Following the notation of [2] we have a self-adjoint linear map:

$$Tf_p : T_pM \to T_pM$$

defined by

$$< Tf_pX, Y > := Hf_p(X, Y) \quad \forall X, Y \in T_pM.$$

Proposition 3.2. *Let V be a Riemannian manifold and let $U \subseteq V$ be a submanifold with the induced metric. If a smooth function g: $V \to \mathbf{R}$ satisfies* $\mathrm{grad}\, g_p \in T_pU$ $\forall p \in U$, *then Tg_q: $T_qU \to T_qU$ at every critical point $q \in U$ of g.*

Proof. Given $X \in T_qU$ and $Y \in T_qV$ extend them to C^∞ vector fields (X and Y, respectively) on a neighbourhood of q in V so that X is tangent to U. If $Z := \mathrm{grad}\, g$ and ∇ is the Riemannian connection on V, then $Z_q = 0$ and

$$\begin{aligned} < Tg_qX, Y > &= X_q(Yg) = X_q < Z, Y > \\ &= < \nabla_{X_q} Z, Y > + < Z_q, \nabla_{X_q} Y > \\ &= < \nabla_{Z_q} X + [X,Z]_q, Y > \\ &= < [X,Z]_q, Y > \quad \forall Y \in T_qV. \end{aligned}$$

Therefore, $Tg_qX = [X, \mathrm{grad}\, g]_q$ $\forall X \in T_qU$. However, $\mathrm{grad}\, g$ is tangent to U so that $[X, \mathrm{grad}\, g]$ is also tangent to U whenever X is. Thus, $Tg_qX \in T_pU$.

Corollary 3.3. *If $p \in A$ is a critical point of f, then Tf_p leaves T_pA invariant.*

Proof. This follows immediately from propositions 3.1 and 3.2.

Corollary 3.4. *If $p \in A$ is a critical point of f, then $(Tf_p)|_{T_pA} = T(f|_A)_p$.*

Proof. For any $X \in T_pA$ and $Y \in T_pM$ let $Y = Y_1 + Y_2$ where $Y_1 \in T_pA$ and $Y_2 \in T_pA^\perp$, the orthogonal complement of T_pA, then

$$
\begin{aligned}
<(Tf_p)|_{T_pA}X, Y> &= <Tf_pX, Y> = <Tf_pX, Y_1 + Y_2> \\
&= <Tf_pX, Y_1> = <T(f|_A)_pX, Y_1> \\
&= <T(f|_A)_pX, Y>
\end{aligned}
$$

and hence the result.

Lemma 3.5. *Let $p \in M$ be a critical point of f and let $k \in K_1$, then*:

1. *kp is a critical point of f and*
2. *$k_* \circ Tf_p = Tf_{kp} \circ k_*$*

Proof.

1. This is trivial by the K_1-invariance of f.
2. Since $f \circ k = f$ we have $\forall X, Y \in T_pM$, that

$$
\begin{aligned}
<Tf_pX, Y> &= Hf_p(X, Y) = H(f \circ k)_p(X, Y) \\
&= X_p\{Y(f \circ k)\} = X_p\{[(k_* Y)f] \circ k\} \\
&= (k_*X)_{kp}\{(k_* Y)f\} = Hf_{kp}(k_*X, k_* Y) \\
&= <Tf_{kp}k_*X, k_* Y> = <k_*^{-1} Tf_{kp}k_*X, Y>
\end{aligned}
$$

and the result follows.

REMARKS.

(i) This lemma is of particular importance when attention is focused on $K_p := \{k \in K_1 : k.p = p\}$. If we split T_pM into the irreducible components of the isotropy representation of K_p at p and use Schur's lemma as modified in (ii) below, then Tf_p is a multiple of the identity on each irreducible component. We will see that on the orthogonal complement of the K_1-orbit of p there are usually only two components.

(ii) Modified Schur's Lemma: If a group $K = K_1 \times \cdots \times K_s$ acts on a metric vector space $V = V_0 \oplus V_1 \oplus \cdots \oplus V_s$ as linear isometries in such a way that for each $1 \leq i \leq s$, K_i acts irreducibly on V_i and acts trivially on $V_j (0 \leq j \leq s,\ j \neq i)$, then any K-invariant symmetric endomorphism T of V leaves each V_i invariant $(0 \leq i \leq s)$ *and* $T|_{V_i} = \lambda_i \cdot (\mathrm{id}_{V_i})$ for $1 \leq i \leq s$.

Theorem 3.6. *If $f|_A$ is a Morse function in the usual sense, then f is a Morse function on M in the sense of Bott* [2] *and the critical submanifolds of f (all of which are nondegenerate) are the K_1-orbits of the critical points of $f|_A$.*

Proof. Let $x \in M$ be a critical point of f and let $C(x)$ be the connected component of the set of critical points of f through x. Then $C(x)$ is the K_1-orbit of $C(x) \cap A$. But, $C(x) \cap A$ is discrete since the critical points of f on A are the same as the critical points of $f|_A$ which are isolated. Hence, $C(x)$ is the K_1-orbit of a point in A and is, therefore, a submanifold of M. For the nondegeneracy; suppose $C(x) = K_1.p$ where $p \in A$. $C(x)$ is nondegenerate $\Leftrightarrow \ker Tf_{kp} = T_{kp}C(x)$ $\forall k \in K_1$ and (since $k_* \circ Tf_p = Tf_{kp} \circ k_*$ $\forall k \in K_1$) this is so $\Leftrightarrow \ker Tf_p = T_pC(x)$. Now, it is clear, by the K_1-invariance of f, that $Tf_p|_{T_pC(x)} = 0$ so for nondegeneracy of $C(x)$ we must show that $\ker Tf_p|_{T_pC^\perp(x)} = \{0\}$ where $T_pC^\perp(x)$ is the normal space to $C(x)$ at p. For the sake of clarity we break the proof into two cases:

1. *Suppose* $p = \exp H.o$ *where* $H \in \mathscr{A}$ *satisfies* $\alpha(H) \notin \mathbf{Z}\pi$ $\forall \alpha \in \mathfrak{R}_M$. Let $h = \exp H \in G$ so, by proposition 2.2, $Y^\alpha_p = Y^\alpha_{h.o} = \sin\alpha(H) h_* X^\alpha_o$ for every root $\alpha \in \mathfrak{R}_M$ and for every pair of conjugate vectors $X^\alpha \in \mathscr{M}_\alpha$ and $Y^\alpha \in \mathscr{K}_\alpha$. Therefore, by proposition 2.3 $\{H_p, Y^\alpha_p$: $H \in \mathscr{A}$, $Y^\alpha \in \mathscr{K}_\alpha$ and $\alpha \in \mathfrak{R}_M\}$ spans T_pM where $\{Y^\alpha_p$: $Y^\alpha \in \mathscr{K}_\alpha$ and $\alpha \in \mathfrak{R}_M\}$ spans $T_pC(x)$ and $T_pC^\perp(x) = T_pA$. However, by corollary 3.4, $(Tf_p)|_{T_pA} = T(f|_A)_p$ which has zero kernel since $f|_A$ has p as a nondegenerate critical point.
2. $\{$(1) *being a special case*$\}$ Here again suppose that $p = \exp H.o$ where $H \in \mathscr{A}$ and let $\mathfrak{R}^H_M := \{\alpha \in \mathfrak{R}_M\colon \alpha(H) \in \mathbf{Z}\pi\}$. Clearly the set

   $\mathscr{K}^H := \mathscr{K}_0 \oplus \sum_{\alpha \in \mathfrak{R}^H_M} \mathscr{K}_\alpha$ is a subalgebra of $\mathscr{K}$ since

   $$[\mathscr{K}_\alpha, \mathscr{K}_\beta] \subseteq \mathscr{K}_{\alpha+\beta} + \mathscr{K}_{\alpha-\beta} \qquad \forall \alpha, \beta \in \mathfrak{R}_M$$

   under the convention: $\mathscr{K}_{-\alpha} = \mathscr{K}_\alpha$, $\alpha \in \mathfrak{R}_M$. Denote by K^H the connected subgroup of K whose Lie algebra is $\mathscr{K}^H$ and observe that since $Y^\alpha_p = \sin\alpha(H) h_* X^\alpha_o = 0$ $\forall \alpha \in \mathfrak{R}^H_M$ it follows that K^H fixes p. Also,

   $$\mathscr{M}^H := \mathscr{A} \oplus \sum_{\alpha \in \mathfrak{R}^H_M} \mathscr{M}_\alpha$$

   is a Lie triple system in $\mathscr{M}$ for which there corresponds a totally geodesic submanifold $M^H \subseteq M$ so that

   (a) $A \subseteq M^H$ and

   (b) $\forall X \in T_pM^H$ $\exists k \in K^H$ such that $k_* X \in T_pA$.

   In this case, $C(x)$ (the K_1-orbit of $p = \exp H.o$) has $T_pC(x)$ spanned by $\{Y^\alpha_p$: $\alpha \notin \mathfrak{R}^H_M\}$ and the normal space is T_pM^H. So $C(x)$ is nondegenerate$\Leftrightarrow\ker(Tf_p|_{T_pM^H})=\{0\}$. Now, let $X\in\ker(Tf_p|_{T_pM^H})$ and choose $k\in K^H$ such that $k_*X\in T_pA$, then

$$0=Tf_pX=Tf_{k^{-1}p}X=k_*^{-1}\circ Tf_p\circ k_*X.$$

Therefore, $k_*X=0$ since $k_*X\in T_pA$ and $(Tf_p)|_{T_pA}=T(f|_A)_p$ has zero kernel. Hence, $X=0$ and $C(x)$ is nondegenerate.

For our convenience, we recall (cf.[7]) that in the decomposition $\mathscr{G}=\mathscr{M}\oplus\mathscr{K}$ where $\mathscr{M}$ is identified with T_oM as in (8) the geodesic in M with initial tangent vector $X\in\mathscr{M}\simeq T_oM$ is given by $\exp tX.o$.

The Morse function on $\mathbf{G_1}$: Let G_1 be the identity component of G and let

$$\rho\colon G_1\rightarrow U(V)$$

be an irreducible representation of G_1 on the complex vector space V. Mostly we choose ρ to be of lowest degree. We define f to be the real part of the character of ρ, that is

$$f\colon G_1\rightarrow\boldsymbol{R}\colon g\mapsto\operatorname{Re}(\operatorname{Tr}(\rho(g)))$$

where Tr is the trace function. Thus f is invariant under conjugation by G_1. Let $\mathscr{A}^{G_1}$ be a maximal abelian subalgebra in $\mathscr{G}$ (not just in $\mathscr{M}$). To obtain a formula for f on $A^{G_1}:=\{\exp H\colon H\in\mathscr{A}^{G_1}\}$ we consider the corresponding representation ρ_*, the derivative of ρ at the identity, on the Lie algebra of G_1. We obtain the weight space decomposition

$$V=\sum_{\lambda\in\Lambda}V_\lambda$$

(where Λ denotes the weights of ρ_*) such that for each $v\in V_\lambda$ and $H\in\mathscr{A}^{G_1}$ we have

$$\rho_*(H)v=\sqrt{-1}\lambda(H)v\ \text{ with }\ \lambda(H)\in\boldsymbol{R}.$$

Thus

$$f(\exp H)=\operatorname{Re}\operatorname{Tr}[\rho(\exp H)]=\operatorname{Re}\operatorname{Tr}[e^{\rho_*(H)}]$$

and if we calculate relative to a basis for V consisting of vectors from the various weight spaces V_λ we find

$$f(\exp H)=\sum_{\lambda\in\Lambda}\mu_\lambda\cos\lambda(H)$$

where μ_λ denotes the multiplicity of the weight λ.

Now to apply the results on symmetric spaces we may view the compact group G_1 as a symmetric space $(G_1 \times G_1)/I$ where $I := \{(g,g): g \in G_1\}$ (cf. [7], ChIV, §6). Indeed we have the bijection

$$\phi\colon (G_1 \times G_1)/I \to G_1\colon (g_1,g_2)I \mapsto g_1 g_2^{-1}$$

Note here that the action of an element $(g,g) \in I$ on the symmetric space $(G_1 \times G_1)/I$, that is $(g,g)\cdot(g_1,g_2)I = (gg_1,gg_2)I$, corresponds under the identification ϕ to conjugation by g in the group G_1. Thus the isotropy-invariant functions on G_1 viewed as a symmetric space (i.e.$(G_1 \times G_1)/I$) correspond to the $\mathrm{Ad}(G_1)$-invariant functions on G_1 viewed as a group. Furthermore, the involution corresponding to the symmetric space G_1 is

$$\sigma\colon G_1 \times G_1 \to G_1 \times G_1\colon (g_1,g_2) \mapsto (g_2,g_1)$$

from which we obtain the Cartan decomposition of $\mathscr{G} \times \mathscr{G}$ into $\mathscr{P} \oplus \mathscr{I}$ where

$$\mathscr{P} := \{(X,-X)\colon X \in \mathscr{G}\} \text{ and } \mathscr{I} := \{(X,X)\colon X \in \mathscr{G}\}$$

are the -1 and $+1$ eigenspaces of σ_*, respectively. As a maximal abelian subalgebra in $\mathscr{P}$ we may take

$$\mathscr{A} := \{(H,-H)\colon H \in \mathscr{A}^{G_1}\}$$

which generates a maximal torus of G_1 viewed as a symmetric space. Now, observe that if X^α and Y^α are conjugate relative to $\mathscr{A}^{G_1}$, from the group view point, then the pairs

$$\begin{gathered}(X^\alpha,-X^\alpha) \in \mathscr{P},\ (Y^\alpha,Y^\alpha) \in \mathscr{I}\\ (-Y^\alpha,Y^\alpha) \in \mathscr{P},\ (X^\alpha,X^\alpha) \in \mathscr{I}\end{gathered}$$

are conjugate relative to $\mathscr{A}$, from the symmetric space view point. Indeed

$$\begin{aligned}\mathrm{ad}(H,-H)(X^\alpha,-X^\alpha) &= \alpha(H)(Y^\alpha,Y^\alpha)\\ \mathrm{ad}(H,-H)(Y^\alpha,Y^\alpha) &= -\alpha(H)(X^\alpha,-X^\alpha)\\ \mathrm{ad}(H,-H)(-Y^\alpha,Y^\alpha) &= \alpha(H)(X^\alpha,X^\alpha)\\ \mathrm{ad}(H,-H)(X^\alpha,X^\alpha) &= -\alpha(H)(-Y^\alpha,Y^\alpha)\end{aligned}$$

In particular, we obtain a one-to-one correspondence between the roots $\{\alpha\}$ of G_1 viewed as a group and the roots $\{\tilde{\alpha}\}$ of G_1 viewed as a symmetric space via: $\tilde{\alpha}(H,-H)=\alpha(H)\ \forall H\in\mathscr{A}^{G_1}$. Finally, from the identification ϕ between G_1 as a symmetric space and G_1 as a group we have that

$$p:=(\exp_{\mathscr{G}\times\mathscr{G}}(H,-H))I=(h,h^{-1})I \overset{\text{identified}}{\leftrightarrow} h^2=\exp_{\mathscr{G}}2H=\phi(p)$$

Proposition 3.7. *Let $p=\exp H$ where $H\in\mathscr{A}^{G_1}$ and let H', $H''\in\mathscr{A}^{G_1}$, then*

$$<\operatorname{grad} f,H'>_p=-\sum_{\lambda\in\Lambda}\lambda(H')\mu_\lambda\sin\lambda(H)$$

and, furthermore, if p is a critical point of f, then

$$<Tf_pH',H''>_p=-\sum_{\lambda\in\Lambda}\lambda(H')\lambda(H'')\mu_\lambda\cos\lambda(H)$$

Proof. To establish the first formula observe that

$$\begin{aligned}
<\operatorname{grad} f,H'>_p&=H'_{\exp H}f\\
&=(d/dt)f(\exp tH'.\exp H)|_{t=0}\\
&=(d/dt)f(\exp(tH'+H))|_{t=0}\\
&=(d/dt)\sum_{\lambda\in\Lambda}\mu_\lambda\cos\lambda(tH'+H)|_{t=0}\\
&=-\sum_{\lambda\in\Lambda}\lambda(H')\mu_\lambda\sin\lambda(H)
\end{aligned}$$

For the second formula we have

$$\begin{aligned}
<Tf_pH',H''>_p&=H'_p(H''f)\\
&=(d/dt)[H''_{\exp tH'.p}f]|_{t=0}\\
&=(d/dt)[(\partial/\partial s)f(\exp sH''\exp tH'.p)|_{s=0}]|_{t=0}\\
&=(d/dt)[(\partial/\partial s)f(\exp\{sH''+tH'+H\})|_{s=0}]|_{t=0}\\
&=(d/dt)[(\partial/\partial s)\sum_{\lambda\in\Lambda}\mu_\lambda\cos\lambda\{sH''+tH'+H\})|_{s=0}]|_{t=0}\\
&=(d/dt)[-\sum_{\lambda\in\Lambda}\lambda(H'')\mu_\lambda\sin\lambda\{tH'+H\}]|_{t=0}\\
&=-\sum_{\lambda\in\Lambda}\lambda(H')\lambda(H'')\mu_\lambda\cos\lambda(H)
\end{aligned}$$

The Morse function on M: The map Q: $M \to G_1$: $x \mapsto s_x \circ s_o$, called the *quadratic representation* of M in G, is a totally geodesic immersion of M into G_1. This map is a homothety, not an isometry, and we note that $Q(x) \in G_1$ because of the connectedness of M. Let $Q(M)$ denote the image of M under Q. We now show that if f: $G_1 \to \mathbf{R}$ is a Morse function, then so also is $(f \circ Q)$: $M \to \mathbf{R}$. We remark that $(f \circ Q)$ is K_1-invariant because the action of K_1 on M corresponds to conjugation in G_1 under Q.

Proposition 3.8. grad$f$ *is tangent to* $Q(M)$ *and if* $f|_{AG_1}$ *has only nondegenerate critical points, then* $(f \circ Q)|_A$ *has only nondegenerate critical points, where* A *is a maximal torus in* M.

Proof. (Compare with [10]). The map

$$\tau:\ G_1 \to G_1:\ g \mapsto \sigma(g^{-1}) = s_o g^{-1} s_o$$

fixes $Q(M)$ pointwise because for any $x \in M$

$$\tau(Q(x)) = \sigma(s_o \circ s_x) = s_o \circ s_o \circ s_x \circ s_o = s_x \circ s_o = Q(x).$$

Also, $(\tau_*)_e = -(\sigma_*)_e$ is $+1$ on $\mathscr{M}$ and -1 on $\mathscr{K}$. Furthermore, $f \circ \tau = f$ since $\forall g \in G_1$, $\rho(g) \in U(V)$ so $\rho(g^{-1}) = (\rho(g))^*$, the adjoint of $\rho(g)$, and hence

$$\begin{aligned}(f \circ \tau)(g) &= f(s_o g^{-1} s_o) = \mathrm{ReTr}(\rho(s_o g^{-1} s_o)) \\ &= \mathrm{ReTr}(\rho(s_o g^{-1} s_o^{-1})) = \mathrm{ReTr}(\rho(g^{-1})) \\ &= \mathrm{ReTr}(\rho(g))^* = \mathrm{ReTr}\rho(g) = f(g).\end{aligned}$$

Now, τ: $G_1 \to G_1$ is an isometry such that $\tau^2 = id_{G_1}$ and, therefore, $(\tau_*)_p$ (the derivative of τ at $p \in Q(M)$) splits $T_p G_1$ into an orthogonal sum of its $+1$ and -1 eigenspaces (E_p^+, E_p^- respectively). The map: $Q(M) \to \boldsymbol{R}$: $p \mapsto \mathrm{Tr}(\tau_*)_p$ is continuous so $\dim E_p^+$ and $\dim E_p^-$ are constant functions of p. Therefore, $\dim E_p^+ = \dim E_e^+ = \dim \mathscr{M} = \dim M$. But, $Q(M)$ is pointwise fixed by τ so $T_p Q(M) \subseteq E_p^+$. Hence, $T_p Q(M) = E_p^+$ for dimension reasons and, therefore, $T_p Q(M)^\perp$ (the normal space to $Q(M)$ at p) is E_p^-. So $\forall X_p \in T_p Q(M)^\perp$ we have

$$\begin{aligned}<\mathrm{grad} f_p, X_p> &= X_p f = X_p(f \circ \tau) = (\tau_* X_p) f \\ &= -X_p f = -<\mathrm{grad} f_p, X_p>\end{aligned}$$

and, therefore, $<\mathrm{grad} f_p,\ X_p> = 0\ \ \forall X_p \in T_p Q(M)^\perp$. Hence, $\mathrm{grad} f_p \in$

$T_pQ(M)$.

To complete the proof we may assume that $Q(A)$ is the identity component of $A^{G_1}\cap Q(M)$. Thus, $\operatorname{grad} f$ is tangent to $Q(A)$ and, therefore,

$$\{\text{critical points of } f \text{ on } G_1\}\cap Q(A)=\{\text{critical points of } f|_{Q(A)}\}.$$

Furthermore, it follows from Proposition 3.2 that for any such critical point $p=Q(a)$, where $a\in A$, the map Tf_p leaves $T_pQ(A)$ invariant. Therefore, if $f|_{A^{G_1}}$ is nondegenetate at p, then so also is $f|_{Q(A)}$ and consequently a is a nondegenerate critical point of $(f\circ Q)|_A$.

4. Polar sets as critical submanifolds

Let γ: $[0,l]\to M$ be a geodesic in M which is parameterized by arc-length and satisfies $\gamma(0)=\gamma(l)=o$. The point $p=\gamma(l/2)$ is said to be *antipodal to o along* γ. For such a point p we note that $s_o.p=p$ and since the isometries $s_o\circ s_p\circ s_o$ and s_p have the same derivative at p (namely $-\mathrm{id}_{T_pM}$) we have that $s_o\circ s_p=s_p\circ s_o$.

To every p which is antipodal to o there is attached a pair of totally geodesic submanifolds $(M^+(p),M^-(p))$ of M (cf. [4]) defined by:

$$M^+(p):=F(s_o,p)=K_1.p \text{ and } M^-(p):=F(s_o\circ s_p,p)$$

where $\forall$ map φ: $M\to M$ we set $\mathrm{F}(\varphi,q):=$ the connected component of the fixed point set of φ through $q\in M$. The space $M^+(p)$ is called the *polar set at p* and in the special case where $M^+(p)=\{p\}$ we say that p is a *pole of M*. We remark that $T_pM=T_pM^+(p)\oplus T_pM^-(p)$ is an orthogonal direct sum. Now, after applying some $k\in K_1$ to p we may assume that p and indeed the entire geodesic γ (along which p is antipodal to o) are contained in the maximal torus A. We will assume this to be the case from now on.

Spaces of Classical Type: For the classical groups and symmetric spaces, we note that the results of T. Frankel [6] and S. Ramanujam [10] are obtained easily using our formulation. We outline the procedure in the case of $M=\mathrm{U}(n)$ with the invariant Riemannian metric $\langle X,Y\rangle=-\mathrm{Tr}(XY)$. As irreducible representation in this case we take the standard action of $\mathrm{U}(n)$ on $\boldsymbol{C}^n$. We choose a maximal torus $A^{\mathrm{U}(n)}$ of M consisting of diagonal matrices of the form

$$e^{\sqrt{-1}\lambda_1}\times\cdots\times e^{\sqrt{-1}\lambda_n}$$

where the real-valued functions $\lambda_1,\cdots,\lambda_n$ on the Lie algebra $\mathscr{A}^{\mathrm{U}(n)}$ of

$A^{U(n)}$ form the weights of this representation. If $\varepsilon_1,\cdots,\varepsilon_n$ denote the standard basis for $\mathscr{A}^{U(n)}$, then

$$\operatorname{grad} f_p = -\sum_{j=1}^{n} \sin(\lambda_j)\varepsilon_j \text{ where } p=\exp(\sum_{j=1}^{n} \lambda_j\varepsilon_j).$$

Therefore, p is a critical point of f if and only if $\sin\lambda_j=0$ for all $1\leq j\leq n$ and this is the case if and only if p is the identity or is antipodal to the identity along some closed geodesic contained in the maximal torus. Thus the critical submanifold of f through p is the polar set $M^+(p)$.

To determine nondegeneracy and to calculate the index we need only consider Tf_p on the space orthogonal to $M^+(p)$ at p, *that is, on* $T_pM^-(p)$. In this case, $M^-(p)=U(m)\times U(n-m)$ where $m=\#\{1\leq j\leq n$: λ_j is an even multiple of $\pi\}$ and its action on $T_pM^-(p)$ is the adjoint action.

Now, $U(m)$ is not simple, but is locally a product of a circle (which is the centre) and $SU(m)$. Accordingly, it follows from the modified Schur's lemma that Tf_p restricted to $T_pM^-(p)$ can have at most four distinct eigenspaces. However, it is easy to check that when $0<m<n$ there are only two since $Tf_p=-I$ on the $U(m)$ component and $+I$ on the $U(n-m)$ component. Otherwise there is only one eigenspace. Thus the index is $\dim U(m)$ and the critical submanifold $M^+(p)$ is the Grassmannian $U(n)/(U(m)\times U(n-m))$

REMARKS.

(i) When G_1 is simple, usually we will find that at a critical point $p\in G_1$ the isotropy subgroup (i.e. the centralizer of p) $G_p=M^-(p)$ is either simple, or splits into a local product of two simple groups, or a simple group and a circle which may be read from the Dynkin diagram by the method of Borel-De Siebenthal (see [1],or Wolf [12] Chapter 8, §10). The action of G_p on $T_pM^-(p)$ is the adjoint action so in the these cases it follows from the modified Schur's lemma that Tf_p restricted to $T_pM^-(p)$ will have either one or two eigenspaces which coincide with this splitting. Thus the negative eigenspace is either empty, or is equal to $T_pM^-(p)$ or else equals one of the factors.

(ii) In view of Proposition 3.8 this splitting is preserved when we pull back to any symmetric space by the quadratic representation of M in G_1.

NOTATION. To see more clearly why this Borel-De Siebenthal type splitting comes about for conjugation-invariant functions we fix the following notation. G will be a compact, connected, simple and semi-simple Lie group of rank r which has maximal torus A^G with Lie algebra $\mathscr{A}^G$ and simple roots $\alpha_1,\cdots,\alpha_r$. For each $1\leq i\leq r$, we define:

(i) $H_i\in\mathscr{A}^G$ by the condition that $\alpha_j(H_i)=2\pi\delta_{ij}$ for all $1\leq j\leq r$.
(ii) $A_i\in\mathscr{A}^G$ by the condition that $\langle A_i,H\rangle=\alpha_i(H)$ for all $H\in\mathscr{A}^G$, where $\langle,\rangle$ is the metric obtained from the Killing form.
(iii) s_i is the simple reflection from the Weyl group corresponding to the simple root α_i

Theorem 4.1. *Let f: $G\rightarrow\boldsymbol{R}$ be a smooth function which is invariant under conjugation. If $t\in\boldsymbol{R}$ and if $p=\exp(tH_i)$ is a critical point of f, then*

$$\{H_i,A_1,\cdots,A_{i-1},\ A_{i+1},\cdots,A_r\}$$

form a basis for T_pA^G consisting of eigenvectors of Tf_p.

Proof. If $j\neq i$, then using Lemma 3.5, we have

$$\begin{aligned} s_jTf_{\exp(tH_i)}A_j &= Tf_{\exp(ts_jH_i)}s_jA_j \\ &= Tf_{\exp(tH_i)}(-A_j). \end{aligned}$$

That is, $s_j(Tf_pA_j)=-Tf_pA_j$ and, therefore, $Tf_pA_j=b_jA_j$ for some $b_j\in\boldsymbol{R}$. This shows that $\{A_1,\cdots,A_{i-1},\ A_{i+1},\cdots,A_r\}$ is a set of eigenvectors of Tf_p and, in particular, its linear span is stabilized by Tf_p. Accordingly, its orthogonal complement (in $\mathscr{A}^G$) is also stabilized by Tf_p, since Tf_p is self-adjoint. But this orthogonal complement is spanned by H_i, so that H_i is also an eigenvector.

REMARKS.

(i) If one deletes the i^{th}-vertex from the Dynkin diagram of G, then there are at most three connected components in what remains. Since the A_j's corresponding to adjacent nodes of the Dynkin diagram are pairwise non-orthogonal it follows that those A_j's corresponding to any one of the connected components all must lie in the same eigenspace. Furthermore, if $t=1/n_i$ where n_i is prime, then H_i must also lie in one of these eigenspaces, because the action of each component in the Borel-De Siebenthal split is irreducible. We note also that there is an obvious generalization of the above theorem to the case where $p=\exp(tH_i+sH_j)$.
(ii) The Borel-De Siebenthal split also holds in the case where n_i is not prime: Let the Lie algebra $\mathscr{G}_p$ of the centralizer of $p\in A^G$

split into $\mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_s$ ($\mathscr{G}_0$ the centre, $\mathscr{G}_i$ simple ideals) and let $\mathscr{A}^G = \mathscr{A}_0 \oplus \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_s$ where $\mathscr{A}_j = \mathscr{G}_j \cap \mathscr{A}^G$ $0 \le j \le s$, be the corresponding decomposition of $\mathscr{A}^G$. Suppose $\mathscr{G}$ is simple with simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ and with the highest root $\tilde{\alpha} = \Sigma_i n_i \alpha_i$. Set $\alpha_0 = -\tilde{\alpha}$ and let $p = \exp(tH_i/n_i)$ where $0 < t \le 1$.

(a) Case $0 < t < 1$: $\dim \mathscr{G}_0 = 1$. Decompose $\Pi - \{\alpha_i\}$ into components:

$$\Pi - \{\alpha_i\} = \Pi_1 \cup \cdots \cup \Pi_s.$$

Then Π_j is a fundamental system for $\mathscr{G}_j$, $\mathscr{A}_j = \mathbf{R}\Pi_j$ and $\mathscr{A}_0 = \mathscr{G}_0 = \mathbf{R}H_i$.

(b) Case $t = 1$: dim $\mathscr{G}_0 = 0$. Decompose $(\Pi - \{\alpha_i\}) \cup \{\alpha_0\}$ into components:

$$(\Pi - \{\alpha_i\}) \cup \{\alpha_0\} = \Pi_1 \cup \cdots \cup \Pi_s.$$

Then Π_j is a fundamental system for $\mathscr{G}_j$ and $\mathscr{A}_j = \mathbf{R}\Pi_j$. In particular, $\mathscr{A}_1 = \mathbf{R}\Pi_1' + \mathbf{R}H_i$ if $\Pi_1 = \Pi_1' \cup \{\alpha_0\}$ contains α_0.

We note that the above also implies Theorem 4.1 immediately.

Spaces of Exceptional Type: We now apply our methods to the groups and symmetric spaces of exceptional type. We also point out the connection between our approach and the Killing vector field approach of R. Hermann [8].

The Group F_4: We will follow the notation for the roots of F_4 as given in Cornwell [5]. If $\alpha_1, \cdots, \alpha_4$ denote the simple roots then the highest root

$$\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

Let ρ be the irreducible representation of degree 26 of F_4 with highest weight the short root $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. The weights then are the 24 short roots (each of multiplicity one) together with the zero weight which has multiplicity two. Motivated by the expression for the long roots of F_4 as given in Bourbaki [3] we use the polarization forms $e_1, \cdots, e_4$ which are related to the simple roots by:

$$\begin{aligned} \alpha_1 &= e_1 - e_2 - e_3 - e_4 \\ \alpha_2 &= 2e_4 \\ \alpha_3 &= e_3 - e_4 \\ \alpha_4 &= e_2 - e_3 \end{aligned}$$

The nonzero weights are now expressed as $\pm(e_i \pm e_j)$ where $1 \leq i < j \leq 4$. Let $\theta_1, \cdots, \theta_4$ denote the coordinates on $\mathscr{A}^{G_1}$ with respect to the basis which is dual to $e_1, \cdots, e_4$. Then for $H \in \mathscr{A}^{G_1}$ our Morse function has the form

$$\begin{aligned} f(\exp H) &= 2 + \sum_{\lambda \neq 0} \cos \lambda(H) \\ &= 2 + 2 \sum_{1 \leq i < j \leq 4} \cos(\theta_i + \theta_j) + \cos(\theta_i - \theta_j) \\ &= 2 + 4 \sum_{1 \leq i < j \leq 4} \cos \theta_i \cos \theta_j \end{aligned}$$

and the equations for a critical point on $\mathscr{A}^{G_1}$ are

$$-4 \sin \theta_i \left(\sum_{j \neq i} \cos \theta_j \right) = 0 \qquad 1 \leq i \leq 4.$$

Since every element of G_1 is conjugate to $\exp H$ for some element H in the fundamental simplex

$$S = \{ H \in \mathscr{A}^{G_1} : \tilde{\alpha}(H) \leq 2\pi \text{ and } \alpha_i(H) \geq 0 \ \forall \ 1 \leq i \leq 4 \}$$

we seek only those solutions which lie in S. To this end we note that

$$\begin{aligned} \alpha_1(H) &= \theta_1 - \theta_2 - \theta_3 - \theta_4 \\ \alpha_2(H) &= 2\theta_4 \\ \alpha_3(H) &= \theta_3 - \theta_4 \\ \alpha_4(H) &= \theta_2 - \theta_3 \end{aligned}$$

and that the nonzero vertices of S are the vectors H_i/n_i where $\alpha_i(H_j) = 2\pi\delta_{ij}$ and $\tilde{\alpha} = \sum_i n_i \alpha_i$.

The solutions of the above equations are easily obtained, and furthermore we can use Proposition 3.7 in accordance with Theorem 4.1 to determine the eigenspaces and corresponding eigenvalues. We now list these solutions together with the eigenspaces.

(A) $\sin \theta_i = 0$ for all $1 \leq i \leq 4$ and the only solutions in the fundamental simplex are:

   (i) $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$ so that $\alpha_i(H) = 0$ for all $1 \leq i \leq 4$ which corresponds to $p = e =$ the identity. $Tf_e = (-1/3)I$ where I is the identity map.

(ii) $\theta_1=\pi$ and $\theta_2=\theta_3=\theta_4=0$ so that $\alpha_1(H)=\pi$ and $\alpha_j(H)=0$ when $j\neq 1$ which corresponds to the $M^+(p)$ at $p=\exp H$ for the vertex $H=H_1/2$. $\{A_2,A_3,A_4\}$ corresponds to the eigenvalue $(-1/9)$ and H_1 has eigenvalue $(1/3)$.

(iii) $\theta_1=\theta_2=\pi$ and $\theta_3=\theta_4=0$, for which $\alpha_4(H)=\pi$ and $\alpha_j(H)=0$ when $j\neq 4$. This corresponds to the $M^+(p)$ at the vertex $H=H_4/2$ and $Tf_p=(1/9)I$.

(B) $\sin\theta_1=0$ and $\sin\theta_j\neq 0$ for $j\neq 1$: Here there is only one solution in the fundamental simplex, which is

$$\theta_1=\pi, \text{ and } \theta_2=\theta_3=\theta_4=\pi/3.$$

Thus, $\alpha_2(H)=2\pi/3$ and $\alpha_j(H)=0$ when $j\neq 2$ which corresponds to the vertex $H=H_2/3$. $\{A_1,H_2\}$ corresponds to the eigenvalue $(1/6)$ while $\{A_3,A_4\}$ corresponds to the eigenvalue $(-1/12)$.

(C) $\sin\theta_1=\sin\theta_4=0$ and $\sin\theta_j\neq 0$ when $j\in\{2,3\}$, we find:

$$\theta_1=\pi,\ \theta_2=\theta_3=\pi/2, \text{ and } \theta_4=0.$$

Thus, $\alpha_3(H)=\pi/2$ and $\alpha_j(H)=0$ when $j\neq 3$ which corresponds to the vertex $H=H_3/4$. $\{A_1,A_2,H_3\}$ corresponds to the eigenvalue $(1/9)$ and A_4 has eigenvalue $(-1/9)$.

Thus we have shown that our function f is a Morse-Bott function whose critical submanifolds include all the polar sets. Furthermore, it seems to be the correct generalization of the trace function in the classical cases, since for these spaces all the n_i's are either 1 or 2 so that the orbit of the exponential of a vertex is an M^+. We remark also that, the exponential of a vertex of S corresponding to an n_i, where n_i is a prime, is well understood as having centralizer which is a maximal subgroup of maximal rank, see Borel-De Siebenthal [1].

The Group E_6: In what follows E_6 will always denote the simply-connected group of type E_6. Again we follow the notation in Cornwell so that if $\alpha_1,\cdots,\alpha_6$ denote the simple roots then the highest root is

$$\tilde{\alpha}=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6.$$

We take ρ to be the irreducible representation with highest weight $(2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6)/3$. The degree of this representation (which is lowest) is 27 and the weights, all of which have multiplicity one, can be calculated using Freudenthal's recursion formula for the weights and their multiplicities. We used the computer to carry out these calculations. The weights are as follows:

$$
\begin{array}{ll}
\lambda_1 = \{2,4,6,5,4,3\} & \lambda_{15} = \{-1,-2,0,2,1,0\} \\
\lambda_2 = \{2,4,6,5,1,3\} & \lambda_{16} = \{-1,1,0,-1,-2,0\} \\
\lambda_3 = \{2,4,6,2,1,3\} & \lambda_{17} = \{-1,-2,0,-1,1,0\} \\
\lambda_4 = \{2,4,3,2,1,3\} & \lambda_{18} = \{-1,-2,0,-1,-2,0\} \\
\lambda_5 = \{2,4,3,2,1,0\} & \lambda_{19} = \{-1,-2,-3,-1,1,0\} \\
\lambda_6 = \{2,1,3,2,1,3\} & \lambda_{20} = \{-1,-2,-3,-1,1,-3\} \\
\lambda_7 = \{2,1,3,2,1,0\} & \lambda_{21} = \{-1,-2,-3,-1,-2,0\} \\
\lambda_8 = \{-1,1,3,2,1,3\} & \lambda_{22} = \{-1,-2,-3,-1,-2,-3\} \\
\lambda_9 = \{2,1,0,2,1,0\} & \lambda_{23} = \{-1,-2,-3,-4,-2,0\} \\
\lambda_{10} = \{-1,1,3,2,1,0\} & \lambda_{24} = \{-1,-2,-3,-4,-2,-3\} \\
\lambda_{11} = \{2,1,0,-1,1,0\} & \lambda_{25} = \{-1,-2,-6,-4,-2,-3\} \\
\lambda_{12} = \{-1,1,0,2,1,0\} & \lambda_{26} = \{-1,-5,-6,-4,-2,-3\} \\
\lambda_{13} = \{2,1,0,-1,-2,0\} & \lambda_{27} = \{-4,-5,-6,-4,-2,-3\} \\
\lambda_{14} = \{-1,1,0,-1,1,0\} &
\end{array}
$$

where $\{m_1,\cdots,m_6\}$ denotes the weight $(\Sigma_i m_i \alpha_i)/3$

We take the (orthogonal) polarization forms $e_1,\cdots,e_6$ which are related to the simple roots by:

$$
\begin{aligned}
\alpha_1 &= (3e_1 - e_2 - e_3 - e_4 - e_5 - e_6)/2 \\
\alpha_2 &= e_5 + e_6 \\
\alpha_3 &= e_4 - e_5 \\
\alpha_4 &= e_3 - e_4 \\
\alpha_5 &= e_2 - e_3 \\
\alpha_6 &= e_5 - e_6
\end{aligned}
\tag{9}
$$

and we let $\theta_1,\cdots,\theta_6$ denote the coordinates on the maximal abelian subalgebra $\mathscr{A}^{G_1}$ with respect to the basis which is dual to $e_1,\cdots,e_6$. Relative to $\theta_1,\cdots,\theta_6$ the highest root $\tilde{\alpha}$ evaluates as

$$\tilde{\alpha} = (3\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 - \theta_6)/2$$

and among those inequalities, obtained from (9), which are used to determine the fundamental simplex we draw particular attention to the following:

$$0 \le \theta_5 \le \theta_4 \le \theta_3 \le \theta_2 \le 4\pi$$
$$-\pi/2 \le \theta_6 \le \pi/2$$

$$|\theta_6| \leq \theta_5 \leq \pi$$

and

$$0 \leq (3\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 - \theta_6)/2 \leq 2\pi.$$

Using these coordinates our Morse function takes the form

$$\begin{aligned} f(\exp H) &= \sum_{\lambda} \cos \lambda(H) \\ &= 16 \prod_{i=1}^{6} \cos\left(\frac{\theta_i}{2}\right) - 16 \prod_{i=1}^{6} \sin\left(\frac{\theta_i}{2}\right) + \\ &\quad \cos 2\theta_1 + 2(\cos\theta_1) \sum_{j=2}^{6} \cos\theta_j. \end{aligned}$$

If we set

$$c_i = \cos\left(\frac{\theta_i}{2}\right) \quad \text{and} \quad s_i = \sin\left(\frac{\theta_i}{2}\right) \ \forall\ 1 \leq i \leq 6$$

then the critical points on $\mathscr{A}^{G_1}$ are determined using the equations $\partial f/\partial\theta_i = 0$, for $1 \leq i \leq 6$, from which we obtain:

$$2[s_1c_2c_3c_4c_5c_6 + c_1s_2s_3s_4s_5s_6] + 2s_1c_1(c_1^2 - s_1^2) + s_1c_1 \sum_{j=2}^{6} (c_j^2 - s_j^2) = 0 \ \text{ for } i = 1 \tag{10}$$

and

$$2[s_ic_1 \cdots \hat{c}_i \cdots c_6 + c_is_1 \cdots \hat{s}_i \cdots s_6] + s_ic_i(c_1^2 - s_1^2) = 0 \ \text{ for } \ 2 \leq i \leq 6 \tag{11}$$

where the hat (^) in the second equation denotes that the term is omitted.

REMARK. It follows from equation (11) that if $(c_1^2 - s_1^2) \neq 0$, then there are at most two distinct s_j^2's (or c_j^2's) for all $2 \leq j \leq 6$. Since these equations are invariant under any permutation of the subscripts $\{2, \cdots, 6\}$ we will assume (after a suitable permutation) that if there are two distinct s_j^2's where $2 \leq j \leq 6$, then $s_2^2 \neq s_3^2$. Furthermore, when $s_2^2 \neq s_3^2$, then the following auxiliary equations are obtained from (11):

$$2c_1c_2c_3c_4c_5c_6 + (c_2c_3)^2(c_1^2 - s_1^2) = 0 \tag{12}$$

$$2s_1s_2s_3s_4s_5s_6 + (s_2s_3)^2(c_1^2 - s_1^2) = 0 \tag{13}$$

We fix the notation

$$D=(c_1^2-s_1^2),\qquad E=(c_2^2-s_2^2),\qquad F=(c_3^2-s_3^2)$$

and draw attention to the identities

$$2c_1^2=(1+D)\quad\text{and}\quad 2s_1^2=(1-D)$$

with similar ones for c_2^2, c_3^2, s_2^2 and s_3^2. We keep in mind also that $s_i^2 \leq 1$ and $c_i^2 \leq 1$ for all $1 \leq i \leq 6$. To solve the above equations we divide our computations into four main cases.

Case 1, $D=0$: In this case $s_1^2=c_1^2=1/2$ and from equations (10) and (11) we obtain

$$4[c_1\cdots c_6+s_1\cdots s_6]+\sum_{j=2}^{6}(c_j^2-s_j^2)=0$$

and

$$s_i^2 c_1\cdots c_6+c_i^2 s_1\cdots s_6=0,\ 2\leq i\leq 6.$$

(A) If $s_j^2 \neq 0$ for all $2 \leq j \leq 6$, then it follows from the second equation that $s_i^2=s_j^2$ for all $2\leq i\leq j\leq 6$. The only solution in this case is $s_i^2=c_i^2$ for all $1\leq i\leq 6$. Thus, $\cos\theta_i=0$ and $\theta_i \in (\text{odd}\,\boldsymbol{Z})(\pi/2)$. Also, from the above equations we have

$$c_1\cdots c_6=-s_1\cdots s_6$$

and the only solution in the fundamental simplex is

$$\theta_1=\theta_2=\cdots=\theta_5=\pi/2 \text{ and } \theta_6=-\pi/2$$

so that

$$\alpha_1(H)=\alpha_2(H)=\cdots=\alpha_5(H)=0 \text{ and } \alpha_6(H)=\pi.$$

This is the M^+ at $H=H_6/2$. $\{A_1,A_2,A_3,A_4,A_5\}$ corresponds to the eigenvalue $(-1/12)$ and H_6 has eigenvalue $(1/4)$.

(B) If there exists $j\in\{2,\cdots,6\}$ such that $s_j^2=0$, then it is easy to check that in this case we must have: exactly two $s_j^2=0$ and exactly two $s_j^2=1$ for $2\leq j\leq 6$ and the remaining $s_i^2=1/2$ for $2\leq i\leq 6$. In this case there is no solution in the fundamental simplex.

Case 2, $D\in\{\pm 1\}$:

(A) $D=1$: Now $s_1=0$, $c_1^2=1$ and from equation (10) we find

$$s_2 s_3 \cdots s_6 = 0.$$

Therefore, at least one $s_j = 0$ for $2 \leq j \leq 6$, so that (after a permutation) we may take $s_2 = 0$.

(i) All $s_i = 0$ for $1 \leq i \leq 6$: In this case $\cos\theta_i = 1$, and $\theta_i \in (\text{even } \boldsymbol{Z})\pi$ for all $1 \leq i \leq 6$. The only solution in the fundamental simplex is

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0.$$

Thus

$$\alpha_1(H) = \alpha_2(H) = \alpha_3(H) = \alpha_4(H) = \alpha_5(H) = \alpha_6(H) = 0$$

and $p = \mathrm{e} =$ the identity. $Tf_e = (-1/4)I$ where I is the identity map.

(ii) At least one $s_j \neq 0$ for $3 \leq j \leq 6$: After a permutation we may take $s_3 \neq 0$ and since $D \neq 0$ we have $s_j^2 \in \{s_2^2, s_3^2\}$ for all $4 \leq j \leq 6$. Let

$$m = \#\{j \in \boldsymbol{N}\colon 4 \leq j \leq 6 \text{ and } s_j^2 = s_3^2\}.$$

Putting $i = 3$ in equation (11), it follows that $2c_3^m \pm c_3 = 0$ and the only solutions possible are when $1 \leq m \leq 3$. However, one can check that none of the resulting solutions lie in the fundamental simplex. This is seen easily, because the condition $\alpha_1 \geq 0$ together with $s_1 = 0$, in this case, implies that $\theta_1 = 2n\pi$ for some $n \in \boldsymbol{N}$ and hence $\tilde{\alpha} > 2\pi$.

(B) $D = -1$: Now $s_1^2 = 1$, $c_1 = 0$ and the argument proceeds as in (A) except that the roles of the c's and s's are interchanged. The solutions are:

(i) All $c_i = 0$ for $1 \leq i \leq 6$: in this case all $\cos\theta_i = -1$ and $\theta_i \in (\text{odd } \boldsymbol{Z})\pi$ for $1 \leq i \leq 6$. In particular, $\theta_6 \notin [-\pi/2, \pi/2]$, so none of these solutions lie in the fundamental simplex.

(ii) With $c_2 = 0$ and $2s_3^m \pm s_3 = 0$, where m is as described in part (A), we list the solutions:

	$\theta_1,\theta_2 \in (\text{odd } \boldsymbol{Z})\pi$
$m=1$	$s_3=s_4=0$ and $c_5=c_6=0$ so that $\theta_3,\theta_4 \in (\text{even } \boldsymbol{Z})\pi$ and $\theta_5,\theta_6 \in (\text{odd } \boldsymbol{Z})\pi$
$m=2$ (a)	$s_3=s_4=s_5=0$ and $c_6=0$ so that $\theta_3,\theta_4,\theta_5 \in (\text{even } \boldsymbol{Z})\pi$ and $\theta_6 \in (\text{odd } \boldsymbol{Z})\pi$
$m=2$ (b)	$s_3^2=s_4^2=s_5^2=1/4$ and $c_6=0$ so that $\theta_3,\theta_4,\theta_5 \in \{\frac{\pi}{3},\frac{5\pi}{3}\}+(\text{even } \boldsymbol{Z})\pi$ and $\theta_6 \in (\text{odd } \boldsymbol{Z})\pi$
$m=3$ (a)	$s_3=s_4=s_5=s_6=0$ so that $\theta_3,\theta_4,\theta_5,\theta_6 \in (\text{even } \boldsymbol{Z})\pi$
$m=3$ (b)	$s_3^2=s_4^2=s_5^2=s_6^2=1/2$ so that $\theta_3,\theta_4,\theta_5,\theta_6 \in (\text{odd}\boldsymbol{Z})(\pi/2)$

REMARK: There is a parity condition in the above list: $s_3=s_1s_2s_4s_5s_6$ and all solutions are obtained by applying a permutation of $\{2,3,4,5,6\}$ to the list and the parity condition.

The only solution from this list which lies in the fundamental simplex is in the case [$m=3$ (a)], when

$$\theta_1=\theta_2=\pi \text{ and } \theta_3=\theta_4=\theta_5=\theta_6=0.$$

That is, $H=(H_1+H_5)/2$ which corresponds to an M^+ since $n_1=n_5=1$, see Nagano [9]. $\{A_2,A_3,A_4,A_6, (H_1+H_5)\}$ corresponds to the eigenvalue (1/12) and (H_1-H_5) has eigenvalue $(-10/72)$.

Case 3, $D\notin\{0,\pm1\}$ and there are two distinct s_j^2 for $2\le j\le 6$: After a permutation we suppose that $s_2^2\ne s_3^2$ and let

$$n=\#\{j\in \boldsymbol{N}\colon 4\le j\le 6 \text{ and } s_j^2=s_2^2\}$$

and put $m=3-n$. After a permutation of $\{2,3\}$ we may assume $n\in\{0,1\}$ and $m\in\{2,3\}$.

(A) $n=0$ and $m=3$: If $s_2=0$ (respectively $c_2=0$) then from equation (11) we find $D=1$ (respectively $D=-1$) which is not allowed in this case. If $s_3=0$ (respectively $c_3=0$) it follows from equations (10) and (11) that D satisfies the equation $2D^3+3D^2+4=0$

(respectively $2D^3-3D^2-4=0$) which has no solution when $D\in[-1,1]$. Thus we may assume $s_2c_2\neq 0$, $s_3c_3\neq 0$ and also, since $D\notin\{\pm 1\}$, that $s_1c_1\neq 0$. Now, $s_2^2\neq s_3^2$ so the auxiliary equations (12) and (13) are valid, and if we put $c_4c_5c_6=\varepsilon c_3^3$ and $s_4s_5s_6=\delta s_3^3$ where $\varepsilon^2=\delta^2=1$, then it follows from these auxiliary equations that

$$c_2D=-2\varepsilon c_1c_3^2 \text{ and } s_2D=-2\delta s_1s_3^2. \tag{14}$$

From these equations we obtain

$$D^2=F^2+2DF+1 \tag{15}$$

$$ED^2=DF^2+2F+D. \tag{16}$$

In this context, equation (10) may be put in the form

$$2[\varepsilon s_1c_2c_3^4+\delta c_1s_2s_3^4]+s_1c_1(2D+E+4F)=0,$$

which together with equations (14),(15) and (16) leads to

$$F=-D/(4D^2+1) \text{ and } 16D^6-6D^2-1=0.$$

The only allowable solution of this latter equation is when $D^2=(1+\sqrt{3})/4$ and the only solution in the fundamental simplex is when $\cos\theta_1=-\sqrt{(1+\sqrt{3})/4}$, $\cos\theta_2=E$ and $\cos\theta_j=F$ for all $3\leq j\leq 6$. At this critical point we find

$$\alpha_1=\alpha_3=\alpha_4=\alpha_6=0 \text{ and } 2\alpha_2+\alpha_5=2\pi.$$

In particular, $\tilde{\alpha}=2\pi$. There are three distinct eigenspaces corresponding to $\{A_1\}$, $\{H_2-2H_5\}$, which are the negative eigenspaces, and $\{A_3,A_4,A_6,(2H_2+H_5)\}$ which is the positive eigenspace.

(B) $n=1$ and $m=2$: Here we put $c_4c_5c_6=\varepsilon c_2c_3^2$ and $s_4s_5s_6=\delta s_2s_3^2$ where $\varepsilon^2=\delta^2=1$.

(i) $s_2=0$: in which case, either $c_3=0$ or $2\varepsilon c_1c_3+D=0$. When $c_3=0$ there is no solution in the fundamental simplex and when $2\varepsilon c_1c_3+D=0$ there is a solution in the fundamental simplex provided $D=F=-1/2$. For this solution we have

$$\theta_1=\theta_2=\theta_3=\theta_4=2\pi/3 \text{ and } \theta_5=\theta_6=0,$$

which corresponds to the vertex $H=H_3/3$. $\{A_1,A_2,A_4,A_5\}$

corresponds to the eigenvalue $(-1/16)$ while $\{H_3,A_6\}$ corresponds to the eigenvalue $(1/8)$.

(ii) $c_2=0$: in which case, either $s_3=0$ or $2\delta s_1 s_3+D=0$. When $s_3=0$ there is one solution in the fundamental simplex given by

$$\theta_1=2\pi/3,\ \theta_2=\theta_3=\pi \text{ and } \theta_4=\theta_5=\theta_6=0,$$

which corresponds to the vertex $H=H_4/2$. This critical submanifold is not an M^+ on the simply-connected E_6, but it projects to an M^+ on the bottom space, $Ad(E_6)$. $\{A_1, A_2,A_3,H_4,A_6\}$ corresponds to the eigenvalue $(1/24)$ and A_5 has eigenvalue $(-1/8)$.

If $2\delta s_1 s_3+D=0$ we obtain no solution in the fundamental simplex.

(iii) If $s_3=0$ but $s_2c_2\neq 0$ we obtain the solution $D=1-\sqrt{3}$ and $E=-(15-8\sqrt{3})/(3-\sqrt{3})$. After a permutation of $\{2,3,4,5,6\}$ we find only one solution in the fundamental simplex given by $\cos\theta_1=D$, $\cos\theta_2=\cos\theta_3=E$ and $\theta_4=\theta_5=\theta_6=0$, which translates to

$$\alpha_2=\alpha_3=\alpha_5=\alpha_6=0 \text{ and } \alpha_1+2\alpha_4=2\pi.$$

There are three distinct eigenspaces corresponding to $\{A_5\}$, $\{2H_1-H_4\}$, which are the negative eigenspaces, and $\{A_2,A_3,A_6, (H_1+2H_4)\}$ which is the positive eigenspace. The eigenvalues coincide with those in case 3, (A).

When $c_3=0$ and $s_2c_2\neq 0$ there is no solution in the fundamental simplex.

(iv) If $s_ic_i\neq 0$ for all $1\leq i\leq 3$, then $D=E=-F=\pm 1/\sqrt{2}$ for which there is no solution in the fundamental simplex.

Case 4, $D\notin\{0,\pm 1\}$ and $s_j^2=s_2^2$ for all $3\leq j\leq 6$:

(A) $E\in\{\pm 1\}$: If $E=1$, then $s_j=0$ for all $2\leq j\leq 6$ and we find that $D=-1/2$ and there are two solutions in the fundamental simplex given by:

(i) $\theta_1=2\pi/3$, $\theta_2=2\pi$ and $\theta_3=\theta_4=\theta_5=\theta_6=0$. This solution corresponds to a pole p at the exponential of the vertex $H=H_5$ and $Tf_p=(1/8)I$.

(ii) $\theta_1=4\pi/3$ and $\theta_2=\theta_3=\theta_4=\theta_5=\theta_6=0$. This also corresponds to a pole p (distinct from that in the previous case) at the exponential of the vertex $H=H_1$ and $Tf_p=(1/8)I$.

In this context we note that the centre of (the simply-connected) E_6 is $\{1, \exp H_1, \exp H_5\}$ so that the two poles above are identified with the identity in $Ad(E_6)$.

If $E=-1$, then $\cos\theta_j=-1$ for all $2\leq j\leq 6$ and, in particular, $\theta_6\in(\text{odd }\boldsymbol{Z})\pi$ so there is no solution in the fundamental simplex.

(B) $E\notin\{\pm 1\}$: In this case we find $E=0$ and $D=\pm\sqrt{3}/2$. The only solution in the fundamental simplex arises when $D=-\sqrt{3}/2$ and this solution is

$$\theta_1=5\pi/6, \text{ and } \theta_2=\theta_3=\theta_4=\theta_5=\theta_6=\pi/2,$$

which corresponds to the vertex $H=H_2/2$. This critical submanifold is not an M^+ on the simply-connected E_6, but it projects to one on $Ad(E_6)$. $\{H_2, A_3, A_4, A_5, A_6\}$ corresponds to the eigenvalue (1/24) and A_1 has eigenvalue $(-1/8)$.

The Symmetric Spaces; EII, EVI, EIX, EIII, EIV, EVII:

The first three spaces have root system of type F_4 and have a unique local isomorphism class. Motivated by our Morse function for the group F_4 we consider the following well defined function on the maximal torus of any of these three spaces

$$f(\exp H.o)=\sum_{\alpha=\text{short}}\cos 2\alpha(H).$$

That this function is well defined follows from the observation (see Proposition 2.2) that on the maximal torus it is the difference of the lengths of the Killing vector fields $\sum_{\alpha=\text{short}}X^\alpha$ and $\sum_{\alpha=\text{short}}Y^\alpha$. Since it is a finite Fourier series on the maximal torus, by Weyl group invariance of the function we may extend it to a K_1-invariant function on the whole space. R-Hermann [8] has considered Morse functions given by the lengths of Killing vector fields but these functions are not K_1-invariant and, in general, agree with ours on the maximal torus only. We note that the determination of the critical points of our K_1-invariant function for these 3 spaces has already been carried out since the effect of the 2 in the definition above is cancelled by the fact that the fundamental simplex for the symmetric spaces extends only to the hyperplane $\tilde{\alpha}=\pi$ and not $\tilde{\alpha}=2\pi$ as is the case for the group. Similarly, we note that the other spaces in the above list have a classical root system of rank ≤ 3 and are easily handled in this way.

References

[1] A. Borel and J. De Siebenthal: *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Commentarii Mathematici Helvetici **23** (1949), 200–221.

[2] R. Bott: *The stable homotopy of the classical groups*, Ann. of Math. **70** (1959), 313–337.

[3] N, Bourbaki: Elements de mathématique, Chapter I-Chapter VIII, Groupes et algèbres de Lie, Hermann, Paris, 1960–1975.

[4] B-Y. Chen and T. Nagano: *Totally geodesic submanifolds of symmetric spaces II*, Duke Math. J. **45** No 2 (1978), 405–425.

[5] J.F. Cornwell: Group theory in physics, Vol II, Academic Press, 1984.

[6] T. Frankel: *Critical submanifolds of the classical groups and Steifel manifolds*, Conference: Differential and combinatorial topology (in honour of M. Morse.), Princeton Univ. Press (1965), 37–53.

[7] S. Helgason: Differential geometry Lie groups and symmetric spaces, Academic Press, 1978.

[8] R. Hermann: *Totally geodesic orbits of groups of isometries*, Nederl. Akad. Wetensch. Proc. Ser. A **65** Indag. Math. **24** (1962), 291–298.

[9] T. Nagano: *The involutions of compact symmetric spaces*, Tokyo J. Math. **11** (1988), 57–79.

[10] S. Ramanujam: *Morse theory of certain symmetric spaces*, J. Diff. Geom. **3** (1969), 213–229.

[11] M. Takeuchi: *Cell decomposition and Morse equalities on certain symmetric spaces*, J. Fact. Sci. Univ. Tokyo Sect. I. **12** (1965), 81–192.

[12] J.A. Wolf: Spaces of constant curvature, Fifth edition, Publish or Perish, 1984.

J. Burns
Department of Mathematics
University College
Galway
Ireland

M. Clancy
School of Mathematical Sciences
Dublin City University
Dublin 9
Ireland