Masuda, M. and Schultz, R.
Osaka J. Math.
31 (1994), 387–402

GENERALIZED ROCHLIN INVARIANTS OF FIXED POINT SETS

MIKIYA MASUDA AND REINHARD SCHULTZ

(Received March 10, 1993)

Introduction

An action of a group G on a space X is said to be *semifree* if for each $x \in X$ either x is fixed under every element of G or else x is not fixed by any element of G except the identity. During the nineteen sixties and seventies it became apparent that the techniques of differential topology had numerous applications to differentiable actions of compact Lie groups (*cf.* [5], [7], [23]). In particular, these and previously developed techniques yielded considerable information on semifree differentiable actions of S^1 and S^3 on spheres. One result was a complete description of the homeomorphism types of the possible fixed point sets. Specifically, these are all closed manifolds with the same integral homology as a sphere of some appropriate dimension (see [12, Ch. V, §4]). On the other hand, questions about the diffeomorphism types of the fixed point sets are more difficult to answer. In this paper we shall prove a result (Theorem B below) that complements previous work on the smooth realization question; this is a special case of a more general result (Theorem A) relating the diffeomorphism type of the fixed point set to the diffeomorphism type of the ambient manifold. Although evidence suggests that an analog of Theorem B holds for semifree S^1-actions (see Proposition 3.2 and [27]), the proof of such an analog seems likely to require additional input. The proofs of Theorems A and B involve a higher dimensional analog of the well known *Rochlin invariant* for closed homology 3-spheres (*e.g.*, see [9], where it is called the μ-invariant).

1. Statement of main results

Let M be a closed oriented manifold such that $H_*(M;\mathbf{Z}_2) \approx H_*(S^n;\mathbf{Z}_2)$;

The first named author would like to thank Purdue University for its support and hospitality during a visit when part of this work was done, and in addition to this the second named author gratefully acknowledges partial support by National Science Foundation Grants DMS 86–02543, DMS89–X02622, and DMS91–02711.

in other words, M is a mod 2 homology n-sphere. We begin with elementary observation:

Fact. *If M^n is a mod 2 homology n-sphere and $n \geq 3$ then M has a unique spin structure. Furthermore, if $n = \dim M$ is* NOT *congruent to* 1 *or* 2 mod 8 *then M^n is a spin boundary.*

Recall that a *spin structure* on an oriented n-manifold is a lifting of the structure group of the oriented orthonormal frame bundle $\mathcal{F}_+(M)$ from $SO(n)$ to the double covering $Spin(n)$; strictly speaking this presupposes some riemannian metric on M but the isomorphism class of $\mathcal{F}_+(M)$ does not depend upon this choice. Spin structures exist if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes, and if they exist the different structures are classified by elements of $H^1(M;\mathbf{Z}_2)$. Since $H^1(M;\mathbf{Z}_2) = H^2(M;\mathbf{Z}_2) = 0$ if M is a mod 2 homology sphere of dimension at least 3, the existence and uniqueness of spin structures follows immediately.

The assertion in the second sentence follows by combining the results of [1, especially p. 15] on classification of homology spheres with standard results relating framed bordism and spin bordism (*e.g.*, see [29, Examples 7 and 9, pp. 44–47]).

DEFINITION. Let s be a fixed nonnegative integer, and let M be an $(8s+3)$-dimensional spin manifold (with a prescribed spin structure δ_M) such that (M,δ_M) is the boundary of (W,δ_W) for some compact $(8s+4)$-dimensional spin manifold W. The *generalized Rochlin invariant* $R(M,\delta_M) \in \mathbf{Z}_{16}$ is defined to be the signature of W modulo 16. If $s=0$ this reduces to the usual definition of the Rochlin invariant as in [9]; a well known theorem of V. Rochlin (see [22, p. 1]) states that the signature of a closed 4-dimensional spin manifold is divisible by 16, and this implies that the 3-manifold invariant does not depend upon the choice of (W,δ_W) (*cf.* [9] again). In order to justify the definition in higher dimensions we need the following generalization of Rochlin's Theorem due to S. Ochanine [21] (also see [22, appendice II]):

Ochanine's Theorem. *Let (N^{8s+4},δ_N) be a closed spin manifold. Then the signature of N is divisible by* 16.

The proof that $R(M,\delta_M)$ is independent of (W,δ_W) now follows from the same sort of argument used in [9] for the 3-dimensional case. ■

NOTATIONAL CONVENTIONS. (1) If (M,δ_M) is a closed spin manifold but the dimension of M is not congruent to 3 mod 8 we shall set $R(M,\delta_M)=0$.

(2) If M is an oriented manifold with $H^1(M;\mathbf{Z}_2)=0$ and $w_2(M)=0$ then there is a unique spin structure and therefore we shall usually suppress the variable δ_M.

By the preceding discussion every closed oriented mod 2 homology sphere M^n has a well-defined generalized Rochlin invariant $R(M)\in\mathbf{Z}_{16}$. If $n\equiv 3$ mod 8 then a standard plumbing construction yields an integral homology sphere Σ^{8s+3} such that $\Sigma=\partial P$ where P is parallelizable with signature equal to 8 (*cf.* [6, Thm. V.2.9, pp. 122–123]); if $n>3$ then one can in fact take Σ to be simply connected. Therefore the invariant R is a nontrivial invariant of integral homology n-spheres for every $n\equiv 3$ mod 8.

In order to apply the generalized Rochlin invariant we need some basic facts about semifree differentiable S^3-actions. If Σ is a closed oriented mod 2 homology sphere and Φ is a semifree differentiable S^3-action on Σ, then cohomological fixed point theory implies that the fixed point set F is also a mod 2 homology sphere; this follows from results on circle actions (*e.g.*, see [3, Thm. III.10.2, p. 159]) because the restricted action $\Phi|S^1$ is also semifree and its fixed point set is also F. Furthermore, local linearity theorems for differentiable actions [3, pp. 171, 308] show that $\dim\Sigma-\dim F$ is divisible by 4 and also present F as a smooth manifold such that the inclusion of F in Σ is a smooth embedding; the normal bundle ξ of this embedding has a quaternionic structure such that the restriction of Φ to some neighborhood of F is equivalent to fiberwise scalar multiplication on an invariant neighborhood of the zero section in ξ. Finally, an orientation of Σ canonically determines an orientation of F.

Our main theorem is the following:

Theorem A. *Let Σ be a closed oriented mod 2 homology sphere with a semifree differentiable S^3-action, and let F be the fixed point set of the action with smooth structure defined by local linearity. Then $R(\Sigma)=R(F)$.*

REMARKS

(A1) The analogous statement for semifree S^1-actions is false. Specifically, let Σ^{8s+3} be one of the homotopy spheres described above such that Σ bounds a parallelizable manifold of index 8 (hence $R(\Sigma^{8s+3})=8$ mod 16). Then by [27] there is a smooth semifree S^1-action on a homotopy $(8s+4t+1)$-sphere N_t with fixed point set Σ^{8s+3} for every $t>0$. For such examples $R(F)\neq 0$ by construction, but dimensional considerations imply that $R(N_t)=0$ *if* t is odd.

(A2) We have already noted that $\dim\Sigma\equiv\dim F$ mod 4. If the difference $\dim\Sigma-\dim F$ is congruent to 4 mod 8 then either $R(\Sigma)$ or $R(F)$ is zero by definition, and in these cases the theorem should be understood as stating that both generalized Rochlin invariants must be zero.

(A3) EXAMPLES. There is a standard construction for semifree S^3-actions on homotopy $(8s+3)$-spheres with generalized Rochlin invariant 8 (mod 16) using isolated singularities of weighted homogeneous polynomials. Specifically, such a group action is given by intersecting the affine variety associated to the equation

$$z_0^3+z_1^5+z_2^2+\cdots+z_{4s+2}^2=0$$

with the unit sphere and taking the restriction of the linear action on $\mathbf{C}^{4s+3}$ given by

$$(\mathbf{R}^3\oplus\mathbf{H}_0^t\oplus\mathbf{H}_1^{s-t})\otimes_{\boldsymbol{R}}\mathbf{C}$$

where S^3 acts trivially on $\mathbf{R}$ and $\mathbf{H}_0$ and by scalar multiplication on $\mathbf{H}_1$. The proof that the manifold in question is a homotopy sphere and the assertion regarding the generalized Rochlin invariant are standard facts in the theory of isolated hypersurface singularities (*cf.* [10]).

If we specialize to actions on integral homology spheres (so that F is also an integral homology sphere) then we obtain the following conclusion:

Theorem B. *Let* F^{8s+3} *be a closed smooth integral homology* $(8s+3)$-*sphere, and suppose that* S^3 *acts differentiably and semifreely on a closed integral homology* $(8k+7)$-*sphere with fixed point set* (*diffeomorphic to*) F. *Then* $R(F)=0$.

REMARKS

(B1) If $k=s$ then F bounds an integrally acyclic manifold (*cf.* the argument in [13]; if the homology $(8k+7)$-sphere is a homotopy sphere then the argument shows that F bounds a contractible manifold—on the other hand, if F is an integral homology sphere such that $\dim F\geq 4$, then F bounds a contractible manifold if and only if it bounds an integrally acyclic manifold by standard considerations as in [12, Ch. V]).

(B2) For all $k\geq s$ there are semifree differentiable S^1-actions on integral homology $(8k+7)$-spheres whose fixed point sets have generalized

Rochlin invariant equal to 8 mod 16 (see [27, Prop. 1.6] for the case $s=0$; extensions of this and the construction in Remark A3 yield the general case).

(B3) The examples in Remark A3 above show that $R(F)$ need not be trivial for differentiable semifree S^3-actions on integral homology $(8l+3)$-spheres.

(B4) Despite Remark A1, there are some results for differentiable semifree S^1-actions that are related to Theorems A and B. These are discussed in Section 3 below.

If we specialize further to homology 3-spheres then Theorem B and the observations of Remarks A3 and B1 yield the following result:

Theorem C. *Let F be a closed manifold that is an integral homology 3-sphere, and let q be a positive integer. Then F is the fixed point set of a semifree differentiable S^3-action on an integral homology $(4q+3)$-sphere if and only if F satisfies the condition below for the given value of q:*

(CASE $q=1$). *F bounds an integrally acyclic manifold.*

(CASE $q<1$ odd). $R(F)=0$.

(CASE q even). *No restriction on F.*

The necessity of the condition for $q=1$ is covered by Remark B1; conversely, if F bounds the acyclic manifold A, one can take the boundary of the action on $A\times D^4$ (with suitably rounded corners) given by the trivial action on the first coordinate and quaternionic multiplication on the second. When $q=2k+1\geq 3$ the necessity of the condition is a special case of Theorem B and sufficiency follows by a straightforward adaptation of the arguments proving [27, Thm. I and (1.4)] to semifree differentiable S^3-actions. In principle the case $q=2k$ is a direct analog of a result for semifree differentiable S^1-actions [5, Thm. 6.1, p. 36]. A crucial step in the proof of that result was to note that if F^{2n-1} is a homotopy sphere that bounds a parallelizable manifold, then $F\times \mathbf{C}P^{2k-1}$ is smoothly h-cobordant to $S^{2n-1}\times \mathbf{C}P^{2k-1}$. Similar considerations to those of [5, pp. 37–38] show that if F^{2n-1} is a closed smooth integral homology sphere that bounds a parallelizable manifold then $F\times \mathbf{H}P^{2k-1}$ is *homologically* smoothly h-cobordant to $S^{2n-1}\times \mathbf{H}P^{2k-1}$. Since the entire discussion of [5, §5] extends in a straightforward manner to semifree differentiable S^3-actions on homotopy spheres modulo minor changes (*e.g.*, $\mathbf{H}P$ instead of $\mathbf{C}P$, quaternionic vector bundles instead of complex vector bundles, homology spheres instead of homotopy spheres as fixed point sets, and homological h-cobordism instead of diffeomorphism or ordinary h-cobordism), one can construct a semifree differentiable S^3-action on a homotopy $(2n+8k-1)$-sphere with F as the fixed point set. The proof can then be completed by noting that every integral homology 3-sphere bounds a parallelizable manifold. ∎

COMPLEMENT TO THEOREM C. In the second and third cases the conditions are in fact equivalent to the existence of a semifree differentiable S^3-action on a *homotopy* $(4q+3)$-sphere whose fixed point set is diffeomorphic to F. There is also a corresponding, but slightly different, result if $q=1$; namely, F is the fixed point set of a semifree differentiable S^3-action on a homotopy 7-sphere if and only if F bounds a contractible manifold (*cf.* [13] again).

2. Proofs of the main results

We begin with some elementary properties of the generalized Rochlin invariant.

(2.1) (Product formula) *Let M^m be a closed spin manifold that bounds a compact $(m+1)$-dimensional spin manifold, and let N be a closed spin manifold. Then the generalized Rochlin invariant satisfies $R(M \times N)=R(M)$ Sign N.*

As usual, the signature of N is understood to be zero if N is not divisible by 4.

(2.2) *R is additive with respect to connected sum and disjoint union.*

(2.3) *Suppose M_1^m and M_2^m are spin manifolds that bound compact spin manifolds, and suppose further that M_1 and M_2 are spin cobordant by a compact spin manifold V^{m+1} such that $H_*(V,M_1;\mathbf{Q})\cong H_*(V,M_2;\mathbf{Q})=0$; then $R(M_1) = R(M_2)$.*

Formula 2.1 follows from the identities $\partial(W\times N)=\partial W\times N$ and $\mathrm{Sign}(W\times N)=\mathrm{Sign}(W)\ \mathrm{Sign}(N)$ which hold if N is a closed manifold. Formula 2.2 follows from the identities $\partial(W_1\# W_2)=\partial W_1\#\partial W_2$ and $\partial(W_1\amalg W_2)=\partial W_1\amalg\partial W_2$. Finally, to prove Formula 2.3 let $M_1=\partial W$; then $M_2=\partial(W\cup_h V)$ where h identifies ∂W with the appropriate piece of ∂V. Now $H^*(V,M_1;\mathbf{Q})\cong H_*(V,M_1;\ \mathbf{Q})=0$ implies that $H^*(W\cup_h V;\ \mathbf{Q})\cong H^*(W;\ \mathbf{Q})$ and $\mathrm{Sign}(W\cup_h V)=\mathrm{Sign}\, W$ (the latter requires a little diagram chasing). It follows that $R(M_1)=R(M_2)$.

Suppose now that we are given a semifree differentiable action on the homology sphere Σ^n and the fixed point set is F^m where $n-m=4q$. Since the conclusion of Theorem A reduces to $0=0$ if n and m are not congruent to 3 mod 4, we make the following

Default Hypothesis. *Unless stated otherwise, the dimensions of F and Σ are congruent to* 3 mod 4 *for the rest of this section.*

Since F is the fixed point set of a semifree S^3-action on Σ, it follows that F has a neighborhood that is equivariantly diffeomorphic to the total space of a quaternionic q-plane bundle ξ over F. If $\mathbf{H}P(\xi)$ is the associated $\mathbf{H}P^{q-1}$-bundle, then basic results on semifree actions imply the following statements:

(2.4) $\mathbf{H}P(\xi)$ *is $\mathbf{Z}_2$-homologically h-cobordant to $S^m\times\mathbf{H}P^{q-1}$ (i.e., there is a cobordism W between them so that the pairs $(W,\mathbf{H}P(\xi))$ and $(W,S^m\times\mathbf{H}P^{q-1})$ have vanishing mod 2 homology).*

(2.5) *The stabilization of the quaternionic vector bundle ξ in $KSp(F)\approx[F,BSp]$ has odd order, and the fiberwise 2-primary localizations $\mathbf{H}P_{(2)}(\xi)$ and $\mathbf{H}P_{(2)}(\xi\oplus\mathbf{H})$ are both fiber homotopically trivial.*

Statement 2.4 is a straightforward variant of [5, last three lines of p. 31] in which homotopy spheres are replaced by mod 2 homology spheres; this will be discussed further in the proof of Proposition 2.8. The second part of statement 2.5 follows from (2.4) and standard properties of fiberwise localization (when the fiber is simply connected), and the first part follows from the second and a 2-local analog of the results in [2, Lemma 11.3 and Cor. 11.4, pp. 23–24]. ■

In order to apply the generalized Rochlin invariant we need one additional property of ξ.

Proposition 2.6. *Let f_ξ: $F\to BSp_q$ be a classifying map for the quaternionic vector bundle ξ. Then (F,f_ξ) represents the trivial class in the 2-localized spin bordism group $\Omega_m^{Spin}(BSp_q)_{(2)}$.*

Proof. We have already noted that the image of f_ξ in $[F,BSp]_{(2)}\cong[F,BSp_{(2)}]$ is trivial. Therefore the stabilization of (F,f_ξ) in $\Omega_m^{Spin}(BSp_{(2)})_{(2)}$ $\Omega_m^{Spin}(BSp)_{(2)}$ lies in the image of the coefficient homomorphism $\Omega_m^{Spin}(\{\text{pt.}\})_{(2)}\to\Omega_m^{Spin}(BSp)_{(2)}$. On the other hand, results of V. Snaith on stable splittings of BSp imply that the stabilization map $\Omega_m^{Spin}(BSp_q)\to\Omega_m^{Spin}(BSp)$ is split injective (*e.g.*, see [28, Part I, § 4]) and therefore the image of (F,f_ξ) in $\Omega_m^{Spin}(BSp_q)_{(2)}$ also lies in the coefficient homomorphism. It follows that the bordism class only depends upon the spin bordism class of the oriented mod 2 homology sphere F. By the observations at the beginning of Section 1, the mod 2 homology sphere F bounds a spin manifold because we are assuming that $\dim F \equiv 3 \bmod 4$; therefore F determines the trivial class in $\Omega_m^{Spin}(\{pt\}.)$. ∎

By the preceding result there is an odd integer r such that a disjoint union of r copies of (F,f_ξ) is the boundary of some (W,f_ω); in other words, if $\amalg^r$ denotes the r-fold disjoint union of something with itself, then $\partial W = \amalg^r F$ and $f_\omega | \partial W = \amalg^r f_\xi$. Since the fiberwise 2-localizations $\mathbf{H}P_{(2)}(\xi)$ and $\mathbf{H}P_{(2)}(\xi \oplus \mathbf{H})$ are both fiber homotopically trivial, the fibrations $\mathbf{H}P_{(2)}(\omega)$ and $\mathbf{H}P_{(2)}(\omega \oplus \mathbf{H})$ over W can be extended to fibrations $\mathbf{E}_0$ and $\mathbf{E}_1$ over

$$W^* = W \cup_{\partial W} \amalg^r \operatorname{Cone}(F)$$

by taking the trivial fibration over each cone.

Although $\mathbf{E}_0$ and $\mathbf{E}_1$ are not necessarily manifolds, they do satisfy rational Poincaré duality with formal dimensions $m+4q$-3 and $m+4q+1$ respectively. Since $m \equiv 3 \bmod 4$ it follows that one can define signatures for each total space in the usual fashion.

Proposition 2.7. *If q is odd then* $\operatorname{Sign}\mathbf{E}_0 = \operatorname{Sign} W^* = \operatorname{Sign} W$ *and* $\operatorname{Sign}\mathbf{E}_1 = 0$. *If q is even then* $\operatorname{Sign}\mathbf{E}_1 = \operatorname{Sign} W^* = \operatorname{Sign} W$ *and* $\operatorname{Sign}\mathbf{E}_0 = 0$.

Proof. Since $\mathbf{H}P(\omega)$ and $\mathbf{H}P(\omega \oplus \mathbf{H})$ are induced from fibrations over the simply connected spaces BSp_q and BSp_{q+1} it follows that these two bundles are always orientable over a generalized cohomology theory. On the other hand, these bundles are the restrictions of the fibrations $\mathbf{E}_0$ and $\mathbf{E}_1$ to W, and by construction the induced homomorphism from $\pi_1(W)$ to $\pi_1(W^*)$ is onto. Therefore both $\mathbf{E}_0$ and $\mathbf{E}_1$ must be orientable fibrations, and consequently the methods of Chern-Hirzebruch-Serre [8] apply to show that $\operatorname{Sign}\mathbf{E}_0 = \operatorname{Sign} W^* \cdot \operatorname{Sign}\mathbf{H}P^{q-1}$ and $\operatorname{Sign}\mathbf{E}_1 = \operatorname{Sign} W^* \cdot \operatorname{Sign}\mathbf{H}P^q$. All the assertions except $\operatorname{Sign}\mathbf{W}^* = \operatorname{Sign} W$ follow from these and the elementary formula

$$\operatorname{Sign}(\mathbf{H}P^l) = \begin{cases} 1 \text{ if } l \text{ is even} \\ 0 \text{ if } l \text{ is odd.} \end{cases}$$

The identity $\operatorname{Sign} W^* = \operatorname{Sign} W$ follows directly from the splitting of W^* into W and a union of cones along the components of ∂W and the

Novikov additivity property of the signature [11] (the cones clearly have trivial signatures). ∎

There is one more property of $\mathbf{H}P(\xi \oplus \mathbf{H})$ that is important for our purposes.

Proposition 2.8. $\mathbf{H}P(\xi \oplus \mathbf{H})$ *is mod* 2 *homologically h-cobordant to* $S^m \times \mathbf{H}P^q \# \Sigma$.

Proof. This is a variant of an argument in [20]; one can also prove this by modifying the argument at the beginning of [25, §3] to deal with S^3-actions rather than S^1-actions. Consider the semifree differentiable S^3-action on $S^m \times \mathbf{H}P^q$ where the S^3-action on S^m is trivial and the action on $\mathbf{H}P^q$ is linear with $\mathbf{H}P^0 \amalg \mathbf{H}P^{q-1}$ as the fixed point set. Form a connected sum of $S^m \times \mathbf{H}P^q$ and Σ equivariantly at points in $S^m \times \mathbf{H}P^0$ and F. Let W be the complement of an invariant open tubular neighborhood of the fixed point set components $F(=S^m \times \mathbf{H}P^0 \# F)$ and $S^m \times \mathbf{H}P^{q-1}$in $S^m \times \mathbf{H}P^q \# \Sigma$. Note that W is a mod 2 homological h-cobordism between $S(\xi)$ and $S^m \times S^{4q-1}$.

Consider the balanced product $W \times_{S^3} D^4$ (with the corners rounded equivariantly). Its boundary contains

$$\partial W \times_{S^3} D^4 \supset S(\xi) \times_{S^3} (S^3 \times [\tfrac{1}{2},1]) = S(\xi) \times [\tfrac{1}{2},1].$$

Let X be the semifree differentiable S^3-manifold obtained from $W \times_{S^3} D^4$ and $D(\xi) \times [\frac{1}{2},1]$ by identifying along $S(\xi) \times [\frac{1}{2},1]$ equivariantly and adjusting the angles at $S(\xi) \times \{\frac{1}{2},1\}$ (also equivariantly). A direct and elementary analysis of the construction shows that the boundary of X is a disjoint union of $\mathbf{H}P(\xi \oplus \mathbf{H})$ and $S^m \times \mathbf{H}P^q \# \Sigma$ and also that the pair $(X, \mathbf{H}P(\xi \oplus \mathbf{H}))$ is $\mathbf{Z}_2$-acyclic. ∎

Proof of Theorem A concluded. The balance of the proof splits into four cases depending on whether $\dim F$ is congruent to 3 or 7 mod 8 and similarly for $\dim \Sigma$. If both dimensions are congruent to 7 mod 8 then $R(F) = R(\Sigma) = 0$ automatically and there is nothing to prove.

Suppose now that $\dim F \equiv 3 \bmod 8$ and $\dim \Sigma \equiv 7 \bmod 8$. Then $R(\Sigma) = 0$ and q is odd. By (2.3) and (2.4) we have $R(\mathbf{H}P(\xi)) = R(S^m \times \mathbf{H}P^{q-1}) = 0$; the latter holds since the signature of $D^{m+1} \times \mathbf{H}P^{q-1}$ is zero. Additivity of the generalized Rochlin invariant implies $R(F) = \frac{1}{r}\operatorname{Sign} W^*$ and from Proposition 2.7 we see that

$$\operatorname{Sign} W^* = \operatorname{Sign}(\mathbf{E}_0) = \operatorname{Sign}(\mathbf{H}P(\omega)) = r \cdot R(\mathbf{H}P(\xi))$$

(the second equation requires Novikov additivity of the signature and the elementary identity Sign(Cone(F) $\times$ anything) $=0$). Combining these, we see that $R(F)=0$.

Consider next the case where $\dim F \equiv 7 \bmod 8$ and $\dim \Sigma \equiv 3 \bmod 8$. Once again q is odd, but in this case $R(F)=0$ holds automatically. On the other hand, by Proposition 2.8 and additivity we know that

$$R(\mathbf{H}P(\xi \oplus \mathbf{H})) = R(S^m \times \mathbf{H}P^q \# \Sigma) = R(S^m \times \mathbf{H}P^q) + R(\Sigma) = R(\Sigma)$$

(using $\mathrm{Sign}(D^{m+1} \times \text{anything}) = 0$ for the final equation). On the other hand we also have

$$R(\mathbf{H}P(\xi \oplus \mathbf{H})) = \frac{1}{r}\ \mathrm{Sign}\mathbf{H}P(\omega \oplus \mathbf{H}) = \frac{1}{r}\ \mathrm{Sign}\mathbf{E}_1$$

and the last of these vanishes by Proposition 2.7. Therefore $R(\Sigma)=0$ also holds in this case.

Finally consider the case where $\dim F = \dim \Sigma \equiv 3 \bmod 8$. Then we have

$$R(F) = \frac{1}{r}\mathrm{Sign}\, W^* = \frac{1}{r}\mathrm{Sign}\,\mathbf{E}_1$$

by Proposition 2.7. But $\mathrm{Sign}\,\mathbf{E}_1 = \mathrm{Sign}\mathbf{H}P(\omega \oplus \mathbf{H})$ by Novikov additivity and the vanishing of $\mathrm{Sign}(\mathrm{Cone}(F) \times \mathbf{H}P^q)$, and therefore $\mathrm{Sign}\,\mathbf{E}_1 = r \cdot R(\mathbf{H}P(\xi \oplus \mathbf{H}))$. In other words we have $R(F) = R(\mathbf{H}P(\xi \oplus \mathbf{H}))$. We can now apply Proposition 2.8 to conclude that

$$R(\mathbf{H}P(\xi \oplus \mathbf{H})) = R(\Sigma \# S^m \times \mathbf{H}P^q) = R(\Sigma) + R(S^m \times \mathbf{H}P^q) = R(\Sigma)$$

in analogy with previous cases. Combining these, we conclude that $R(F)=R(\Sigma)$ in the final case. ■

3. Related results

Theorems A and B are complementary to several known restrictions on

(*a*) the fixed point sets of semifree differentiable S^1- and S^3-actions on integral homology spheres,

(*b*) the homology spheres that can support actions with fixed point sets of a given codimension.

In order to explain this we need to recall some background. For many purposes the most tractable integral homology spheres are the boundaries of contractible manifolds. In particular, if F^m bounds a contractible manifold and $k>0$, then there are semifree differentiable S^1- and S^3-actions on S^{m+2k} and S^{m+4k} respectively such that the fixed point sets are diffeomorphic to F^m (see [12, Ch. V]). In general there are obstructions to finding a contractible manifold K^{m+1} such that $\partial K = F$ and these are generally studied in two phases; namely, the obstruction to finding a parallelizable manifold with boundary F and, if F does bound a parallelizable manifold, the obstruction to making the parallelizable manifold contractible.

In [25] the obstruction to bounding a parallelizable manifold is studied for numerous cases of (*b*). On the other hand, if F is the fixed point set of a semifree differentiable S^1- or S^3-action on an integral homology sphere then some restrictions on the obstruction to F bounding a parallelizable manifold are given by the results of [24]. In particular, if $\dim F$ is odd then F must bound a spin manifold. One can combine the methods of [24] with computations of K. Knapp [15, Thm. 6, p.21] to obtain further restrictions. Although there are examples in [24] for which the fixed point set F of a semifree differentiable S^1-action does not bound a parallelizable manifold, if one strengthens (*a*) to ask which F can be realized as the the fixed point sets of semifree differentiable S^1- and S^3-actions on integral homology spheres *in infinitely many dimensions*, the results of [26] yield the following conclusion:

Proposition 3.1. *Let m be a fixed positive integer. Then there is an integer $N(m)$ such that if F is the fixed point set of a differentiable semifree S^1-action on an integral homology sphere of dimension $\geq N(m)$, then*

(i) *F bounds a contractible manifold if m* is even,

(ii) *F bounds a parallelizable manifold if m is odd.*

Sketch of proof. The obstruction group for determining if F bounds a parallelizable manifolds is finite [14]. For each prime p dividing the order of this group consider the restriction of the original semifree action to $\boldsymbol{Z}_p$. The results of [26] then yield an integer $N_p(m)$ such that if F is the fixed point set of a differentiable semifree S^1-action on an integral homology sphere of dimension $\geq N_p(m)$, then the p-primary component of the obstruction is trivial. If we take $N(m)$ to be the maximum of the numbers $N_p(m)$ over all p dividing the order of the obstruction group, then (ii) follows immediately. In fact the conclusion of (ii) also holds if $\dim F$ is odd. But if F is even dimensional and bounds a parallelizable manifold then the results of [14] and [12, Ch. V] combine to show that F bounds a contractible manifold. ∎

This completely answers the strengthened version of (*a*) in even dimensions. Furthermore, if $m=\dim F$ is odd and F^m bounds a parallelizable manifold then [5, Thm. 6.1, p. 36] and the proof of Theorem C yield differentiable semifree S^1- and S^3-actions on homotopy $(m+2dk)$-spheres with fixed point set F where $d=1$ for S^1 and $d=2$ for S^3 (strictly speaking [5] only does this when F is a homotopy sphere, but as in the proof of Theorem C one can use the same ideas to realize arbitrary homology spheres bounding parallelizable manifolds for actions of S^1). Theorem B shows that the analogous examples of semifree S^3-actions on homology $(m+8k+4)$-spheres do not necessarily exist if $\dim F \equiv 3 \bmod 8$. It seems likely that a partial analog of this holds for semifree S^1-actions on homology $(m+4k+2)$-spheres under the same condition on $\dim F$. The results of [27] can be viewed as a result in the case $s=0$. One can also prove similar results for small values of s by combining the methods of [24,25,26] and [2] with the known results on the homotopy groups of spheres (*e.g.*, as summarized in Kochman's book [16]).

Proposition 3.2. *Let F be a closed integral homology $(8s+3)$-sphere that bounds a parallelizable manifold, where $1 \le s \le 6$, and let $N(m)$ be given as in the preceding result. If F is the fixed point set of a semifree differentiable S^1-actions on an integral homology sphere of dimension $4l+1 \ge N(m)$, then $R(F)=0$.*

The methods of this paper also yield a result analogous to Theorem A on the generalized Rochlin invariants of homology spheres with semifree differentiable S^1-actions satisfying the same codimension condition.

Proposition 3.3. *Let Σ^{8s+3} be a closed integral homology sphere that admits a differentiable semifree S^1-action with an $(8l+7)$-dimensional fixed point set. Then $R(\Sigma)=0$.*

In principle one uses the same techniques as in Section 2, substituting $\mathbf{C}$ for $\mathbf{H}$, $2q$ for $4q$, and so forth. The most crucial technical points are that $\mathbf{C}P^{odd}$ is a spin manifold and the work of Snaith yields corresponding stable splittings of BU. ■

If F is an integral homology sphere that bounds a parallelizable manifold, then the obstruction to bounding a contractible manifold is carried by a torsion valued invariant that is given in terms of the signature and can be viewed as a refinement of the generalized Rochlin invariant (but it is defined for a narrower class of manifolds). It seems clear that further restrictions on fixed point sets should be obtainable in terms of this refinement and the dimensions of F and Σ, but it also seems likely that any general statements will be relatively complicated.

An application to diffeomorphism groups. The properties of the generalized Rochlin invariant yield a simple proof that certain self-diffeomorphisms of closed manifolds are not pseudo-isotopic to the identity; recall that two diffeomorphisms $f_0, f_1 : X \to Y$ are said to be (smoothly) pseudo-isotopic if there is a diffeomorphism $H : X \times [0,1] \to Y \times [0,1]$ such that H sends $X \times \{i\}$ to $Y \times \{i\}$ by f_i for $i = 0,1$.

Let M^n be a closed oriented smooth manifold and let $\pi_0(\mathrm{Diff}^{\sim}_{+} M)$ be the group of pseudo-isotopy classes of orientation preserving diffeomorphisms from M to itself. If N^n is a compact oriented smooth manifold with boundary define $\pi_0(\mathrm{Diff}^{\sim}_{+}(N,\partial))$ for orientation preserving diffeomorphisms that are the identity on the boundary. If we choose and fix an embedding : $D^n \to M^n$ then there is a canonical homomorphism $\mathrm{Diff}^{\sim}_{+}(D^n,\partial) \to \mathrm{Diff}^{\sim}_{+} M$ that sends f to f_M where

$$f_M = \begin{cases} \mathrm{id}_M & \text{on } M - \mathrm{Int}\, D^n \\ f & \text{on } D^n \end{cases}$$

This induces a homomorphism

$$\Phi_M : \pi_0(\mathrm{Diff}^{\sim}_{+}(D^n, \partial)) \to \pi_0(\mathrm{Diff}^{\sim}_{+} M).$$

If $n \geq 4$ it is well known (*cf.* [19, pp. 243–244]) that $\pi_0(\mathrm{Diff}^{\sim}_{+}(D^n,\partial)) \cong \Theta_{n+1}$, where the latter is the Kervaire-Milnor group of homotopy spheres; denote by $\Sigma(f)$ the homotopy sphere corresponding to $f \in \mathrm{Diff}^{\sim}_{+}(D^n,\partial)$. A standard argument as in [4] shows that $M \times S^1 \# \Sigma(f) \approx M(f_M)$, where $M(f_M)$ denotes the mapping torus of f_M. If f_M is pseudo-isotopic to the identity map, then $M(f_M) \approx M \times S^1$.

Theorem 3.4. *If M is a closed simply connected spin boundary of dimension $8k+2$ ($k \geq 1$), then the center of $\pi_0(\mathrm{Diff}^{\sim}_{+} M)$ contains a non-trivial element of even order.*

Proof. If M is a spin manifold and f_M is isotopic to the identity map, then it follows from (2.2) that $R(\Sigma(f)) = 0$ (in fact, there are two possible spin structures on $M(f_M) \approx M \times S^1$ and for each the generalized Rochlin invariant is zero). But the generalized Rochlin invariant R is non-trivial on Θ_{n+1} if $n \equiv 2 \pmod 8$ and $n \neq 2$, so Φ_M is non-trivial. Since it is a standard exercise to show that the image of Φ_M is contained in the center of $\pi_0(\mathrm{Diff}_+^{\sim}M)$ (*cf.* [18, §2]), the class of Φ_M has the required properties. ■

FINAL REMARKS.

(1) A result similar to Theorem 3.4 holds for M of dimension $8k$ or $8k+1$ ($k\geq 1$) using KO-characteristic numbers instead of the generalized Rochlin invariant. The methods are essentially the same as in the preceding argument.

(2) Let M be a closed spin null cobordant manifold with a semifree $S^3=G$ action such that the equivariant normal bundle of M^G is trivial. Then one can show that M^G is also spin null cobordant. It would be useful to understand the relation between $R(M)$ and $R(M^G)$ more generally in such cases. NOTE. There is an example such that $R(M)\neq R(M^G)$. Specifically, let V^{8s+4} be a compact spin manifold with boundary such that Sign $V\not\equiv 0$ (mod 16) and take $M=\partial(V\times D(\mathbf{H}^r))$. Then $M^G=\partial V$ and $R(M^G)=\mathrm{Sign}\,V\neq 0\in\mathbf{Z}_{16}$. But $R(M)=0$ since $\mathrm{Sign}(V\times D(\mathbf{H}^r))=0$.

(3) If Σ^{8k+3} is an integral homology sphere with a free involution T, then one has a Browder-Livesay invariant $\sigma(T,\Sigma)\in\mathbf{Z}$. A result of T. Yoshida [30] shows that $R(\Sigma)=\sigma(T,\Sigma)$ mod 16 if $k=0$. It would be interesting to know if this holds for other values of k; the difference $R(\Sigma)$-$\sigma(T,\Sigma)$ mod 16 is easily shown to be an invariant of free $\mathbf{Z}_2$-equivariant normal cobordism, so it should be possible to analyze this in terms of $[\mathbf{R}P^{8k+3}, F/O]$.

(4) There are other instances in transformation groups where Ochanine's result is applicable. One example involves a conjecture of H.-T. Ku and M.-C. Ku [17]; namely, if $a_r(f)$ and $s_r(g)$ are the splitting invariants of homotopy equivalences $f{:}X\to\mathbf{H}P^n$ and g: $Y\to\mathbf{C}P^m$ (for the respective submanifolds $\mathbf{H}P^r$ and $\mathbf{C}P^{2r}$), then these invariants are even. If f is transverse to $\mathbf{H}P^r$ then $8\cdot a_r(f)\in\mathbf{Z}$ is the difference between the signatures of $\mathbf{H}P^r$ and $f^{-1}(\mathbf{H}P^r)$. Since both of these are spin manifolds, the theorem of Ochanine implies that $a_r(f)$ is even if r is odd.

References

[1] J. P. Alexander, G. C. Hamrick, and J. W. Vick: *Involutions on homotopy spheres*, Invent. Math. **24** (1974), 15–50.

[2] J. C. Becker and R. E. Schultz: *Equivariant function spaces and stable homotopy theory*, Comment. Math. Helv. **49** (1974) 1–34.

[3] G. Bredon: Introduction to Compact Transformation Groups, Pure and Applied

Mathematics Vol. 46, Press, New York, 1972.

[4] W. Browder: *Diffeomorphisms of 1-connected manifolds*, Trans. Amer. Math. Soc. **128** (1967), 155–163.

[5] ————: *Surgery and the theory of differentiable transformation groups*, Proceedings of the Conference on Transformation Groups (Tulane, 1967), Springer, Berlin-Heidelberg-New York, 1968, pp. 3–46.

[6] ————: Surgery on Simply Connected Manifolds, Ergeb. der Math. (2) Bd. 65, Springer Berlin-Heidelberg-New York, 1972.

[7] W. Browder and T. Petrie: *Diffeomorphisms of manifolds and semifree actions on homotopy spheres*, Bull. Amer. Math. Soc. **77** (1971), 160–163.

[8] S.-S. Chern, F. Hirzebruch, and J.-P. Serre: *On the index of a fibered manifold*, Proc. Amer. Math. Soc. **8** (1957), 587–596.

[9] J. Eells and N. Kuiper: *An invariant for certain smooth manifolds*, Ann. Mat. Pura Appl. (4) **60** (1962), 93–110.

[10] F. Hirzebruch and K. H. Mayer: *O(n)-Mannigfaltigkeiten, exotische Sphären, und Singularitäten*, Lecture Notes in Math. Vol. 57, Springer, Berlin-Heidelberg-New York, 1968.

[11] F. Hirzebruch, W. Neumann, and S. Koh: Differentiable Manifolds and Quadratic Forms, Lecture Notes in Pure Math. Vol. 4, Marcel Dekker, New York, 1971.

[12] W.-C. Hsiang and W.-Y. Hsiang: *Differentiable actions of compact connected classical groups I*, Amer. J. Math. **89** (1967), 705–786.

[13] W.-Y. Hsiang: *On the unknottedness of the fixed point sets of differentiable circle actions on spheres—P. A. Smith Conjecture*, Bull. Amer. Math. Soc. **70** (1964), 678–680.

[14] M. Kervaire and J. Milnor: *Groups of homotopy spheres*, Ann. of Math. **77** (1963), 514–537.

[15] K. Knapp: *On the bistable J-homomorphism*, Algebraic Topology, Aarhus 1978 (Proc. Sympos., Aarhus Univ.), Lecture Notes in Mathematics Vol. 763, Springer, Berlin-Heidelberg-New York, 1979, pp. 13–22.

[16] S. O. Kochman: *Stable Homotopy Groups of Spheres: A Computer-Assisted Approach*, Lecture Notes in Mathematics Vol. 1423, Springer, Berlin-Heidelberg-New York *et al*, 1990.

[17] H.-T. Ku and M.-C. Ku: *Characteristic invariants of free differentiable actions of S^1 and S^3 on homotopy spheres*, Proceedings of the Second Conference on Compact Transformation Groups (Univ. of Massachusetts, Amherst, 1971) Part 1, Lecture Notes in Mathematics Vol. 298, Springer, Berlin-Heidelberg-New York, 1972, pp. 19–40.

[18] J. Levine: *Self-equivalences of $S^n \times S^k$*, Trans. Amer. Math. Soc. **143** (1969), 523–544.

[19] ————: *Inertia groups of manifolds and diffeomorphisms of spheres* Amer. J. Math. **92** (1970), 243–258.

[20] M. Masuda: *A product formula for connected sum*, Transformation Groups (Proceedings, Osaka, 1987), Lecture Notes in Math. Vol. 1375, Springer, Berlin-Heidelberg-New York-Tokyo, 1989, pp. 231–239.

[21] S. Ochanine: *Signature et invariants de Kervaire généralisés*, C. R. Acad. Sci. Paris Sér. A **285** (1977), 211–213.

[22] ————: *Signature modulo 16, invariants de Kervaire genéralisés et nombres caractéristiques dans la K-théorie reélle*, Mem. Soc. Math. France (N. S.) no. 5 (1980/81).

[23] M. Rothenberg and J. Sondow: *Nonlinear smooth representations of compact Lie groups*, Pac. J. Math. **84** (1980), 427–444.

[24] R. Schultz: *Homotopy sphere pairs admitting semifree differentiable actions*, Amer. J. Math. **96** (1974), 308–323.

[25] ————: *Differentiable group actions on homotopy spheres I: Differentialstructure and the knot invariant*, Invent. Math. **31** (1975), 105–128.

[26] ————: *Differentiability and the P. A. Smith theorem for spheres I: Actions of prime order groups*, Conference on Current Trends in Algebraic Topology (Univ. of Western Ontario, 1981), Canad. Math. Soc. Conf. Proc. **2 Pt. 2** (1982), 235–273.

[27] ————: *Homology spheres as stationary sets of circle actions*, Mich. Math. J. **34** (1987), 83–100.

[28] V. P. Snaith : *Algebraic Cobordism and K-Theory*, Memoirs Amer. Math. Soc. **21** (1979), No. 221.

[29] R. Stong: Lectures on Cobordism Theory, Princeton Mathematical Notes No. 7, Princeton University Press, Princeton, N. J. , 1969.

[30] T. Yoshida: *On the Browder-Livesay invariant for free involutions on homology 3-spheres*, Okayama Univ. Math. J. **22** (1980), 91–106.

Mikiya Masuda
Department of Mathematics
Osaka City University
Osaka 558 JAPAN

Reinhard Schultz
Department of Mathematics
Purdue University
West Lafayette, Indiana 47907 USA