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A CHARACTERIZATION OF TRANSLATION PLANES AND DUAL TRANSLATION PLANES OF CHARACTERISTIC≠2.*

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Introduction

In [7], A. Wagner has introduced a special class of finite affine planes. These are called W-planes by M.J. Kallaher (See [3], p. 106). These planes admit plenty of involutory homologies. A. Wagner proved (lemma 3, [7]) that a W-plane is either a translation plane or a dual translation plane or has certain property. In this paper, we study a finite affine plane, in which only the first of Wagner's condition is satisfied, called weak W-plane. We show a stronger theorem that a weak W-plane is a translation plane of characteristic $\neq 2$ or a dual translation plane of characteristic $\neq 2$. Towards this end, we study the implications of a weak W-plane admitting a (P, l)-transitivity for some point-line pair (P,l) and prove our assertion under this extra hypothesis. In the third section of this paper, we relax this condition and show our main theorem.

1. Previous Results

For basic definitions and theorems, we refer to [2] and [3]. We also make frequent use of the following well known theorems on collineations of projective planes.

Therem 1.1. ([2], p. 98 Corollary). *Let* Π *be a finite projective plane of order n and let G be a collineation group of* Π. *If* $|G_{(A,l)}|>1$ *for at least two choices of A on l, then* $G_{(l,l)}$ *is an elementary abelian p-group where p is a prime divisor of n.*

Theorem 1.2. ([2], p. 104, Corollary 1 to Theorem 4.25). *Let* π *be a finite projective plane and* α, β *be two non-trivial homologies with distinct centers* A, B *and the same axis l. Then* $\langle\alpha, \beta\rangle$ *contains an* $(AB \cap l, l)$*-elation mapping*

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A onto B.

Theorem 1.3. (Gleasen) ([2], p. 104, Corollary 1 to Theorem 4.26). *Let Π be a finite projective plane and let G be a collineation group of Π. If, for some line l, $|G_{(X,l)}|=h>1$ for all x on l, then l is a translation line.*

Theorem 1.4. ([2], p. 101. Lemma 4.22). *Let π be a projective plane. If α is an involutory (A, a)-homology of π and β is an involutory (B, b)-homology of π such that $B\in a$ and $A\in b$, then $\alpha\beta$ is an involutory $(a\cap b, AB)$-homology of π.*

Theorem 1.5. (Gleason) ([3], p. 30, Lemma 3.6). *Let G be a permutation group on a finite set Ω of order $n>1$, and let p be a prime. If, for every point $x\in\Omega$, the group G contains an elememt α of order p fixing x and no other point of Ω, then G is transitive on Ω.*

Theorem 1.6. ([5]). *If a projective plane π is (A, l)-transitive for every line l through B and $A\neq B$, then the plane π is (A, B)-transitive (i.e., π is (A, l)-transitive for every line l through B). In particular AB is a translation line.*

Theorem 1.7. ([4] and [6]). *If a projective plane π is (p_i, l_i)-transitive for $i=1, 2$ where $P_1\notin l_1, P_2\notin l_2$ and $l_1\cap l_2\notin P_1P_2$, then π is Desarguesian or it is the plane over the near-field of order* 9 *or its dual.*

Theorem 1.8. ([1]). *This is no finite plane of the class I_6 or the class III_1.*

Theorem 1.9. (Ostrom-Wagner) ([3], p. 43, Theorem 4.3). *Let Σ be a finite affine plane, and G, a collineation group of Σ. The following statements are equivalent:*

(1) *The group G is transitive on the affine points of Σ.*

(2) *For every point $U\in l_\infty$, the group G_U operates transitively on the affine lines of Σ through U.*

Theorem 1.10. ([2]), p. 104, Theorem 4.26). *Let π be a finite projective plane of order n and let G be a collineation group of π. Suppose there is a line l and a point Q on l such that $|G_{(A,l)}|=h>1$ for all A in l, $A\neq Q$. Then $|G_{(Q,l)}|=n$ i.e. π is (Q, l)-transitive.*

Theorem 1.11. (Ostrom) ([3], p. 51, Theorem 4.6). *Let π be a finite projective plane of order n, let G be a collineation group of π, and let l be a line of π. If $|G_{(l,l)}|>n$, then $|G_{(R,l)}|>1$ for every point $P\in l$.*

2. Definitions and basic lemmas

DEFINITION: ([3], p. 106). Let Σ be a finite affine plane and let G be a collineation group of Σ. Then the plane Σ is called a W-plane and G, a W-group if G has the following properties:

(i) For every affine flag (Q, l) of Σ, the group G contains an involutory homology fixing the flag (Q, l).

(ii) Let P and Q be any two points on the line l_∞ of Σ, and let l and m be two affine lines of Σ through Q. If G contains an involutory homology fixing P, Q and l, then G contains an involutory homology fixing P, Q and m.

DEFINITION. Let Σ be a finite affine plane and G be a collineation group of Σ. If G has property (i) of the above definition, then Σ is called a weak W-plane with respect to G. We will also describe this by saying that the pair (Σ, G) is a weak W-plane.

In the rest of this paper, whenever a weak W-plane (Σ, G) and any collineation α is considered, it is tacitly assumed that $\alpha \in G$. This is to avoid repeated mention of G.

We may note here that any translation plane or dual translation plane of characteristic $\neq 2$ is a W-plane and hence a weak W-plane (with respect to the full collineation group).

Since we are looking at affine planes, every collineation of G fixes the line l_∞. Hence l_∞ is fixed by every (P, l)-perspectivity. Hence, if further $P \in l$, then necessarily $P \in l_\infty$.

Lemma 2.1. *Let Σ be a weak W-plane with respect to a collineation group G of Σ. Then either G does not have any fixed point in Σ or Σ is a Moufang plane.*

Proof. Let $\pi = \Sigma \cup l_\infty$ be the projective closure of Σ. Let, if possible, *Q be an affine point fixed by G.*

Let P be a point on l_∞ and l be any affine line such that $P \in l$ and $Q \notin l$. Let L be any affine point on l. Since Σ is a weak W-plane, there exists an involutory homology α fixing the flag (L, l). Now there are three possibilities, regarding the center and the axis of α.

(i) L is the center of α and so l_∞ is the axis of α. (ii) l is the axis of α and some point on l_∞ is the center of α. (iii) $l \cap l_\infty$ is the center of α and some line through L is the axis of α.

Since α fixes Q, we conclude that α must be an involutory $(l \cap l_\infty, LQ)$-homology. Since L is an arbitrary affine point on l, there exists an involutory $(l \cap l_\infty, \mathrm{D}Q)$-homology for every affine point X on l. By dual of (1.2), the group of all $(l \cap l_\infty, (l \cap l_\infty)Q)$-elations is transitive on the affine points of l. Thus the plane π is $(P = l \cap l_\infty, (l \cap l_\infty)\, Q = PQ)$-transitive. Since l is an arbitrary affine line, not through Q, P can be taken to be an arbitrary point on l_∞. Therefore the plane π belongs to Lenz-Barlotti class III_1 with (Q, l_∞) as the distinguished point-line pair. This contradicts theorem (1.8) and our assertion stands proved.

Lemma 2.2. *Let (Σ, G) be a weak W-plane which is not a translation plane. Then G has at most one fixed point P on l_∞ and in this case P is a dual translation point.*

Proof. Let $\pi=\Sigma\cup l_\infty$ the projective closure of Σ and P *be a point on* l_∞, fixed by every collineation of G. Let n be the order of π. Since π admits involutory homologies, n is odd.

We consider *two cases separately* and prove *our assertion in each case.*

Case (i) Suppose there exists an affine line l, not through P, which is not the axis of any involutory homology.

Let L be an affine point on l. Since Σ is a weak W-plane, there exists an involutory homology α fixing the flag (L, l). Now either the center of α is L and the axis of α is l_∞ or the center of α is $l\cap l_\infty$ and the axis of α is LP (Since α fixes P also). Let Ω_1 be the set of all affine points X on l such that there exists an involutory (X, l_∞)-homology and let Ω_2 be the set of all affine points Y on l such that $Y\notin\Omega_1$. Note that $|\Omega_1\cup\Omega_2|=n=$odd.

For any two distinct affine points R_1 and $R_2\in\Omega_1$, there exists an involutory (R_1, l_∞)-homology and an involutory (R_2, l_∞)-homology. Thus there exists a $(l\cap l_\infty, l_\infty)$-elation mapping R_1 onto R_2 by (1.2). Therefore the group of all $(l\cap l_\infty, l_\infty)$-elations is transitive on the points of Ω_1.

For any two distinct affine points Q_1 and $Q_2\in\Omega_2$, there exists an involutory $(l\cap l_\infty, Q_1P)$-homology and an involutory $(l\cap l_\infty, Q_2P)$-homology. By dual of (1.2), there exists a $(l\cap l_\infty, l_\infty)$-elation β mapping Q_1P onto Q_2P and so β maps Q_1 onto Q_2. Thus the group of all $(l\cap l_\infty, l_\infty)$-elations is transitive on the points of Ω_2. Since $\Omega_1\cap\Omega_2=\emptyset$ the group of all $(l\cap l_\infty, l_\infty)$-elations divides the line l into two orbits, namely Ω_1 and Ω_2.

If either $\Omega_1=\emptyset$ or $\Omega_2=\emptyset$, then the plane π is $(l\cap l_\infty, l_\infty)$-transitive by (1.2) or its dual. Since l_∞ has at least three distinct points, let Z be a point on l_∞ such that $P\neq Z\neq l\cap l_\infty$. Let r be an affine line through Z, and R be an affine point on r. Let λ be an involutory homology fixing the flag (R, r). Since λ fixes P, λ has to be an involutory (R, l_∞)-homology. Otherwise, λ shifts the point $l\cap l_\infty$ and fixes the line l_∞ and so the plane π is $((l\cap l_\infty)\lambda\neq l\cap l_\infty, l_\infty\lambda=l_\infty)$-transitive and thus l_∞ is a translation line which contradicts our hypothesis. Since R is an arbitrary affine point on r, the plane π is $(Z, l\cap l_\infty)$-transitive by (1.2) and so l_∞ is a translation line which again contradicts our hypothesis.

Hence $\Omega_1\neq\emptyset\neq\Omega_2$.

We now observe that the plane π is $(l\cap l_\infty, l_\infty)$-transitive if there exists a non-trivial elation which takes a point of Ω_1 onto a point of Ω_2. Therefore every non-trivial $(l\cap l_\infty, l_\infty)$-elation fixes Ω_1 and Ω_2 setwise. Since a non-trivial

$(l \cap l_\infty, l_\infty)$-elation is uniquely determined by the image of any single point, not on l_∞, it is clear that $|G_{(l \cap l_\infty, l_\infty)}| = |\Omega_1|$ and $||G_{(l \cap l_\infty, l_\infty)}| = |\Omega_2|$. Thus $|\Omega_1| = |\Omega_2|$.

Hence n must be even which is not possible.

Case (ii) Every affine line, not through P, is the axis of an involutory homology. Since G fixes P, the center of all such involuotry homologies is P. Thus P is a dual translation point by dual of (1.2). □

Theorem 2.3. *Let* (Σ, G) *be a weak W-plane and* Σ *be* (P, l)*-transitive for some incident point-line pair* (P, l). *Then* Σ *is either a translation plane or a dual translatin plane.*

Proof. Let $\pi = \Sigma \cap l_\infty$ be the projective closure of Σ. We consider the following *two cases separately* and prove *our assertion in each case.*

Case (i): $l \neq l_\infty$.

Case (ii): $l = l_\infty$.

Case (i). Clearly $P \in l_\infty$. Suppose Σ is a translation plane or P is a dual translation point. Then there is nothing to prove. Otherwise, by (2.2), there exists a collinearion α which shifts P. Thus the plane π is $(P\alpha \neq P, l\alpha \neq l)$-transitive. Therefore the plane π belongs to Lenz-Barlotti class III_1 with $(l \cap l_\infty, PP\alpha = l_\infty)$ as the special point-line pair. But there is no finite plane of the class III_1 by (1.8). So the plane Σ must be Moufang and hence Paipian.

Case (ii) Clearly $P \in l_\infty$. If either Σ is a translation plane or P is a dual translation point, then our assertion stands proved. Otherwise, by (2.2), there exists a collineation λ which shifts P. Then the plane π is $(P\lambda \neq P, l_\infty\lambda = l_\infty)$-transitive. Therefore the line l_∞ is a translation line.

Theorem 2.4. *Let* (Σ, G) *be a weak W-plane and* Σ *be* (P, l)*-transitive for some non-incident point-line pair* (P, l). *Then* Σ *is the plane over a near-field or a dual near-field.*

Proof. Let π be the proejctive closure of Σ. We consider the following *two cases separately* and prove our assertion *in each case.*

Case (i): $l = l_\infty$.

Case (ii): $l \neq l_\infty$.

Case (i). If Σ is Moufang, we are done. Otherwise by (2.1), there exists a collineation α of Σ such that $P\alpha \neq P$. Then the plane π is $(P\alpha \neq P, l_\infty\alpha = l_\infty)$-transitive. Thus the plane π is $(PP\alpha, l_\infty)$-transitive by (1.6) and in particular $PP\alpha \cap l_\infty$ is a dual translation point. Hence the plane π belongs to the plane over a dual near-field.

Case (ii) Subcase (a). Clearly $P \in l_\infty$. *Suppose G fixes* $l \cap l_\infty$. By (2.2), either $l \cap l_\infty$ is a dual translation point or l_∞ is a translation line. In the first case, there exists a non-trivial $(l \cap l_\infty, l)$-elation β which fixes l and shifts P. Thus the plane

π is $(P\beta \neq P, l\beta = l)$-transitive and so the plane π is $(PP\beta = l_\infty, l)$-transitive by dual of (1.6). Therefore the plane π belongs to the plane over a dual near-field. In second case, there exists a non-trivial (P, l_∞)-elation ω which fixes P and shifts l. Thus the plane π is $(P\omega = P, l\omega \neq l)$-transitive. Therefore the plane π is $(P, l \cap l\omega = l \cap l_\infty)$-transitive by (1.6). Hence the plane π belongs to the plane over a near-field.

Subcase (b). *Supppse G does not fix* $l \cap l_\infty$. Then there exists a collineation δ which shifts $l \cap l_\infty$. If δ fixes P, then the plane π is $(P\delta = P, l\delta \neq l_\infty)$-transitive and so the plane π is $(P, l \cap l\delta)$-transitive by (1.6) where $l \cap l\delta \notin l_\infty$. In particular the plane π is $(l \cap l\delta, l_\infty)$-transitive. By case (i), we are done. Hence δ does not fix P. So the plane π is $(P\delta \neq P, l\delta \neq l)$-transitive with $l \cap l\delta \notin PP\delta = l_\infty$. Thus the plane π is Desarguesian or the plane over the near-field of order 9 or its dual by (1.7). □

3. On weak W-planes

Lemma 3.1. *Let* (Σ, G) *be a weak W-plane. It further* Σ *is neither a translation plane nor a dual translation plane, then a point* $X \in l_\infty$, *which is the center of a non-trivial homology, is also the center of a non-trivial translation and a non-trivial affine elation (elation with affine line as axis).*

Proof. Let α be a non-trivial (X, x)-homology where $x \neq l_\infty$. Let π be the projective closure of Σ. Let n be the order of the plane π.

Case (i) Suppose there is an affine line l through x, which is not the axis of any involutory homology. Let Ω_1 be the set of all affine points Q on l such that there exists an involutory (Q, l_∞)-homology and let Ω_2 be the set of all affine points Q on l such that $Q \in \Omega_1$.

Suppose $\Omega_1 = \emptyset$. Then for every $Q_i \in l$, $Q_i \neq x$, the plane π has an involutory (x, l_i)-homology α_i for some affine line l_i through Q_i, $i=1$ to n and $|\Omega_2| = n$. If any two of axes of all such involutory homologies, namely l_i's intercept at a point on l_∞, then the group of all (X, l_∞)-elations is transitive on the affine points on l by dual of (1.2). Hence the plane Σ is (X, l_∞)-transitive and hence Σ is a translation plane or a dual translation plane by (2.3) which contradicts our assumption. It follows that there exists a Z_1 and $Z_2 \in \Omega_2$ such that π admits an involutory (X, q_1)-homology and an involutory (X, q_2)-homology with $Z_1 \in q_1$, $Z_2 \in q_2$, $q_1 \cap q_2 \notin l_\infty$. Let $(q_1 \cap q_2)X = q$. Now if $l_i \cap l_j \in l_\infty$ for some $i, j \in \{1, 2, \cdots, n\}$, then we are done by dual of (1.2). So, we may assume that $l_i \cap l_j \notin l_\infty$ for every $i, j \in \{1, 2, \cdots, n\}$. If for every $i, j \in \{1, 2, \cdots, n\}$ $l_i \cap l_j \in q$, then Σ is (X, q)-transitive, and so we are done by (2.3). Thus we may assume $l_i \cap l_j \notin q$ for some $i, j \in \{1, 2, \cdots, n\}$]. In this case, G has an affine elation β whose axis $\neq q$. Therefore $|G(X, q)| = P^a$ for some prime p dividing n and for some integer $a \geq 1$ by dual of (1.1). Let Ω be the set of lines among l_i's such that they are concurrent at $q_1 \cap q_2$. Clearly $|\Omega| = P^a$. If l_i's meet q at dintinct points, let identity$\neq \beta_0 \in G(X, q)$. Then $\beta_0^{-1}\alpha_i\beta_0$ is an involutory $(X, l_i\beta_0)$-homology. Note that $l_i\beta_0 \neq l_j, j \in \{1, 2, \cdots, n\}$. So we may apply the dual of (1.2), and (1.11), and conclude that there exists a non-trivial (X, l_∞)-elation.

Hence we may assume that there exists an affine point $Z \in q$, and $Z \neq q \cap l_i$, $i \in \{1, 2, \cdots, n\}$. We consider the flag (Z, q). Let α_0 be the involutory homology fixing the flag (Z, q). Then it is clear that $\alpha_0 \neq \alpha_i$ for any i. If X is the center of α_0, then the axis of α_0 passes through Z, and hence is different from any l_i. So we may apply the dual of (1.2), and (1.11). If α_0 is an involutory (Z, l_∞)-homology, then $\beta^{-1}\alpha_0\beta$ is also an involutory $(Z\beta \neq Z, l_\infty\beta = l_\infty)$-homology. So $|G(X, l_\infty)| > 1$ by (1.2). Finally suppose α_0 is an involutory (D, q)-homology where $D \in l_\infty$. If $D \in l_i$ for some i where l_i is an element of Ω, then $\alpha_0\alpha_i$ is an involutory $(q \cap l_i, DX = l_\infty)$-homology. Then $\beta^{-1}\alpha_0\alpha_i\beta$ is also an involutory $((q \cap l_i)\beta \neq l \cap l_\infty, l_\infty)$-homology. Thus G contains a non-trivial (X, l_∞)-elation. Therefore we may assume $D \notin l_i$ for any $l_i \in \Omega$. Here we observe that fixes Ω setwise and note that α does not fix any element in Ω. Therefore 2 divides $|\Omega| = P^a$. This implies $2 = p$, p divides n implies that 2 divides n, a contradiction.

Hence $|\Omega_1| > 1$ by (2.1). We observe that $|G_{(X, l_\infty)}| > 1$ by (1.2). If $\Omega_2 = \emptyset$, then $|\Omega_1| = n$ and for every point P on l, there exists an involutory (P, l_∞)-homology. By (1.2), the group of all (X, l_∞)-elations is transitive on the affine points of l. So the plane π is (X, l_∞)-transitive which contradicts our assumption by (2.3). So $|\Omega_2| > 1$. Also for every point $Q \in \Omega_2$, there exists an involutory (X, q)-homology with $Q \in q$. If any two axes of all such involutory homologies intercept at a point on l_∞, then the group of all (X, l_∞)-elations divides the line l into exactly two orbits, namely Ω_1 and Ω_2 by (1.2) and its dual. Following the same method as (2.2), we observe that n is even. But the plane π has involutory homologies and so n has to be odd. Hence this case cannot arise. So, there exists Q_1 and $Q \in \Omega_2$ such that π admits an involutory (X, q_1)-homology and an involutory (X, q_2)-homology with $Q_1 \in q_1$, $Q_2 \in q_2$ and $q_1 \cap q_2 \notin l_\infty$. By dual of (1.2), there exists a non-trivial $(X, (q_1 \cap q_2)X)$-elation which proves our assertion in this case.

Case (ii) Let us assume that every affine line through X is the axis of an involutory homology. If the center of all such involutory homologies is the point $x \cap l_\infty$, then the plane π is $(x \cap l_\infty, l_\infty)$-transitive by dual of (1.2) which cannot happen by (2.3). Therefore π has an involutory (D, z)-homology γ with $D \neq x \cap l_\infty$ and $X \in z$. Then $\alpha^{-1}\gamma\alpha$ is an involutory $(D\alpha \neq D, z\alpha = z)$-homology. Then the group $\langle \alpha, \alpha^{-1}\gamma\alpha \rangle$ contains a non-trivial (X, z)-elation, say β by a dual of (1.2). Then let q be an affine line through X. Then by our assumption, π has an involutory (D_1, q)-homology δ for some point $X \neq D_1 \in l_\infty$. Thus the group $\langle \delta, \beta^{-1}\delta\beta \rangle$ has a non-trivial (X, q)-elation by dual of (1.2). Since q is an arbitrary affine line through X, we observe that $|G_{(X,X)}|>n$. By (1.11), $|G_{(X,l_\infty)}|>1$.

Corollary 3.2. *Let (Σ, G) be a weak W-plane and $\pi=\Sigma\cup l_\infty$ the projective closure of Σ. Then there exists a point on l_∞ which is not the center of any involutory homology if Σ is neither a translation plane nor a dual translation plane.*

Proof. Suppose π is neither a translation plane nor a dual translation plane.

Suppose every point on l_∞ is the center of an involutory homology. By (3.1), for every point X on l_∞, we observe that $|G_{(X,l_\infty)}|>1$ and $|G_{(X,l)}|>1$ for at least one affine line l through X. By (1.1), $G_{(l_\infty,l_\infty)}$ is an elementary abelian p-group for some prime number p dividing n and for every point X on l_∞ $|G_{(X,X)}|>1$ is an elementary abelian p-group by dual of (1.1). Let X be a point on l_∞. By (3.1), there exists a non-trivial (X, l)-elation β_1 of order p and the collineation β_1 fixes the point X and acts on other points on l_∞. Since X is an arbitrary point on l_∞, the collineation group G is transitive on l_∞ by (1.5). Also $|G_{(X,l_\infty)}|>1$ for every point X on l_∞. Therefore $|G_{(X,l_\infty)}|$ is independent of X. Then $|G_{(X,l_\infty)}|=h>1$ for all points X on l_∞. By (1.3), l_∞ is a translation line which contradicts our hypothesis. Hence our assertion stands proved.

Corollary 3.3. *Any weak W-plane, which contains an involutory homology with an affine center, is a translation plane or dual translation plane.*

Proof. Let (Σ, G) be a weak W-plane and $\pi=\Sigma\cup l_\infty$ the projective closure of Σ.

Let α be an involutory (Z, l_∞)-homology. We prove that the plane Σ has to be a translation plane or a dual translation plane.

Let Ω_1 be the set of all points X on l_∞ such that $|G_{(X,l_\infty)}|>1$ and $|G_{(X,l)}|>1$ for at least one affine line l through X and let Ω_2 be the set of all points Y on l_∞ such that $Y\notin\Omega_1$. We claim that $\Omega_2=\emptyset$.

Suppose $Y\in\Omega_2$. Consider the line YZ. If YZ is not the axis of any involutory homology, then the involutory homology δ fixing the affine flag (Q, YZ) is an involutory (Q, l_∞)-homology. Otherwise, Y is the center of δ or δ is an involutory (D, YZ)-homology. In the second case, $\alpha\delta$ is an involutory (Y, DZ)-homology by (1.4). So in each case, Y is the center of some involutory homology. Thus, by (3.1), $Y\in\Omega_1$ which contradicts our assumption. It follows that for every affine point $Q\in YZ$ there exists an involutory (Q, l_∞)-homology. By (1.2), the group of all (Y, l_∞) ellations is transitive on the affine points of YZ. Hence the plane π is (Y, l_∞)-transitive from which our assertion follows by (2.3).

Hence $\Omega_2=\emptyset$ and so $|\Omega_1|=n$. Following the same proof as in (3.2), we see that l_∞ is a translation line these by proving our theorem.

Theorem 3.4. *Under the assumption of* (3.1), *the collineation group G of Σ is transitive on the affine points of Σ.*

Proof. Let Ω_1 be the set of all points X on l_∞ such that X is the center of an involutory homology and let Ω_1 be the set of all points Y on l_∞ such that $Y \notin \Omega_1$.

We observe that for every point $X \in \Omega_1$, $|G_{(X,l_\infty)}|>1$ and $|G_{(X,l)}|>1$ for at least one affine line l through X by (3.1). If $\Omega_2=\emptyset$, by the same argument as in (3.1), we observe that l_∞ is a translation line which contradicts our assumption. Thus $|\Omega_2|>1$ by (2.2). Let $Y \in \Omega_2$. Let m be an affine line through Y and let M be an affine point on m. Let α be an involutory homology fixing the flag (M, m). By (3.3), M is not the center of α. Since $Y \in \Omega_2$, Y is not the center of α. Therefore m is the axis of α. Also since m is an arbitrary affine line throgh Y, every affine line through Y is the axis of an involutory homology. We observe that α fixes Ω_2 setwise. Further the collineation α also fixes Y in Ω_2 and fixes no other point in Ω_2. So 2 divides $|\Omega_2|-1$.

Clearly $\Omega_1 \neq \emptyset$ and so $|\Omega_1|>1$ by (2.2). Let $X \in \Omega_1$. Let l be an affine line through X and T be an affine point on l. Let γ be an involutory homology fixing the flag (T, l). By (3.3), the point T is not the center of γ. If the axis of γ is the line l, then the center of γ is a point in Ω_1 by our assumption. We observe the collineation γ fixes Ω_2 setwise and fixes no point in Ω_2. So 2 divides $|\Omega_2|$ which contradicts to 2 divides $(|\Omega_2|-1)$. Hence $l \cap l_\infty = X$ is the center of γ and the axis of γ is an affine line l_0 through T and further $l_0 \cap l_\infty \in \Omega_2$. Since T is an arbitrary point on l, for every affine point $Z \in l$, there exists an involutory (X, z)-homology where z is an affine line through Z. Let Z_1 and $Z_2 \in t$. Then there exists an involutory (X, z_1)-homology and an involutory (X, z_2)-homology with $Z_1 \in z_1$ and $Z_2 \in z_2$. Then there exists a non-trivial $(X, (z_1 \cap z_2)X)$-elation mapping z_1 onto z_2 (hence mapping Z_1 onto Z_2) by dual of (1.2). Thus the group $G_{(X,X)}$ is transitive on the affine point of l. Since l is an arbitrary affine line through X, the group $G_{(X,X)}$ is transitive on the affine points of l for every affine line l through X. The above assertion follows for every point X on Ω_1. Since $|\Omega_1|>1$, there exists $X_1, X_2 \in \Omega_1$ and $X_1 \neq X_2$. We clear that $\langle G_{(X_1,X_1)}|, G_{(X_2,X_2)}\rangle$ is transitive on the affine points of Σ. □

We now come to our main theorem.

Theorem 3.5. *A weak W-plane is either a translation plane or a dual translation plane of chracteristic $\neq 2$.*

Proof. Let (Σ, G) be a weak W-plane. Let $\pi = \Sigma \cup l_\infty$ the piojective closure of Σ.

Suppose the plane π is neither a translation plane nor a dual translation plane. Then the group G is transitive on the affine points of Σ by (3.4). Let Ω_1 and

Ω_2 be subsets of points on l_∞, defined as in (3.4). Also we observe that for every point $X\in\Omega_1$, $|G_{(X,l)}|>1$ for at least one affine line l through X. By (1.9), for every point $U\in l_\infty$, the group G_U operates transitively on the affine lines of Σ through U. Therefore for every point $X\in\Omega_1$, $|G_{(X,l)}|=h>1$ for all affine line l through X. So the plane π is (X, l_∞)-transitive by (1.10) for every point $X\in\Omega_1$. By (2.3), the plane π is either a translation plane or a dual translation plane which contradicts our assumption.

The assertion about characteristic easily follows and so it stands proved. □

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