Koiso, N.
Osaka J. Math.
30 (1993), 559–565

CONVERGENCE TO A GEODESIC

Dedicated to Professor Masaru Takeuchi on his 60th birthday

NORIHITO KOISO

(Received April 9, 1992)

0. Introduction

For a closed curve $\gamma(s)$ in a riemannian manifold M we define its energy $E(\gamma)$ by $||\dot{\gamma}||^2$. The first variation formula of E is given by $-2\langle\delta\gamma, D_{\dot{\gamma}}\dot{\gamma}\rangle$. Therefore, its Euler-Lagrange equation is $D_{\dot{\gamma}}\dot{\gamma}=0$, the equation of geodesics. We consider the corresponding parabolic equation

$$\text{(EP)} \qquad \frac{d}{dt}\gamma_t = D_{\dot{\gamma}_t}\dot{\gamma}_t\,.$$

This is locally expressed as

$$\frac{\partial}{\partial t}\gamma^i = \frac{\partial^2}{\partial s^2}\gamma^i + \Gamma_j{}^i{}_k \frac{\partial}{\partial s}\gamma^j \frac{\partial}{\partial s}\gamma^k\,,$$

which is a semi-linear heat equation.

This equation was studied by Eells and Sampson [ES], in higher dimensional case. They proved that if the manifold (M, g) is compact and has non-positive sectional curvature, then a solution γ_t exists for all time, and a *subsequence* γ_{t_i} converges to a geodesic. And it is not so difficult to show that if the manifold (M, g) has negative sectional curvature, then the solution γ_t itself converges to the geodesic.

Physically, equation (EP) represents the equation of motion of a rubber band in high viscous liquid. Therefore, it seems that the above curvature restriction is not necessary. More precisely, we have the following

Conjecture A. *If the manifold M is compact then Cauchy problem* (EP) *has a unique solution γ_t for all time.*

Conjecture B. *The solution γ_t converges to a geodesic when $t\to\infty$.*

In this paper we will show that this conjecture holds "almost always", with "a few" exceptions.

Theorem A. *If the manifold M is compact then Cauchy problem* (EP) *with C^∞ initial data has a unique solution γ_t for all time.*

Theorem B. *Moreover, if the riemannian manifold (M, g) is real analytic, then the solution γ_t converges to a geodesic when $t \to \infty$.*

Thoerem C. *There exists a compact riemannian manifold (M, g) such that for certain C^∞ initial data the solution γ_t of Cauchy problem* (EP) *does not converge.*

1. Preliminaries

Throughout in this paper, we use the following notations. The parameter of a curve γ is denoted by s and the velocity vector $d\gamma/ds$ is denoted by $\dot{\gamma}$ or v. We treat curves γ_t depending on time t and denote by $\dot{\gamma}_t$ or v_t their velocity vectors. But we usually omit the subscript t in them.

The riemannian covariant derivation is denoted by D. The norm $|*|$, the L_2 norm $||*||$ and the L_2 inner product $\langle *, * \rangle$ are defined by $|*|^2 = g(*, *)$, $\langle *, * \rangle = \oint g(*, *) ds$ and $||*||^2 = \langle *, * \rangle$.

We start from results in [ES].

Theorem 1.1. [ES, Theorem 10A, 10B] *For any closed C^1 curve γ_0, there is a positive constant T depending only on the energy density $|v_0|^2$ such that* (EP) *has a unique solution γ_t on $0 \leq t \leq T$.*

Let T be the largest number such that a solution with initial data γ_0 exists on $0 \leq t < T$, and *suppose that the energy density $|v_t|^2$ is bounded on* $\{(s, t)\} = S^1 \times [0, T)$. Then by Theorem 1.1 there exists a positive number T_1 such that any γ_t can be continued as a solution onto the interval $(t, t + T_1)$. This implise that T is infinite. Therefore, the proof of Theorem A is reduced to the following

Proposition 1.2. *Let γ_t be a solution of* (EP) *on $0 \leq t < T$, where T is a finite positive number. Then the energy density $|v_t|^2$ is bounded from above by a constant C on* $\{(s, t)\} = S^1 \times [0, T)$. *Here, the constant C dependends only on the initial data γ_0 and the time T.*

To prove this, we need some basic inequalities. As usual, we use symbols $D_t = D_{d/dt}$ and $D_v = D_{d/ds}$. First, for a solution γ_t on $0 \leq t < T$ we see

$$\frac{d}{dt} ||v||^2 = 2\langle v, D_t v \rangle = 2\left\langle v, D_v \frac{d}{dt} \gamma \right\rangle = 2\langle v, D_v^2 v \rangle = -2||D_v v||^2 .$$

It implies that $||v||$ is non-increasing. Therefore, we have a positive constant C_1 such that $||v|| \leq C_1$ on $0 \leq t < T$.

Lemma 1.3. *For any vecotr field ξ along γ, we have*

$$\max_s |\xi|^2 \leq 2||\xi|| (||\xi|| + ||D_v \xi||)$$

Proof.

$$\max_s |\xi|^2 \leq \min_s |\xi|^2 + \oint \left| d|\xi|^2 \right| ds \leq \frac{1}{2\pi} \|\xi\|^2 + 2\langle |\xi|, |D_v\xi| \rangle$$
$$\leq 2(\|\xi\|^2 + \|\xi\| \, \|D_v\xi\|)$$

Q.E.D.

Lemma 1.4. *For any positive integers $p \leq q$, we have a constant C_2 depending only on (the constant C_1 and) p and q such that*

$$\|D_v^p v\| \leq C_2 \|D_v^q v\|^{p/q} .$$

Proof. Since

$$\|D_v^p v\|^2 = -\langle D_v^{p-1} v, D_v^{p+1} v \rangle \leq \|D_v^{p-1} v\| \, \|D_v^{p+1} v\| ,$$

we see that the function $\log \|D_v^p v\|$ is concave with respect to $p \geq 0$. Therefore,

$$\|D_v^p v\| \leq \|v\|^{1-(p/q)} \|D_v^q v\|^{p/q} \leq C_1^{1-(p/q)} \|D_v^q v\|^{p/q} .$$

Q.E.D.

Lemma 1.5. *For any non-negative integers $p < q$, we have a constant C_3 depending only on (C_1 and) p and q such that*

$$\max_s |D_v^p v| \leq C_3 (1 + \|D_v^q v\|^{(2p+1)/(2q)}) .$$

Proof. From Lemma 1.3, we know

$$\max_s |D_v^p v| \leq \sqrt{2} \|D_v^p v\|^{1/2} (\|D_v^p v\| + \|D_v^{p+1} v\|)^{1/2} .$$

By Lemma 1.4, the right hand side

$$\leq \text{const} \cdot \|D_v^q v\|^{p/(2q)} (\|D_v^q v\|^{p/q} + \|D_v^q v\|^{(p+1)/q})^{1/2}$$
$$\leq \text{const} \cdot (1 + \|D_v^q v\|^{(2p+1)/(2q)}) .$$

Q.E.D.

2. Proof of Theorem A

Now we have to see more closely equation (EP). For the solution γ_t on $0 \leq t < T$, we see

$$D_t D_v v = R(\frac{d}{dt}\gamma, v) v + D_v D_t v = D_v^3 v + R(D_v v, v) v .$$

Therefore, by induction, we get for $n \geq 2$,

$$D_t D_v^{n-1} v = R(\frac{d}{dt}\gamma, v) D_v^{n-2} v + D_v D_t D_v^{n-1} v$$
$$= D_v^{n+1} v + \sum^A A_{ijkl} (D_v^i R)(D_v^j v, D_v^k v) D_v^l v ,$$

where A's are universal constants and the sum $\sum^A$ is taken over all $i, k, l \geq 0$, $j \geq 1$ with $i+j+k+l=n-1$. This holds also for $n=1$, taking $A=0$. Thus, we get

$$\frac{1}{2}\frac{d}{dt} ||D_v^{n-1} v||^2 = \langle D_v^{n-1} v, D_t D_v^{n-1} v \rangle$$
$$= \langle D_v^{n-1} v, D_v^{n+1} v + \sum^A A_{ijkl} (D_v^i R)(D_v^j v, D_v^k v) D_v^l v \rangle .$$

Here the term $D_v^i R$ is expanded into

$$\sum^B B^i_{m p_1 \cdots p_m} (D^m R)(D_v^{p_1} v, \cdots, D_v^{p_m} v) ,$$

where B's are universal constants and the sum $\sum^B$ is taken over all $m, p_1, \cdots, p_m \geq 0$ with $m + \sum_{1 \leq a \leq m} p_a = i$.

Lemma 2.1. *There is a positive constant C_4 depending only on C_1 and non-negative integer n such that*

$$\frac{d}{dt} ||D_v^n v||^2 \leq C_4$$

Proof. Let n be a positive integer. From the above equality and Lemmas 1.4, 1.5, we see

$$\frac{1}{2}\frac{d}{dt} ||D_v^{n-1} v||^2$$
$$= -||D_v^n v||^2 + \sum^C \langle D_v^{n-1} v, B^i_{m p_1 \cdots p_m} (D^m R)(D_v^{p_1} v, \cdots, D_v^{p_m} v)(D_v^l v, D_v^k v) D_v^j v \rangle$$
$$\leq -||D_v^n v||^2 + \text{const} \cdot \sum^C (\prod_q \max_s | D_v^q v |) ||D_v^j v|| \, ||D_v^{n-1} v||$$
$$\leq -||D_v^n v||^2 + \text{const} \cdot \sum^C (\prod_q (1 + ||D_v^n v||^{(2q+1)/(2n)})) \, ||D_v^n v||^{j/n} ||D_v^n v||^{(n-1)/n}$$
$$\leq -||D_v^n v||^2 + \text{const} \cdot \sum^C (1 + ||D_v^n v||^{((\sum_q 2q)+m+2+2j+2(n-1))/(2n)})$$
$$\leq -||D_v^n v||^2 + \text{const} \cdot \sum^C (1 + ||D_v^n v||^{(4n-2)/(2n)})$$
$$\leq \text{const} ,$$

where $\sum^C *$ denotes $\sum^A (\sum^B *)$ and q runs in the set $\{p_1, \cdots, p_m, k, l\}$. Q.E.D.

Proof of Theorem A. Lemma 2.1 and Lemma 1.3 imply that we can estimate each C^n norm of the solution γ_t only by the initial data γ_0. This completes the proof of Proposition 1.2, hence Theorem A holds by the remark above Proposition 1.2. Q.E.D.

Before proceeding to Theorem B and C, we derive the following

Lemma 2.2. *For any positive integer n, the integral* $\int_0^\infty \|D_v^n v\|^2 dt$ *is finite, and* $\|D_v^n v\| \to 0$ *when* $t \to \infty$.

Proof. Since $\frac{1}{2}\frac{d}{dt}\|v\|^2 = -\|D_v v\|^2$, we see

$$\int_0^\infty \|D_v v\|^2 dt = -\frac{1}{2}[\|v\|^2]_0^\infty \leq \frac{1}{2}\|v_0\|^2 < \infty .$$

Combining it with Lemma 2.1, we get the result for $n=1$. Suppose that the assertion holds for any positive integer less than n. Note that $q \leq n-2$ in the third line of the inequality in the proof of Lemma 2.1. Therefore, by Lemma 1.3, all $\max_s |D_v^q v|$ are already bounded by a constant. Thus,

$$\frac{1}{2}\frac{d}{dt}\|D_v^{n-1}v\|^2 \leq -\|D_v^n v\|^2 + \text{const} \cdot \sum \|D_v^j v\| \, \|D_v^{n-1} v\| ,$$

where the sum is taken for $1 \leq j \leq n-1$. By integration, we see

$$\frac{1}{2}[\|D_v^{n-1}v\|^2]_0^\infty \leq -\int_0^\infty \|D_v^n v\|^2 dt + \text{const} \cdot \sum \int_0^\infty \|D_v^j v\| \, \|D_v^{n-1}v\| dt$$

$$\leq -\int_0^\infty \|D_v^n v\|^2 dt + \text{const} \cdot \sum \left(\int_0^\infty \|D_v^j v\|^2 dt \int_0^\infty \|D_v^{n-1} v\|^2 dt \right)^{1/2} .$$

Thus, $\int_0^\infty \|D_v^n v\|^2 dt$ is finite by the assumption of induction. Combining it with Lemma 2.1, we get the result for n. Q.E.D.

3. Proof of Theorem B

The next Lemma is a direct consequence of a result of [S, Theorem 3].

Lemma 3.1. *Let* (M, g) *be a real analytic riemannian manifold and* η *a closed geodesic. Then there are positive constants* $\mu \in (0, 1)$, $\theta \in (0, 1/2)$, *and a* $C^{2+\mu}$ *neighbourhood* U *of* η *such that if a closed curve* γ *is in* U, *then*

$$\|D_v v\| \geq |E(\gamma) - E(\eta)|^{1-\theta} .$$

Again, let γ be a solution of equation (EP). If the manifold M is compact, then γ_t are C^0 bounded and Lemma 2.2 implies that γ_t are C^4 bounded, and so has a C^3 convergent subsequence. Let γ_∞ be its limiting closed curve. Since $\|D_{v_t} v_t\| \to 0$, γ_∞ is a closed geodesic. We apply Lemma 3.1 to $\eta = \gamma_\infty$. Fix a geodesic coordinate system around a point $\gamma_\infty(s_0)$. Take sufficiently large T so that $D_{v_t} v_t$ is sufficiently small for any $t \geq T$. If $t_1 \geq T$ and $\gamma_{t_1}(s_0)$ is close to $\gamma_\infty(s_0)$, then $(\frac{d}{ds})^2 \gamma_{t_1}(s)$ is sufficiently small in the coordinate. It means that if $t_1 \geq T$ and γ_{t_1} is close to γ_∞ in L_2 topology, then they are close in C^3 toplogy. Thus, Lemma 3.1 can be rewritten as the following

Lemma 3.2. *Let (M, g) and γ_∞ be as above. Then there are positive constants $\theta \in (0, 1/2)$, T and an L_2 neighbourhood V of γ_∞ such that if $t \geq T$ and $\gamma_t \in V$, then*

$$\|D_{v_t} v_t\| \geq (\|v_t\|^2 - \|v_\infty\|^2)^{1-\theta}.$$

Proof of Theorem B. Suppose that on a time interval (t_1, t_2), γ_t is in V and satisfies the above inequality. Then, for γ_t,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 &= -\|D_v v\|^2 = -\|D_v v\| \left\| \frac{d}{dt}\gamma \right\| \\ &\leq -(\|v\|^2 - \|v_\infty\|^2)^{1-\theta} \left\| \frac{d}{dt}\gamma \right\|. \end{aligned}$$

Therefore,

$$\begin{aligned} -\left\| \frac{d}{dt}\gamma \right\| &\geq \frac{1}{2}(\|v\|^2 - \|v_\infty\|^2)^{\theta-1} \frac{d}{dt}(\|v\|^2 - \|v_\infty\|^2) \\ &= \frac{1}{2\theta}\frac{d}{dt}(\|v\|^2 - \|v_\infty\|^2)^{\theta}. \end{aligned}$$

Thus, we get

$$\int_{t_1}^{t_2} \left\| \frac{d}{dt}\gamma \right\| dt \leq \frac{1}{2\theta}\left[(\|v_t\|^2 - \|v_\infty\|^2)^{\theta}\right]_{t_2}^{t_1}.$$

Let B_r be the L_2 ball in V centered at γ_∞ with radius r. If γ_t enters in $B_{r/2}$ at $t = t_1$ and leaves from B_r at $t = t_2$, we have $\int_{t_1}^{t_2} \|d\gamma/dt\| dt \geq r/2$. Thus, if γ_t repeats entering and leaving infinitely many times, we get $\int_I \|d\gamma/dt\| dt = \infty$, where $I = \{t; \gamma_t \in B_r\}$. This contradicts to the above inequality. Therefore, there exists a time T so that γ_t stays in B_r on $t \geq T$. Since r can be taken arbitrarily small, we conclude that γ_t converges to γ_∞ in L_2 topology. Thus, γ_t converges to γ_∞ in C^∞ topology by the remark below Lemma 3.1. Q.E.D.

4. A counter example

We recall Theorem 1.1. The uniqueness of the solution implies that if all initial data are invariant under a group action, then so is the solution γ_t.

Let f be a C^∞ function on $\boldsymbol{R}^2$ defined by the polar coordinate (r, θ) as

$$f(r, \theta) = \begin{cases} 0 & (r \leq 1) \\ (r-1)\left(2 + \sin\left(\dfrac{1}{r-1} + \theta\right)\right) e^{-1/(r-1)} & (r > 1) \end{cases}$$

We take a point h_0 outside the circle $r = 1$. Then the integral curve h_t of the gradient vector field $-\operatorname{grad} f$ closes to the circle $r=1$ when $t\to\infty$, but does not converge. This example is suggested by Professor O. Kobayashi.

We define a C^∞ riemannian metric g on the manifold $S^1\times \boldsymbol{R}^2=\{(u,x,y)\}$ as

$$\begin{cases} g(\partial_u, \partial_x) = g(\partial_u, \partial_y) = g(\partial_x, \partial_y) = 0\,, \\ g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = 1\,, \\ g(\partial_u, \partial_u) = 1+\phi(x,y) \quad (\phi(x,y) = f(r,\theta))\,. \end{cases}$$

We solve equation (EP) with initial data $\gamma_0(s)=(s, a, b)$, where a and b are constants satisfying $a^2+b^2>1$. Since the initial data are S^1 invariant, so is the solution γ_t. It means that the solution γ_t behaves like the integral curve h_t. In fact we easily compute that the solution $\gamma_t(s)=(s, x(t), y(t))$ is given by a solution of the equation: $\frac{d}{dt}(x,y)=-\frac{1}{2}\operatorname{grad}\phi$. We can easily relpace the manifold $S^1\times\boldsymbol{R}^2$ by a compact manifold, say $S^1\times T^2$.

References

[ES] J. Eells, Jr. and J.H. Sampson: *Harmonic mappings of riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[S] L. Simon: *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. **118** (1983), 525–571.

College of General Education
Osaka University
Toyonaka, Osaka 560
Japan