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THE SECOND VARIATION OF THE BERGMAN KERNEL OF ELLIPSOIDS

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Introduction. In this paper we shall study Fefferman's asymptotic expansion of the Bergman kernel of (real) ellipsoids in $\boldsymbol{C}^n$, $n\geq 2$. Regarding ellipsoids as perturbations of the ball, we compute the variations of the Bergmen kernel, and give the Taylor expansion of the log term coefficient to the second order in Webster's invariants. (The ellipsoids in normal form are parametrized by n real numbers, which we call Webster's invariants, and we shall consider the ellipsoids with small parameters.) As a consequence, we show that the vanishing of the log term of the Bergmen kernel characterizes the ball among these ellipsoids. In addition, we derive, from the procedure of computing the variation, a relation among the Bergmen kernels of different dimensional ellipsoids.

Let Ω be a smoothly bounded strictly pseudoconvex domain in $\boldsymbol{C}^n$, with defining function $f>0$ in Ω. It has been known since the work of Fefferman [5] that the Bergman kernel of Ω is written in the form

$$K(z,\bar{z})=\varphi(z)f(z)^{-n-1}+\psi(z)\log f(z)\,,\quad \text{where }\ \varphi,\psi\in C^\infty(\bar{\Omega})\,.$$

The coefficients φ, ψ were studied in Fefferman [6] and Graham [7], where they wrote down parts of the expansions of φ, ψ by using invariant polynomials in Moser's normal form coefficients. Fefferman expressed φ mod $O(f^{n-19})$ in terms of the Weyl invariants, and Graham in case $n=2$ determined φ explicitly. For further information on φ, in case $n\geq 3$, see [1].

In 2-dimensional case, Graham [7] further expressed the log term coefficient ψ mod $O(f^2)$ in an invariant manner, and showed that ψ vanishes if and only if the boundary is spherical, that is, locally biholomorphically equivalent to the sphere. (It is mentioned in [7] that an unpublished computation of D. Burns plays an essential role in characterizing spherical boundaries in terms of ψ.) The situation is rather simple in this 2-dimensional case, because the first invariant polynomial in ψ is linear. That polynomials is no longer linear in the higher dimensional case $n\geq 3$, and the analysis of Graham only gives its linear part. We are thus interested in getting information on the non-linear part.

We shall here compute the second variation of the log term for the class of ellipsoids of arbitrary dimension; the first variation always vanishes. This class provides a typical example of strictly pseudoconvex perturbations of the ball.

In computing these variations, we shall make use of Kashiwara's micro-local calculus for the Bergman kernel in [10], which is summarized in §3. He reduced the study of the Bergman kernel to that of a system of micro-differential equations, a simple holonomic system, of which a unique solution is given by the Bergman kernel. In order to study the parameter dependence of the Bergman kernel, we need to modify this characterization. Our idea is to consider a new simple holonomic system with more variables, regarding the parameters as additional variables, see §§4 and 5. From that system, we derive an algorithm of computing the variations of the Bergman kernel in §6, and apply it to the ellipsoids in §7. The first variational formula was obtained in [10] by a formal calculus. We generalize the calculus to our algorithm; a justification is also given. Finally in §8 we examine the dimension dependence of the variational formula.

We would like to mention that our algorithm of computing the variation is inspired by an article of Boutet de Monvel [2]; our Proposition 3 is obtained as an analogy of Theorem 3 in [2], see also Remark 5.3 below. Applying that Theorem 3, in [2, 3], he derived results similar to that of Graham [7] independently.

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1. Webster's classification of ellipsoids. In [13], Webster showed that two ellipsoids in $\boldsymbol{C}^n$, $n\geq 2$, are biholomorphically equivalent only when they are already equivalent by a complex linear transformation and that any ellipsoid other than the ball has no biholomorphic self-map except for linear transformations. As a corollary of these facts, he gave the following normal form for ellipsoids: After a linear change of coordinates, any ellipsoid

$$\sum_{i,j=1}^{n}(a_{ij}x_ix_j+b_{ij}y_iy_j+c_{ij}x_iy_j)<1\,, \quad \text{where } z_j=x_j+\sqrt{-1}\,y_j\,,$$

is written in the form

$$E(A_1,\cdots,A_n)=\{z\in\boldsymbol{C}^n: f_A(z,\bar{z})=1-|z|^2-\sum_{j=1}^{n}A_j(z_j^2+\bar{z}_j^2)>0\}\,,$$

where $-1/2<A_j<1/2$, and two ellipsoids $E(A)$ and $E(A')$ are biholomorphically equivalent if and only if $A=(A_1,\cdots,A_n)$ is equivalent to $A'=(A'_1,\cdots,A'_n)$ up to changes of signatures and a permutation.

2. Results. Let $K_A(z,\bar{z})$ be the Bergman kernel of $E(A)$ on the diagonal. Then,

Main Theorem. *The Bergman kernel K_A has the expansion $K_A=\varphi_A f_A^{-n-1}+\psi_A \log f_A$, where φ_A and ψ_A are real-analytic functions in a neighborhood of $\partial E(A)$. The log term coefficient ψ_A depends real-analytically on the parameters $A_1,\cdots,A_n$ and has the Taylor expansion*

$$\psi_A(z,\bar{z})=\frac{1}{\pi^n}\left(\frac{2n!\langle A^2\rangle}{(f_0-1)^{n+1}}+\frac{4(n+1)!\langle A^2z\bar{z}\rangle}{(f_0-1)^{n+2}}+\frac{(n+2)!\langle Az^2\rangle\langle A\bar{z}^2\rangle}{(f_0-1)^{n+3}}\right)$$
$$+(\text{terms of degree} \geq 3 \text{ in } A_1,\cdots,A_n)\,,$$

where

$$\langle A^2\rangle=\sum_{j=1}^n A_j^2\,,\quad \langle A^2z\bar{z}\rangle=\sum_{j=1}^n A_j^2 z_j\bar{z}_j\,,\quad \langle Az^2\rangle=\overline{\langle A\bar{z}^2\rangle}=\sum_{j=1}^n A_j z_j^2\,.$$

From this formula we can see, for small A_j, that ψ_A vanishes identically if and only if $E(A)$ is the ball, i.e., all $A_j=0$. See Remark 7.8.

Note also that the second order term of ψ_A, for n-dimensional ellipsoid, can be considered a function ψ_n of five variables $\langle Az^2\rangle$, $\langle A\bar{z}^2\rangle$, $\langle A^2z\bar{z}\rangle$, $\langle A^2\rangle$, and f_0. If we regard them as formal independent variables, then ψ_n satisfies the following relation

$$\psi_n=\left(\frac{-1}{\pi}\frac{\partial}{\partial f_0}\right)^{n-2}\psi_2\,.$$

In Proposition 8.1, we generalize this relation to the one among the whole expansions in A of the Bergman kernels of different dimensional ellipsoids. In that formula, the kernel functions are regarded as functions of infinite number of variables

$$\langle A^p z^2\rangle,\ \langle A^p\bar{z}^2\rangle,\ \langle A^p z\bar{z}\rangle,\ \langle A^p\rangle,\ p=1,2,\cdots,\quad\text{and}\quad f_0\,.$$

See Lemma 7.7 below for their definition. If we consider the convergence of the formula, we can get a relation in micro-local sense, see Corollary 8.4 below.

We prove the real-analytic dependence of the log term coefficient on an arbitrary finite diemsnional real-analytic perturbation of strictly pseudoconvex domains $\{\Omega_t\}_{t\in\boldsymbol{R}^m}$: Let $f(z,\bar{z},t)$ be a (complex-valued) real-analytic function on an open set $U\times\{t\colon|t|<\varepsilon\}\subset\boldsymbol{C}^n\times\boldsymbol{R}^m\cong\boldsymbol{R}^{2n}\times\boldsymbol{R}^m$ such that $\partial\Omega_t=\{z\in U\colon f(z,\bar{z},t)=0\}$ and $d_z f\neq 0$ on $\partial\Omega_t\times\{t\}$ for each t, and let $K_t(z,\bar{z})$ be the Bergman kernel of Ω_t on the diagonal.

Proposition 1. *There exist real-analytic functions φ, ψ defined in a neighborhood of $f(z,\bar{z},t)=0$ in $\boldsymbol{C}^n\times\boldsymbol{R}^m$ such that, for each fixed t,*

$$K_t(z,\bar{z})\equiv\varphi(z,\bar{z}\,t)f^{-n-1}(z,\bar{z},t)+\psi(z,\bar{z},t)\log f(z,\bar{z},t)\tag{2.1}$$

modulo real-analytic functions in a neighborhood of $\partial\Omega_t$ *in* C^n.

Namely, the Bergman kernel regarded as a (holomorphic) microfunction depends analytically on parameters.

In particular, the log term coefficient of K_t depends real-analytically on t, because ψ is uniquely determined by the modulo class, whereas φ is not.

The proof of Proposition 1 will be done (in §§4 and 5) by finding the simple holonomic system which characterizes $K_t(z, w)$ as a function of (z, w, t)-variables; the right side of (2.1) is obtained by solving that system. See Proposition 2 in §4; there (2.1) is written in an intrinsic form (4.5).

3. Kashiwara's analysis. In this section we shall recall Kashiwara's analysis [10] of the Bergman kernel. In [10], he used some fundamental facts from the theory of simple holonomic systems, which are contained in Sato, Kawai and Kashiwara [11]. A self-contained introduction to the theory is now given e.g. by Chap. I of [12] we shall cite this book when we use those facts.

We begin with the following observation: Let Ω_1, Ω_2 be strictly pseudoconvex domains with real-analytic boundaries which coincide near a point z^0. Then their Bergman kernels $K_1(z, w)$, $K_2(z, w)$ differ by a holomorphic function in a neighborhood of $(z^0, \overline{z^0})\in C^n\times\overline{C^n}$. Namely, the Bergman kernel is determined, modulo holomorphic functions in a neighborhood of the complexification of the boundary, by local information of the boundary. (This fact is implicitly contained in [10]; for a proof of it see [9]). Since we are now interested in Fefferman's asymptotic expansion of the Bergman kernel, we shall work locally near a boundary point and consider the modulo classes. Such a class is called a holomorphic microfunction.

Let us recall the definition of holomorphic microfunctions. Let Y be a complex hypersurface in a complex manifold X defined by $f=0$. For a point (x, ξ) in $\dot{T}^*_Y X$, the conormal bundle of Y with the zero section deleted, a germ of holomorphic microfunction at (x, ξ) is a class of a multi-valued function $af^{-k}+b\log f \bmod \mathcal{O}_{X,x}$, where $a, b\in\mathcal{O}_{X,x}$, and $k\in \boldsymbol{N}$. The sheaf of holomorphic microfunctions with support $\dot{T}^*_Y X$ is denoted by $\mathcal{C}_{Y|X}$. In the case of the Bergman kernel, we take $f(z, w)$ to be the complexification of a defining function of a domain and let $Y=\{(z, w): f(z, w)=0\}\subset X=C^n\times\overline{C^n}$; so that the Bergman kernel $K(z, w)=\varphi(z, w)f^{-n-1}(z, w)+\psi(z, w)\log f(z, w)$, mod $\mathcal{O}_X$, defines a section of $\mathcal{C}_{Y|X}$ (see Theorem 0 [10] and Remark 3.3 below).

By the general theory of holonomic systems, a holomorphic microfunction is characterized as a unique solution, up to constant multiple, of a system of micro-differential (i.e., classical analytic pseudo-differential) equations, which is called a simple holonomic system. See Lemma 2.1 [10] and §4.2 of [12]. In the case of the Bergman kernel, the corresponding simple holonomic system is given by the following theorem.

Kashiwara's Theorem ([10]). *Let Ω be a strictly pseudoconvex domain with real-analytic defining function $f(z, \bar{z})$. Then the Bergman kernel of Ω satisfies simple holonomic system*

$$(3.1)\qquad \begin{cases} (z_j+P_j(w, D_w))K(z, w) = 0\,, & j=1, \cdots, n\,, \\ (\partial/\partial z_j+Q_j(w, D_w))K(z, w) = 0\,, & j=1, \cdots, n\,, \end{cases}$$

where P_j, Q_j are the micro-differential operators uniquely determined by the relations

$$\begin{cases} (z_j+P_j^*(w, D_w)) \log f(z, w) = 0\,, & j=1, \cdots, n\,, \\ (\partial/\partial z_j-Q_j^*(w, D_w)) \log f(z, w) = 0\,, & j=1, \cdots, n\,. \end{cases}$$

Here the Bergman kernel and $\log f$ *are regarded as holomorphic microfunctions and P_j^*, Q_j^* denote the adjoint operators of P_j, Q_j.*

REMARK 3.2. In [10], this theorem is stated as

$$(P(z,D_z)+Q(w,D_w))K=0, \quad \text{whenever } (P^*(z,D_z)+Q^*(w,D_w)) \log f=0\,.$$

The $2n$ equations in (3.1) give generators of this system.

REMARK 3.3. The theorem above is stated in Kashiwara [10] with only a heuristic proof. A justification of the proof is now available in Kaneko [9], which is based on Kashiwara's lectures at Kyushu University in 1979; this article also contains a proof of Theorem 0 [10].

4. Parameter dependence of the Bergman kernel. In this and next sections we prove Proposition 1.

We begin by recalling Hörmander's formula (Theorem 3.5.1 [8]) which determines φ mod $O(f)$, the principal part of K_t:

$$(4.1)\qquad K_t(z, \bar{z}) = \frac{n!}{\pi^n} J(f) f^{-n-1}+O(f^{-n})\,,$$

where J denotes the complex Monge-Ampère operator

$$(4.2)\qquad J(f) = (-1)^n \det\begin{pmatrix} f & \partial f/\partial \bar{z}_k \\ \partial f/\partial z_j & \partial^2 f/\partial z_j \partial \bar{z}_k \end{pmatrix}.$$

Using this formula and Kashiwara's theorem, we shall find φ, ψ satisfying (2.1) by the following process: By Kashiwara's theorem, we have $2n$ equations (3.1) which characterize K_t, up to a constant multiple, for each t. We add m equations containing t derivatives to (3.1) in such a way that the resulting $2n+m$ equations form a simple holonomic system and a solution of the system has the principal part given by (4.1). Then that solution gives the right side of (2.1), because it satisfies (3.1) for each fixed t.

In order to state the $2n+m$ equations, we set $\widetilde{X}=\boldsymbol{C}^n\times\overline{\boldsymbol{C}^n}\times\boldsymbol{C}^m$ and continue $f(z,\bar{z},t)$ to a holomorphic function $f(z,w,t)$ on an open set of $\widetilde{X}$. We then regard $\log f(z,w,t)$ as a holomorphic microfunction, a section of $\mathcal{C}_{\widetilde{Y}|\widetilde{X}}$, where $\widetilde{Y}=\{(z,w,t)\in\widetilde{X}: f(z,w,t)=0\}$.

Proposition 2. *The Bergman kernel $K_t(z,w)$ satisfies the simple holonomic system*

$$(4.3)\qquad\begin{cases}(z_j+P_j(w,t,D_w))K_t(z,w)=0\,, & j=1,\cdots,n\,,\\ (\partial/\partial z_j+Q_j(w,t,D_w))K_t(z,w)=0\,, & j=1,\cdots,n\,,\\ (\partial/\partial t_k+R_k(w,t,D_w))K_t(z,w)=0\,, & k=1,\cdots,m\,,\end{cases}$$

where P_j, Q_j, R_k are the micro-differential operators in w-variable with holomorphic parameter t which are uniquely determined by the relations

$$(4.4)\qquad\begin{cases}(z_j+P_j^*(w,t,D_w))\log f(z,w,t)=0\,, & j=1,\cdots,n\,,\\ (\partial/\partial z_j-Q_j^*(w,t,D_w))\log f(z,w,t)=0\,, & j=1,\cdots,n\,,\\ (\partial/\partial t_k-R_k^*(w,t,D_w))\log f(z,w,t)=0\,, & k=1,\cdots,m\,.\end{cases}$$

More precisely, (4.3) *means that there exists a solution $u(z,w,t)$ of the system* (4.3) *in $\mathcal{C}_{\widetilde{Y}|\widetilde{X}}$ which satisfies*

$$(4.5)\qquad u(z,w,t)=K_t(z,w)\quad\text{in } \mathcal{C}_{Y_t|X} \text{ for each fixed } t\in\boldsymbol{R}^m\,,$$

where $Y_t=\{(z,w): f(z,w,t)=0\}\subset X=\boldsymbol{C}^n\times\overline{\boldsymbol{C}^n}$.

If u in (4.5) is the class modulo $\mathcal{O}_{\widetilde{X}}$ of $\varphi f^{-n-1}+\psi\log f$, then the restriction of (4.5) on the diagonal $w=\bar{z}$ gives (2.1).

It is possible to prove this proposition without using Hörmander's formula, by generalizing the proof of Kashiwara's theorem; however, the proof requires some preparations from the theory of holonomic systems, which are out of the aim of this paper. We thus give here a proof which uses (4.1) and the calculus in Boutet de Monvel [2].

5. Proof of Proposition 2.

We first examine the relations (4.4).

Lemma. *For any micro-differential operator $P(z,w,t,D_z,D_w,D_t)$ defined in a neighborhood of a point of $\dot{T}^*_{\widetilde{Y}}\widetilde{X}$, there exists a unique micro-differential operator $Q(w,t,D_w)$ in w-variable with holomorphic parameter t such that $(P-Q)\log f=0$.*

There is no difficulty in generalizing the proof of the existence of quantizations of contact transformations to this case, see [12] Chap. I §5. The key is the following fact. The projection $p: \dot{T}^*_{\tilde{Y}}\tilde{X} \to T^*_{C^m}(\overline{C^n}\times C^m)$ induced by $T^*\tilde{X} = T^*(C^n\times\overline{C^n}\times C^m) \to T^*(\overline{C^n}\times C^m)$ is an open immersion, i.e., dp is bijective; this follows from the strict pseudoconvexity of Ω_t. The surjectivity and the injectivity of dp imply the existence and the uniqueness of Q respectively.

By the Lemma above, we can find the operators

$$z_j + P_j^*(w, t, D_w),\ \partial/\partial z_j - Q_j^*(w, t, D_w), \quad j = 1, \cdots, n,$$
$$\partial/\partial t_k - R_k^*(w, t, D_w), \quad k = 1, \cdots, m,$$

satisfying (4.4). In order to show that their adjoints (4.3) form a simple holonomic system, let us recall a sufficient condition for a system of microdifferential equations to be holonomic, that is, a system $P_1u = \cdots = P_nu = 0$ is holonomic if P_j commute each other and if the principal symbols of P_j form a defining system of a Lagrangian manifold, see Propostion 4.1.5 of [12]. In the present case, the operators in (4.3) commute, because their adjoint operators (4.4) commute, and their principal symbols form a definning system of the conormal bundle $\dot{T}^*_{\tilde{Y}}\tilde{X}$. Thus (4.3) is a simple holonomic system.

If we take a solution $u \in \mathcal{C}_{\tilde{Y}|\tilde{X}}$ of the system (4.3), then for each fixed t we have $K_t(z, w) = c_t u(z, w, t)$ in $\mathcal{C}_{Y_t|X}$ with a constant c_t. This is because the first $2n$ equations of (4.3) coincide with the holonomic system for the Bergman kernel of Ω_t given by Kashiwara's theorem and its solution is unique up to a constant multiple. It therefore suffices to show that c_t does not depend on t.

First we consider the case in which $\{\partial\Omega_t\}_{t\in R^m}$ is locally in Moser's *partial* normal form (we de not require the trace conditions):

$$2\,\mathrm{Re}\, z_1 + |z'|^2 + F(z', \overline{z'}, \mathrm{Im}\, z_1, t) = 0. \tag{5.1}$$

Here F is a real-analytic function in a neighborhood of the origin of $C^{n-1}\times R\times R^m$ such that $F(z', w', (z_1 - w_1)/2i, t) = O(|z'|^2|w'|^2)$, where $z' = (z_2', \cdots, z_n)$, $w' = (w_2, \cdots, w_n)$. If we replace $\bar{z}$ in (5.1) by w and solve the resulting equation for z_1-variable, we obtain a defining function f of the complexification of $\partial\Omega_t$ such that

$$f(z, w, t) = z_1 + w_1 + z'\cdot w' + \rho(z', w, t), \quad \text{with} \quad \rho(z', w, t) = O(|z'|^2|w'|^2).$$

In this case, we can compute R_k^* in (4.4) by using the calculus in §5 of Boutet de Monvel [2], see also Remark 5.3 below. Differentiating (31) of [2] with respect to our parameter t, we get

$$R_k^*(w, t, D_w) = [\partial/\partial t_k, A(w, t, D_w)] \circ A^{-1}(w, t, D_w), \tag{5.2}$$

where $A(w, t, D_w)$ is the formal micro-differential operator of infinite order with the total symbol

$$A(w, t, \omega) = \exp(\rho(\omega'/\omega_1, w, t)\omega_1)\,.$$

Thus $\rho(z', w, t)=O(|w'|^2)$ implies $R_k^*(w, t, \omega)=O(|w'|^2)$. So, in view of $R_k(w, t, \omega)=e^{\langle D_w, D_\omega\rangle}R_k^*(w, t, -\omega)$, we see that $R_k(w_1, 0, \cdots, 0, t, D_w)$ has order at most -1. Hence $\frac{\partial}{\partial t_k}u=-R_k u$ gives $\frac{\partial}{\partial t_k}u|_{z'=w'=0}=O((z_1+w_1)^{-n})$. In particular, writing $u=\varphi f^{-n-1}+\psi \log f$, we have $\frac{\partial}{\partial t_k}\varphi|_{z=w=0}=0$. Therefore (4.1) and $J(f)|_{z=w=0}=(-1)^{n+1}$ imply that c_t is independent of t.

In order to reduce the general case to the case we studied above, it suffices to construct a real-analytic family of partial normal forms for a given family of domains. This is because the Bergman kernel and the solution of the system (4.3) satisfy the same transformation rule under a change of coordinates. It is almost clear by inspecting Moser's construction in [4] that the partial normal form (and in fact also the normal form) can be chosen so that they vary real-analytically with a parameter. To explain it more precisely, let us first recall that a partial normal form was uniquely specified in [4] by giving a curve γ in the boundary $\partial\Omega$, where γ is to be transformed into the Im z_1-axis. In [4], it was shown that the partial nromal form depends real-analytically on a parameter t if so does the curve γ_t in $\partial\Omega$. And, the proof was done by using only the implicit function theorem and Schmidt's orthognalization procedure. Therefore, the same proof applies to the present case, in which both the boundary $\partial\Omega_t$ and the curve γ_t in $\partial\Omega_t$ depend real-analytically on a parameter t.

REMARK 5.3. In order to explain the meaning of the formula (5.2) above, we shall birefly recall the calculus in §5 of Boutet de Monvel [2]. For strictly pseudoconvex domains which are locally written in Moser's normal form, he introduced a weight — similar to the one used by Moser in [4] — on micro-differential operators and holomorphic microfunctions, and considered the asymptotic expansions of the micro-differential operators with respect to the weight. He also considered the formal micro-differential operators of infinite order which admit such asymptotic expansions (e.g. $A(w, t, D_w)$ above) and showed that the compositions and the adjoints for these operators are defined naturally. By using these operators, he derived an asymptotic expansion formula of the Bergman kernel with respect to the weight (Theorem 3 [2]).

The formula (5.2) means that the asymptotic expansion of R_k^* with respect to the weight is given by the right side: the existence of R_k^* as a micro-differential operator is known from the lemma stated in this section, and its asymptotic expansion with respect to the weight can be explicitly computed by the calculus explained above.

6. Variational formula. Using Proposition 2, we shall derive an algorithm of computing the variations of the Bergman kernel.

The first variation is obtained easily: Substituting $t=0$ to the last m equations in (4.3) and (4.4), we get

$$\frac{\partial}{\partial t_j} K_t(z, w)|_{t=0} = -R_j(z, 0, D_z)K_0(z, w)\,, \quad j = 1, \cdots, m\,, \tag{6.1}$$

$$\frac{\partial}{\partial t_j} \log f(z, w, 0) = R_j^*(z, 0, D_z) \log f(z, w, 0)\,, \quad j = 1, \cdots, m\,. \tag{6.2}$$

We can determine $R_j(z, 0, D_z)$ by the relation (6.2), and then (6.1) gives $\frac{\partial}{\partial t_j} K_t(z, w)|_{t=0}$ from K_0. If we consider the Taylor expansions of K_t and $\log f$ in t, these formulas can be written as

$$K_t(z, w) \equiv \Big(1-\sum_{j=1}^m t_j R_j(z, 0, D_z)\Big)K_0(z, w) \quad \mathrm{mod}\ O(t^2)\,,$$

$$\log f(z, w, t) \equiv \Big(1+\sum_{j=1}^m t_j R_j^*(z, 0, D_z)\Big) \log f(z, w, 0) \quad \mathrm{mod}\ O(t^2)\,.$$

These formulas were obtained in [10] (Proposition 7.1); the arguments above gives a justification of the formal calculus used there.

In order to generalize these formulas to a higher order, let us introduce the notion of asymptotic expansions of holomorphic microfunctions in parameters. For a germ $u(z, w, t) \in \mathcal{C}_{\tilde{Y}|\tilde{X}}$, we consider the formal power series $\sum_\alpha u_\alpha(z, w)\frac{t^\alpha}{\alpha!}$ with coefficients $u_\alpha(z, w) = \left(\frac{\partial}{\partial t}\right)^\alpha u(z, w, 0) \in \mathcal{C}_{Y_0|X}$. This power series is called the asymptotic expansion of u in t and expressed as

$$u(z, w, t) \sim \sum_\alpha u_\alpha(z, w)\frac{t^\alpha}{\alpha!}\,.$$

We do not discuss the convergence of this series. However, we can easily show that the map $u \mapsto \sum_\alpha u_\alpha \frac{\alpha!}{t^\alpha}$ is injective, and thus the expansion uniquely determines u.

Proposition 3. *If $P_\alpha(z, D_z)$ are the micro-differential operators satisfying*

$$\log f(z, w, t) \sim \sum_\alpha \frac{t^\alpha}{\alpha!} P_\alpha(z, D_z) \log f_0(z, w), \quad \textit{where } f_0(z, w) = f(z, w, 0), \tag{6.3}$$

then the Bergman kernel has the asymptotic expansion

$$K_t \sim \sum_{l=0}^\infty \Big(1-\sum_\alpha \frac{t^\alpha}{\alpha!} P_\alpha^*\Big)^l K_0\,. \tag{6.4}$$

In particular, if Ω_0 is the unit ball, then

(6.5) $$K_t(z, w) \sim \frac{n!}{\pi^n} \sum_{l=0}^{\infty} \Big(1 - \sum_\alpha \frac{t^\alpha}{\alpha!} P_\alpha^*(z, D_z)\Big)^l (1 - z \cdot w)^{-n-1}.$$

In the right side of (6.4), since $P_0 = 1$, the Neumann series of $\sum_\alpha \dfrac{t^\alpha}{\alpha!} P_\alpha^*$ defines a formal power series in t with coefficients in micro-differential operators. Note that the set of formal sums of the form $\sum_\alpha P_\alpha(z, w, D_z, D_w)t^\alpha$ naturally forms a ring, where each P_α is a micro-differential operator in (z, w). This ring contains the subring consist of micro-differential operators in (z, w) with holomorphic parameter t, and acts on formal power series in t with coefficients in $C_{Y_0|X}$.

Proof of Proposition 3. The micro-differential operators P_α are determined by the relations $\left(\dfrac{\partial}{\partial t}\right)^\alpha \log f|_{t=0} = P_\alpha(z, D) \log f_0$. We set $\boldsymbol{P}(z, t, D_z) = \sum_\alpha \dfrac{t^\alpha}{\alpha!} P_\alpha(z, D_z)$ and write (6.3) as $\log f \sim \boldsymbol{P} \log f_0$. Substituting this expansion into $(\partial/\partial t_k - R_k^*) \log f = 0$, we have $\left(\dfrac{\partial}{\partial t_k}\boldsymbol{P} - R_k^* \circ \boldsymbol{P}\right) \log f_0 = 0$, or,

(6.6) $$\Big(\boldsymbol{P}^{-1}(z, t, D_z) \circ \frac{\partial}{\partial t_k} \boldsymbol{P}(z, t, D_z) - R_k^*(w, t, D_w)\Big) \log f_0(z, w) = 0\,,$$

where $\boldsymbol{P}^{-1}$ is defined by the Neumann series $\boldsymbol{P}^{-1} = \sum_{l=0}^\infty (1 - \boldsymbol{P})^l$.

Similarly, if we write $K_t(z, w) \sim \boldsymbol{Q}(z, t, D_z) K_0(z, w)$, the last m equations in (4.3) are written as

(6.7) $$\Big(\boldsymbol{Q}^{-1}(z, t, D_z) \circ \frac{\partial}{\partial t_k} \boldsymbol{Q}(z, t, D_z) + R_k(w, t, D_w)\Big) K_0(z, w) = 0\,.$$

Thus, applying Kashiwara's theorem (see also Remark 3.2) to each coefficient of t^α in (6.6) and (6.7), we get $(\boldsymbol{P}^{-1} \circ \dfrac{\partial}{\partial t_k} \boldsymbol{P})^* = -\boldsymbol{Q}^{-1} \circ \dfrac{\partial}{\partial t_k} \boldsymbol{Q}$, or, $(\dfrac{\partial}{\partial t_k} \boldsymbol{P}^*) \circ (\boldsymbol{P}^*)^{-1} = -\boldsymbol{Q}^{-1} \circ \dfrac{\partial}{\partial t_k} \boldsymbol{Q}$. Hence $\dfrac{\partial}{\partial t_k}(\boldsymbol{Q} \circ \boldsymbol{P}^*) = \dfrac{\partial}{\partial t_k} \boldsymbol{Q} \circ \boldsymbol{P}^* + \boldsymbol{Q} \circ \dfrac{\partial}{\partial t_k} \boldsymbol{P}^* = 0$. Therefore $\boldsymbol{Q}(z, 0, D_z) \circ \boldsymbol{P}^*(z, 0, D_z) = 1$ implies that $\boldsymbol{Q} = (\boldsymbol{P}^*)^{-1}$, which proves the proposition.

7. Computation of the log term for ellipsoids. In this section we shall apply Proposition 3 to ellipsoids and prove Main Theorem.

In the case of ellipsoids, the defining function $f_A(z, w)$ is a symmetric polynomial in the triples (A_j, z_j, w_j), i.e., $f_A(z, w) = f_{\sigma A}(\sigma z, \sigma w)$ for any permutation σ of n elements. Thus the algorithm of computing the Bergman kernel can be carried out by using only the operators having the same symmetry, so we set

$$\langle zw\rangle=\sum_{j=1}^{n} z_j w_j\,,\quad \langle Az^2\rangle=\sum_{j=1}^{n} A_j z_j^2\,,\quad \langle Aw^2\rangle=\sum_{j=1}^{n} A_j w_j^2\,,$$

and, replacing w_j by $D_j=\partial/\partial z_j$, we also set

$$\langle zD\rangle=\sum_{j=1}^{n} z_j D_j\,,\quad \langle AD^2\rangle=\sum_{j=1}^{n} A_j D_j^2\,.$$

Then we can write down the explicit formulas of the operators appearing in Proposition 3.

Proposition 7.1. *The micro-differential operator of infinite order*

$$\boldsymbol{P}(z,A,D)=1+\sum_{p+q\geq 1}\frac{\langle Az^2\rangle^p}{p!}\frac{\langle AD^2\rangle^q}{q!}L_{p-q}(\langle zD\rangle)$$

satisfies $\log f_A\sim \boldsymbol{P}(z,A,D)\log f_0$. *Here $L_m(x)$ are rational functions*

$$L_m(x)=\begin{cases} x(x+1)\cdots(x+m-1), & m>0,\\ 1, & m=0,\\ (x-1)^{-1}(x-2)^{-1}\cdots(x+m)^{-1}, & m<0.\end{cases}$$

Then the Bergman kernel K_A of ellipsoids $E(A)$ satisfies $K_A\sim\dfrac{n!}{\pi^n}\boldsymbol{P}^{*-1}f_0^{-n-1}$, and the operator $\boldsymbol{P}^{*-1}$ can be written

$$\begin{aligned}\boldsymbol{P}^{*-1}=1+\sum_{l=1}^{\infty}\sum_{p_j+q_j\geq 1}&\langle AD^2\rangle^{q_1}\langle Az^2\rangle^{p_1}\langle AD^2\rangle^{q_2}\langle Az^2\rangle^{p_2}\cdots\\ &\cdots\langle AD^2\rangle^{q_l}\langle Az^2\rangle^{p_l}\Phi_{p,q}(\langle zD\rangle^*)\,,\end{aligned}\tag{7.2}$$

where $\Phi_{p,q}(x)$ are rational functions

$$\Phi_{p,q}(x)=\frac{1}{p!q!}\prod_{j=1}^{l}-L_{m_j}(x-2(m_j+\cdots+m_l))\,,\quad m_j=p_j-q_j\,.$$

Proof. The expansion of $\log f_A$ in A is expressed as

$$\begin{aligned}\log f_A&\sim\log f_0-\sum_{l=0}^{\infty}\frac{1}{l}(\langle Az^2\rangle+\langle Aw^2\rangle)^l f_0^{-l}\\ &=\log f_0-\sum_{p+q\geq 1}\frac{(p+q-1)!}{p!q!}\langle Az^2\rangle^p\langle Aw^2\rangle^q f_0^{-p-q}\,,\end{aligned}$$

and the sum in p,q can be written

$$\begin{aligned}&\sum_{p>q}\frac{(p-q-1)!}{p!q!}\langle Az^2\rangle^p\langle AD^2\rangle^q f_0^{-p+q}\\ &\qquad-\sum_{p\leq q}\frac{\langle Az^2\rangle^p\langle AD^2\rangle^q}{p!q!(q-p)!}(-f_0)^{-p+q}\log f_0.\end{aligned}$$

Therefore, in view of

$$L_m(\langle zD\rangle)\log f_0=\begin{cases}-(m-1)!\,f_0^{-m}, & m>0,\\ \dfrac{(-f_0)^{-m}}{(-m)!}\log f_0, & m\leq 0,\end{cases}$$

we get

$$\log f_A\sim\Big[1+\sum_{p+q\geq 1}\frac{\langle Az^2\rangle^p}{p!}\frac{\langle AD^2\rangle^q}{q!}L_{p-q}(\langle zD\rangle)\Big]\log f_0 .$$

The formula (7.2) is obtained by expanding the Neumann series $P^{*-1}=\sum_{l=0}^{\infty}(1-P^*)^l$ and using the relation

$$\Phi(\langle zD\rangle^*)\langle AD^2\rangle^q\langle Az^2\rangle^p=\langle AD^2\rangle^q\langle Az^2\rangle^p\,\Phi(\langle zD\rangle^*-2p+2q),$$

where $\Phi(x)$ is any rational function. □

In order to compute the operation of each term in the right side of (7.2), we prepare two lemmas. First, in order to compute

$$\Phi_{p,q}(\langle zD\rangle^*)n!\,f_0^{-n-1}=\Phi_{p,q}(-\langle zD\rangle-n)n!\,(1-\langle zw\rangle)^{-n-1} \tag{7.3}$$

we derive the following formulas in the one-variable case.

Lemma 7.4. (i) *Let $\Phi(x)$ be a rational function and $D_x=\frac{d}{dx}$. Then*

$$\Phi(xD_x+n)n!\,(1-x)^{-n-1}=D_x^n\Phi(xD_x)(1-x)^{-1}\ \textit{in}\ \mathcal{C}_{\{1\}|C}. \tag{7.5}$$

(ii) *If $l, m\in\mathbf{Z}$ and $l\geq 1$, then*

$$(xD_x-m)^{-l}(1-x)^{-1}=-\frac{x^m(\log x)^{l-1}}{(l-1)!}\log(1-x)\ \textit{in}\ \mathcal{C}_{\{1\}|C}. \tag{7.6}$$

Proof. (i) The equation (7.5) follows from

$$D_x^n\Phi(xD_x)=\Phi(D_x^n(xD_x)D_x^{-n})D_x^n=\Phi(xD_x+n)D_x^n .$$

(ii) Let $\psi(x)$ be a holomorphic function near 1 satisfying

$$(xD_x-m)^{-l}(1-x)^{-1}=\psi(x)\log(1-x).$$

Then we must have $(xD_x-m)^l\psi(x)=0$, because the logarithmic singularity at $x=1$ of $(xD_x-m)^l(\psi(x)\log(1-x))$ is equal to that of $((xD_x-m)^l\psi(x))\log(1-x)$ and it must vanish. Thus ψ is written in the form

$$\psi(x)=\sum_{k=0}^{l-1}c_kx^m(\log x)^k .$$

To determine the constants c_k, we compare $(1-x)^{-1}$ with

$$(xD_x-m)^l\psi(x)\log(1-x) = -\sum_{k=0}^{l-1} k!\,c_k(xD_x-m)^{l-k-1}(1-x)^{-1}.$$

Then we get $c_0=c_1=\cdots=c_{l-2}=0$ and $-(l-1)!\,c_{l-1}=1$. □

The operation of $\langle AD^2\rangle$ and $\langle Az^2\rangle$ can be computed by using:

Lemma 7.7. *Define*

$$\langle A^p\rangle = \sum_{j=1}^n A_j^p\,,\quad \langle A^p z^2\rangle = \sum_{j=1}^n A_j^p z_j^2\,,\quad \langle A^p zw\rangle = \sum_{j=1}^n A_j^p z_j w_j$$

and, with z_j replaced by w_j or $D_j=\partial/\partial z_j$, define also $\langle A^p w^2\rangle$, $\langle A^p zD\rangle$, $\langle A^p wD\rangle$, $\langle A^p D^2\rangle$ in the same manner. Then the following commutation relations hold

$$[\langle A^pD^2\rangle, \langle A^qz^2\rangle] = 4\langle A^{p+q}zD\rangle+2\langle A^{p+q}\rangle,\quad [\langle A^pD^2\rangle, \langle A^qzw\rangle] = 2\langle A^{p+q}wD\rangle,$$
$$[\langle A^pzD\rangle, \langle A^qz^2\rangle] = 2\langle A^{p+q}z^2\rangle,\qquad [\langle A^pzD\rangle, \langle A^qzw\rangle] = \langle A^{p+q}zw\rangle,$$
$$[\langle A^pwD\rangle, \langle A^qz^2\rangle] = 2\langle A^{p+q}zw\rangle,\qquad [\langle A^pwD\rangle, \langle A^qzw\rangle] = \langle A^{p+q}w^2\rangle,$$

and, for any $\psi(x)\in C_{\{1\}|C}$, we have

$$\langle A^pD^2\rangle\psi(\langle zw\rangle) = \langle A^pw^2\rangle\psi''(\langle zw\rangle),$$
$$\langle A^pzD\rangle\psi(\langle zw\rangle) = \langle A^pzw\rangle\psi'(\langle zw\rangle),$$
$$\langle A^pwD\rangle\psi(\langle zw\rangle) = \langle A^pw^2\rangle\psi'(\langle zw\rangle).$$

The proof is straightforward, and we will omit it.

Now we are ready to compute the explicit formula of the Bergman kernel for ellipsoids.

Proof of Main Theorem. If we differentiate both sides of the formula $K_A=\varphi_A f_A^{-n-1}+\psi_A\log f_A$ in A, we have $\left(\frac{\partial}{\partial A}\right)^\alpha K_A|_{A=0}=\left(\frac{\partial}{\partial A}\right)^\alpha \psi_A|_{A=0}\log f_0+$(terms without logarithmic singularity). It follows that ψ_A is equal to the sum of the coefficients of $\log f_0$ in the expansion of K_A in A, and thus the computation of ψ_A is reduced to that of the log terms in $\frac{n!}{\pi^n}P^{*-1}f_0^{-n-1}$.

When we compute the log terms in $P^{*-1}f_0^{-n-1}$, it suffices to examine the terms in (7.2) with indices $p=(p_1,\cdots,p_l)$, $q=(q_1,\cdots,q_l)$ satisfying:

(i) There exists a number j such that $p_j<q_j$.

This is because $\Phi_{p,q}(\langle zD\rangle^*)f_0^{-n-1}$ contains no log term if $\Phi_{p,q}(x)$ has no pole. Moreover, since the Bergman kernel is hermitian symmetric, we can restrict our attention to the terms of which the degree in z is equal or greater than that in w, i.e., the terms with indices satisfying:

(ii) $p_1+p_2+\cdots+p_l\geq q_1+q_2+\cdots+q_l$.

As long as we look at the terms having order less than 3, i.e., the terms with indices satisfying

$$\text{(iii)} \qquad p_1+p_2+\cdots+p_l+q_1+q_2+\cdots+q_l<3\,,$$

the conditions (i) (ii) are satisfied by only two pairs of indices: $p=(1, 0)$, $q=(0, 1)$ and $p=(0, 1)$, $q=(1, 0)$. In each case $\Phi_{p,q}$ is given by

$$\Phi_{(1,0),(0,1)}(x) = L_1(x)L_{-1}(x+2) = \frac{x}{x+1} = 1-\frac{1}{x+1}\,,$$
$$\Phi_{(0,1),(1,0)}(x) = L_{-1}(x)L_1(x-2) = \frac{x-2}{x-1} = 1-\frac{1}{x-1}\,.$$

Therefore Lemma 7.4 implies that the log term of the Bergman kernel modulo $O(A^3)$ is contained in

$$\frac{-1}{\pi^n}\Big[\langle Az^2\rangle\langle AD^2\rangle D_x^n|_{x=\langle zw\rangle}(x\log(1-x)) + \langle AD^2\rangle\langle Az^2\rangle D_x^n|_{x=\langle zw\rangle}(x^{-1}\log(1-x))\Big]\,.$$

We can omit the first term because $D_x^n(x\log(1-x))$, with $n\geq 2$, contains no logarithmic singularity at $x=1$. In the second term, its logarithmic singulairty at $\langle zw\rangle=1$ equals to $(\langle AD^2\rangle\langle Az^2\rangle D_x^n|_{x=\langle zw\rangle}x^{-1})\log f_0$. Therefore, using Lemma 7.7, we have

$$\begin{aligned}
\pi^n\psi_A(z,w) &= -\langle AD^2\rangle\langle Az^2\rangle D_x^n|_{x=\langle zw\rangle}x^{-1}+O(A^3)\\
&= -(\langle Az^2\rangle\langle AD^2\rangle+4\langle A^2zD\rangle+2\langle A^2\rangle)D_x^n|_{x=\langle zw\rangle}x^{-1}+O(A^3)\\
&= -(\langle Az^2\rangle\langle Aw^2\rangle D_x^2+4\langle A^2zw\rangle D_x+2\langle A^2\rangle)D_x^n x^{-1}|_{x=\langle zw\rangle}+O(A^3)\\
&= \langle Az^2\rangle\langle Aw^2\rangle(n+2)!\,(-\langle zw\rangle)^{-n-3}+4\langle A^2zw\rangle(n+1)!\,(-\langle zw\rangle)^{-n-2}\\
&\qquad +2\langle A^2\rangle n!\,(-\langle zw\rangle)^{-n-1}+O(A^3)\,.
\end{aligned}$$

Note that the last formula is real, and thus no term is omitted by the condition (ii). Therefore, substituting $\langle zw\rangle=1-f_0$, we get Main Theoıem.

REMARK 7.8. If we substitute, for example, $z^0=(1, 0, \cdots, 0)$ in ψ_A, we get

$$\begin{aligned}
(-1)^{n+1}\frac{\pi^n}{n!}\psi_A(z^0, \bar{z}^0) &= 2\langle A^2\rangle+(n-2)(n+1)A_1^2+O(A^3)\\
&\geq 2\langle A^2\rangle+O(A^3)\,.
\end{aligned}$$

This implies that $(-1)^{n+1}\psi_A(z^0, \bar{z}^0)$ is positive if $\langle A^2\rangle>0$ small.

8. Relation among the Bergman kernels of different dimensional ellipsoids.

In the procedure of computing the Bergman kernel of ellipsoids, the dimension n only appears in the formula (7.3). In (7.3), replacing n by $n+1$ is equivalent to differentiating both sides formally with respect to the variable f_0, see (7.5). Namely, we can write the relation among the Bergman kernels of different dimensional ellipsoids by utilizing the f_0-derivatives.

Proposition 8.1. *Let K_n be the Bergman kernel of ellipsoid $E(A_1, \cdots, A_n)$ in $\boldsymbol{C}^n$.*
(i) *Consider the asymptotic expansion of K_n in A*

$$K_n \sim \sum_{l=0}^{\infty} K_{n,l}\,,$$

where $K_{n,l}$ is homogeneous degree l in $A_1, \cdots, A_n$. Then each $K_{n,l}$ is written

$$K_{n,l} = \sum_{j=-1}^{-\infty} \varphi_{n,l,j} f_0^j + \sum_{j=0}^{\infty} \varphi_{n,l,j} f_0^j \log f_0\,, \tag{8.2}$$

where $\varphi_{n,l,j}$ are polynomials in $\langle A^p\rangle, \langle A^p z^2\rangle, \langle A^p zw\rangle, \langle A^p w^2\rangle, p=1, 2, \cdots$.
(ii) *Let us identify $K_n, n=2, 3, \cdots$ with their asymptotic expansions in A and regard them as functions of $f_0, \langle A^p\rangle, \langle A^p z^2\rangle, \langle A^p zw\rangle, \langle A^p w^2\rangle$ which are now considered as formal independent variables. Then*

$$K_n = \left(\frac{-1}{\pi}\frac{\partial}{\partial f_0}\right)^{n-2} K_2\,. \tag{8.3}$$

If we evaluate the formal variables and consider the convergence of (8.3), we get a micro-local relation between the Bergman kernel of $E(A_1, \cdots, A_{n-1})$ in $\boldsymbol{C}^{n-1}$ and that of $E(A_1, \cdots, A_{n-1}, 0)$ in $\boldsymbol{C}^n$.

Corollary 8.4. *Let F: $\boldsymbol{C}^{n-1}\times\overline{\boldsymbol{C}^{n-1}}\to\boldsymbol{C}^n\times\overline{\boldsymbol{C}^n}$ be an embedding defined by $F(z, w)=(z, 0, w, 1)$. Then*

$$K_{n-1}(z, w) = \pi\left(\frac{\partial}{\partial z_n}\right)^{-1} K_n\circ F(z, w)\,|_{A_n=0} \tag{8.5}$$

in the sense of holomorphic microfunctions of $3(n-1)$ variables z, w, and $A_1, \cdots, A_{n-1}$.

Proof of Proposition 8.1. For each pair of indices p, q, we examine

$$\langle AD^2\rangle^{q_1}\langle Az^2\rangle^{p_1}\cdots\langle AD^2\rangle^{q_l}\langle Az^2\rangle^{p_l}\ \Phi_{p,q}(\langle zD\rangle^*)\frac{n!}{\pi^n}f_0^{-n-1}\,. \tag{8.6}$$

Let us take a holomorphic microfunction $\varphi_{n,p,q}(x)\in\mathcal{C}_{\{1\}|C}$ such that $\varphi_{n,p,q}(\langle zw\rangle) =\Phi_{p,q}(\langle zD\rangle^*)\frac{n!}{\pi^n}f_0^{-n-1}$. Then, using Lemma 7.7, we can write (8.6) in the form

$$\sum_{m=0}^{2(q_1+\cdots+q_l)} F_m D_x^m \varphi_{n,p,q}(x)\,|_{x=\langle zw\rangle}\,, \tag{8.7}$$

where F_m is a polynomial in $\langle A^p\rangle$, $\langle A^p z^2\rangle$, $\langle A^p zw\rangle$, $\langle A^p w^2\rangle$, $p=1, 2, \cdots$ which has homogeneous degree m in w. So, substituting $\langle zw\rangle = 1-f_0$ into (8.7), we get the expansion of the form (8.2).

On the other hand, in view of (7.3) and (7.5), we have

$$\varphi_{n,p,q}(x) = (\pi^{-1}D_x)^{n-2}\varphi_{2,p,q}(x)\,.$$

Thus, noting that F_m is independent of n, we can write (8.7), as

$$(\pi^{-1}D_x)^{n-2}\sum_{m=0}^{2|q|} F_m D_x^m \varphi_{2,p,q}(x)\,|_{x=1-f_0}\,.$$

If we regard $\langle A^p\rangle$, $\langle A^p z^2\rangle$, $\langle A^p zw\rangle$, $\langle A^p w^2\rangle$, and f_0 as independent variables, the formula above can be expressed as

$$\left(\frac{-1}{\pi}\frac{\partial}{\partial f_0}\right)^{n-2}\sum_{m=0}^{2|q|} F_m\cdot(D_x^m\varphi_{2,p,q})(1-f_0)\,.$$

Therefore summing up over indices p,q, we get (8.3). □

Proof of Corollary 8.4. From (8.3), we get

$$K_{n-1} = -\pi\left(\frac{\partial}{\partial f_0}\right)^{-1} K_n \tag{8.8}$$

in the sense of expansions in formal variables $\langle A^p\rangle$, $\langle A^p z^2\rangle$, $\langle A^p zw\rangle$, $\langle A^p w^2\rangle$, f_0.

Now we regard the variables as functions on $\boldsymbol{C}^n\times\overline{\boldsymbol{C}^n}$. Then we have

$$-(D_x\varphi)\circ f_0\circ F = \left(\frac{\partial}{\partial z_n}(\varphi\circ f_0)\right)\circ F \quad \text{for any} \quad \varphi(x)\in \boldsymbol{C}_{\{0\}|\boldsymbol{C}}\,,$$

and

$$\frac{\partial}{\partial z_n}\langle A^p z^2\rangle\,|_{A_n=0} = \frac{\partial}{\partial z_n}\langle A^p zw\rangle\,|_{A_n=0} = 0\,.$$

If we apply these rules to each term in the formal relation (8.8), we get

$$K_{n-1} = \pi\left(\frac{\partial}{\partial z_n}\right)^{-1} K_n\circ F\,|_{A_n=0} \tag{8.9}$$

in the sense that the asymptotic expansions in $A_1, \cdots, A_{n-1}$ of each side coincide. Here $\pi\left(\frac{\partial}{\partial z_n}\right)^{-1} K_n\circ F\,|_{A_n=0}$ is defined as a holomorphic microfunction which has the same support as that of the Bergman kernel K_{n-1}. Since the expansion in A uniquely determines a holomorphic microfunction, (8.9) implies the corollary.

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