Ozawa, S.
Osaka J. Math.
30 (1993), 303–314

ASYMPTOTIC PROPERTY OF EIGENFUNCTION OF THE LAPLACIAN AT THE BOUNDARY

SHIN OZAWA

(Received November 27, 1991)

1. Introduction.

Let M be a bounded region in $\boldsymbol{R}^n$ with smooth boundary ∂M. Let $\{\varphi_j(x)\}_{j=1}^{\infty}$ be a complete orthonomal basis of eigenfunctions of the Laplacian in M under the Dirichlet condition on ∂M. Let λ_j be the j-th eigenvalue of $-\Delta$ in M under the Dirichlet condition.

We shall prove the following:

Theorem 1. *The formula*

$$\sum_{j=1}^{\infty} e^{-\lambda_j t}(\partial\varphi_j/\partial\nu_x)(x)^2 \tag{1.1}$$

$$\sim C_1(x)t^{-(n/2)-1}+C_2(x)t^{-(n/2)-(1/2)}+\cdots+C_k(x)t^{-(n/2)-(3/2)+(k/2)}+\cdots \tag{1.2}$$

holds when $t\to 0$. Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction.

In Ozawa [9] the author determined the structure of $C_k(x)$ by the geometric invariant theory due to Gilkey [4]. As a corollary of (1.2) we have the following Proposition by Tauberian theorem.

Proposition 1. *Fix $z\in\partial M$. Then,*

$$\sum_{\lambda_j<\lambda}(\partial\varphi_j/\partial\nu_x)(z)^2|_{z\in\partial M} = \tilde{D}\lambda^{1+(n/2)}+o(\lambda^{(n/2)+1}) \quad as\ \lambda\to\infty, \tag{1.3}$$

where

$$\tilde{D} = (4\pi)^{-n/2}\Gamma(2+(n/2))^{-1}.$$

It is natural to ask the sharp asymptotic remainder estimate. Melrose [8] and Ivrii [6] proved that

$$\sum_{\lambda_j<\lambda} 1 = C'|M|\lambda^{(n/2)}+C''|\partial M|\lambda^{(n-1)/2}+o(\lambda^{(n-1)/2}) \tag{1.4}$$

holds under some condition on ∂M using wave equation and micro local analysis. Here the author would like to offer a conjecture.

Conjecture. We have an asymptotic for (1.3)

$$D_1\lambda^{(n/2)+1}+D_2H_1(z)\lambda^{(n+1)/2}+o(\lambda^{(n+1)/2}).$$

Here $H_1(z)$ denotes the first mean curvature of $z\in\partial M$ with respect to the exterior normal direction.

We want to make a comment. It is desirable to get more delicate asymptotics of eigenfunction. Zelditch [12] and Y. Colin de Verdiere [3] showed the following. For details see [3], [12].

When the geodesic flow of a compact Riemannian manifold M is ergodic, then there exists a subsequence $\{\varphi_{i_k}\}$ of $\{\varphi_i\}$ such that

$$\lim_{i_k\to\infty}\int_D \varphi_{i_k}(x)^2dx=\mathrm{Vol}(D)/\mathrm{Vol}(M). \tag{1.5}$$

The author gives another conjecture here.

Conjecture. When $\dim M=2$, then there exists a subsequence k_j of k such that

$$\lim_{k_j\to\infty}\mu_{k_j}^{-1}\int_{D^*}(\partial\varphi_{k_j}/\partial\nu)^2d\sigma_x$$

exists and it is equal to $C|D^*|/|\partial M|$ for $D^*\subset\partial M$, where $d\sigma_x$ denotes the surface element of ∂M.

The author would like to add one more question. What can one say the off diagonal saymptotic for

$$\sum_{j=1}^{\infty}e^{-\lambda_jt}(\partial\varphi_j/\partial\nu)(z)(\partial\varphi_j/\partial\nu)(w)?$$

For the off diagonal asymptotic for the heat kernel the reader may refer to Kannai [7], Taylor [11].

2. Seeley's work.

We recall Seeley [10]. We discuss general second order elliptic operators with the Dirichlet condition. To prove (1.3) we use diffeomorphism near the boundary point which transform $\partial\Omega$ to the boundary of $\boldsymbol{R}^n_+=\{x_n>0, (x_1,\cdots,x_n)\in\boldsymbol{R}^n\}$. Thus, the Laplacian is transformed into second order elliptic operator of variable coefficients. Thus, it is better to discuss general frame work of Seeley, if the author wants to discuss the Laplacian.

Let $\alpha=(\alpha_1,\cdots,\alpha_n)$ be multi indices and $D^\alpha=\left(-i\dfrac{\partial}{\partial x_i}\right)^{\alpha_1}\cdots\left(-i\dfrac{\partial}{\partial x_n}\right)^{\alpha_n}$,

$\alpha!=\alpha_1!\cdots\alpha_n!$. Let $A(x, D)=\sum_{|\alpha|\leq 2} a_\alpha(x)D^\alpha$ be the second order elliptic operators on $\boldsymbol{R}^n$ with smooth coefficients. Let $\sigma(A)(x, \xi)=\sum_{|\alpha|\leq 2} a_\alpha(x)\xi^\alpha$. We set $\sigma_2(A)(x, \xi) = \sum_{|\xi|=2} a_\alpha(x)\xi^\alpha$. Here ξ denotes the dual variable of x. We say that $R_\theta=\{\lambda\in \boldsymbol{C}; \arg\lambda=\theta\}$ is a ray.

We say that A satisfies Agmon's condition with respect to the ray R_θ at x, when $\sigma_2(A)(x, \xi)-\lambda$ is invertible for $\xi\in\boldsymbol{R}^n\setminus\{0\}$, $\lambda\in R_\theta$. See Agmon [1].

Next we consider the general theory on $\boldsymbol{R}^n_+=\{x=(x_1, \cdots, x_{n-1}, x_n); x_n>0\}$. Then, A is written as

$$A=\sum_{j=0}^{2} A_j(x_n)D_n^{2-j},$$

where $A_j(x_n)$ is a differential operator of order $\leq j$ with smooth coefficients. Let $x'=(x_1, \cdots, x_{n-1})$ and $\xi'=(\xi_1, \cdots, \xi_{n-1})$ be its dual variable. Then,

$$\sigma(A)(x, \xi)=\sum_{j=0}^{2}\sigma(A_j)\xi_n^{2-j}$$
$$\sigma_2(A)(x, \xi)=\sum_{j=0}^{2}\sigma_j(A_j)\xi_n^{2-j}.$$

Here $\sigma_j(A_j)$ is a symbol of A_j with homoegneous order j. We put

$$\sigma_2'(A)(x', \xi', D_{x_n})=\sum_{j=0}^{2}\sigma_j(A_j(0))D_{x_n}^{2-j}.$$

We consider the Dirichlet boundary value problem in $\boldsymbol{R}^n_+$. We write B as the Dirichlet boundary condition symbolically.

We say that (A, B) satisfies Agmon's condition on $\boldsymbol{R}^n_+$, if the following is filled.

(Agm): There exists a ray R_θ such that the ordinary differential equation (2.1) has the unique solution for any $x'\in\boldsymbol{R}^{n-1}$, $(\xi', \xi_n)\neq 0$, and for any $g(x')\in C_0^\infty(\boldsymbol{R}^{n-1})$, $\lambda\in R_\theta$.

$$\begin{aligned}
&\sigma_2'(A)(x, \xi', D_{x_n})u(x', x_n)=\lambda u(x', x_n) && x_n>0.\\
&\lim_{x_n\to\infty} u(x', x_n)=0, && x'\in\boldsymbol{R}^{n-1}.\\
&u(x', 0)=g(x') && x'\in\boldsymbol{R}^{n-1}.
\end{aligned}\tag{2.1}$$

We can use the localized version of Agmon's condition in the sense that (A, B) satisfies Agmon's condition for a neighbourhood of $x_0'\in\boldsymbol{R}^{n-1}$.

Hereafter we assume that A satisfies Agmon's condition for the ray R_θ and (A, B) satisfies Agmon's condition for the ray R_θ.

We follow Seeley's paper to construct formal calculus of parametrices. We put

$$a_2(x, \xi, \lambda) = -\lambda + \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$$
$$a_j(x, \xi. \lambda) = \sum_{|\alpha|=j} a_\alpha(x)\xi^\alpha\,, \qquad j = 0, 1\,.$$

We put

$$\sigma(A-\lambda) = \sum_{j=0}^{2} a_j(x, \xi, \lambda)\,.$$

It should be remarked that λ can be thought as the second order term.

We construct inner parametrices $c_{-2-j}(x, \xi, \lambda)$ inductively:

$$\text{(2.2)} \qquad \begin{aligned} &C_{-2}(x, \xi, \lambda) = a_2(x, \xi, \lambda)^{-1} \\ &a_2 c_{-2-j} + \sum_{\substack{k-|\alpha|-2-m \\ =-j, m<j}} (D_\xi^\alpha a_k)(iD_x)^\alpha c_{-2-m}/\alpha! = 0\,, \end{aligned}$$

$j=1, 2, \cdots$. It is easy to see that $c_{-2-j}(x, \xi, \lambda)$ is homogeneous of order $-2-j$ for $(\xi, \lambda^{1/2})$ variable.

Next we construct boundary compensating parametrices. We put

$$\text{(2.3)} \qquad a^{(j)}(x, \xi', D_{x_n}, \lambda) = \sum_{m-k=j} x_n^k \left(\left(\frac{\partial}{\partial x_n}\right)^k a_m\right)(x', 0, D_{x_n}, \lambda)/k!\,.$$

We put $\tilde{\sigma}(A-\lambda) = \sum_{j=2}^{-\infty} a^{(j)}$.

We solve the equations (2.4).

$$\text{(2.4)} \qquad \begin{aligned} &a^{(2)}(x, \xi', D_{x_n}, \lambda) d_{-2-j}(x, \xi, \lambda) \\ &\quad + \sum_{\substack{k-|\alpha|-2-m=-j \\ m<j}} [D_{\xi'}^\alpha a^{(k)}(x, \xi', D_{x_n}, \lambda)](iD_{x'})^\alpha d_{-2-m}(x, \xi, \lambda)/\alpha! = 0 \\ &d_{-2-j}(x', 0, \xi, \lambda) = -c_{-2-j}(x', 0, \xi', \lambda) \\ &\lim_{x_n \to \infty} d_{-2-j}(x', x_n, \xi, \lambda) = 0\,. \end{aligned}$$

By Agmons' condition (2.4) is uniquely solvable.

We follow Seeley's paper [10] to construct operators $O_p(c_{-2-j})$, $O_p'(d_{-2-j})$. When $\lambda \in R_\theta$,

$$\text{(2.5)} \qquad O_p(c_{-2-j})f(x', x_n) = (2\pi)^{-n} \iint e^{i(x'\xi'+x_n\xi_n)} c_{-2-j}(x, \xi, \lambda)\hat{f}(\xi)d\xi' d\xi_n\,.$$

$$\text{(2.6)} \qquad O_p'(d_{-2-j})f(x', x_n) = (2\pi)^{-n} \iint e^{ix'\xi'} d_{-2-j}(x, \xi, \lambda)\hat{f}(\xi)d\xi' d\xi_n\,,$$

where $f(x', x_n) \in C_0^\infty(\boldsymbol{R}_+^n)$ and $\hat{f}(\xi) = \int_{\boldsymbol{R}^n} e^{-iy\xi} f(y)dy$.

We put

$$\tilde{d}_{-2-j}(x', x_n, \xi', y_n, \lambda) = -\oint_{\mathcal{P}_-^\infty} e^{-iy_n\xi_n} d_{-2-j}(x', x_n, \xi', \xi_n, \lambda) d\xi_n,$$

Here $\mathcal{P}_-$ is an anti-clock wise simple closed curve such that $\mathcal{P}_-$ includes all poles of $d_{-2-j}(x', x_n, \xi', \xi_n, \lambda)$ in $\xi_n^+ = \{\xi_n \in \boldsymbol{C}: \text{Im } \xi_n < 0\}$ with respect to ξ_n variable. Then,

$$(2.7) \quad O_p'(d_{-2-j})f(x', x_n) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{ix'\xi'} \tilde{d}_{-2-j}(x, \xi', y_n, \lambda) \hat{f}(\xi', y_n) dy_n d\xi'.$$

Here

$$\hat{f}(\xi', y_n) = \int_{-\infty}^{\infty} e^{-\xi' z'} f(z', y_n) dz'$$

If $A = -\Delta$ with the Dirichlet condition in a half space, then $c_{-2}(x, \xi, \lambda) = (|\xi|^2 - \lambda)^{-1}$ and $d_{-2}(x, \xi, \lambda) = -(|\xi|^2 - \lambda)^{-1} \exp(-{}^+\sqrt{|\xi'|^2 - \lambda}\, x_n)$ where $|\xi'|^2 = \sum_{j=1}^{n-1} \xi_j^2$, $|\xi|^2 = |\xi'|^2 + \xi_n^2$,
Here ${}^+\sqrt{a}$ is the square of a whose real part is positive. In this case

$$\tilde{d}_{-2}(x, \xi', y_n, \lambda) = -({}^+\sqrt{|\xi'|^2 - \lambda}) \exp((-{}^+\sqrt{|\xi'|^2 - \lambda})(x_n + y_n)).$$

We list here some facts from Seeley [10].

Fix $\mu > 0$. We put

$$\mathcal{S}_\mu^k = \{(\zeta, \eta) \in \boldsymbol{C}^k \times \boldsymbol{C};\ \text{Im } \eta < \mu(|\text{Re } \eta| + |\text{Re } \zeta|^2 - \mu^{-1}|\text{Im } \zeta|^2).$$

Then, $\mathcal{S}_\mu^k$ is an open real cone with respect to $(\xi, \eta^{1/2})$ variables.

Lemma 2.1. *Let V be a compact set in R^n. Then, there exists $\mu > 0$ and $M_{j,\alpha} > 0$ such that* (2.8) *holds.*

(2.8) *For any fixed $x \in V$, $D_x^\alpha c_{-2-j}(x, \xi, -i\eta)$ is holomorphic on $\mathcal{S}_\mu^n$ with respect to (ξ, η)-variable and it satisfies*

$$\sup_{x \in V} |D_x^\alpha c_{-2-j}(x, \xi, -i\eta)| \leq |M_{j,\alpha}| (|\xi| + |\eta|^{1/2})^{-2-j}$$

holds for any j, α.

Lemma 2.2. *Let W be a compact set in $\boldsymbol{R}^{n-1}$. Then, there exist positive constants μ, $M_{j,\alpha,\beta}$, $S_{\alpha,\beta}$ such that* (2.9) *holds.*

For any fixed $x' \in W$, we see that $x_n^\alpha D_{x_n}^\beta \tilde{d}_{-2-j}(x, \xi', y_n, -i\eta)$ is holomorphic with respect to (ξ', η)-variable in $\mathcal{S}_\mu^{n-1}$ and it satisfies

$$\sup_{x' \in W} |x_n^\alpha D_{x_n}^\beta \tilde{d}^{-2-j}(x, \xi', y_n, -i\eta)|$$

$$(2.9) \quad \leq M_{j,\alpha,\beta} (|\xi|' + |\eta|^{1/2})^{-1-j-|\alpha|-|\beta|} \exp(-\delta_{\alpha,\beta}(x_n + y_n)(|\xi'| + |\eta|^{1/2})$$

for any $x_n, y_n \geq 0$. *And it satisfies*

$$(D^{\beta}_{x_n}\tilde{d}_{-2-j})(x', px_n, p^{-1}\xi', py_n, -i\eta p^{-2})$$
$$= p^{1+j-|\beta|}(D^{\beta}_{x_n}\tilde{d}_{-2-j})(x', x_n, \xi', y_n, -i\eta)\,.$$

Here we recall the fact about Fourier-Laplace transformation. Let $g(\xi, \eta)$ be a function holomorphic in $\mathcal{S}^k_\mu$ satisfying

$$|g(\xi, \eta)| \leq C(|\xi|+|\eta|^{1/2})^k$$

for some $k<0$. Then, we define the distribution $\mathcal{D}'(\boldsymbol{R}^{n+1})G(x, t)$ by

$$\langle G(x, t), f(x, t)\rangle = (2\pi)^{-n-1}\int_{\boldsymbol{R}^n} e^{ix\xi}d\xi\int_{-\infty-iq}^{\infty-iq} e^{it\eta}g(\xi, \eta)\hat{f}(\xi, \eta)d\eta\,.$$

Here q is a fixed positive number.

We have the following.

Lemma 2.3. Arima [2].
Support of $G(x, t)$ *is in* $t \geq 0$ *and the following estimate holds*

$$|D^{\alpha}_x D^{\beta}_t\, G(x, t)| \leq C_{\alpha,\beta}t^{-((n+k+2+|\alpha|)/2)-\beta}\exp(-\delta_{\alpha,\beta}t^{-1}|x|^2)$$

for $t>0$, *for some* $C_{\alpha,\beta}, \delta_{\alpha,\beta}>0$.

Let $g(\xi', \eta, s)$ be a holomorphic function of $(\xi', \eta)\in\mathcal{S}^{n-1}_\mu$ satisfying

$$g(\xi', \eta, s) \leq C(|\xi'|+|\eta|^{1/2})^k \exp(-cs(|\xi'|+|\eta|^{1/2}))$$

holds for any $(\xi', \eta)\in\mathcal{S}^{n-1}_\mu$, $s>0$.

Lemma 2.4. Arima [2].
We put

$$G(x', t, s) = (2\pi)^{-n}\int_{\boldsymbol{R}^{n-1}} e^{ix'\xi'}d\xi'\int_{-\infty-iq}^{\infty-iq} e^{it\eta}g(\xi', \eta, s)d\eta\,.$$

Here q *is a fixed positive number. Then*

$$|G(x', t, s)| \leq Ct^{-(n+k+1)/2}\exp(-ct^{-1}(|x'|^2+|s|))$$

holds for any $t>0$, $s>0$, $x'\in\boldsymbol{R}^{n-1}$.

3. Construction of heat parametrix.

Let us recall the construction of parametrix of the heat equations. Let $\{\Omega_k\}^m_{k=1}$ be an open covering of $\bar{\Omega}$. And let, $\{\varphi_k\}^m_{k=1}\{\psi_k\}^s_{k=1}$ be partion of unity which belongs to coverings $\{\Omega_k\}$ such that $\psi_k=1$ on support of φ_k. We assume that $\partial\Omega\cap\Omega_k\neq\phi$ for $k=1,\cdots,m'$ and $\partial\Omega\cap\Omega_k=\phi$ for $k=m'+1,\cdots,m$. Let $\Psi_k:\bar{\Omega}\cap\bar{\Omega}_k\to\boldsymbol{R}^n_+$ be an injective diffeomorphism such that

$$\Psi_k(\Omega_k\cap\partial\Omega)\subset\boldsymbol{R}^{n-1}=\{x=(x_1,\cdots,x_{n-1},x_n)\,|\,x_n=0\}\ .$$

We have the second order elliptic differential operator $\Psi_k^*\Delta$ which is strongly elliptic and it astisfies Agmon's condition at the boundary. We can construct parametrices $c_{-2-d}^{(k)}$, $\tilde{d}_{-2-j}^{(k)}$ as in section two.

We put

$$U_j^{(k)}(x,y,t) \tag{3.1}$$
$$=(2\pi)^{-n-1}\int_{\boldsymbol{R}^n}e^{i(x-y)\xi}\int_{-\infty-iq}^{\infty-iq}e^{it\eta}c_{-2-j}^{(k)}(x,\xi,-i\eta)d\eta)d\xi$$

and

$$\tilde{U}_j^{(k)}(x,y,t) \tag{3.2}$$
$$=(2\pi)^{-n-1}\int_{\boldsymbol{R}^{n-1}}e^{i(x'-y')\xi'}\int_{-\infty-iq}^{\infty-iq}e^{it\eta}\tilde{d}_{-2-j}^{(k)}(x,\xi',y_n,-i\eta)\,d\eta\,d\xi'\ ,$$

We put

$$U_j^{(k)}=U_j^{(k)}+\tilde{U}_j^{(k)}\ .$$

Then, we put

$$U_{\gamma,N}(w,z,t)=\sum_{k=1}^{m'}\sum_{j=0}^{N}\psi_k(w)\hat{U}_j^{(k)}(\Psi_k(w),\Psi_k(z),t)\,|(\Psi_k^{-1})'(\Psi_k(z))|^{-1}\varphi_k(z)\ .$$

Here $|(\Psi_k^{-1})'(\Psi_k(z))|$ is a Jacobian of Ψ_k^{-1} at $\Psi_k(z)$. Let $U_0(w,z,t)$ be the fundamental solution of the heat equation in $\boldsymbol{R}^n$, that is

$$U_0(w,z,t)=(4\pi t)^{-n/2}\exp\left(-|w-z|^2/4t\right).$$

The following Lemma is well known.

Lemma 3.1. Greiner [5].
We put

$$R_N(w,z,t)=U(w,z,t)-(U_{\gamma,N}(w,z,t)+\sum_{k=m'+1}^{m}\psi_k(w)U_0(w,z,t)\varphi_k(z))\ .$$

Then, there exists positive constant δ *and* C_N *such that*

$$|D_w^\alpha D_z^\beta R_N(w,z,t)|\leq C_N t^{-(n+|\alpha|+|\beta|-N-1)/2}e^{-\delta|w-z|^2/t}$$

holds for $|\alpha|+|\beta|\leq 1$, $t>0$, $w,z\in\Omega$.

We fix $w\in\partial\Omega$. Without loss of generalities we suppose that $w\in\Omega_1$, $w\notin\cup_{j=2}\Omega_j$. Assume that $\psi_1=1$. Now z is contained in a little neighbourhood and $\varphi_1(z)=1$. By Lemma 3.1 we see that there exist $\delta>0$, $C_N>0$, such that

$$D_w^\alpha U(w, z, t)-\sum_{j=0}^{N} D_w(U_j^{(1)}(\Psi_1(w), \Psi_1(z), t)\,|(\Psi_1^{-1})'(\Psi_1(z))|^{-1}\varphi_1(z)$$
$$\leq C_N t^{-(n+|\alpha|-N-1)/2}\, e^{-\delta|w-z|^2/t}\,.$$

We fix $w\in\gamma$, $\Psi_1(w)=x$. We assume that ν_w corresponds to the exterior unit normal vector at $x\in \boldsymbol{R}^{n-1}$.

We abbreviate $U_j^{(1)}$ as U_j and Ψ_1 as Ψ. We have

$$\text{(3.3)}\qquad \frac{\partial U}{\partial \nu_w}(w, z, t)+\sum_{j=0}^{N}\left(\frac{\partial \hat{U}_j}{\partial x_n}\right)(\Psi(w), \Psi(z), t)\,|(\Psi^{-1})'(\Psi(z))|^{-1}\varphi_1(z)$$
$$\leq C_N t^{-(n-N)/2} e^{-\delta|w-z|^2/t}\,.$$

4. Proof of asymptotics (1.2).

We put

$$B(w, t)=\int_0^t d\tau\int_{\boldsymbol{R}_+^n}\frac{\partial U(w, z, t-\tau)}{\partial \nu_w}\frac{\partial U(w, z, \tau)}{\partial \nu_w}dz\,.$$

We need some Lemma.

Lemma 4.1.

$$\int_0^t d\tau\int_{\boldsymbol{R}_+^n}\tau^{-(n+1+N')/2}(t-\tau)^{-(n+1+N'')/2}e^{-((1/\tau)+(1/(t-\tau)))|w-z|^2}dz\leq Ct^{(N'+N''-n)/2}$$

Lemma 4.2. *For any $x, y\in V\Subset R_+^n$. We have*

$$\max\left(|(\partial U_j/\partial x_n)(x, y, t)|,\ |(\partial \hat{U}_j/\partial x_n)(x, y, t)|\right)\leq C_i t^{-(n+1-j)/2}e^{-|x-y|^2/C_j t}\,.$$

Proof. These are deduced by Lemmas 2.1, 3.3, 3,4, (3.1), (3.2).

Proof of asymptotic (1.2):

Let Ω_1 be one of open covering $\{\Omega_k\}_{k=1}^m$. We put $w=\Psi_1^{-1}(x)\in\partial\Omega$, $x\in\partial\boldsymbol{R}_+^n=\boldsymbol{R}^{n-1}$. Then, we put

$$\tilde{\omega}(w, t)=\int_0^t d\tau\int_\Omega\frac{\partial U(w, z, t-\tau)}{\partial \nu_w}\frac{\partial U(w, z, \tau)}{\partial \nu_w}dz$$
$$\tilde{\omega}(w, t)=\int_0^t d\tau\int_{\Omega_1}\frac{\partial U(w, z, t-\tau)}{\partial \nu_w}\frac{\partial U(w, z, \tau)}{\partial \nu_w}dz\,.$$

Then, there exists $c_k>0$ such that

$$|\tilde{\omega}(w, t)-\tilde{\tilde{\omega}}(w, t)|\leq C_k t^k$$

for any $t>0$. We put

$$(4.1)\qquad I_{j,h}(w, t)=\int_0^t d\tau\int_{R_+^n}(\partial\hat{U}_j/\partial x_n)(x, y, t-\tau) \times(\partial\hat{U}_h/\partial x_n)(x, y, \tau)\,|(\Psi^{-1})'(y)|^{-1}\phi_1(\Psi_1^{-1}(y))^2\,dy$$

We know that $I_{j,h}(w, t)$ is an approximation of $\tilde{\tilde{\omega}}(w, t)$. We see by Lemma 4.2, (4.1) that

$$|\tilde{\tilde{\omega}}(w, t)-\sum_{j,h=0}^{N}I_{j,h}(w, t)|\leq C_N t^{-(n-1-N)/2}\,.$$

Now we want to expand $\varphi_1(\Psi^{-1}(y))^2|(\Psi^{-1})'(y)|^{-1}$ at $x\in R^{n-1}$. We put $v(y)=|(\Psi^{-1})'(y)|^{-1}$. Then,

$$(4.2)\quad \varphi_1(\Psi^{-1}(y))^2 v(y)-(v(x)+\sum_{1\leq|\alpha|\leq k}(y-x)^\alpha(\alpha!)^{-1}(D^\alpha v)(x)\leq C_k|x-y|^{k+1}$$

holds for any $y\in\Psi(\Omega_1)$.

Now we put

$$(4.3)\quad I_{j,h,\alpha}(w, t) =\int_0^t d\tau\int_{R_+^n}(\partial\hat{U}_j/\partial x_n)(x, y, t-\tau)(\partial\hat{U}_h/\partial x_n)(x, y, \tau)\frac{(y-x)^\alpha}{\alpha!}D^\alpha v(y)dy.$$

Note that α is a multi index. The following inequality is easy to see

$$(4.4)\quad |x-y|^{k+1}\exp(-\delta((1/\tau)+(1/(t-\tau))|x-y|^2) \leq C_k((1/\tau)+(1/(t-\tau))^{-(k+1)/2}\exp(-(\delta/2)((1/\tau)+(1/(t-\tau))|x-y|^2)\,.$$

Therefore, we have

$$(4.5)\qquad I_{j,h}(w, t)-\sum_{|\alpha|\leq k}I_{j,h,\alpha}(w, t)\leq C_{j,h,k}t^{(-n+j+h+k+1)/2}$$

An approximation scheme

$$\tilde{\omega}\leftarrow\tilde{\tilde{\omega}}\leftarrow\sum_{j,h=0}^{N}I_{j,h}\leftarrow\sum_{j,b=0}^{N}\sum_{|\alpha|\leq k}I_{j,h,\alpha}$$

is very well, since when we tends $N, k\to\infty$, the remainder is of order $O(t^s)$, for any s. Thus, if we prove that $I_{j,h,\alpha}(w, t)$ has an asymptotic expansion of t when $t\to 0$, we get (1.2).

We have the following.

Lemma 4.3. *The representation*

$$I_{j,h,\alpha}(w, t)=t^{\beta/2}a_{j,h,\alpha}^{(0)}(w)+t^{(\beta+1)/2}a_{j,h,\alpha}^{(1)}(w)+t^{(\beta+2)/2}a_{j,h,\alpha}^{(2)}(w)$$

holds, when $\beta=-n+j+h+|\alpha|$.

REMARK. If we put $a_{n-k}(w)=\sum_{j+h+|\alpha|=k-\beta} a^{(\beta)}_{j,h,\alpha}(w)$, then we get (1.2).

Proof of Lemma 4.3. We put

$$I^{(1)}_{j,h,\alpha}(w,t)=\int_0^t d\tau\int_{R^n_+}\left(\frac{\partial U_j}{\partial x_n}\right)(x,y,t-\tau)\left(\frac{\partial U_h}{\partial x_n}\right)(x,y,\tau)P_\alpha dy\,,$$

where $P_\alpha=(x-y)^\alpha D^\alpha v(x)/\alpha!$. We put

$$I^{(2)}_{h,j,\alpha}(w,t)=\int_0^t d\tau\int_{R^n_+}\left(\frac{\partial U_j}{\partial x_n}\right)(x,y,t-\tau)\left(\frac{\partial \tilde{U}_h}{\partial x_n}\right)(x,y,\tau)P_\alpha dy$$

$$I^{(3)}_{j,h,\alpha}(w,t)=\int_0^t d\tau\int_{R^n_+}\left(\frac{\partial \tilde{U}_j}{\partial x_n}\right)(x,y,t-\tau)\left(\frac{\partial U_h}{\partial x_n}\right)(x,y,\tau)P_\alpha dy$$

$$I^{(4)}_{j,h,\alpha}(w,t)=\int_0^t d\tau\int_{R^n_+}\left(\frac{\partial \tilde{U}_j}{\partial x_n}\right)(x,y,t-\tau)\left(\frac{\partial \tilde{U}_h}{\partial x_n}\right)(x,y,\tau)P_\alpha dy\,.$$

Then, $I_{j,h,\alpha}(w,t)$ is the sum of these terms. Of course we have $\Psi(w)=x$, $x_n=0$.

Now,

$$\text{(4.6)}\qquad \left(\frac{\partial U_j}{\partial x_n}\right)(x,y,t)|_{x_n=0}$$
$$=(2\pi)^{-n-1}\int_{R^n} e^{i(x-y)\xi}\left(\left(\int_{-\infty-iq}^{\infty-iq} e^{it\eta} i\xi_n c_{-2-j}(x',0,\xi,-i\eta)d\eta\right)d\xi\right.$$
$$+(2\pi)^{-n-1}\int_{R^n} e^{i(x-y)\xi}\left(\int_{-\infty-iq}^{\infty-iq} e^{it\eta}\left(\frac{\partial c_{-2-j}}{\partial x_n}\right)(x',0,\xi,-i\eta)d\eta\right)d\xi\,.$$

Here q is a fixed positive number. Recall that c_{-2-j} is of homogeneous degree $-2-j$ with respect to $(\xi,\lambda^{1/2})$-variables. $D^\alpha{}_{-2-j}$ has the same homogeneity. The first term, the second term in the right hand side of (4.6) is denoted by $W^{(1)}_j(x,y,t)$, $W^{(2)}_j(x,y,t)$, respectively. We make a change of variables $\eta\to\hat{\eta}$, $\xi\to\hat{\xi}$ so that $\eta=t^{-1}\hat{\eta}$, $\xi=t^{-1/2}\hat{\xi}$, that is

$$\hat{\xi}=(\hat{\xi}_1,\cdots,\hat{\xi}_n)=(t^{1/2}\xi_1,\cdots,t^{1/2}\xi_n)\,.$$

Now we put $W^{(1)}_j$ as

$$W^{(1)}_j(x,y,t)=(2\pi)^{-n-1}t^{-(n+1-j)/2}B\,,$$

where

$$B=e^{i(x-y)t^{-1/2}\xi}\,i\hat{\xi}_n\int_{-\infty-iq}^{\infty-iq} e^{i\hat{\eta}}c_{-2-j}(x',0,\hat{\xi},-i\hat{\eta})d\hat{\eta}d\hat{\xi})\,.$$

Here we change the integral path from $(-\infty-iqt,\infty-iqt)$ to $(-\infty-iq,\infty-iq)$.

We put

$$H_j^{(1)}(u, x') = (2\pi)^{-n-1}\int_{R^n} e^{iu\hat{\xi}} i\hat{\xi}_n \int_{-\infty-iq}^{\infty-iq} e^{i\eta} c_{-2-j}(x', ,0, \hat{\xi}, -i\hat{\eta})d\hat{\eta}d\hat{\xi}\,.$$

Then,

$$W_j^{(1)}(x, y, t) = H_j^{(1)}((x-y)t^{-1/2}, x')t^{-(n+1-j)/2}\,.$$

Similarly, we have

$$H_j^{(2)}(u, x') = (2\pi)^{-n-1}\int_{R^n} e^{iu\hat{\xi}}\Big(\int_{-\infty-iq}^{\infty-iq} e^{i\hat{\eta}}\Big(\frac{\partial c_{-2-j}}{\partial x_n}\Big)(x', 0, \hat{\xi}, -i\hat{\eta})d\hat{\eta}d\hat{\xi}\,.$$

We change variables $y-x=t^{-1/2}\tau^{-1/2}(t-\tau)^{1/2}\tilde{y}$, $\tau=rt$. Then,

$$I_{j,h,\alpha}^{(1)}(t) = t^{(j+h+|\alpha|-n)/2}P_\alpha(B_1+B_2t^{1/2}+B_3t^{1/2}+B_4t)\,, \tag{4.8}$$

where

$$\begin{aligned}
B_1 &= \int_0^1 dr\,\Big(\int_{R_+^n} H_j^{(1)}((1-r)^{1/2}y, x')H_h^{(1)}(r^{1/2}y, x')\tilde{y}^\alpha d\tilde{y}\\
&\qquad\times r^{(j+|\alpha|-1)/2}(1-r)^{(h+|\alpha|-1)/2}\\
B_2 &= \int_0^1 dr\,\Big(\int_{R_+^n} H_j^{(1)}((1-r)^{1/2}y, x')H_h^{(2)}(r^{1/2}y, x')\tilde{y}^\alpha d\tilde{y}\\
&\qquad\times r^{(j+|\alpha|-1)/2}(1-r)^{(h+|\alpha|)/2})\\
B_3 &= \int_0^1 dr\,\Big(\int_{R_+^n} H_j^{(2)}((1-r)^{1/2}y, x')H_h^{(1)}(r^{1/2}y, x')\tilde{y}^\alpha d\tilde{y}\\
&\qquad\times r^{(j+|\alpha|)/2}(1-r)^{(h+|\alpha|-1)/2})\\
B_4 &= \int_0^1 dr\,\Big(\int_{R_+^n} H_j^{(2)}((1-r)^{1/2}y, x')H_h^{(2)}(r^{1/2}y, x')\tilde{y}^\alpha d\tilde{y}\\
&\qquad\times r^{(j+|\alpha|)/2}(1-r))^{(h+|\alpha|)/2}\,.
\end{aligned}$$

Thus, $I_{j,h,\alpha}^{(1)}(w, t)$ has the form in Lemma 5.3. We do not discuss precisely, however we see that $I_{j,h,\alpha}^{(2)}(w, t)=t^{(j+h+|\alpha|-n)/2}(b_{j,h,\alpha}(w)+t^{1/2}\tilde{b}_{j,h,\alpha}(w))$, $I_{j,h,\alpha}^{(3)}(w, t) = t^{(j+h+|\alpha|-n)/2}(C_{j,h,\alpha}(w)+t^{1/2}\tilde{c}_{j,hj\alpha}(w))$, etc, holds. Thus, we get the desired result.

5. Comment.

In section 1 we give some conjecture and problems. We add some problem.

Can one give constants C, C' which are independent of Ω such that

$$C\lambda_j \leq \int_{\partial\Omega} (\partial\varphi_j/\partial\nu)(x)^2 d\sigma_x \leq C'\lambda_j\,? \tag{5.1}$$

References

[1] S. Agmon: *On the eigenfunction and on the eigenvalues of general elliptic boundary value problems,* Comm. Pure Appl. Math., **15** (1962), 119–147.

[2] R. Arima: *On general boundary value problems for parabolic equations,* J. Math. Kyoto Univ., **4** (1964), 207–243.

[3] Y. Colin de Verdiere: *Ergodicité et fonctions propres du laplacian,* Commun. Math Phys., **102** (1985), 497–502.

[4] P. Gilkey: *The boundary integrand in the formula for the signature and Euler characteristic of a Riemannian manifold with boundary,* Advances in Math., **15** (1975), 344–360.

[5] P. Greiner: *An asymptotic expansion for the heat equations,* Arch. Rational Mech. Anal., **41** (1971), 163–218.

[6] V.Ja. Ivrii: *On the second term in the spectral asymptotique for the Laplace-Beltrami operators on a manifold with boundary.* Functional Anal. i prilozen, **14**:2, (1980). 25–34.

[7] Y. Kannai: *Off diagonal short time asymptotiques for fundamental solutions of diffusion equations*, Comm. in P.D.E., **2** (1977), 781.

[8] R.B. Melrose: *Weyl's conjecture for manifolds with concave boundary,* Proceeding of Symposia in Pure Math., 36 (1980), 257–274.

[9] S. Ozawa: *Remarks on Hadamard's variation of eigenvalues of the Laplacian,* Proc. Japan Acad., **55A** (1979), 328–333.

[10] R. Seeley: *The resolvent of an elliptic boundary problem,* Amer. J. Math., **91** (1969), 889–919.

[11] T.J.S. Taylor: *Off diagonal asymptotics of hypoelliptic diffusion equations and singular Riemannian geometry,* Pacific J. of Math., **136** (1989), 379–399.

[12] S. Zelditch: *Uniform distribution of eigenfunctions on compact hyperbolic surfaces,* Duke Math. J., **55** (1987), 919–941.

Department of Mathematics
Faculty of Sciences
Tokyo Institute of Technology
O-okayama, Meguro-ku
Tokyo, 152, Japan