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INVARIANT DIFFERENTIAL OPERATORS ON THE GRASSMANN MANIFOLD $G_{2,n-1}(C)$

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0. Introduction. The present paper is the latter one of twin papers on invariant linear differential operators of Grassmann manifolds. In the former one [9] we determined and clarified the structure of the algebra $\boldsymbol{D}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ of invariant linear differential operators on the Grassmann manifold $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ of oriented 2-planes in $\boldsymbol{R}^{n+1}$ by exhibiting a set of generators with their simultaneous eigenspace decompositions.

The complex Grassmann manifold $\boldsymbol{G}_{2,n-1}(\boldsymbol{C})$ defined as the totality of complex 2-planes passing through the origin of $\boldsymbol{C}^{n+1}$, is known to be a symmetric space of rank 2. Hence, the algebra $\boldsymbol{D}(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ of invariant linear differential operators acting on $C^{\infty}(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}), \boldsymbol{R})$ is generated by two differential operators, where $C^{\infty}(M, \boldsymbol{K})$ denotes the algebra of $\boldsymbol{K}$-valued C^{∞}-functions defined on a complex manifold M and $\boldsymbol{K}$ denotes either the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$.

The aim of the present paper lies, as in [9], in exhibiting a simultaneous eigenspace decomposition of an explicit set of generators $\Delta_0^{\wedge}$ and $\Delta_1^{\wedge}$ of the algebra $\boldsymbol{D}(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$.

Define

$$\boldsymbol{S}^{p}(\boldsymbol{P}_n(\boldsymbol{C})) := \sum_{k+l=p} \boldsymbol{S}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C})) \quad \text{(direct sum)},$$

$$\boldsymbol{S}^{*}(\boldsymbol{P}_n(\boldsymbol{C})) := \sum_{p\geqq 0} \boldsymbol{S}^{p}(\boldsymbol{P}_n(\boldsymbol{C})) \quad \text{(direct sum)},$$

$$\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C})) := \sum_{k,l\geqq 0} \boldsymbol{S}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C})) \quad \text{(direct sum)},$$

where $\boldsymbol{S}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$ is the $C^{\infty}(\boldsymbol{P}_n(\boldsymbol{C}), \boldsymbol{C})$-module of complex (contravariant) symmetric tensor fields of bidegree (k, l) on the complex projective space $\boldsymbol{P}_n(\boldsymbol{C})$. $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ is a bigraded algebra over $C^{\infty}(\boldsymbol{P}_n(\boldsymbol{C}), \boldsymbol{C})$. We obtained in [8] the following about the complex projecitve space $(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ with prescribed standard Riemannian mteric g_0:

(1) The eigenspace decomposition of Δ_0 restricted to $\boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ is given, Where Δ_0 is the Lichnerowicz operator acting on $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ and $\boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ is the bigraded $\boldsymbol{C}$-subalgebra of $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ defined as

$$K^{**}(P_n(C), g_0) = \sum_{k,l\geqq 0} K^{k,l}(P_n(C), g_0),$$

where $K^{k,l}(P_n(C), g_0)=S^{k,l}(P_n(C)) \cap K^p(P_n(C), g_0)$ for $p=k+l$ and $K^p(P_n(C), g_0)$ is the C-submodule in $S^p(P_n(C))$ linearly generated by the totality of p-th symmetric tensor products of Killing vector fields on $(P_n(C), g_0)$.

(2) Denote by $\Delta_0^\wedge$ the Laplace-Beltrami operator on $(G_{2,n-1}(C), g_1)$, where g_1 is the standard metric on $G_{2,n-1}(C)$. Then $\Delta_0^\wedge$ is related to the Lichnerowicz operator Δ_0 through the Radon transform

$$\wedge : S^{**}(P_n(C)) \rightarrow C^\infty(G_{2,n-1}(C))$$

by the formula:

$$(\Delta_0 \xi)^\wedge = \Delta_0^\wedge \xi^\wedge$$

for $\xi \in S^{**}(P_n(C))$.

(3) The eigenspace decomposition in (1) is transferred to that of $\Delta_0^\wedge$ by means of the Radon transform.

In the present paper a new differential operator Δ_1 on $S^{**}(P_n(C))$ with properties analogous to (1), (2) and (3) above is constructed. Especially, it is shown that $\Delta_0^\wedge$ togeher with $\Delta_1^\wedge$ generates the algebra $D(G_{2,n-1}(C))$. In the section **1** we recall the results obtained in [8] with some improvements. Δ_1 is defined at the end of the section **1**. In the section **2** the eigenspace decomposition of Δ_1 restricted to $K^{**}(P_n(C), g_0)$ is obtained. $\Delta_0^\wedge$ and $\Delta_1^\wedge$ together with their simultaneous eigenspace decomposition are studied in the section **3**.

1. Fundamental operators. Let M be a complex manifold of complex dimension n. Denote by $E^p(M)$ the $C^\infty(M, C)$-module of complex linear differential operators of order at most p. Put

$$E^*(M) := \bigcup_{p\geqq 0} E^p(M).$$

$E^*(M)$ will be abbreviated as $E(M)$.

Let $S^{k,l}(M)$ be the $C^\infty(M, C)$-module of complex symmetric tensor fields of bidegree (k, l) on M.

Define

$$S^p(M) := \sum_{k+l=p} S^{k,l}(M) \quad \text{(direct sum)},$$

$$S^*(M) := \sum_{p\geqq 0} S^p(M) \quad \text{(direct sum)},$$

$$S^{**}(M) := \sum_{k,l\geqq 0} S^{k,l}(M) \quad \text{(direct sum)}.$$

$S^{**}(M)$ is a bigraded $C^\infty(M, C)$-algebra.

Denote the symbol operator of degree p by

$$\sigma^p : E^p(M) \ni D \mapsto \sigma^p(D) \in S^p(M),$$

where $\sigma^p(D)$ is the symbol tensor field of D.

Let

$$\iota^p \colon \boldsymbol{E}^{p-1}(M) \to \boldsymbol{E}^p(M)$$

be the canonical injection. Then we obtain a short exact sequence of $C^\infty(M, \boldsymbol{C})$-modules:

$$0 \to \boldsymbol{E}^{p-1}(M) \xrightarrow{\iota_p} \boldsymbol{E}^p(M) \xrightarrow{\sigma^p} \boldsymbol{S}^p(M) \to 0 .$$

Put

$$\boldsymbol{L}^*(M) := \bigcup_{q \in \boldsymbol{Z}} \boldsymbol{L}^q(M) ,$$

where we set

$$\boldsymbol{L}^q(M) := \begin{cases} \boldsymbol{E}^{q+1}(M) & \text{for } q \geqq -1 , \\ \{0\} & \text{for } q \leqq -2 . \end{cases}$$

$\boldsymbol{L}^*(M)$ is not only a filtered associative algebra over $\boldsymbol{C}$ with respect to the product of operators, it is a filtered Lie algebra over $\boldsymbol{C}$ (cf. [8]) for the bracket product $[D_1, D_2] := D_1 D_2 - D_2 D_1$. In fact we have

$$[\boldsymbol{E}^p(M), \boldsymbol{E}^q(M)] \subset \boldsymbol{E}^{p+q-1}(M) .$$

$\boldsymbol{S}^*(M)$ is canonically $\boldsymbol{C}$-isomorphic to the associated graded Lie algebra $\boldsymbol{Gr}(\boldsymbol{L}^*(M))$:

$$\begin{aligned} \boldsymbol{S}^*(M) &\cong \sum_{q \geqq 0} \boldsymbol{E}^{q-1}(M)/\boldsymbol{E}^{q-2}(M) \text{ (direct sum)} \\ &= \sum_{q \in \boldsymbol{Z}} \boldsymbol{L}^q(M)/\boldsymbol{L}^{q-1}(M) \text{ (direct sum)} = \boldsymbol{Gr}(\boldsymbol{L}^*(M)) , \end{aligned}$$

as $\boldsymbol{S}^p(M) \cong \boldsymbol{E}^{p-1}(M)/\boldsymbol{E}^{p-2}(M)$ for $p \geqq 0$. Hence, the bracket product in $\boldsymbol{S}^*(M)$ inherited from that of $\boldsymbol{L}^*(M)$ through the isomorphism $\boldsymbol{S}^*(M) \cong \boldsymbol{Gr}(\boldsymbol{L}^*(M))$ is given by

$$[\xi, \eta] = \sigma^{p+q-1}[D_1, D_2] ,$$

where $D_1 \in \boldsymbol{E}^p(M)$ and $D_2 \in \boldsymbol{E}^p(M)$ are chosen so that $\xi = \sigma^p(D_1)$ and $\eta = \sigma^q(D_2)$.

For a compact Kahlerian manifold (M, g), $\boldsymbol{S}^{k,l}(M)$ is equipped with a positive definite Hermitian inner product defined by

$$(\xi, \eta) = k!\, l! \int \langle \xi, \eta \rangle \, d\sigma \textit{ for } \xi, \eta \in \boldsymbol{S}^{k,l}(M) , \tag{1.1}$$

where $\langle \, , \, \rangle$ is the pointwise inner product associated with the metric g and $d\sigma$ is the canonical volume element.

Let $P = P(M, G)$ be a differentiable principal bundle on a differentiable manifold M with Lie group G as its fibre. Let $\boldsymbol{E}^G(P)$ be the totality of G-invariant complex linear differential operators on P. $\boldsymbol{E}^G(P)$ is a $\boldsymbol{C}$-subalgebra of $\boldsymbol{E}(P)$ if we regard $\boldsymbol{E}(P)$ as an algebra over $\boldsymbol{C}$.

Lemma 1.1. (cf. [5] and [7]).

$$\boldsymbol{E}(M)\cong\boldsymbol{E}^G(P)/\boldsymbol{J},$$

where $\boldsymbol{J}$ is the two-sided ideal in $\boldsymbol{E}^G(P)$ generated by G-invariant vertical vector fields on P.

Applying Lemma 1.1 to the Hopf fibering

$$\varphi\colon S^{2n+1}\to\boldsymbol{P}_n(\boldsymbol{C})$$

with fibre S^1, we obtain an isomorphism

$$(1.2)\qquad \pi_H\colon \boldsymbol{E}^{S^1}(S^{2n+1})/\boldsymbol{J}\cong\boldsymbol{E}(\boldsymbol{P}_n(\boldsymbol{C})),$$

where $\boldsymbol{J}$ is as in Lemma 1.1 the two-sided ideal in $\boldsymbol{E}^{S^1}(S^{2n+1})$ generated by S^1-invariant vertical vector fields.

Lemma 1.2 ([6]). *Let M_i $(i=1,2)$ be differentiable manifolds. There are subalgebras $\tilde{\boldsymbol{E}}(M_i)$ $(i=1,2)$ of $\boldsymbol{E}(M_1\times M_2)$ canonically isomorphic to $\boldsymbol{E}(M_i)$ respectively, each one of which is the centralizer of the other in $\boldsymbol{E}(M_1\times M_2)$.*

Let

$$(1.3)\qquad \iota\colon S^{2n+1}\to\boldsymbol{C}^{n+1}\text{-}\{0\}$$

be the canonical imbedding whose image is the unit sphere: $\{z=(z^0, z^1, \cdots, z^n)\in\boldsymbol{C}^{n+1}\text{-}\{0\}\mid r^2=1\}$, where $r^2=\sum_{a=0}^n z^a\bar{z}^a$.

$\boldsymbol{C}^{n+1}\text{-}\{0\}$ can be regarded as a product bundle on S^{2n+1} with $\boldsymbol{R}$ as its fibre. Thus as an application of Lemma 1.2, the existence of

$$\tilde{\boldsymbol{E}}(S^{2n+1}) := \{D\in\boldsymbol{E}(\boldsymbol{C}^{n+1}\text{-}\{0\})\mid[D, r^2]=0 \textit{ and } [D, \partial/\partial(r^2)]=0\}$$

as a subalgebra of $\boldsymbol{E}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ and of an isomorphism

$$\tilde{\iota}\colon \boldsymbol{E}(S^{2n+1})\to\tilde{\boldsymbol{E}}(S^{2n+1})$$

is assured.

Connecting π_H in (1.2) with $\tilde{\iota}$ above, we obtain an isomorphism

$$(1.4)\qquad \pi^\dagger\colon \boldsymbol{E}(\boldsymbol{P}_n(\boldsymbol{C}))\to\boldsymbol{E}^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))/(\tau),$$

where $\boldsymbol{E}^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$ is a subalgebra of $\boldsymbol{E}(S^{2n+1})$ defined as the image of $\boldsymbol{E}^{S^1}(\boldsymbol{P}_n(\boldsymbol{C}))$ by the isomorphism $\tilde{\iota}$. Here (τ) is a two-sided ideal in $\boldsymbol{E}^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$ defined as the image of $\boldsymbol{J}$ in (1.2) by $\tilde{\iota}$. (τ) is generated by the S^1-invariant vertical vector field:

$$\tau^\dagger=\sqrt{-1}(\zeta-\bar{\zeta})\in\boldsymbol{E}^\dagger(\boldsymbol{P}_n(\boldsymbol{C})),$$

where $\zeta=\sum_{a=0}^n z^a\partial/\partial z^a$ and $\bar{\zeta}=\sum_{a=0}^n \bar{z}^a\partial/\partial\bar{z}^a$ [8]. Here we have

$$E^{\dagger}(P_n(C)) = \bigcup_{p \geqq 0} (E^{\dagger})^p(P_n(C))$$

with

$$(E^{\dagger})^p(P_n(C)) = E^p(C^{n+1}\text{-}\{0\}) \cap E^{\dagger}(P_n(C)) .$$

Notice that $\tau^{\dagger}$ is an infinitensimal generator of the S^1-action of isometries on S^{2n+1} given by the multiplication of $z \in C$ with $|z|=1$.

Put

$$(S^{\dagger})^p(P_n(C)) := \sigma^p(E^{\dagger})^p(P_n(C)) ,$$
$$(S^{\dagger})^*(P_n(C)) := \sum_{p \geqq 0} \sigma^p(E^{\dagger})^p(P_n(C)) \quad \text{(direct sum)} .$$

Then we have an isomorphism

$$(1.5) \qquad \pi_s^{\dagger}\colon S^*(P_n(C)) \to (S^{\dagger})^*(P_n(C))/(\tau)_s ,$$

where $(\tau)_s$ is the two-sided ideal in $(S^{\dagger})^*(P_n(C))$ generated by

$$\tau_s^{\dagger} = \sqrt{-1}(\zeta_s - \bar{\zeta}_s) \in (S^{\dagger})^1(P_n(C))$$

with $\zeta_s = \sum_{a=0}^n z^a \partial/\partial z^a$ and $\bar{\zeta}_s = \sum_{a=0}^n \bar{z}^a \partial/\partial \bar{z}^a$.

REMARK. When we regard an element $\zeta \in (E^{\dagger})^1(P_n(C))$ as an element of $(S^{\dagger})^1(P_n(C))$, we distinguish it from ζ just by putting a subscript s as ζ_s above. We need such a distinction specifically in (3) and (4) of Definition 1.3.

A representative in $E^{\dagger}(P_n(C))$ of $D \in E(P_n(C))$ under the identification (1.4) will be denoted by $D^{\dagger}$ in the following. Similarly, a representative in $(S^{\dagger})^*(P_n(C))$ of $\xi \in S^*(P_n(C))$ will be denoted by $\xi^{\dagger}$.

Lemma 1.3. *$\xi \in S^*(C^{n+1}\text{-}\{0\})$ belongs to $(S^{\dagger})^*(P_n(C))$ if and only if*

$$[\xi, r^2] = 0, [\xi, \zeta_s] = 0 , \quad [\xi, \bar{\zeta}_s] = 0 .$$

Proof. This is obvious from the construction of $(S^{\dagger})^*(P_n(C))$. Q.E.D.

DEFINITION 1.1. Define

$$(S^{\dagger})^{**}(P_n(C)) := \sum_{k,l \geqq 0} (S^{\dagger})^{k+l}(P_n(C)) \cap S^{k+l}(C^{n+1}\text{-}\{0\}) \quad \text{(direct sum)} .$$

Lemma 1.4. *There is a canonical isomorphism ϕ of the bigraded algebras*

$$\phi\colon (S^{\dagger})^{**}(P_n(C)) \to S^{**}(P_n(C)) ,$$

where the map ϕ is the restriction of the inverse of the isomorphism $\pi_s^{\dagger}$ in (1.5) *to* $(S^{\dagger})^{**}(P_n(C))$.

Proof. Both of the surjectivity and the triviality of the kernel of ϕ are proved in [8; Lemma 1.3].

Q.E.D.

From now on, every $\xi \in \boldsymbol{S}^{k,l}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ with components $\xi^{a_1\cdots a_k\bar{b}_1\cdots\bar{b}_e} \in C^\infty(\boldsymbol{C}^{n+1}\text{-}\{0\})$ will be identified with a function on the cotangent bundle $T^*(\boldsymbol{C}^{n+1}\text{-}\{0\})$:

$$\xi = \frac{1}{k!\, l!} \sum \xi^{a_1\cdots a_k\bar{b}_1\cdots\bar{b}_l}\, w_{a_1}\cdots w_{a_k}\overline{w}_{b_1}\cdots\overline{w}_{b_l}\,, \tag{1.6}$$

where w_i's$(0\leqq i\leqq n)$ together with $\overline{w}_j$'s$(0\leqq j\leqq n)$ are regarded as the current coordinates in

$$T^*(\boldsymbol{C}^{n+1}\text{-}\{0\})_{(z,\bar{z})} = \{\sum_{i=0}^{n} (w_i dz_i|_{(z,\bar{z})} + \overline{w}_i d\bar{z}_i|_{(z,\bar{z})})\}\,,$$

where $T^*(\boldsymbol{C}^{n+1}\text{-}\{0\})_{(z,\bar{z})}$ is the cotangent space at $(z,\bar{z})\in\boldsymbol{C}^{n+1}\text{-}\{0\}$. Namely, we regard a contravariant symmetric tensor field of bidegree (k, l) as a homogeneous polynomial of bidegree (k, l) with respect to the variables w_i's and $\overline{w}_j$'s.

Denote by $\check{\boldsymbol{E}}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ the set of all linear differential operators of $4(n+1)$ variables $z^0, \cdots, z^n, \bar{z}^0, \cdots, \bar{z}^n, w_0, \cdots, w_n, \overline{w}_0, \cdots, \overline{w}_n$, the coefficients of which are C^∞ with respect to the variables z^i' s and $\bar{z}^j$'s on $\boldsymbol{C}^{n+1}\text{-}\{0\}$ and are homogeneous polynomials with respect to the variables w_i's and $\overline{w}_j$'s $(0\leqq i, j\leqq n)$. An element of $\check{\boldsymbol{E}}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ can be regarded as a linear differential operatoı acting on $\boldsymbol{S}^*(\boldsymbol{C}^{n+1}\text{-}\{0\})$ in virtue of the identification (1.6).

We also remark that $\boldsymbol{E}(\boldsymbol{C}^{n+1}\text{-}\{0\})\subset\check{\boldsymbol{E}}(\boldsymbol{C}^{n+1}\text{-}\{0\})$.

EXAMPLES. ζ_s and $\bar{\zeta}_s$ in (1.4) and $\tau_s^\dagger$ in (1.5) are reexpressed as follows:

$$\zeta_i = \sum_{a=0}^{n} z^a w_a,\ \bar{\zeta}_s = \sum_{a=0}^{n} \bar{z}^a \overline{w}_a$$

and

$$\tau_s^\dagger = \sqrt{-1}\,(\sum_{a=0}^{n} z^a w_a - \sum_{a=0}^{n} \bar{z}^a \overline{w}_a)\,.$$

Lemma 1.5. (1) *$\zeta\in(\boldsymbol{S})^{k,l}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ belongs to $(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$ if and only if*

(i) $\sum_{a=0}^n \bar{z}_a\partial\xi/\partial w_a=0$, (ii) $\sum_{a=0}^n z^a\partial\xi/\partial\overline{w}_a=0$,

(iii) $\sum_{a=0}^n z^a\partial\xi/\partial z^a=k\xi$, (iv) $\sum_{a=0}^n \bar{z}^a\partial\xi/\partial\bar{z}^a=l\xi$.

(2) *If $\xi\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$, then we have necessarily*

$$\sum_{a=0}^{n} w_a\partial\xi/\partial w_a = k\xi\,,\quad \sum_{a=0}^{n} \overline{w}_a\partial\xi/\partial\overline{w}_a = l\xi\,.$$

Proof. (1) (i)$\sim$(iv) follow from Lemma 1.3 if $\xi\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$. In virtue of (1.6) we can regard $\xi\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$ as a homogeneous function of homogeneous degree k and l with respect to the variables w and $\overline{w}$, respectively. The two identities in (2) follow from this fact by Euler's theorem. Q.E.D.

DEFINITION 1.2. (1) Denote by $\check{\boldsymbol{I}}$ the left ideal in $\check{\boldsymbol{E}}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ generated by the following four linear differential operators:

(i) $\zeta-\sum_{a=0}^{n} w_a\partial/\partial w_a$, (ii) $\bar{\zeta}-\sum_{a=0}^{n} \overline{w}_a\partial/\partial \overline{w}_a$,

(iii) $(1/r^2)\sum_{a=0}^{n} \bar{z}^a\partial/\partial w_a$, (iv) $(1/r^2)\sum_{a=0}^{n} z^{\alpha}\partial/\partial \overline{w}_a$.

(2) We denote by $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$ the normalizer of $\check{\boldsymbol{I}}$ in $\check{\boldsymbol{E}}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ viewed as a Lie algebra, i.e.,

$$\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))=\{D\,|\,[D,\check{\boldsymbol{I}}]\subset\check{\boldsymbol{I}}\}.$$

Lemma 1.6. *$D\in\boldsymbol{E}(\boldsymbol{C}^{n+1}\text{-}\{0\})$ preserves $(\boldsymbol{S}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ if and only if $D\in\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$.*

Proof. The assertion is obtained by expressing Lemma 1.5 (1) in terms of $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$. Q.E.D.

Put

$$\check{\boldsymbol{I}}_0:=\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))\cap\check{\boldsymbol{I}}.$$

Then $\check{\boldsymbol{I}}_0$ is easily proved to be a two-sided ideal in $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$ and

$$\boldsymbol{EO}(\boldsymbol{P}_n(\boldsymbol{C})):=\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))/\check{\boldsymbol{I}}_0$$

is regarded as an algebra of linear differential operators acting on $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$.

DEFINITION 1.3. Put

(1) $(T^*)^{\dagger}:=2\sum_{a,b=0}^{n}(r^2\delta_{ab}-z^a\bar{z}^b)w_a\overline{w}_a$,

(2) $T^{\dagger}:=(1/2r^2)\sum_{a=0}^{n}\partial^2/\partial w_a\partial\overline{w}_a$,

(3) $(\partial^*)^{\dagger}:=2r^2\sum_{a=0}^{n}w_a\partial/\partial\bar{z}^a+2\zeta_s(\zeta-\bar{\zeta})$,

(4) $(\bar{\partial}^*)^{\dagger}:=2r^2\sum_{a=0}^{n}\overline{w}_a\partial/\partial z^a-2\bar{\zeta}_s(\zeta-\bar{\zeta})$,

(5) $\partial^{\dagger}:=-\sum_{a=0}^{n}\partial^2/\partial z^a\partial w_a-\sum_{a=0}^{n}(\bar{\zeta}_s/r^2)\partial^2/\partial\overline{w}_a\partial w_a$,

(6) $\partial^{\dagger}:=-\sum_{a=0}^{n}\partial^2/\partial\bar{z}^a\partial\overline{w}_a-\sum_{a=0}^{n}(\zeta_s/r^2)\partial^2/\partial\overline{w}_a\partial w_a$,

(7) $\kappa_{a,b}:=\sqrt{-1}(z^a\partial/\partial z^b-\bar{z}^b\partial/\partial\bar{z}^a)$,

(8) $\kappa_{a,b}:=\sqrt{-1}(z^a\partial/\partial z^b-\bar{z}^b\partial/\partial\bar{z}^a+\overline{w}^a\partial/\partial\overline{w}^b-w^b\partial/\partial w^a)$,

where $0\leqq a,\ b\leqq n$.

Lemma 1.7. (1) *$(T^*)^{\dagger}$ is an element of $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))\cap(\boldsymbol{S}^{\dagger})^2(\boldsymbol{P}_n(\boldsymbol{C}))$. $T^{\dagger}$, $(\partial^*)^{\dagger}$, $(\bar{\partial}^*)^{\dagger}$, $\partial^{\dagger}$, $\bar{\partial}^{\dagger}$, ζ and $\bar{\zeta}$ are element of $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$,*

(2) *$\kappa_{a,b}$ is an elememt of $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))\cap\boldsymbol{E}^1(\boldsymbol{P}_n(\boldsymbol{C}))$,*

(3) *$\check{\kappa}_{a,b}$ is an element of $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$,*

where $0\leqq a,\ b\leqq n$.

Proof. (1) These properties can be verified immediately. (2) is an immediate consequence of Lemma 1.5 (1). (3) is verified by examining the bracket products of $\check{\kappa}_{a,b}$ with the four generators of $\check{\boldsymbol{I}}$, respectively. Q.E.D.

Denote by $\kappa^*_{a,b}$ and $\check{\kappa}^*_{a,b}$ the adjoint operator of $\kappa_{a,b}$ with respect to the Hermitian inner product defined on $C^\infty(\boldsymbol{P}_n(\boldsymbol{C}))$ and the adjoint operator of $\check{\kappa}_{a,b}$ with respect to the canonical Hermitian inner product defined on $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$, respectively.

Lemma 1.8.
(1) $\kappa^*_{a,b}=-\kappa_{b,a}$ *and* (2) $\check{\kappa}^*_{a,b}=-\check{\kappa}_{b,a}$.

Proof. These follow immediately from their definitions, respectively.
Q.E.D.

Lemma 1.9. (1) $\tau^\dagger$ *can be expressed as follows:*

$$\tau^\dagger=\sum_{a=0}^{n}\kappa_{a,a}\,.$$

(2) *Each of* $\check{\kappa}_{a,b}(0\leqq a,\ b\leqq n,\ a\neq b)$ *satisfies*

$$[\kappa_{a,b},\xi^\dagger]=\check{\kappa}_{a,b}(\xi^\dagger)$$

for $\xi^\dagger\in(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$, *where the bracket in the left-hand side is the bracket product in* $(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$.

(3) *Put*

$$\check{\tau}:=\sum_{a=0}^{n}\check{\kappa}_{a,a}\,.$$

Then

$$\check{\tau}\in\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))\cap\check{\boldsymbol{I}}\,.$$

Proof. This can be verified immediately. Q.E.D.

DEFINITION 1.4. (1) Define $\Delta_0^\dagger\in(\boldsymbol{E}^\dagger)^2(\boldsymbol{P}_n(\boldsymbol{C}))$ by

$$\Delta_0^\dagger=\sum_{a,b=0}^{n}\kappa^*_{a,b}\kappa_{a,b}+\sum_{a,b=0}^{n}\kappa_{a,b}\kappa^*_{a,b}\,.$$

(2) Define $\Delta_0^\dagger\in\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$ by

$$\Delta_0^\dagger=\sum_{a,b=0}^{n}\check{\kappa}^*_{a,b}\check{\kappa}_{a,b}+\sum_{a,b=0}^{n}\check{\kappa}_{a,b}\check{\kappa}^*_{a,b}\,.$$

Lemma 1.10. (1) $\Delta_0^\dagger$ *in Definition* 1.4(1) *is a representative in* $\boldsymbol{E}^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$ *modulo* (τ) *of the Laplace-Beltrami operator* Δ_0 *on* $(\boldsymbol{P}_n(\boldsymbol{C}),g_0)$.

(2) $\Delta_0^\dagger$ *in Definition* 1.4 (2) *is a representative in* $\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$ *modulo* $\check{\boldsymbol{I}}_0$ *of the Lichnerowicz operator* Δ_0 *acting on* $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ (*cf.* [8] *pp.* 123~129 *for the definition of the Lichnerowicz operator*).

Proof. (1) By a direct calcluation we have

$$\sum_{a,b=0}^{n}\kappa_{a,k}^{*}\kappa_{a,b}+\sum_{a,b=0}^{n}\kappa_{a,b}\kappa_{a,b}^{*}=-4\sum_{a,b=0}^{n}(r^2\delta^{ab}-z^a\bar{z}^b)\partial^2/\partial z^a\partial\bar{z}^b$$

$$+2n\sum_{a=0}^{n}z^a\partial/\partial z^a+2n\sum_{b=0}^{n}\bar{z}^b\partial/\partial\bar{z}^b-2(\tau^{\dagger})^2 .$$

This operator satisfies the following three conditions:

(i) its symbol tensor field coincides with $-g_0^{\dagger}$ modulo $(\tau)_s$; (ii) it is a selfadjoint linear differential operator; (iii) it annihilates constant functions. Such an operator must be a representative of the Laplace-Beltrami operator.

(2) A representative $\Delta_0^{\dagger}\in\widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$ of the Lichnerowicz operator Δ_0 is given by (c.f. [8] Lemma 2.13)

$$[\delta^{\dagger},(\delta^*)^{\dagger}]+4\{(\zeta+\bar{\zeta})(n-1)+2\zeta^2+2\bar{\zeta}^2-2\zeta\bar{\zeta}\}-8T^*T ,$$

where $(\delta^*)^{\dagger}:=(\partial^*)^{\dagger}+(\bar{\partial}^*)^{\dagger}$ and $\delta^{\dagger}:=\partial^{\dagger}+\bar{\partial}^{\dagger}$ are representatives of $\delta^*:=\partial^*+\bar{\partial}^*$ resp. $\delta:=\partial+\bar{\partial}$. (Compare with [8] pp. 137~139 for the representatives of Δ_0, δ^*, and δ, where representatives of these operators are treated in a slightly different manner from the present paper). By direct calculations we can verify

$$\Delta_0^{\dagger}=\sum_{a,b=0}^{n}(\check{\kappa}_{a,b}^{*}\check{\kappa}_{a,b}+\check{\kappa}_{a,b}\check{\kappa}_{a,b}^{*})\ \text{modulo}\ \boldsymbol{I}_0 .$$

Q.E.D.

DEFINITION 1.5. Define

(1) an endomorphism S of bidegree $(-1,-1)$ on the bigraded algebra $(\boldsymbol{S}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ by

$$S=\Delta_0^{\dagger}T^{\dagger}-\lambda_{k,l,1}T^{\dagger}+6(T^*)^{\dagger}(T^{\dagger})^2-\partial^{\dagger}T^{\dagger}(\partial^*)^{\dagger}+(\partial^*)^{\dagger}T^{\dagger}\partial^{\dagger}$$

on $(\boldsymbol{S}^{\dagger})^{kl}(\boldsymbol{P}_n(\boldsymbol{C}))$, where in general

$$\lambda_{k,l,m}=4\{(2k-m)n+3k^2+l^2-2kl-(m+1)(k+l)+m^2+2m\}$$

for $k,l,m\geqq 0$, $(k,l,m\in\boldsymbol{Z})$,

(2) $B_m^*:=4m(m+1)(T^*)^{\dagger}+2(\partial^{\dagger})^*(\bar{\partial}^{\dagger})^*$ for $m\geqq 1$ $(m\in\boldsymbol{Z})$,

(3) $A_m^*:=(\prod_{i=1}^{m}B_i^*)(T^{\dagger})^m$ $(A_0^*=id.)$ for $m\geqq 1$ $(m\in\boldsymbol{Z})$.

DEFINITION 1.6. (1) $(\boldsymbol{K}^{\dagger})^*(\boldsymbol{P}_n(\boldsymbol{C}),g_0)$ is the graded $\boldsymbol{C}$-subalgebra of $(\boldsymbol{S}^{\dagger})^*(\boldsymbol{P}_n(\boldsymbol{C}))$ generated by $\kappa_{a,b}$ $(0\leqq a,b\leqq n)$, i.e.,

$$(\boldsymbol{K}^{\dagger})^*(\boldsymbol{P}_n(\boldsymbol{C}),g_0):=\sum_{p=0}^{\infty}(\boldsymbol{K}^{\dagger})^p(\boldsymbol{P}_n(\boldsymbol{C}),g_0)\ \text{(direct sum)},$$

where

$$(\boldsymbol{K}^{\dagger})^p(\boldsymbol{P}_n(\boldsymbol{C}),g_0):=(\boldsymbol{K}^{\dagger})^*(\boldsymbol{P}_n(\boldsymbol{C}),g_0)\cap(\boldsymbol{S}^{\dagger})^p(\boldsymbol{P}_n(\boldsymbol{C})) .$$

(2) Define

$$(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) := \sum_{k,l\geqq 0} (\boldsymbol{K}^{\dagger})^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \text{ (direct sum)},$$

where

$$(\boldsymbol{K}^{\dagger})^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) := (\boldsymbol{K}^{\dagger})^{k+l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \cap (\boldsymbol{S}^{\dagger})^{k,l}(\boldsymbol{P}_n(\boldsymbol{C})).$$

$(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ is a bigraded $\boldsymbol{C}$-subalgebra of $(\boldsymbol{S}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$.

Theorem 1.1. ([8] p. 136) (1) *We have*

$$(\boldsymbol{K}^{\dagger})^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = \{0\}$$

for $k \neq l$.

(2) $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *is generated by* $\kappa_{ab,cd} \in (\boldsymbol{K}^{\dagger})^{1,1}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ $(0 \leqq a, b, c, d \leqq n)$, *where*

$$\kappa_{a,\bar{b}} := z^a \partial/\partial \bar{z}^b - z^b \partial/\partial \bar{z}^a,$$
$$\kappa_{\bar{c},d} := \bar{z}^c \partial/\partial z^d - \bar{z}^d \partial/\partial z^c,$$
$$\kappa_{ab,cd} := \kappa_{a,d}\kappa_{b,c} - \kappa_{a,c}\kappa_{b,d} = \kappa_{a,\bar{b}}\kappa_{\bar{c},d}.$$

DEFINITION 1.7. (1) Denote by $(T^*)_0^{\dagger}$ the restriction of $(T^*)^{\dagger}$ to $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. Notice that $(T^*)_0^{\dagger}$ *preserves* $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

(2) Denote the image of $(T^*)_0^{\dagger}$ by Im $(T^*)_0^{\dagger}$ $(\subset (\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0))$ and denote the orthogonal complement of Im $(T^*)_0^{\dagger}$ in $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ by $\boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

Thus we have

$$(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = \mathrm{Im}(T^*)_0^{\dagger} \oplus \boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0),$$

and $\boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ has a bigradation:

$$\boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = \sum_{k,l=0}^{\infty} \boldsymbol{P}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \text{ (direct sum)},$$

where

$$\boldsymbol{P}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) := \boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \cap (\boldsymbol{K}^{\dagger})^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0).$$

Lemma 1.11. ([8] p. 147, Lemma 4.2.) (1) *The endomorphism S leaves* $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *invariant.*

(2) $A_k^*(k \geqq 0)$ *also leaves* $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *invariant.*

Denote the canonical projection by

$$\Pi_0 \colon (\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \to \boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0).$$

Π_0 can be proved to be commutative with Δ_0.

Put

$$C_m^* := \Pi_0 A_m^* \ (m \geqq 0).$$

C_m^*'s satisfy

$$\Delta_0^\dagger C_m^* - \lambda_{k,m} C_m^* + \frac{1}{(m+1)^2} C_{m+1}^* = 0 \ (k \geqq m+1 > m \geqq 0)$$

on $(\boldsymbol{K}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$,
where $\lambda_{k,m} := \lambda_{k,k,m} = 4\{(2k-m)n + 2k^2 - 2(m+1)k + m^2 + 2m\}$.

DEFINITION 1.8. (1) Define an operator

$$P_{k,m} := \frac{n+2k-2m-2}{m!\,(n+2k-m-2)!} \sum_{i=m}^{k} \frac{(-1)^{i-m}(n+2k-i-m-3)!}{2^{2i}(i!)^2\,(i-m)!} C_{m+1}^*$$
$$: \boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C})) \to \boldsymbol{P}^{**}(\boldsymbol{P}_n(\boldsymbol{C})) \text{ for } k \geqq m \geqq 0 .$$

(2) Denote the image of the map $P_{k,m}$ by $E_{k,m}$. Notice that $\Delta_0^\dagger$ preserves $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

Theorem 1.2. (*c.f.* [8]) *Let k and m be as in Definition* 1.8, *then*

(1) $\Delta_0 P_{k,m} = \lambda_{k,m} P_{k,m}$ *on* $(\boldsymbol{K}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_n)$.

(2) *Each* $E_{k,m}$ *is non-trivial under the assumpiton* $n \geqq 3$.

(3) *We have direct sum decompositions:*

$$(\boldsymbol{K}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = \sum_{h=0}^{k} ((T^*)^\dagger)^h \boldsymbol{P}^{k-h,k-h}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$$

and

$$\boldsymbol{P}^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = \sum_{m=0}^{k} E_{k,m} .$$

(1)~(3) *yield the eigenspace decomposition of the restriction of* $\Delta_0^\dagger$ *on* $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

DENINITION 1.9. Define

(1) $D_{abcd} := \frac{1}{8} \sum_{e,f,g,h=0}^{n} \delta_{abcd}^{efgh} \check{\kappa}_{e,f} \check{\kappa}_{g,h} \in \widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$.

(2) $\Delta_1^\dagger := \frac{1}{4!} \sum_{a,b,c,d=0}^{n} (D_{abcd}^* D_{abcd} + D_{abcd} D_{abcd}^*) \subset \widetilde{\boldsymbol{EO}}(\boldsymbol{P}_n(\boldsymbol{C}))$.

(3) The linear differential operator on $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ corresponding to $\Delta_1^\dagger$ is denoted by Δ_1.

Theorem 1.3.

(1) $[\check{\kappa}_{a,b}, \Delta_0^\dagger] = 0$. (2) $[\check{\kappa}_{a,b}, \Delta_1^\dagger] = 0$.

Proof. These identities follow easily from their definitions. Q.E.D.

Theorem 1.4. $\Delta_0^\dagger$ *and* $\Delta_1^\dagger$ *commute with the operators introduced in Definition* 13 *as follows:*

(1) $[(T^*)^\dagger, \Delta_i^\dagger] = 0$, (2) $[T^\dagger, \Delta_i^\dagger] = 0$,

(3) $[(\partial^*)^\dagger, \Delta_i^\dagger] = 0$, (4) $[\partial^\dagger, \Delta_i^\dagger] = 0$,

(5) $[(\bar{\partial}^*)^\dagger, \Delta_i^\dagger] = 0$, (6) $[\bar{\partial}^\dagger, \Delta_i^\dagger] = 0$,

($i=0, 1$).

Proof. From Definition 1.3 we obtain these formulae immediately (cf. [8] pp. 54∼55 and p. 59). Q.E.D.

2. The simultaneous eigenspace decomposition of Δ_0 and Δ_1 on $\boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. In this section we assume $n+1\geqq 4$, where n is the complex dimension of $\boldsymbol{P}_n(\boldsymbol{C})$.

Theorem 2.1. *$\Delta_1^\dagger$ can be expressed as follows:*

$$\begin{aligned}\Delta_1^\dagger = &-2(T^*)^\dagger T^\dagger\Delta_0^\dagger+(pn+2kl-2p)\Delta_0^\dagger-8(T^*)^\dagger)^2(T^\dagger)^2\\ &-2(T^*)^\dagger(\partial^\dagger\bar\partial^\dagger+\bar\partial^\dagger\partial^\dagger)-2((\partial^*)^\dagger(\bar\partial^*)^\dagger+(\bar\partial^*)^\dagger(\partial^*)^\dagger)T^\dagger-\\ &2(n+2l-2)(\bar\partial^*)^\dagger\bar\partial^\dagger-2(n+2k-2)(\bar\partial^*)^\dagger\bar\partial^\dagger+8((p-2)n+k^2\\ &+l^2-3p+4)(T^*)^\dagger T^\dagger-4k(n+l-2)((k-1)n+k^2\\ &l^2-kl-2k+1)-4l(n+k-2)((l-1)n+k^2+l^2-kl-2l+1)\end{aligned}$$

($p=k+l$) *on* $(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

Proof. From the definition of $\Delta_1^\dagger$ in **1** and Definition 1.3 we can obtain the required relation by direct calculations. Q.E.D.

Corollary. *Restricting the action of $\Delta_1^\dagger$ to $(\boldsymbol{S}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}))$, we obtain the reduced form of Theorem 2.1:*

$$\begin{aligned}\Delta_1 = &-2(T^*)^\dagger T^\dagger\Delta_0^\dagger+2k(n+2k-2)\Delta_0^\dagger-8((T^*)^\dagger)^2(T^\dagger)^2-\\ &2(T^*)^\dagger(\bar\partial^\dagger\partial^\dagger+\partial^\dagger\bar\partial^\dagger))-2((\partial^*)^\dagger(\bar\partial^*)^\dagger+(\bar\partial^*)^\dagger(\partial^*)^\dagger)T-\\ &2(n+2k-2)((\partial^*)^\dagger\partial^\dagger-(\bar\partial^*)^\dagger(\bar\partial)^\dagger)+16(k-1)(n+k-2)(T^*)^\dagger T^\dagger\\ &-8k(n+k-2)(k-1)(n+k-1)\,.\end{aligned}$$

Lemma 2.1 *$\Delta_1^\dagger$ and S satisfy*

$$\begin{aligned}\Delta_1^\dagger+4(T^*)^\dagger S = &\ 2(k+1)(n+k-1)\Delta_1^\dagger-8k(n+k-1)(k+1)(n+k)\\ &-2(n+2k-1)(\partial^\dagger(\partial^*)^\dagger+(\bar\partial)^\dagger(\bar\partial^*)^\dagger)-2((\bar\partial^*)^\dagger T\,(\partial^*)^\dagger\\ &+(\partial^*)^\dagger T^\dagger(\bar\partial^*)^\dagger) \qquad on\ \ (\boldsymbol{S}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)\,.\end{aligned}$$

Proof. From Definition 1.5 (1), we can express $(T^*)^\dagger S$ in terms of fundamental operators. Eliminating the first term of the right-hand side in the formula in Theorem 2.1, we obtain the required relation. Q.E.D.

Theorem 2.2. *We have*

$$\Delta_1^\dagger = \sum_{m=0}^{k}\mu_{k,m}P_{k,m}$$

as the eigenspace decomposition of $\Delta_1^\dagger$ restricted to $(\boldsymbol{K}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$, where $\mu_{k,m}=$

$8(k-m)(k+1)(n+k+1)(n+k-m-2)$.

Proof. Restricting $\Delta_1^\dagger$ on $E_{k,m}$, we obtain in virtue of Lemma 2.1

$$\begin{aligned}\Delta_1^\dagger &= 2(k+1)(n+k-1)\{\Delta_0^\dagger-4k(n+k)\}\\ &= 2(k+1)(n+k-1)\{4(2k-m)n+2k^2-2(m+1)k+m^2+2m-4k(n+k)\},\end{aligned}$$

which coincides with the desired value $\mu_{k,m}$. Q.E.D.

3. The Radon transform and $D(G_{2n-1}(C))$. In this section we also assume $n+1\geqq 4$. Denote by $W_2(C^{n+1})$ the Stiefel manifold of all 2-frames in C^{n+1} and denote by $V_2(C^{n+1})$ the submanifold of $W_2(C^{n+1})$ defined as the totality of orthonormal 2-frames with respect to the standard Hermitian metric g on C^{n+1}. $V_2(C^{n+1})$ is identified with a homogeneous space:

$$U(n+1)/U(n-1)\,.$$

Denote by $G_{2,n-1}(C)$ the Grassmann manifold of all complex 2-planes passing through the origin of C^{n+1}. As is well known, $G_{2n-1}(C)$ is identified with a homogeneous space

$$U(n+1)/U(n-1)\times U(2)\,.$$

$V_2(C^{n+1})$ can be regarded as a principal bundle on the complex Grassmann manifold $G_{2,n-1}(C)$ with structure group $U(2)$, where the projection π_V is defined canonically.

Applying Lemma 1.1 to the principal bundle

$$\pi_V\colon V_2(C^{n+1})\to G_{2,n-1}(C)$$

with $U(2)$ as its fibre, we obtain an isomorphism:

$$(3.1)_V \qquad E(G_{2,n-1}(C))\cong E^{U(2)}(V_2(C^{n+1}))/J\,,$$

where J is the two-sided ideal in $E^{U(2)}(V_2(C^{n+1}))$ generated by $U(2)$-invariant vertical vector fields. On the other hand, there is a polar decomposition of the Stiefel manifold $W_2(C^{n+1})$:

$$(3.2) \qquad W_2(C^{n+1})\cong H_2^+\times V_2(C^{n+1})\,,$$

where H_2^+ is the space of positive definite 2×2 Hermitian matrices [8]. Denote by $\pi_V\colon W_2(C^{n+1})\to V_2(C^{n+1})$ the canonical projection to the second factor of (3.2).

Put $\rho_{\alpha\beta}^2 := \langle q_\alpha, q_\beta\rangle$, where $0\leqq\alpha, \beta\leqq 1$, $q=(q_0, q_1)\in W_2(C^{n+1})$ and $\langle\,,\,\rangle$ denotes the pointwise inner product as introduced in (1.1). The positive definite square root matrix $(\rho_{\alpha,\beta})$ of $(\rho_{\alpha\beta}^2)$ is called the *radial part* of q, which can be regarded as the H_2^+ part of q in the polar decomposition (3.2). In virtue of Lemma 1.2 the polar decomposition (3.2) assures the existence of two subalgebras, each one of which is the centralizer of the other in $\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ and the second one of which is canonically isomorphic to $\boldsymbol{E}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$. Thus a linear differential operator $D\in\boldsymbol{E}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ can be represented by a linear differential operator $D^{\dagger\dagger}\in\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ satisfying

$$(3.3)\qquad [D^{\dagger\dagger}, \rho_{\alpha\beta}^2] = 0 \text{ and } [D^{\dagger\dagger}, \frac{\partial}{\partial\rho_{\alpha\beta}^2}] = 0\ (0\leqq\alpha, \beta\leqq 1)\,.$$

The totality of such operators in $\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ is designated as $\boldsymbol{E}^{\dagger\dagger}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$.

Similarly, we have an isomorphism:

$$(3.1)_W\qquad \boldsymbol{E}(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))\cong(\boldsymbol{E}^{U(2)})^{\dagger\dagger}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1})))/\boldsymbol{J}^{\dagger\dagger}\,,$$

where $(\boldsymbol{E}^{U(2)})^{\dagger\dagger}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ is the subalgebra of $\boldsymbol{E}^{\dagger\dagger}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$, which is canonically isomorphic to $\boldsymbol{E}^{U(2)}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ and $\boldsymbol{J}^{\dagger\dagger}$ is the ideal in $(\boldsymbol{E}^{U(2)})^{\dagger\dagger}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ corresponding to $\boldsymbol{J}$ in $(3.1)_V$.

To an arbitrary element $q\in\boldsymbol{W}_2(\boldsymbol{C}^{n+1})$ corresponds a linear isometric imbedding ι_q: $(\boldsymbol{C}^2\text{-}\{0\}, \iota^* g_0)\hookrightarrow(\boldsymbol{C}^{n+1}\text{-}\{0\}, g_0)$, where g_0 is the metric in $\boldsymbol{C}^{a+1}\text{-}\{0\}$ defined as

$$g_0=\frac{1}{r^2}g\,,$$

where g is the canonical flat metric in $\boldsymbol{C}^{n+1}$ and r^2 is as in (1.3).

Let ξ be a contravariant tensor field on a Riemannian manifold. We denote by ξ^* the corresponding covariant tensor field. Conversely, if ξ is a covariant tensor field, we denote by ξ^* the coresponding contravariant tensor field. A contravariant symmetric tensor field ξ defined on $(\boldsymbol{C}^{n+1}\text{-}\{0\}, g_0)$ induces a contravariant symmetric tensor field $((\iota_q)^*\xi_*)^*$ on $(\boldsymbol{C}^2\text{-}\{0\}, \iota_q^* g_0)$ through the above imbedding ι_q. Fundamental differential operators of $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_1(\boldsymbol{C}))$ will be denoted by lower index 1, e.g., T_1, (δ_1^*).

DEFINITION 3.1. Define the *Radon transform*

$$\wedge: (\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))\to C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}), \boldsymbol{C})$$

by

$$(\xi)^\wedge(\Gamma)=\begin{cases}(2^k/\mathrm{Vol}(S^3))\int_{S^2}(T_1^\dagger)^k((\iota_q)^*(\xi)_*)^* d\sigma & (k=l)\\ 0 & (k\neq l)\end{cases}$$

for $\xi\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$, where $\Gamma=\pi_V\pi_W(g)$ for $q\in W_2(\boldsymbol{C}^{n+1})$ and Vol (S^3) is the total volume of the standard sphere S^3. The corresponding map

$$\wedge: \boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))\to \boldsymbol{C}^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}), \boldsymbol{C})$$

is also called the *Radon transform*.

Notice that behind the naturality of such a definition of the Radon transform lies a fact that the Hopf fibering φ in (1.2) is a Riemannian submersion.

Let $p=(p_0, p_1)\in V_2(C^{n+1})$. Put $P^{ab}=p_0^a p_1^b - p_1^a p_0^b$, where $p_\alpha=\sum_{a=0}^n p_\alpha^a e_a$ for a fixed orthonormal basis $(e_0, \cdots, e_n)$ in (C^{n+1}, g_0). We can easily verify that $\sum_{a,b=0}^n P^{ab}\bar{P}^{ab}=2$.

$\{P^{ab}\mid 0\leqq a, b\leqq n, a\neq b\}$ determined by a frame $p\in V_2(C^{n+1})$ is called a system of *normalized Plücker coordinates* of the 2-plane spanned by the frame p.

Theorem 3.1. (1) *The image of $(K^\dagger)^{**}(P_n(C), g_0)$ by the Radon transform is the subalgebra of $C^\infty(G_{2,n-1}(C), (C)$ generated by the totality of products $P^{ab}\bar{P}^{cd}$'s $(0\leqq a<b\leqq n$ and $0\leqq c<d\leqq n)$. It is uniformly dense in $C^\infty(G_{2,n-1}(C), C)$.*

(2) *The kernel of the Radon transform restricted to $K^{**}(P_n(C), g_0)$ is the ideal generated by $g_0^*/2-1$.*

Proof. Both of (1) and (2) were proved in [8] as the basic properties of the Radon transform (cf. p. 150~p. 153 in [8]). Q.E.D.

DEFINITION 3.2. Define

(1) $${}_w\hat{\kappa}_{a,b} := \sqrt{-1}(q_0^a\partial/\partial q_0^b - \bar{q}_0^b\partial/\partial\bar{q}_0^a + q_1^a\partial/\partial q_1^b - \bar{q}_1^b\partial/\partial\bar{q}_1^a) \in E^1(W_2(C^{n+1})).$$

(2) $${}_w\Delta_0^\wedge := \sum_{a,b=0}^n {}_w\hat{\kappa}_{a,b}\, {}_w\hat{\kappa}_{a,b} + \sum_{a,b=0}^n {}_w\hat{\kappa}_{a,b}\, {}_w\hat{\kappa}_{a,b}^* \in E^2(W_2(C^{n+1})).$$

(3) $$D_{abcd}^\wedge := \frac{1}{2^3}\sum_{e,f,g,h=0}^n \delta_{abcd}^{efgh}\, {}_w\hat{\kappa}_{e,f}\, {}_w\hat{\kappa}_{g,h}.$$

(4) $${}_w\Delta_1^\wedge := \frac{1}{4!}\sum_{a,b,c,d=0}^n \{(D_{abcd}^\wedge)^* D_{abcd}^\wedge + D_{abcd}^\wedge (D_{abcd}^\wedge)^*\} \in E^4(W_2(C^{n+1})).$$

Lemma 3.1. *${}_w\hat{\kappa}_{a,b}(0\leqq a<b\leqq n)$ and ${}_w\Delta_i^\wedge(i=0, 1)$ belong to $(E^{U(2)})^{\dagger\dagger}(V_2(C^{n+1}))$ and each of them is a representative of some linear differential operator in $E(G_{2,n-1}(C))$.*

Proof. We can verify by routine calculations that these operators satisfy the equations (3.3) . Moreover, these operators can be proved to be $GL(2, C)$-invariant by direct caluclations. Thus our second assertions follow from $(3.1)_W$. Q.E.D.

DEFINITION 3.3. Denote by $\hat{\kappa}_{a,b}(0\leqq a<b\leqq n)$ and $\Delta_i^\wedge(i=0.1)$ the linear differential operators belonginsg to $E(G_{2,n-1}(C))$ whose representatives in $E(W_2(C^{n+1}))$ are ${}_w\hat{\kappa}_{a,b}$ and ${}_w\Delta_i^\wedge$ $(i=0, 1)$, respectively. Denote by g_1 the canonical $U(n+1)$-invariant metric on $G_{2,n-1}(C)$.

Notice that

(1) $\hat{\kappa}_{a,b}$ is a Killing vector field on $(G_{2,n-1}(C), g_1)$. (2) $\Delta_0^\wedge$ is the Laplace-Beltrami operator on $(G_{2,n-1}(C), g_1)$. In virtue of Lemma 3.1, for (1) it is fuffi-

cient to show that ${}_w\hat{\kappa}_{a,b}$'s are infinitesimal generators of the action of $U(n+1)$ on $\boldsymbol{W}_2(\boldsymbol{C})$ and this can be immediately chekced. The second proof of them to be Killing is obtained in conjunction with (2) as follows: ${}_w\Delta_0^\wedge$ is expressed explicitly as

$$
\begin{aligned}
{}_w\Delta_0^\wedge = -4 \sum_{\alpha,\beta,\gamma,\delta=0}^{1} \sum_{a,b=0}^{n} (\delta^{ab} - \bar{q}_\alpha^a q_\beta^b (\rho^2)^{\alpha\beta}) \rho_{\gamma\delta}^2 \partial^2/\partial \bar{q}_\gamma^a \partial q_\delta^b \\
+2n \sum_{\alpha=0}^{1} \sum_{a=0}^{n} (q_\alpha^a \partial/\partial q_\alpha^a + \bar{q}_\alpha^a \partial/\partial \bar{q}_\alpha^a) ,
\end{aligned}
$$

where $((\rho^2)^{\alpha\beta})$ is the inverse matrix of $(\rho_{\alpha\beta}^2)$. ${}_w\Delta_0^\wedge$ can be proved to be a representative of the Laplacian on $(\boldsymbol{G}_{2,n-1}(\boldsymbol{C})), g_1)$, for the proof of which we refer to Lemma 3.4 of [8]. A vector field on a Riemannian manifold is Killing if and only if it commutes with the Laplace Beltrami operator (cf. [7]) and we have $[{}_w\hat{\kappa}_{a,b}, {}_w\Delta_0^\wedge]=0$ by direct calculation. The second proof of (1) follows from this fact immediately.

Theorem 3.2. *The $U(n+1)$-actions commute with the Radon transform. Namely,*

$$\wedge\ \rho_0 = \rho_1 \wedge ,$$

where ρ_0 and ρ_1 are the natural representation of $U(n+1)$ on $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ and on $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}), \boldsymbol{C})$, respectively.

Proof. This follows from the definition of the Radon transform obviously. Q.E.D.

Corollary.

$$\hat{\kappa}_{a,b}(\eta^\dagger)^\wedge = (\check{\kappa}_{a,b}\eta^\dagger)^\wedge ,$$

where $\eta^\dagger \in (\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ is the unique representative of $\eta \in \boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$.

Proof. In virtue of Lemma 1.9 (2) our assertion is the infinitesimal version of Theorem 3.2. Note that the uniqueness of the representative follows from Lemma 1.4. Q.E.D.

Theorem 3.3. *Let $\eta^\dagger \in (\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ be a representative of $\eta \in \boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ and $(\eta^\dagger)^\wedge$ the Radon transform of $\eta^\dagger$. Then*

(1) $((\Delta_0\eta)^\dagger)^\wedge = \Delta_0^\wedge (\eta^\dagger)^\wedge$.

(2) $((\Delta_1\eta)^\dagger)^\wedge = \Delta_1^\wedge (\eta^\dagger)^\wedge$.

Proof. In virtue of Definition 3.1, Definition 3.2 and Definition 3.3, the assertions follow from Corollary above. Q.E.D.

DEFINITION 3.4. (1) Denote by $E_{k,m}^\wedge$ the image of $E_{k,m}$ by the Radon transform.

(2) Denote by $D(G_{2,n-1}(C))$ the algebra of the totality of $U(n+1)$-invariant differential operators acting on $C^{\infty}(C_{2,n-1}(C), R)$ [3].

The eigenvalues of $\Delta_0^{\wedge}$ and $\Delta_1^{\wedge}$ restricted to $E_{k,m}^{\wedge}$ coincide with $\lambda_{k,m}$ and $\mu_{k,m}$, respectively. These are direct consequences of Theorem 3.3.

Main theorem. (1) $\Delta_0^{\wedge}$, *together with* $\Delta_1^{\wedge}$ *generates* $D(G_{2,n-1}(C))$. (2) *Each* $E_{k,m}^{\wedge}(k\geqq m\geqq 0)$ *is a* $U(n+1)$*-irreducible representation subspace of* $C^{\infty}(G_{2,n-1}(C), R)$.

Proof. Notice first that $\Delta_i^{\wedge}(i=0, 1)$ preserve $C^{\infty}(G_{2,n-1}(C), R)$ and they can be regarded as elements belonging to $D(G_{2,n-1}, (C))$.

(1) It is known that $D(G_{2,n-1}(C))$ is generated by two invariant linear differential operators of order 2 and 4, respectively (cf. [3]). It remains to show that $\Delta_i^{\wedge}$ $(i=1, 2)$ are algebraically independent over the field of real numbers.

Now suppose that

$$f(\Delta_0^{\wedge}, \Delta_1^{\wedge}) = 0 ,$$

where $f(x, y)$ is an irreducible real polynomial in two variables. Then we have

$$f(\Delta_0^{\wedge}, \Delta_1^{\wedge})\xi = f(\lambda_{k,m}, \mu_{k,m}) = 0 ,$$

where ξ is a non-trivial element of $E_{k,m}^{\wedge}$. Therefore we have

$$f(\lambda_{k,m}, \mu_{k,m}) = 0, k\geqq m\geqq 0\ (k, m\in Z) .$$

We can deduce from this that the left-hand side of the equality above vanishes as a polynomial of two real variables k and m. By the chain rule, we obtain

$$\frac{\partial\lambda_{k,m}}{\partial k}\frac{\partial f}{\partial x}(\lambda_{k,m}, \mu_{k,m})+\frac{\partial\mu_{k,m}}{\partial k}\frac{\partial f}{\partial y}(\lambda_{k,m}, \mu_{k,m}) = 0 ,$$

$$\frac{\partial\lambda_{k,m}}{\partial m}\frac{\partial f}{\partial x}(\lambda_{k,m}, \mu_{k,m})+\frac{\partial\mu_{k,m}}{\partial m}\frac{\partial f}{\partial y}(\lambda_{k,m}, \mu_{k,m}) = 0 ,$$

As we can prove the non-vanishing of the determinant of the coefficient matrix of the simultaneous euqations above for sufficiently large values of the indices k and m by direct calculations, we can conclude that there exist k_0 and m_0 such that

$$\frac{\partial f(\lambda_{k,m}, \mu_{k,m})}{\partial x} = 0 \text{ and } \frac{\partial f(\lambda_{k,m}, \mu_{k,m})}{\partial y} = 0$$

for $k\geqq k_0$ and $m\geqq m_0$. This means that the real algebraic curve defined by $f(x, y)=0$ has an infinite number of singular points in virtue of the following lemma. This is a contradiction.

In ordr to prove (2) we also need the following

Lemma 3.2. *$\lambda_{k,m}=\lambda_{k',m'}$ and $\mu_{k,m}=\mu_{k',m'}$ if and only if*

$$k = k' \text{ and } m = m' .$$

Proof. Assume that $\lambda_{k,m}=\lambda_{k'm'}$ and $\mu_{k,m}=\mu_{k',m'}$,
Put $\varphi(t)=t^2+(n-2)t$, Then

$$\lambda_{k,m} = 4(\varphi(k-m)+k(n+k))$$

and

$$\mu_{k:m} = \varphi(k-m)\varphi(k+1) = \varphi(k+1)\{4\lambda_{k,m}-k(n+k)\}.$$

If we substitute this into

$$\mu_{k,m} = \mu_{k',m'} ,$$

we obtain

$$(k-k')(n+k+k')\{4\lambda_{k',m'}-\varphi(k+1)-k'(n+k')\} = 0 .$$

From this we can verify the assertion. Q.E.D.

Proof of (2) in Main theorem. From Lemma 3.2, $E^{\wedge}_{k,m}$ is concluded to be a maximal simultaneous eigenspace with eigenvalue $\lambda_{k,m}$ and $\mu_{k,m}$ of $\Delta^{\wedge}_0$ and $\Delta^{\wedge}_1$, respectively. The irreducibility of the space follows from a known result (cf. [3]). Q.E.D.

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