Kowalski, Z.S.
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THE EXACTNESS OF GENERALIZED SKEW PRODUCTS

ZBIGNIEW S. KOWALSKI

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0. Introduction

Recently, it appears several papers concerning ergodic properties of random maps i.e. skew products. See T. Morita [5], S. Pelikan [7], etc. The main proof tool in their considerations is the so-called Perron-Frobenius operator. In the present paper the autor proves the theorem about exactness of generalized skew products using the Pinkser algebra.

Let $\sigma: X \to X$ be the shift endomorphism in a space $X \subset \{1, \cdots, s\}^N$ preserving μ. Let $(\tilde{\sigma}, \tilde{\mu})$ be the natural extension of (σ, μ) to the automorphism. The automorphism $\tilde{\sigma}$ is the shift automorphism on the set $\tilde{X} \subset \{1, \cdots, s\}^Z$. Let $\tilde{A}_i = \{\tilde{x} \in \tilde{X}: \tilde{x}(0) = i\}$, $\tilde{\alpha} = \{\tilde{A}_1, \cdots, \tilde{A}_s\}$ and $\tilde{\alpha}_m^n = \bigvee_{k=m}^{n} \tilde{\sigma}^k \tilde{\alpha}$.

DEFINITION. The endomorphism σ is called discrete if $\tilde{\mu}(\tilde{C}) > 0$, for some atom $\tilde{C}$ in $\tilde{\alpha}_{-\infty}^0 \wedge \tilde{\alpha}_0^\infty$.

EXAMPLES: If σ is one-sided Markov shift then it is discrete. If σ is given by Lasota-Yorke type map, then it is also discrete (see [8]).

Let p be a Borel measure on $[0, 1]$ which is positive on open sets. Moreover, let $T_1, \cdots, T_s$ be piecewise monotonic and continuous transformations of $[0, 1]$ into itself so that there exists the partition $\beta_0 = \{I_1, I_2, \cdots\}$ of finite entropy given by $I_i = (t_{i-1}, t_i)$ with $0 = t_0 < t_1 < \cdots$, $\lim t_i = 1$, such that $T_j | (t_i, t_{i+1})$ is continuous and strictly monotonic, for $j = 1, \cdots, s$, $i = 0, 1, \cdots$. We assume that the transformation

$$\bar{T}(x, y) = (\sigma(x), T_{x(0)} y) \tag{1}$$

preserves the product measure $\mu \times p$. Such a transformation is called generalized skew product [2]. The following theorem provides sufficient conditions for $\bar{T}$ to be an exact transformation.

Theorem 1. *Let σ be a discrete endomorphism. If the transformations T_i are 1–1 p a.e., for $i = 1, \cdots, s$, and T_i does not preserve the measure p for some i, then the transformation $\bar{T}$ is exact or $\bar{T}^m$ is not ergodic, for some $m \neq 0$.*

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The above theorem will be applied to show the exactness of random transformations considered in [3] (see section 3). Moreover, section 4 contains conclusions about exactness of random perturbations of Lasota-Yorke maps.

1. Preliminary facts and lemmas which are used to prove Theorem 1

The following property results immediately from Theorem 2 [2].

Property 1. *$\bar{T}$ has not any one-sided generator with finite entropy.*

Let $A_i=\{x\in X: x(0)=i\}$, $\alpha=\{A_1, \cdots, A_s\}$. Moreover, let γ be a partition of the set $X\times[0, 1]$ defined as follows: $\gamma=\alpha\times\delta$, where δ is a countable partition of $[0, 1]$ into intervals such that $\beta_0\leqslant\delta$ and $H(\delta)<\infty$. Here $\alpha\times\delta=\{A_i\times B: i=1, \cdots, s, B\in\delta\}$ and $H(\delta)$ denotes the entropy of δ.

Lemma 1. *If $\bar{T}$ is ergodic, then almost every atom of the partition $\gamma_{\to\infty}=\bigvee_{i=0}^{\infty}\bar{T}^{-i}\gamma$ has the form $x\times U$, where U is a nonempty interval.*

Proof. The atoms of the partition $\gamma_{-\infty}$ have form $x\times U$, where $U\in\lim_{n\to\infty}(\delta\vee T_{x(0)}^{-1}\delta\vee\cdots\vee T_{x(0)}^{-1}\circ\cdots\circ T_{x(n)}^{-1}\delta)$. Therefore $\mathrm{card}(U)\leqslant 1$ or U is a nonempty interval. By Property 1, the set V of atoms $x\times U$ such that U is non-empty interval has positive measure. We note that if $\bar{T}_{\gamma_{-\infty}}(x, U)=(\sigma(x), U^*)$ then $T_{x(0)}U\subset U^*$ so that $\bar{T}_{\gamma_{-\infty}}V\subset V$. By ergodicity of $\bar{T}_{\gamma_{-\infty}}$ we get $\mu\times p_{|\gamma_{-\infty}}(V)=1$. □

Let $(\tilde{T}, \tilde{m})$ be the natural extension of $(\bar{T}, \mu\times p)$ to the automorphism. The automorphism $\tilde{T}$ is defined on the set $M\subset\tilde{X}\times[0, 1]^N$ of pairs $(\tilde{x}, \tilde{y})$, where $\tilde{x}$ is a two sided sequence and $\tilde{y}=(y_0, y_1, \cdots)$ where $T_{\tilde{x}(-i)}(y_i)=y_{i-1}$, for $i=1, 2, \cdots$. Therefore $\tilde{T}$ is given by $\tilde{T}(\tilde{x}, \tilde{y})=(\tilde{\sigma}(\tilde{x}), \tilde{T}_{\tilde{x}(0)}(\tilde{y}))$ where $\tilde{T}_{\tilde{x}(0)}(\tilde{y})=(T_{x(0)}y_0, y_0, y_1, \cdots)$.

Hence the automorphism $(\tilde{\sigma}, \tilde{\mu})$ is a factor of $(\tilde{T}, \tilde{m})$. Now, let $\beta_n=\{[0, \frac{1}{n}], (\frac{1}{n}, \frac{2}{n}], \cdots, (\frac{n-1}{n}, 1]\}\vee\beta_0^n$, $n=1, 2, \cdots$ and $\beta_0^n=\{I_1, \cdots, I_n, J_{n+1}\}$, $J_{n+1}=(t_n, 1]$. Let $\tilde{\gamma}_n$ be the extension of the partition $\gamma_n=\alpha\times\beta_n$ on the space M. Let $\tilde{\gamma}_{n_\infty}=\bigvee_{-\infty}^{\infty}\tilde{T}^i\tilde{\gamma}_n$, $\tilde{\gamma}_{n_{-\infty}}^0=\bigvee_0^{\infty}\tilde{T}^{-1}\tilde{\gamma}_n$, $\tilde{\gamma}_{n_0}^{\infty}=\bigvee_0^{\infty}\tilde{T}^i\tilde{\gamma}_n$ and let $\tilde{T}_n=\tilde{T}_{\tilde{\gamma}_{n_\infty}}$ be the factor automorphism.

Lemma 2. *The pair $\{\tilde{\gamma}_{n_{-\infty}}^0, \tilde{\gamma}_{n_0}^{\infty}\}$ is discrete i.e. there exists an atom $\tilde{D}$ in $\tilde{\gamma}_{n_{-\infty}}^0\wedge\tilde{\gamma}_{n_0}^{\infty}$ such that $\tilde{m}(\tilde{D})>0$.*

Proof. To see this, let $\bar{\alpha}=\{\bar{A}_1, \cdots, \bar{A}_s\}$ where $\bar{A}_i=\{(\tilde{x}, \tilde{y}): \tilde{x}(0)=i\}$. Let $\tilde{C}\in\bar{\alpha}_{-\infty}^0\wedge\bar{\alpha}_0^{\infty}$ be an atom of positive $\tilde{m}$ measure and let $C=\{x\in X: (\tilde{x}, \tilde{y})\in\tilde{C}$ for some $\tilde{y}\}$. There exists an atom $\tilde{D}\in\tilde{\gamma}_{n_{-\infty}}^0\wedge\tilde{\gamma}_{n_0}^{\infty}$ such that $\bigcup_{y\in[0,1]} D_y=C$. Here

$D_y=\{x: (x, y)\in D\}$ and $D=\{(x, y): (\tilde{x}, \tilde{y})\in\tilde{D}\}$. By Lemma 1 we have $\tilde{m}(\tilde{D})=\mu\times p(D)=\int_C p(U_x)\,d\mu(x)>0$, where U_x is the union of sets U such that $x\times U$ is an atom of $\gamma_{n_{-\infty}}$ and $x\times U\subset D$. □

2. Proof of Theorem 1

Assume that $\tilde{T}^m$ is ergodic for every $m\neq 0$. Then the same conditions holds for $\tilde{T}$ and $\tilde{T}_n$ $n=1, 2, \cdots$. By Theorem 1 [8] and by Lemma 2 the transformation $\tilde{T}_n$ is K-automorphism, for $n=1, 2, \cdots$. Let $\mathcal{A}_n=\sigma(\tilde{\gamma}_{n_\infty})$. Then $\tilde{T}^{-1}\mathcal{A}_n=\mathcal{A}_n$ and $\mathcal{A}_n\uparrow\mathcal{B}$ where $\mathcal{B}$ is the σ-algebra of $\tilde{m}$ measurable sets. By Theorem 13 ([6] p. 69) $P(\tilde{T})\cap\mathcal{A}_n\uparrow P(\tilde{T})$ where $P(\tilde{T})$ denotes Pinsker algebra for $\tilde{T}$. The equalities $P(\tilde{T})\cap\mathcal{A}_n=P(\tilde{T}_n)=\{M, \phi\}$ imply $P(\tilde{T})=\{M, \phi\}$. Therefore $\tilde{T}$ is K-automorphism and hence $\tilde{T}$ is the exact endomorphism. □

3. An application to some class of random maps

Now we proceed to consider the exactness of generalized skew products considered in [3]. Let $\{T_\varepsilon\}_{\varepsilon\in(a,b)}$ be the one-parameter family of transformations of the interval [0, 1] into itslef such that

$$T_\varepsilon^{-1}(y)=(1-\varepsilon)y+\varepsilon g(y)\,, \tag{2}$$

where $g\in C^2[0, 1]$, $g(0)=0$, $g(1)=1$, and $a=(1-\sup g')^{-1}$, $b=(1-\inf g')^{-1}$. Moreover, assume that there exists exactly one point y_0, for which $g'(y_0)=1$.

Let T be an infinite interval exchange transformation of [0, 1] of the following type:

(i) there exists a partition $\beta_0=\{I_1, I_2, \cdots\}$ given by $I_i=(t_{i-1}, t_i)$ with $0=t_0<t_1<\cdots$, $\lim t_i=1$, $H(\beta_0)<\infty$
(ii) there exist real constants a_i, so that for $t\in I_i$, $T(t)=t+a_i$
(iii) the only accumulation point of $\{t_{i-1}+a_i\}\cup\{t_i+a_i\}$ is 1
(iv) T is one-to-one.

The above transformations have been considered in [1]. Let σ be the one-sided $(p_1, \cdots, p_s)$-Bernoulli shift and let p be the Lebesgue measure. Using the endomorphism σ we will randomly perturb the automorphism T by s elements of the family (2). Namely we take s functions $T_{\varepsilon_1}, \cdots, T_{\varepsilon_s}$, $\varepsilon_i\neq\varepsilon_j$ for $i\neq j$, and we define the transformation

$$\bar{T}(x, y)=(\sigma(x),\ T_{\varepsilon_{x(0)}}T(y))\,. \tag{3}$$

In addition we postulate that $\bar{T}$ preserves the product measure, which is equivalent to $\sum_{i=1}^{s}\varepsilon_i p_i=0$. For the rest of the paper we shall denote $T_{\varepsilon_i}T$ by T_i, $i=1, \cdots, s$. The transformation $\bar{T}$ has the following property.

Property 2. ([3]) *$\bar{T}$ is weakly mixing, for $s\geqslant 3$. If $T=I$ where $I(x)=x$,* *then $\bar{T}$ is mixing for $s \geqslant 2$* ([4]).

By Theorem 1 and by Property 2 we get

Theorem 2. *The endomorphism $\bar{T}$ is exact for $s \geqslant 3$ (or $s \geqslant 2$ if $T=I$).*

As the conclusion of Theorem 2 we obtain the convergence of the iterations of some double stochastic operators which arise from Frobenius-Perron operator for $\bar{T}$. The Frobenius-Perron operator for $\bar{T}$ is given by the formula

$$P_{\bar{T}}(r(x)f(y)) = \sum_{i=1}^{s} p_i r(ix)(T_i^{-1})'(y) f(T_i^{-1}y),$$

where $ix=(i, x(0), x(1), \cdots)$. Here $r \in L_1(\mu)$ and $f \in L_1(m)$. For $r=1$ we get

$$P_{\bar{T}} f(y) = \sum_{i=1}^{s} p_i (T_{\varepsilon_i}^{-1})'(y) f(T^{-1}T_{\varepsilon_i}^{-1}y).$$

Conclusion 1. $\lim_{n\to\infty} P_{\bar{T}}^n f = \int_0^1 f\, dm$ *in L_1 norm, for every $f \in L_1(m)$.*

REMARK. The results of this paper are still true if it is only assumed that $g \in C^2[0, 1]$, $g(0)=0$, $g(1)=1$, $g(y) \neq y$ for every $y \in (0, 1)$ and there exists $y_0 \in (0, 1)$ such that $g'(y_0)=1$ and $g'(y) \neq 1$ in some neighbourhood of the point y_0. In this case the family of transformations T_ε should be defined by (2) for ε from suitable smaller interval.

4. Exactness of random perturbations of Lasota-Yorke type maps

Let T be piecewise monotone and C^2. Piecewise monotone and C^2 means that there is a partition of $[0, 1]$, $0=a_0 < \cdots < a_k=1$, so that for each $i=0, 1, \cdots, k-1$, $T|_{(a_i, a_{i+1})}$ is monotone and extends to a C^2 map on $[a_i, a_{i+1}]$.
Assume that T preserves the Lebesgue measure m. Denote by $\bar{T}$ the transformation defined for T by equality (3) and let $T_i = T_{\varepsilon_i} T$ for $i=1, \cdots, s$.

Theorem 3. *If, for all $x \in [0, 1]$*

$$\sum_{i=1}^{s} \frac{p_i}{|T_i'(x)|} < 1,$$

then $\bar{T}$ is exact, for $s \geqslant 3$.

Proof. By Theorem 1 [7] some power of Frobenius-Perron operator $P_{\bar{T}} f = \sum_{i=1}^{s} p_i P_{T_i} f$ is quasicompact on $BV[0, 1]$. By Property 2 all iterations of $\bar{T}$ are ergodic. Due to the uniqueness of absolutely continuous invariant measure we obtain $\lim_{n\to\infty} P_{\bar{T}}^n f = \int f\, dm$ in L_1 norm for every $f \in L_1(m)$. Therefore $\lim_{n\to\infty} P_{\bar{T}}^n (rf) = \int r(x) f(y)\, d\mu \times m$ in L_1 norm for every $r \in L_1(\mu)$ and $f \in L_1(m)$ which implies the exactness of $\bar{T}$. □

Conclusion 2. *If* inf $|T'(x)|>1$, *then there exists* $\eta>0$ *such that for* $|\varepsilon_i|<\eta$, $i=1, \cdots, s$, *the transformation* $\bar{T}$ *is exact.*

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Institute of Mathematics
Wroclaw Technical University
Wybrzeze St. Wyspianskiego 27
50–370 Wroclaw, Poland