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ON QUASI-HOMOGENEOUS FOURFOLDS OF *SL*(3)

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Introduction

We recall that a quasi-homogeneous variety of an algebraic group G is an algebraic variety with a regular G-action which has an open dense orbit. A general theory of quasi-homogeneous varieties has been presented in Luna-Vust [5], and in particular, quasi-homogeneous varieties of $\boldsymbol{SL}(2)$ have been studied by Popov [9], Jauslin-Moser [2]. On the other hand, the geometry of smooth projective quasi-homogeneous threefolds of $\boldsymbol{SL}(2)$ has been thoroughly studied in Mukai-Umemura [7] and Nakano [8] by means of Mori theory.

In this note, we shall study and classify the smooth irreducible complete quasi-homogeneous fourfolds of $\boldsymbol{SL}(3)$. The motivation for this research comes from Mabuchi's work [6], in which the smooth complete n-folds with a non-trivial $\boldsymbol{SL}(n)$-action have been completely classified. Since $\boldsymbol{SL}(n)$-varieties of dimension less than n are obvious ones, we are interested in $\boldsymbol{SL}(n)$-varieties of dimension $n+1$. Let X be a smooth complete $\boldsymbol{SL}(n)$-variety of dimension $n+1$, and let d be the maximum of the dimensions of all orbits of X. It turns out that, if $d\leq n-1$, then $\boldsymbol{SL}(n)$-actions on X are easy, and essential problems occur when (1) $d=n+1$ (quasi-homogeneous case) and (2) $d=n$ (codimension 1 case). We hope that the investigation of the case (1) for $n=3$ in this note will be a good example toward the understanding of the structure of $\boldsymbol{SL}(n)$-varieties of dimension $n+1$.

Our main result is the classification theorem 11 of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$, which turns out extermely simple compared to the $\boldsymbol{SL}(2)$-case. Indeed, all the varieties appearing in the classification are rational 4-folds of very simple type.

This note is organized as follows. First in §1, we classify the closed subgroups of $\boldsymbol{SL}(3)$ of codimension 4. The author is indebted to Prof. Ariki for Proposition 1. In §2, examples of quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ are constructed by rather ad-hok methods. Finally, in §3, the classification will be done.

In this note, algebraic varieties, algebraic groups and Lie algebras are all defined over a fixed algebraically closed field k of characteristic 0. An algebraic variety is always assumed to be reduced and irreducible, and an (algebraic)

n-fold is an algebraic variety of dimension n. The symbol $*$ in a matrix stands for any element in k, or some element in k which we do not need to specify.

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1. Classification of closed algebraic subgroups of $SL(3)$ of codimension 4

This section is devoted to the proof of the following proposition due to Ariki. We denote by $\boldsymbol{SL}(3)$ the special linear group of degree 3 defined over k.

Proposition 1. *Let $G\subset \boldsymbol{SL}(3)$ be a closed algebraic subgroup of codimension 4. Then G is one of the following subgroups up to conjugation.*

$$G_0=\left\{\begin{bmatrix} A & 0 \\ 0 & b \end{bmatrix} \mid A\in \boldsymbol{GL}(2),\ b\in k^\times,\ \det(A)\cdot b=1\right\}$$

$$G_1=\left\{\begin{bmatrix} x & * & * \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \mid xyz=1\right\}$$

$$N(G_1)=G_1\cdot\left\langle\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right\rangle$$

$$G_2=\left\{\begin{bmatrix} x & 0 & * \\ 0 & y & * \\ 0 & 0 & z \end{bmatrix} \mid xyz=1\right\}$$

$$N(G_2)=G_2\cdot\left\langle\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\rangle$$

$$G_{p,q}=\left\{\begin{bmatrix} x & * & * \\ 0 & y & * \\ 0 & 0 & 1/(xy) \end{bmatrix} \mid x^p y^q=1\right\}\ \text{for}\ p, q\in \boldsymbol{Z},\ q\geq 0,$$

$(p, q)\neq(0, 0)$.

Proof. (1) Let $\mathfrak{sl}(3)$ be the Lie algebra of $\boldsymbol{SL}(3)$. We first determine the Lie subalgebras of $\mathfrak{sl}(3)$ of dimension 4 and the corresponding connected closed subgroup of $\boldsymbol{SL}(3)$. Let $\mathfrak{g}\subset\mathfrak{sl}(3)$ be a Lie subalgebra of dimension 4. Then $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{r}$ (semi-direct sum), where $\mathfrak{s}$ is a semi-simple Lie subalgebra and $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$, by Levi-Malcev's theorem. Since the rank of $\mathfrak{s}\leq 2$, we have $\mathfrak{s}\simeq\mathfrak{sl}(2)$ or 0. In fact, if the rank of $\mathfrak{s}=2$, then $\mathfrak{s}\simeq A_1\oplus A_1$, A_2, B_2 or G_2 and hence $\dim_k \mathfrak{s}\geq 5$, which is impossible.

(a) First, we assume $\mathfrak{s}=\mathfrak{sl}(2)$. Consider the faithful representation of $\mathfrak{s}$ on k^3 which is the restriction of the natural representation of $\mathfrak{sl}(3)$ on k^3. We decompose this representation into irreducible ones and may asume that $\mathfrak{s}$ is one of the following two fomrs up to conjugation.

$$\mathfrak{s}=k\cdot\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\qquad\text{(type 1)}$$

or

$$\mathfrak{s}=k\cdot\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}0&0&0\\2&0&0\\0&2&0\end{bmatrix}\oplus k\cdot\begin{bmatrix}2&0&0\\0&0&0\\0&0&-2\end{bmatrix}\qquad\text{(type 2)}\,.$$

Consider the adjoint representation of $\mathfrak{s}$ on $\mathfrak{r}$: $(\mathfrak{r}, ad|_{\mathfrak{s}})$. Since $\dim\mathfrak{r}=1$, this is trivial and we find that $\mathfrak{r}=k\cdot R$, where R commutes with any element of $\mathfrak{s}$. Assume that $\mathfrak{s}$ is of type 1. Then a simple calculation shows that $R=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}$ up to scalar multiplication. The corresponding connected closed subgroup is

$$G_0=\left\{\begin{bmatrix}g&&0\\&&0\\0&0&1\end{bmatrix}\mid g\in \boldsymbol{SL}(2)\right\}\cdot\left\{\begin{bmatrix}x&0&0\\0&x&0\\0&0&x^{-2}\end{bmatrix}\mid x\in k^{\times}\right\}$$

$$=\left\{\begin{bmatrix}g&&0\\&&0\\0&0&1/\det g\end{bmatrix}\mid g\in \boldsymbol{GL}(2)\right\}.$$

Assume that $\mathfrak{s}$ is of type 2. Then a simple calcualtion shows that there is no nonzero R which commutes with every element of $\mathfrak{s}$. Hence the type 2 never occurs.

(b) Second, we assume that $\mathfrak{s}=\{0\}$. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{n}$, where $\mathfrak{t}$ is a maximal abelian subalgebra consisting of semi-simple elements and $\mathfrak{n}$ is the ideal of all nilpotent elements in $\mathfrak{g}$. We set

$$\mathfrak{b}:=\left\{\begin{bmatrix}*&*&*\\0&*&*\\0&0&*\end{bmatrix}\right\}\text{ and }\mathfrak{h}:=\left\{\begin{bmatrix}0&*&*\\0&0&*\\0&0&0\end{bmatrix}\right\}.$$

Then we may assume $\mathfrak{g}\subset\mathfrak{b}$ and $\mathfrak{n}=\mathfrak{g}\cap\mathfrak{h}$ by Lie's theorem.

If $\dim\mathfrak{n}=3$, then $\mathfrak{g}\supset\mathfrak{h}=\mathfrak{n}$. Then we have

$$\mathfrak{g}=\mathfrak{h}\oplus k\cdot\begin{bmatrix}a&0&0\\0&b&0\\0&0&-a-b\end{bmatrix}\text{ for some } a, b\in k.$$

The corresponding algebraic subgroup G is of the form

$$G=\left\{\begin{bmatrix} x^a & * & * \\ 0 & x^b & * \\ 0 & 0 & x^{-a-b} \end{bmatrix} \mid x\in k^\times\right\} \text{ for } a, b\in \boldsymbol{Z}.$$

Since G is connected, we conclude that $G=G_{b,a}$ for coprime $a, b\in \boldsymbol{Z}$ in this case.

If $\dim \mathfrak{n}=2$, then $\dim \mathfrak{t}=2$ and $\mathfrak{g}$ is full-rank in $\mathfrak{sl}(3)$. Hence we may assume that $\mathfrak{t}=\left\{\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}\right\}$, and then,

$$\mathfrak{n}=\left\{\begin{bmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \text{ or } \left\{\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

by root-decomposition of $\mathfrak{n}$ with respect to $\mathfrak{t}$. The corresponding connected subgroup is

$$G_1:=\left\{\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}\right\} \text{ or } G_2:=\left\{\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\right\}.$$

If $\dim \mathfrak{n}\leq 1$, then $\dim \mathfrak{t}\geq 3$ which is impossible.

(2) Let G be a connected closed subgroup of codimension 4 determined in (1). In order to determine not necessarily connected such subgroups, we calculate $N_{SL(3)}(G)/G$, where $N_{SL(3)}(G)$ is the normalizer of G in $\boldsymbol{SL}(3)$. In the following, we set $N:=N_{SL(3)}(G)$.

(a) Suppose $G=G_0$. We consider the linear N-action on k^3 induced by the natural $\boldsymbol{SL}(3)$-action on k^3. Let $[x, y, z]$ be the coordinates of k^3, and set $P=[0, 0, 0]$, $l=\{x=y=0\}$ and $S=\{z=0\}$. Then the orbit decomposition of k^3 with respect to the G-action is given by

$$k^3=\{P\}\cup\{l-P\}\cup\{S-P\}\cup\{k^3-(l\cup S)\}.$$

For any $g\in N$, $g\circ l$ and $g\circ S$ are G-stable. Since l (resp. S) is the unique G-stable line (resp. plane), $g\circ l=l$ and $g\circ S=S$. It follows that $g\in G$ and hence $N=G$.

(b) Suppose $G=G_1$. We set $l=\{y=z=0\}$, $S_1=\{z=0\}$ and $S_2=\{y=0\}$. Then the orbit decomposition of k^3 with respect to the G-action is given by

$$k^3=\{P\}\cup\{l-P\}\cup\{S_1-l\}\cup\{S_2-l\}\cup\{k^3-(S_1\cup S_2)\}.$$

For any $g\in N$, $g\circ l$ and $g\circ S_1$ is G-stable, and hence we have $g\circ l=l$, $g\circ S_1=S_1$ or S_2. Therefore we may assume that g is of the following 2 types modulo G:

$$g_1=\begin{bmatrix}1&*&*\\0&1&*\\0&0&1\end{bmatrix} \text{ or } g_2=\begin{bmatrix}-1&*&*\\0&0&1\\0&1&*\end{bmatrix}.$$

Since $g_1Gg_1^{-1}\subset G$, a direct computation shows that $g_1\in G$ in this case. Similarly, $g_2=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ modulo G. Hence we conclude that $N/G=\langle\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}\rangle\cong\boldsymbol{Z}_2$, and $N(G_1):=G_1\cdot\langle\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}\rangle$ is the only non-connected closed subgroup whose connected component containing the identity is G_1.

(c) Suppose $G=G_2$. Similar calculations as in (b) show that $N(G_2):=G_2\cdot\langle\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\rangle$ is the only non-connected closed subgroup which has G_2 as the identity component.

(d) Suppose $G=G_{p,q}$ (p, q are coprime). Then $N=B:=$ the Borel subgroup of all the upper triangular matrices. In fact, $N\supset B$ is obvious. Conversely, if $g\in N$, then $g\in N_{SL(3)}(U)=B$, where U is the unipotent radical of B. Hence we find $N/G\simeq B/G_{p,q}$. Now, let $\varphi\colon B\to k^\times$ be the character of B defined by $\varphi\left(\begin{bmatrix}x&*&*\\0&y&*\\0&0&z\end{bmatrix}\right)=x^py^q$. Then $\mathrm{Ker}(\varphi)=G_{p,q}$, and we have $B/G_{p,q}\simeq k^\times$. Since any finite subgroup of $k^\times$ is a group of roots of unity, we conclude that

$$G_{np,nq}=\{\begin{bmatrix}x&*&*\\0&y&*\\0&0&z\end{bmatrix}\mid(x^py^q)^n=1,\ xyz=1\}\qquad(n\in\boldsymbol{N})$$

are the subgroups whose identity component is $G_{p,q}$. □

2. Examples of quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$

In this section, we construct various types of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ by rather ad-hok methods. We use the following notations for some standard closed subgroups of $\boldsymbol{SL}(3)$:

$$B:=\{\begin{bmatrix}a&b&c\\0&e&f\\0&0&i\end{bmatrix}\mid aei=1\},\quad B':=\{\begin{bmatrix}a&0&0\\d&e&0\\g&h&i\end{bmatrix}\mid aei=1\},$$

$$H:=\{\begin{bmatrix}a&b&c\\0&e&f\\0&h&i\end{bmatrix}\mid a(ei-fh)=1\},\quad H':=\{\begin{bmatrix}a&0&0\\d&e&f\\g&h&i\end{bmatrix}\mid a(ei-fh)=1\}.$$

We note that B and B' are conjugate in $\boldsymbol{SL}(3)$, whereas H and H' are not. Now, for the construction of examples, we need to know the explicit description of $\boldsymbol{SL}(3)/B$.

Let $\boldsymbol{SL}(3)$ act on $\boldsymbol{P}^2$ in the standard way. Namely, for $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \boldsymbol{SL}(3)$ and $P=[x:y:z]\in \boldsymbol{P}^2$, $A\circ P:=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{bmatrix}$. We also consider the dual projective plane $(\boldsymbol{P}^2)^*$ with the induced $\boldsymbol{SL}(3)$-action. Namely, for $Q=[u:v:w]\in(\boldsymbol{P}^2)^*$, $A\circ Q={}^t\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$. We define an $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^2 \times(\boldsymbol{P}^2)^*$ by $A\circ(P,Q)=(A\circ P, A\circ Q)$ for $(P,Q)\in\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$, and we set $W:=\{xu+yv+zw=0\}\subset\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$. W is a flag manifold $\{(x,l)\in\boldsymbol{P}^2\times(\boldsymbol{P}^2)^* \mid x\in L\}$, where $L\subset\boldsymbol{P}^2$ is a line corresponding to l. The following lemma is standard and well-known. However, we give a proof since the calculation in it is frequently referred to later in this note.

Lemma 2. (1) *W is $\boldsymbol{SL}(3)$-stable and isomorphic to $\boldsymbol{SL}(3)/B$.*

(2) *Let $p_1\colon W\to\boldsymbol{P}^2$ (resp. $p_2\colon W\to(\boldsymbol{P}^2)^*$) be the projection to the first (resp. second) factor. Then $p_1\colon W\to\boldsymbol{P}^2$ (resp. $p_2\colon W\to(\boldsymbol{P}^2)^*$) is isomorphic to the projectivized tangent bundle $\boldsymbol{P}(T_{\boldsymbol{P}^2})\to\boldsymbol{P}^2$ (resp. $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*})\to(\boldsymbol{P}^2)^*$).*

(3) *Let $\mathcal{O}_{\boldsymbol{P}}(1)$ (resp. $\mathcal{O}_{\boldsymbol{P}^*}(1)$) be the tautological line bundle of $\boldsymbol{P}(T_{\boldsymbol{P}^2})$ (resp. $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*})$). Then $\mathcal{O}_{\boldsymbol{P}}(1)\simeq\mathcal{O}_W(-2,1)$ and $\mathcal{O}_{\boldsymbol{P}^*}(1)\simeq\mathcal{O}_W(1,-2)$, where $\mathcal{O}_W(a,b)=p_1^*(\mathcal{O}_{\boldsymbol{P}^2}(a))\otimes p_2^*(\mathcal{O}_{(\boldsymbol{P}^2)^*}(b))$.*

Proof. (1) It is clear that W is $\boldsymbol{SL}(3)$-stable. Take a point $R:=([1:0:0],[0:0:1])\in W$. Then the isotropy group $\boldsymbol{SL}(3)_R$ at R is B. In fact, it is clear that $\boldsymbol{SL}(3)_R\subset H$. Take $A=\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}\in H$. Since ${}^t(A)^{-1}=\begin{bmatrix} 1/a & 0 & 0 \\ * & ai & -ah \\ * & -af & ae \end{bmatrix}$, A fixes R if and only if $h=0$, namely $A\in B$. Hence W contains a 3-dimensional orbit $O(R)$ isomorphic to $\boldsymbol{SL}(3)/B$ which is complete. It follows that $W=O(R)\simeq\boldsymbol{SL}(3)/B$.

(2) We show that $p_1\colon W\to\boldsymbol{P}^2$ is isomorphic to $\boldsymbol{P}(T_{\boldsymbol{P}^2})\to\boldsymbol{P}^2$. Let $(k^3)^*$ be an affine 3-space endowed with the dual $\boldsymbol{SL}(3)$-action. We set $W':=\{xu'+yv'+zw'=0\}\subset\boldsymbol{P}^2\times(k^3)^*$, $([x:y:z],[u',v',w'])\in\boldsymbol{P}^2\times(k^3)^*$. Then $p_1'\colon W'\to\boldsymbol{P}^2$ (p_1' is the projection to the first factor) is an $\boldsymbol{SL}(3)$-vector bundle of rank 2 whose projectivization is $p_1\colon W\to\boldsymbol{P}^2$. We note that $\boldsymbol{SL}(3)$-vector bundles over the homogeneous space $\boldsymbol{P}^2\simeq\boldsymbol{SL}(3)/H$ are determined by the slice representations of H on the fiber over $P=[1:0:0]\in\boldsymbol{P}^2$ (Kraft [3; 6.3.]). Now, take $A=$

$\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} \in H$. Then A acts on the fiber W'_P over P by $\begin{bmatrix} v' \\ w' \end{bmatrix} \mapsto \begin{bmatrix} ai & -ah \\ -af & ae \end{bmatrix} \begin{bmatrix} v' \\ w' \end{bmatrix}$.
On the other hand, let $\eta = y/x$, $\zeta = z/x$ be the inhomogeneous coordinates around P. Since $A^*\eta = (e\eta + f\zeta)(a + b\eta + c\zeta)^{-1}$, $A^*\zeta = (h\eta + i\zeta)(a + b\eta + c\zeta)^{-1}$, we get $A^* d\eta = (e/a)d\eta + (f/a)d\zeta$, $A^* d\zeta = (h/a)d\eta + (i/a)d\zeta$. It follows that $A_*: T_{\boldsymbol{P}^2,P} \to T_{\boldsymbol{P}^2,P}$ is represented by $\begin{bmatrix} e/a & f/a \\ h/a & i/a \end{bmatrix}$ with respect to the basis $\{\partial/\partial\eta, \partial/\partial\zeta\}$. Let $\mathcal{O}_{\boldsymbol{P}^2}(-1) \subset \boldsymbol{P}^2 \times k^3$ be the universal subbundle. Since H acts on the line $\mathcal{O}_{\boldsymbol{P}^2}(-1)_P$ by multiplication by a, we find that $W' \simeq T_{\boldsymbol{P}^2} \otimes \mathcal{O}_{\boldsymbol{P}^2}(-2)$. Hence $p_1: W = \boldsymbol{P}(W') \to \boldsymbol{P}^2$ is isomorphic to $\boldsymbol{P}(T_{\boldsymbol{P}^2}) \to \boldsymbol{P}^2$. We can verify that $p_2: W \to (\boldsymbol{P}^2)^*$ is isomorphic to $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*}) \to (\boldsymbol{P}^2)^*$ similarly.

(3) We take a point $S = [1:0] \in \boldsymbol{P}(T_{\boldsymbol{P}^2})_P$ whose isotropy group is B: $\boldsymbol{SL}(3)_S = B$. Let $\mathcal{O}_{\boldsymbol{P}}(-1) \subset \pi_1^*(T_{\boldsymbol{P}^2})$ be the universal subbundle over $\boldsymbol{P}(T_{\boldsymbol{P}^2}) \simeq W$, where $\pi_1: \boldsymbol{P}(T_{\boldsymbol{P}^2}) \to \boldsymbol{P}^2$ is the projection. Then $\mathcal{O}_{\boldsymbol{P}}(-1)_S = k \cdot [1, 0] \subset T_{\boldsymbol{P}^2,P} \simeq k^2$. Since for $A = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix} \in B$, $A_*: T_{\boldsymbol{P}^2,P} \to T_{\boldsymbol{P}^2,P}$ is represented by $\begin{bmatrix} e/a & f/a \\ 0 & i/a \end{bmatrix}$, A acts on the line $\mathcal{O}_{\boldsymbol{P}}(-1)_S$ by multiplication by e/a. On the other hand, take a point $R = (P, Q) = ([1:0:0], [0:0:1]) \in W$ at which the isotropy group is B. Since A acts on the line $\mathcal{O}_{\boldsymbol{P}^2}(-1)_P$ (resp. $\mathcal{O}_{(\boldsymbol{P}^2)^*}(-1)_Q$) by multiplication by a (resp. ae), A acts on the line $\mathcal{O}_W(p, q)_R \simeq \mathcal{O}_{\boldsymbol{P}^2}(-1)_P^{\otimes(-p)} \otimes \mathcal{O}_{(\boldsymbol{P}^2)^*}(-1)_Q^{\otimes(-q)}$ by multiplication by $a^{-(p+q)}e^{-q}$. Therefore we get $\mathcal{O}_{\boldsymbol{P}}(1) \cong \mathcal{O}_W(-2, 1)$. Similar calculations show that $\mathcal{O}_{\boldsymbol{P}^*}(1) \simeq \mathcal{O}_W(1, -2)$. □

Now, we construct 9 types of examples of smooth complete (actually projective) quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$. The examples (a), (b), (c), (d) deal with quasi-homogeneous 4-folds whose open orbits are of the form $\boldsymbol{SL}(3)/G_{p,q}$.

(a) Let $W = \boldsymbol{SL}(3)/B$ be as in Lemma 2. The $\boldsymbol{SL}(3)$-line bundles on W are in one-to-one correspondence with the characters of B. Let $\varphi_{p,q}: B \to k^\times$ be the character of B defined by $\begin{bmatrix} a & * & * \\ 0 & e & * \\ 0 & 0 & i \end{bmatrix} \mapsto a^p e^q$, and $L_{p,q}$ be the $\boldsymbol{SL}(3)$-line bundle corresponding to $\varphi_{p,q}$. We note $L_{p,q} \simeq \mathcal{O}_W(-p+q, -q)$ in view of the proof of Lemma 2. Consider the $\boldsymbol{SL}(3)$-action on the total space of $L_{p,q}$. If we take a non-zero vector v of the fiber of $L_{p,q}$ over $I_3 B \in W = \boldsymbol{SL}(3)/B$ (I_3 is the identity matrix of degree 3), then the isotropy group at v is equal to $G_{p,q}$. Hence $L_{p,q}$ contains a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_{p,q}$. We projectivize $L_{p,q}$ equivariantly to a $\boldsymbol{P}^1$-bundle by adding the infinite section. More precisely, let $\mathcal{O}_W$ be the trivial bundle of rank 1 over W, where $\boldsymbol{SL}(3)$ acts on the fiber trivially, and we set $X_{p,q}:=\boldsymbol{P}(L_{p,q}\oplus\mathcal{O}_W)$ endowed with the induced $\boldsymbol{SL}(3)$-action. The orbit decomposition of $X_{p,q}$ is given by $X_{p,q}=X^4_{p,q}\cup U_0\cup U_\infty$, where $X^4_{p,q}$ is the open dense orbit isomorphic to $\boldsymbol{SL}(3)/G_{p,q}$, U_0 is the 0-section of $L_{p,q}$ isomorphic to $\boldsymbol{SL}(3)/B$, and U_∞ is the infinite section of $X_{p,q}$ isomorphic to $\boldsymbol{SL}(3)/B$.

Lemma 3. *Let $X_{p,q}$ be as above, and let the notation be the same as in Lemma 2.*

(1) *$X_{p,q}$ can be blown-down to a smooth algebraic space along $U_0\simeq W$ in the p_1-direction (resp. p_2-direction) if and only if $q=1$ (resp. $p-q=1$).*

(2) *$X_{p,q}$ can be blown-down to a smooth algebraic space along $U_\infty\simeq W$ in the p_1-direction (resp. p_2-direction) if and only if $q=-1$ (resp. $q-p=1$).*

Proof. (1) Let l_1 (resp. l_2) $\subset W$ be a fiber of p_1 (resp. p_2), and $N(U_0/X_{p,q})$ be the normal bundle of U_0 in $X_{p,q}$. Then we have

$$(N(U_0/X_{p,q}), l_1) = (L_{p,q}, l_1) = (\mathcal{O}_W(-p+q, -q), l_1) = -q\,,$$

and similarly, $(N(U_0/X_{p,q}), l_2)=-p+q$. Hence (1) holds from the criterion for smooth blow-downs.

(2) Since $N(U_\infty/X_{p,q})\simeq L^{-1}_{p,q}$, (2) follows from (1). □

(b) Let $\boldsymbol{SL}(3)$ act on $\boldsymbol{P}^2$ in the standard way. Take a point $P=[1:0:0]\in\boldsymbol{P}^2$ at which the isotropy group is H. Let $\rho_\alpha: H\to\boldsymbol{GL}(2)$ be a 2-dimensional representation of H defined by $\begin{bmatrix} a & b & c\\ 0 & e & f\\ 0 & h & i\end{bmatrix}\mapsto a^\alpha\begin{bmatrix} e & f\\ h & i\end{bmatrix}$, and E_α be the $\boldsymbol{SL}(3)$-vector bundle of rank 2 corresponding to $\rho_\alpha(E_\alpha\simeq T_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(-\alpha-1))$. If we take a point $Q=[1, 0]\in E_{\alpha,P}=k^2$, then $\boldsymbol{SL}(3)_Q=\{A\in H\,|\,a^\alpha e=1,\ a^\alpha h=0\}=G_{\alpha,1}$. We projectivize E_α to a $\boldsymbol{P}^2$-bundle by adding infinite lines. More precisely, let $\mathcal{O}_{\boldsymbol{P}^2}$ be the trivial bundle of rank 1, where $\boldsymbol{SL}(3)$ acts on the fiber trivially, and we set $Y_\alpha:=\boldsymbol{P}(E_\alpha\oplus\mathcal{O}_{\boldsymbol{P}^2})$. Since H acts on the infinite line by $\begin{bmatrix} v\\ w\end{bmatrix}\mapsto\begin{bmatrix} e & f\\ h & i\end{bmatrix}\begin{bmatrix} v\\ w\end{bmatrix}$, the isotropy group at $[1:0]$ on the infinite line is B. Hence we have a following orbit decomposition of Y_α: $Y_\alpha=Y^4_\alpha\cup Y^3_\alpha\cup Y^2_\alpha$, where Y^4_α is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_{\alpha,1}$, Y^3_α is a 3-dimensional orbit consisting of infinite lines isomorphic to $W=\boldsymbol{SL}(3)/B$, and Y^2_α is the 0-section of E_α isomorphic to $\boldsymbol{SL}(3)/H$.

Lemma 4. *Y_α cannot be blown-down to a smooth algebraic space along $Y^3_\alpha\simeq W$ in the p_1-direction, and can be blown-down in the p_2-direction if and only if $\alpha=0$.*

Proof. An easy calculation shows that $A=\begin{bmatrix} a & b & c\\ 0 & e & f\\ 0 & 0 & i\end{bmatrix}\in B$ acts on $N(Y^3_\alpha/Y_\alpha)_P$

($P:=I_3B\in SL(3)/B\simeq Y_\alpha^3$) by multiplication by $ia^{1-\alpha}$. Hence we have $N(Y_\alpha^3/Y_\alpha)\simeq\mathcal{O}_W(\alpha-1,1)$ (see the proof of Lemma 2). Now, $(N(Y_\alpha^3/Y_\alpha),l_1)=(\mathcal{O}_W(\alpha-1,1),l_1)=1$, and $(N(Y_\alpha^3/Y_\alpha),l_2)=\alpha-1$. Therefore our assertion is verified by the criterion for smooth blow-downs. □

(c) We consider the standard $SL(3)$-action on the dual projective plane $(P^2)^*$. The isotropy group at $P=[1:0:0]\in(P^2)^*$ is H'. Take the 2-dimensional representation $\lambda_\alpha\colon H'\to GL(2)$ given by $\begin{bmatrix} a & 0 & 0\\ d & e & f\\ g & h & i\end{bmatrix}\mapsto a^\alpha\begin{bmatrix} e & f\\ h & i\end{bmatrix}$, and let $F_\alpha\to(P^2)^*$ be the $SL(3)$-bundle of rank 2 corresponding to λ_α. If we take a point $R=[0,1]\in E_{\alpha,P}=k^2$, then $SL(3)_R=\{A\in H'\mid a^\alpha i=1, f=0\}=\{\begin{bmatrix} a & 0 & 0\\ d & e & 0\\ g & h & i\end{bmatrix}\mid a^\alpha i=1\}=C^{-1}G_{-\alpha+1,-\alpha}C$, where $C=\begin{bmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$. Hence the isotropy group $SL(3)_{C\circ R}$ at $C\circ R$ is equal to $G_{-\alpha+1,-\alpha}$. We projectivize F_α to a P^2-bundle $Z_\alpha:=P(F_\alpha\oplus\mathcal{O}_{(P^2)^*})$. The orbit decomposition of Z_α is given by $Z_\alpha=Z_\alpha^4\cup Z_\alpha^3\cup Z_\alpha^2$, where Z_α^4 is an open dense orbit isomorphic to $SL(3)/G_{-\alpha+1,-\alpha}$, Z_α^3 is a 3-dimensionl orbit consisting of the infinite lines isomorphic to $SL(3)/B$, and Z_α^2 is the 0-section of F_α isomorphic to $SL(3)/H'$.

Lemma 5. *Z_α cannot be blown-down to a smooth algebraic space along $Z_\alpha^3\simeq W$ in the p_2-direction, and can be blown-down in the p_1-direction if and only if $\alpha=1$.*

Proof. We have $N(Z_\alpha^3/Z_\alpha)\simeq\mathcal{O}_W(1,\alpha-2)$. The rest of the proof is similar to Lemma 4. □

(d) Let $[x_0:x_1:x_2:y_0:y_1:y_2]$ be the homogeneous coordinates of P^5, and define an $SL(3)$-action on P^5 by $A\circ[x_0:x_1:x_2:y_0:y_1:y_2]=[x_0':x_1':x_2':y_0':y_1':y_2']$ for $A\in SL(3)$, where ${}^t[x_0':x_1':x_2']=A\cdot{}^t[x_0:x_1:x_2]$ and ${}^t[y_0':y_1':y_2']=({}^tA)^{-1}\cdot{}^t[y_0:y_1:y_2]$. We set $Q:=\{x_0y_0+x_1y_1+x_2y_2=0\}\subset P^5$. Q is an $SL(3)$-stable nonsingular quadric 4-fold. If we take a point $P:=[1:0:0:0:0:1]\in Q$, then $SL(3)_P=G_{0,1}$. In fact, it is clear that $H\supset SL(3)_P$. Take $A=\begin{bmatrix} a & b & c\\ 0 & e & f\\ 0 & h & i\end{bmatrix}\in H$. Since $({}^tA)^{-1}=\begin{bmatrix} * & * & 0\\ * & * & -ah\\ * & * & ae\end{bmatrix}$, $A\circ P=[a:0:0:0:-ah:ae]$. Hence $SL(3)_P=\{A\in H\mid h=0, e=1\}=G_{0,1}$. Set $Q^2:=\{y_0=y_1=y_2=0\}\simeq P^2$ and $Q^{2\prime}:=\{x_0=x_1=x_2=0\}\simeq(P^2)^*$. Then Q^2 (resp. $Q^{2\prime}$) is a closed orbit isomorphic to $SL(3)/H$ (resp.

$SL(3)/H'$). The orbit decomposition of Q is given by $Q=Q^4\cup Q^2\cup Q^{2'}$, where $Q^4=Q-(Q^2\cup Q^{2'})$ is a 4-dimensional orbit isomorphic to $SL(3)/G_{0,1}$. In fact, take any point $R=[p:q:r:s:t:u]\in Q^4$. If, for instance, $p\neq 0$, then $A\circ P=R$ for $A:=\begin{bmatrix} p & 0 & * \\ q & u/p & * \\ r & -t/p & * \end{bmatrix}\in SL(3)$. Thus we find that Q^4 is an orbit.

Lemma 6. $N(Q^2/Q)\simeq T_{P^2}\otimes\mathcal{O}_{P^2}(-1)$, $N(Q^{2'}/Q)\simeq T_{(P^2)^*}\otimes\mathcal{O}_{(P^2)^*}(-1)$.

Proof. We consider the following exact sequence of normal bundles:

$$(*)\quad 0\to N(Q^2/Q)\to N(Q^2/P^5)\to N(Q/P^5)|_{Q^2}\to 0\,.$$

Since $N(Q^2/P^5)\simeq\mathcal{O}_{P^2}(1)^{\oplus 3}$ and $N(Q/P^5)|_{Q^2}\simeq\mathcal{O}_{P^2}(2)$, we have $N(Q^2/Q)\cong\Omega_{P^2}\otimes\mathcal{O}_{P^2}(2)\simeq T_{P^2}\otimes\mathcal{O}_{P^2}(-1)$ by comparing $(*)$ with the dual of the standard Euler sequence. □

The relation of quasi-homogeneous 4-folds in examples (a)∼(d) is given in the following proposition. We denote by $B_Z(X)$ the blowing-up of a variety X along a subvariety Z.

Proposition 7. *$B_{Y_p^2}(Y_p)\simeq X_{p,1}$, $B_{Z_q^2}(Z_q)\simeq X_{q-1,q}$ $(q\geq 1)$, $B_{Z_{-q}^2}(Z_{-q})\simeq X_{q+1,q}$ $(q\geq 0)$, and $B_{Q^2}(Q)\simeq Y_0$, $B_{Q^{2'}}(Q)\simeq Z_1$.*

Proof. We show $B_{Y_p^2}(Y_p)\simeq X_{p,1}$. In fact, the exceptional divisor $C\subset B_{Y_p^2}(Y)$ is isomorphic to $W\simeq P(T_{P^2})$ since $N(Y_p^2/Y_p)\simeq E_p\simeq T_{P^2}\otimes\mathcal{O}_{P^2}(-p-1)$. Let $F\colon B_{Y_p^2}(Y_p)\cdots\to X_{p,1}$ be a birational map induced by identifying the open dense orbits $\simeq SL(3)/G_{p,1}$. Let I (resp. J) be the indeterminacy locus of F (resp. F^{-1}). Then, since I and J are $SL(3)$-stable closed subsets of codimension equal to or larger than 2, we find that I and J are empty, and F is an isomorphism. The other isomorphisms are proved similarly. □

(e) G_1-case. We consider the standard $SL(3)$-action on the dual projective plane $(P^2)^*$ and set $M_1:=(P^2)^*\times(P^2)^*$ endowed with the diagonal $SL(3)$-action. If we take a point $P:=([1:0:0],[0:1:0])\in M_1$, then clearly $H'\supset SL(3)_P$. Take $A:=\begin{bmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}\in H'$. Then, since ${}^t(A)^{-1}=\begin{bmatrix} * & fg-di & * \\ * & ai & * \\ * & -af & * \end{bmatrix}$, $A\in SL(3)_P$ if and only if $f=d=0$. Hence $SL(3)_P$ consists of the matrices of the form $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ * & * & * \end{bmatrix}$. It follows that $D^{-1}G_1D=SL(3)_P$, where $D=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and we get $SL(3)_{D\circ P}=G_1$. The orbit decomposition of M_1 is given by $M_1=\Delta\cup(M_1-\Delta)$, where Δ is the diagonal isomorphic to $SL(3)/H'$ and $M_1-\Delta$ is a 4-dimensional orbit isomorphic to $SL(3)/G_1$.

Let $\pi: \bar{M}_1 \to M_1$ be the blowing-up of M_1 along Δ. Since Δ is a closed orbit, we can define a regular $\boldsymbol{SL}(3)$-action on $\bar{M}_1$ such that π is $\boldsymbol{SL}(3)$-equivariant. Since $N(\Delta/M_1) \simeq T_\Delta \simeq T_{(\boldsymbol{P}^2)^*}$, the exceptional divisor $E \subset \bar{M}_1$ is isomorphic to $\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*}) \simeq W$, and the orbit decomposition of $\bar{M}_1$ is given by $\bar{M}_1 = \bar{M}_1^4 \cup E$, where $\bar{M}_1^4 = \bar{M}_1 - E$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_1$. We note that $\bar{M}_1$ cannot be blown-down to a smooth algebraic space along $E \simeq W$ in the p_1-direction since $(N(E/\bar{M}_1), l_1) = (\mathcal{O}_W(-1, 2), l_1) = 2$ (notations are the same as in Lemma 2).

(f) $N(G_1)$-case. We consider the standard $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^2$. Let $S^2(T_{\boldsymbol{P}^2})$ be the symmetric tensor bundle of degree 2 of $T_{\boldsymbol{P}^2}$, and we set $N_1 := \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))$ endowed with the induced $\boldsymbol{SL}(3)$-action. Take a point $P := [1:0:0] \in \boldsymbol{P}^2$ at which the isotropy group is H. Take $A = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} \in H$ and let $[y, z]$ be the inhomogeneous affine coordinates around the origin P. We recall that the H-action on $T_{\boldsymbol{P}^2, P}$ is represented by $a^{-1}\begin{bmatrix} e & f \\ h & i \end{bmatrix}$ with respect to the basis $\{\partial/\partial y, \partial/\partial z\}$ (cf. Lemma 2). Hence the H-action on $S^2(T_{\boldsymbol{P}^2})_P$ is represented by $a^{-2}\begin{bmatrix} e^2 & ef & f^2 \\ 2eh & ie+fh & 2fi \\ h^2 & ih & i^2 \end{bmatrix}$ with respect to the basis $\{(\partial/\partial y)^{\otimes 2}, (\partial/\partial y) \otimes (\partial/\partial z), (\partial/\partial z)^{\otimes 2}\}$. Thus the isotropy group at $[0:1:0] \in \boldsymbol{P}_P := \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))_P$ is given by $\{A \in H \mid ef = ih = 0\} = \{A \in H \mid e = i = 0 \text{ or } f = h = 0\} = N(G_1)$. The orbit decomposition of $\boldsymbol{P}_P$ with respect to the H-action is given by $\boldsymbol{P}_P = C \cup (\boldsymbol{P}_P - C)$, where C is a conic defined by $\{\eta^2 - 4\xi\zeta = 0\}$ and $[\xi : \eta : \zeta]$ are the homogeneous coordinates of $\boldsymbol{P}_P$. C is the orbit through $[1:0:0] \in \boldsymbol{P}_P$ and hence isomorphic to H/B. Therefore the orbit decomposition of N_1 with respect to the $\boldsymbol{SL}(3)$-action is given by $N_1 = N_1^4 \cup F$, where N_1^4 is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/N(G_1)$ and F is a 3-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/B \simeq W$.

Proposition 8. *Let $\bar{M}_1$ and N_1 be as in (e), (f).*

(1) *There exists an $\boldsymbol{SL}(3)$-equivariant finite morphism $\varphi: \bar{M}_1 \to N_1$ of degree 2. The ramification locus of φ is $E \subset \bar{M}_1$ and the branch locus is $F \subset N_1$.*

(2) *Let l_1 (resp. l_2) be a fiber of $p_1: F = W \to \boldsymbol{P}^2$ (resp. $p_2: F \to (\boldsymbol{P}^2)^*$). Then $(F, l_1) = 4$ and $(F, l_2) = -2$. In particular, N_1 cannot be blown-down to a smooth algebraic space along F in neither directions.*

Proof. (1) From the inclusion $G_1 \subset N(G_1)$, an $\boldsymbol{SL}(3)$-equivariant étale morphism $\nu: \bar{M}_1^4 \simeq \boldsymbol{SL}(3)/G_1 \to N_1^4 \simeq \boldsymbol{SL}(3)/N(G_1)$ of degree 2 is induced. We note that ν is the unique $\boldsymbol{SL}(3)$-equivariant morphism from $\bar{M}_1^4$ to N_1^4 since $\{a \in \boldsymbol{SL}(3) \mid aG_1a^{-1} \subset N(G_1)\} = N(G_1)$. Let $\varphi: \bar{M}_1 \cdots\to N_1$ be a rational map induced by ν with the indeterminacy locus I. Since I is an $\boldsymbol{SL}(3)$-stable closed subset of codimension≥ 2, I is empty and φ is a morphism. Since φ is $\boldsymbol{SL}(3)$-equivariant, $\varphi(E)=F$. We note that $\varphi|_E: E\to F$ is an isomorphsm. In fact, since $N_{SL(3)}(B)=B$, identity is the unique $\boldsymbol{SL}(3)$-equivariant morphism from $W=\boldsymbol{SL}(3)/B$ to W. The assertion (1) is thus proved.

(2) We note $N(E/\bar{M}_1)\simeq \mathcal{O}_{\boldsymbol{P}(T_{(\boldsymbol{P}^2)^*})}(-1)\simeq \mathcal{O}_W(-1, 2)$. Hence $(E, l_1)=((N(E/\bar{M}_1), l_1)=(\mathcal{O}_W(-1, 2), l_1)=2$, and $(E, l_2)=-1$ similarly. Now, we have $(F, l_1)=(\varphi^*(F), l_1)=(2E, l_1)=4$, and $(F, l_2)=-2$ similarly. The assertion (2) is proved. □

REMARK. We have $[F]\simeq \mathcal{O}_{\boldsymbol{P}}(2)\otimes\pi^*(\mathcal{O}_{\boldsymbol{P}^2}(6))$, where $[F]$ is the line bundle associated to the divisor F, $\mathcal{O}_{\boldsymbol{P}}(1)$ is the tautological line bundle of $\boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))$, and $\pi: \boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))\to \boldsymbol{P}^2$ is the projection. Indeed, if we take a point $R=[1:0:0]\in \boldsymbol{P}_P$, then $B=\boldsymbol{SL}(3)_R$ acts on the line $\mathcal{O}_{\boldsymbol{P}}(-1)_R\subset \pi^*(S^2(T_{\boldsymbol{P}^2}))_R$ by multiplication by e^2/a^2. Hence we find that $\mathcal{O}_{\boldsymbol{P}}(1))|_F\simeq \mathcal{O}_W(-4, 2)$. Now, if we set $[F]\simeq \mathcal{O}_{\boldsymbol{P}}(2)\otimes\pi^*(\mathcal{O}_{\boldsymbol{P}^2}(\alpha))$ $(\alpha\in \boldsymbol{Z})$, then $-2=(F, l_2)=2(\mathcal{O}_{\boldsymbol{P}}(1), l_2)+(\pi^*(\mathcal{O}_{\boldsymbol{P}^2}(\alpha)), l_2)=2(\mathcal{O}_W(-4, 2), l_2)+(\mathcal{O}_{\boldsymbol{P}^2}(\alpha), \text{line})=-8+\alpha$. Hence $\alpha=6$.

(g) G_2-case. Consider the standard $\boldsymbol{SL}(3)$-action on $\boldsymbol{P}^2$ and let $\boldsymbol{SL}(3)$ act on $M_2:=\boldsymbol{P}^2\times\boldsymbol{P}^2$ diagonally. If we take a point $S:=([1:0:0], [0:1:0])\in M_2$, then it is clear that $\boldsymbol{SL}(3)_S=\left\{\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\right\}=G_2$. The orbit decomposition of M_2 is given by $M_2=(M_2-\Delta)\cup\Delta$, where $M_2-\Delta$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_2$ and Δ is the diagonal isomorphic to $\boldsymbol{SL}(3)/H$.

Next, we denote by $\bar{M}_2$ the blowing-up of M_2 along the diagonal Δ. The orbit decomposition of $\bar{M}_2$ is given by $\bar{M}_2=\bar{M}_2^4\cup E'$, where E' is the exceptional divisor isomorphic to $\boldsymbol{SL}(3)/B$, and $\bar{M}_2^4=\bar{M}_2-E'$ is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_2$. We note that $\bar{M}_2$ cannot be blown-down to a smooth algebraic space along $E'\simeq W$ in the p_2-direction. Details are similar to (e).

(h) $N(G_2)$-case. We consider the dual projective plane $(\boldsymbol{P}^2)^*$. Let $S^2(T_{(\boldsymbol{P}^2)^*})$ be the symmetric tensor bundle of degree 2 of $T_{(\boldsymbol{P}^2)^*}$, and we set $N_2:=\boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*}))$. Take a point $P:=[1:0:0]\in(\boldsymbol{P}^2)^*$ at which the isotropy group is H', and take $A=\begin{bmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}\in H'$. Let $[y, z]$ be the inhomogeneous affine co-ordinates around the origin P. Since $({}^tA)^{-1}=\begin{bmatrix} 1/a & fg-di & eg-dh \\ 0 & ai & -ah \\ 0 & -af & ae \end{bmatrix}$, an easy calculation shows that the H'-action on $T_{(\boldsymbol{P}^2)^*,P}$ is represented by $a^2\begin{bmatrix} i & -h \\ -f & e \end{bmatrix}$ with respect to the basis $\{\partial/\partial y, \partial/\partial z\}$. Hence the H'-action on $S^2(T_{(P^2)^*})_P$ is represented by $a^4\begin{bmatrix} i^2 & -ih & h^2 \\ -2if & ie+fh & -2he \\ f^2 & -fe & e^2 \end{bmatrix}$ with respect to the basis $\{(\partial/\partial y)^{\otimes 2}, (\partial/\partial y)\otimes(\partial/\partial z), (\partial/\partial z)^{\otimes 2}\}$. Thus the isotropy group at $[0:1:0]\in \boldsymbol{P}(S^2(T_{(P^2)^*})_P)$ is given by $\{A\in H' \mid ih=fe=0\}=\{A\in H' \mid i=e=0 \text{ or } f=h=0\}=N(G_2)$. The orbit decomposition of N_2 is given by $N_2=N_2^4\cup F'$, where N_2^4 is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/N(G_2)$ and F' is a 3-dimensional closed orbit isomorphic to $\boldsymbol{SL}(3)/B$ such that $[F']\simeq \mathcal{O}_{P^*}(2)\otimes\pi^*(\mathcal{O}_{(P^2)^*}(6))$, where $\mathcal{O}_{P^*}(1)$ is the tautological line bundle of $\boldsymbol{P}(S^2(T_{(P^2)^*}))$ and $\pi\colon \boldsymbol{P}(S^2(T_{(P^2)^*}))\to(\boldsymbol{P}^2)^*$ is the projection. Details are similar to (f).

Proposition 9. *Let $\bar{M}_2$ and N_2 be as in (g), (h).*

(1) *There exists an $\boldsymbol{SL}(3)$-equivariant finite morphism $\psi\colon \bar{M}_2\to N_2$ of degree 2. The ramification locus of ψ is $E'\subset\bar{M}_2$ and the branch locus is $F'\subset N_2$.*

(2) *Let l_1 (resp. l_2) be a fiber of $p_1\colon F'=W\to\boldsymbol{P}^2$ (resp. $p_2\colon F'\to(\boldsymbol{P}^2)^*$). Then $(F', l_1)=-2$ and $(F', l_2)=4$. In particular, N_2 cannot be blown-down to a smooth algebraic space along F' in neither directions.*

The proof of this proposition is similar to that of Proposition 8.

(i) G_0-case. Consider the standard $\boldsymbol{SL}(3)$-actions on $\boldsymbol{P}^2$ and $(\boldsymbol{P}^2)^*$ and set $X_0:=\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$. Define an $\boldsymbol{SL}(3)$-action on X_0 by $A\circ(P,Q)=(A\circ P, A\circ Q)$ for $(P,Q)\in X_0$, $A\in\boldsymbol{SL}(3)$. Take a point $P:=([0:0:1],[0:0:1])\in X_0$. Then an easy calculation shows that $\boldsymbol{SL}(3)_P=G_0$. The orbit decomposition of X_0 is given by $X_0=X_0^4\cup X_0^3$, where X_0^4 is a 4-dimensional orbit isomorphic to $\boldsymbol{SL}(3)/G_0$, and X_0^3 is a closed orbit isomorphic to $\boldsymbol{SL}(3)/B$, which is defined by $x_0y_0+x_1y_1+x_2y_2=0$, $([x_0:x_1:x_2],[y_0:y_1:y_2])\in\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$.

3. Classification of quasi-homogeneous 4-folds of $SL(3)$

In this section, we classify smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ up to isomorphisms. First, we need a lemma:

Lemma 10. *Let V be a smooth complete quasi-homogeneous 4-fold of $\boldsymbol{SL}(3)$. Then V has no fixed points, no 1-dimensional orbits. The possible 2-dimensional orbits are isomorphic to $\boldsymbol{P}^2$ or $(\boldsymbol{P}^2)^*$ with the standard actions.*

Proof. Assume that $x\in V$ is a fixed point. We consider the induced linear action ρ of $\boldsymbol{SL}(3)$ on $T_{V,x}$. Since V is smooth, dim $T_{V,x}=4$ and ρ is represented as one of the following three types:

$$A\mapsto\begin{bmatrix} A & 0 \\ 0 & 1\end{bmatrix},\ \begin{bmatrix} ({}^tA)^{-1} & 0 \\ 0 & 1\end{bmatrix} \text{ or } I_4 \text{ (identity matrix) for } A\in\boldsymbol{SL}(3).$$

Now, by Luna's étale slice theorem [4], there exists an $\boldsymbol{SL}(3)$-stable affine subvariety S containing x such that there is an étale $\boldsymbol{SL}(3)$-equivariant morphism $\nu\colon S\to T_{V,x}$. But then, S has a 4-dimensional orbit, whereas $T_{V,x}$ has no 4-dimensional orbits in any case. Thus, we got a contradiction and V has no fixed points. Since $\boldsymbol{SL}(3)$ has no closed subgroups of codimension 1, and any closed subgroup of codimension 2 is conjugate to H or H' (Mabuchi [6; Theorem 2.2.1]), V has no orbits of dimension 1 and any 2-dimensional orbit is isomorphic to $\boldsymbol{P}^2$ or $(\boldsymbol{P}^2)^*$. □

Now, we state the main theorem of this note. For a closed subgroup $G\subset \boldsymbol{SL}(3)$ of codimension 4, we denote by $\mathcal{C}(G)$ the set of all isomorphism classes of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ whose open dense orbit is of the form $\boldsymbol{SL}(3)/G$.

Theorem 11. *Let X be a smooth complete quasi-homogeneous 4-fold of $\boldsymbol{SL}(3)$. Then X is classified completely as follows:*

(1) *Assume $X\in\mathcal{C}(G_{p,q})$. Then $X\simeq X_{p,q}$ if $|p-q|\neq 1$, $q\neq 1$; $X\simeq X_{p,1}, Y_p$ if $q=1$; $X\simeq X_{q-1,q}, Z_q$ if $q-p=1$ $(q\geq 1)$; $X\simeq X_{q+1,q}, Z_{-q}$ if $p-q=1$ $(q\geq 0)$; $X\simeq X_{0,1}, Y_0, Z_1, Q$ if $p=0$, $q=1$.*

(2) *If $X\in\mathcal{C}(G_1)$, then $X\simeq(\boldsymbol{P}^2)^*\times(\boldsymbol{P}^2)^*$, $B_\Delta((\boldsymbol{P}^2)^*\times(\boldsymbol{P}^2)^*)$.*

(3) *If $X\in\mathcal{C}(N(G_1))$, then $X\simeq\boldsymbol{P}(S^2(T_{\boldsymbol{P}^2}))$.*

(4) *If $X\in\mathcal{C}(G_2)$, then $X\simeq\boldsymbol{P}^2\times\boldsymbol{P}^2$, $B_\Delta(\boldsymbol{P}^2\times\boldsymbol{P}^2)$.*

(5) *If $X\in\mathcal{C}(N(G_2))$, then $X\simeq\boldsymbol{P}(S^2(T_{(\boldsymbol{P}^2)^*}))$.*

(6) *If $X\in\mathcal{C}(G_0)$, then $X\simeq\boldsymbol{P}^2\times(\boldsymbol{P}^2)^*$.*

Proof. We verify the assertion (1). Let X be a smooth complete quasi-homogeneous 4-fold of $\boldsymbol{SL}(3)$ which belongs to $\mathcal{C}(G_{p,q})$. Let $\nu\colon X\cdots\to X_{p,q}$ be a birational map induced by identifying the open dense orbits isomorphic to $\boldsymbol{SL}(3)/G_{p,q}$. By Hironaka [1], we resolve the indeterminacy locus I of ν by successive blowing-ups along smooth centers. Since I is an $\boldsymbol{SL}(3)$-stable closed subset of codimension≥ 2, each center is isomorphic to $\boldsymbol{P}^2$ or $(\boldsymbol{P}^2)^*$ by Lemma 10. Let $\sigma\colon \tilde{X}\to X$ be the composition of these blowing-ups and $\mu=\nu\circ\sigma\colon \tilde{X}\to X_{p,q}$ be the resolution of ν. Since the indeterminacy locus J of μ^{-1} is $\boldsymbol{SL}(3)$-stable and has codimension greater than or equal to 2, J is empty and μ is an isomorphism. Therefore, X is isomorphic to $X_{p,q}$ or its smooth blow-downs. (1) is thus proved by Lemmas 3, 4, 5 and Proposition 7. Assertions (2)~(6) can be proved similarly. □

REMARK. We note that in the $\boldsymbol{SL}(2)$-case, some interesting minimal rational 3-folds are constructed as smooth projective quasi-homogeneous 3-folds of $\boldsymbol{SL}(2)$ (Mukai-Umemura [7]). Here, a rational n-fold X is called minimal if the identity component $\mathrm{Aut}^\circ(X)$ of the automorphism group of X is maximal in the Cremona group $\mathrm{Bir}(\boldsymbol{P}^n)$ of n variables. Therefore, to determine whether our quasi-homogeneous 4-folds of $SL(3)$ are minimal rational 4-folds or not will be an interesting problem, which we plan to discuss elsewhere.

As an easy corollary to our theorem, the Picard groups of 4-dimensional homogeneous spaces of $SL(3)$ are determined from the orbit decomposition of these quasi-homogeneous 4-folds.

Corollary. $\mathrm{Pic}(SL(3)/G_{p,q}) \simeq Z \oplus Z/(g.c.d.(p, q))$, $\mathrm{Pic}(SL(3)/G_i) \simeq Z^2$ $(i=1, 2)$, $\mathrm{Pic}(SL(3)/N(G_i)) \simeq Z \oplus Z/(2)$ $(i=1, 2)$ and $\mathrm{Pic}(SL(3)/G_0) \simeq Z$.

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