Kaiser, U.
Osaka J. Math.
**29** (1992), 573–593

# HOMOLOGY BOUNDARY LINKS AND FUSION CONSTRUCTIONS

UWE KAISER

(Received August 12, 1991)

## Introduction

It is well-known [6] that a smooth link of spheres $L^n \subset S^{n+2}$ is trivial if and only if

i) the link group $\pi=\pi_1(S^{n+2}\backslash L)$ is free on the set of meridians of $L$, and

ii) the homotopy groups $\pi_j(S^{n+2}\backslash L)$ are trivial for $2\leq j\leq[(n+1)/2]$,

at least when $n\neq 2$. It is of considerable interest to study classes of links for which the link group satisfies weaker "freeness" conditions than i). The following three consecutively enlarged classes turn out to be of special importance:

(I) *Boundary links*. These are links whose components bound disjoint oriented manifolds in $S^{n+2}$ (Seifert surfaces). Equivalently, there is an epimorphism $\pi\rightarrow F$ ($F$=free group of rank equal to the number of components of $L$), which maps meridians to generators.

(II) *Homology boundary links* (*HBLs*). Here we drop the condition on meridians. A geometric interpretation in terms of "singular" Seifert surfaces is known (compare [15]).

(III) *Sublinks of homology boundary links* (*SHB-links*). This class arises since class (II) is not closed with respect to sublinks.

Interest in these classes comes mostly from the study of link concordance. In higher dimensions a classification of boundary links up to boundary link concordance is known (compare for example [9]). Recently T. Cochran and J. Levine proved that each *HBL* is *concordant* to a fusion of a boundary link [3]. Roughly, a *fusion* of an $r$-component link is an $(r-j)$-component link, which is formed by attaching $j$ 1-handles (bands) to the link. The question whether in the last statement *concordance* can be replaced by *isotopy* is in fact related to the Andrews Curtis Conjecture (see [3]). Cochran and Levine define an obstruction for a *HBL* to be a boundary link, the *pattern*, an isotopy invariant, which in the author's opinion has not yet been studied adequately. In 1989, T. Cochran and K. Orr proved the result, surprising to the experts, that there are *HBL*s, which are not even *concordant* to boundary links [4]. Their examples arise from a "completed" fusion construction, which preserves the number of components:

Canonical unknotted $n$-spheres linking the bands are added to a fusion to form a *strong fusion link*. Finally, the class (III) naturally appears in the study of disk link concordance via J. Ledimets exact sequence and in Levine's discussion of $\hat{F}$-links ([11] and [10]).

In this paper we shall study (strong) fusion and the classes (I)-(III) up to isotopy.

J. Hillman gave several examples of ribbon links in $S^3$ to distinguish the three classes [7]. His proofs use Wirtinger calculus and the ribbon property. It is our main aim to find the actural geometric reason for why these examples work. The first step in this direction was made by T. Cochran. He proved that each fusion (resp. strong fusion) of a boundary link is a *SHB*-link (resp. *HBL*). It is easy to see that both fusion constructions preserve class (III). This suggests the following questions:

a) When is a (strong) fusion of a *HBL* a *HBL*?

b) When is a (strong) fusion of a boundary link a boundary link?

We will apply techniques of [2], [3] and [10] to translate both questions into pure problems in combinatorial group theory. In case b) the hard work is contained in [3], namely the reduction to a property of the pattern of the (strong) fusion. Our contribution is to actually compute the pattern. We think that it is important to recognize that the answer for strong fusion in case a) only depends on the pattern of the original link, in case b) only depends on the bands.

Our methods work in all dimensions. No reference to the fundamental group of a specific given link is needed. From our computations we deduce the following result:

**Theorem 1.** *A strong fusion of a boundary link along a band is a boundary link if and only if one can choose Seifert surfaces which intersect the band only in its boundary.*

Here the main point is to have sufficient control over the relation between bands and Seifert surfaces. The necessary algebraic gadget is the band word (which is well defined up to a certain action of an extension of the special automorphism group of the free group): The transverse oriented intersection of the oriented band and an arbitrary choice of Seifert surfaces (whose components are labelled by generators of the free group) represents an element of the free group in the obvious way.

Theorem 1 may look technical, but gives a very convenient method not only to construct large classes of *HBL*s, which are not boundary links, but actually to *decide* when a strong fusion of a boundary link is a boundary link.

For example we prove:

**Theorem 2.** *A strong fusion of a two-component link is not a boundary link if and only if the corresponding (reduced) band word contains an occurence*

$x_2^{\varepsilon} x_1^{\eta} (\varepsilon, \eta \in \{\pm 1\})$. *(Here the band is oriented from the first to the second component)*.

Many results (for example 2.4 and 3.4) admit obvious generalizations to "several" bands, but to reduce technical details we shall keep to the case of a single band.

The author would like to thank the Mathematics Department at Brandeis University for its hospitality, in particular Prof. J. Levine for his help and encouragement in numerous discussions.

## 1. Basic Notions

A link $L$ is a smooth oriented submanifold of $S^{n+2}$ diffeomorphic to a disjoint union of $n$-spheres. Throughout isotopy means smooth (in particular ambient) isotopy.

Let $N(L)$ be a closed tubular neighbourhood of $L$ and $X(L) := S^{n+2} \backslash int(N(L))$ be the exterior. For any subset $A \subset S^{n+2}$ let $\pi(A) := \pi_1(S^{n+2} \backslash A)$.

We consider pairs $(L, b)$, where $L = (K_1, \cdots, K_r)$ is an $r$-component link and $b = (\beta, u)$ is a smooth arc $\beta \subset S^{n+2}$ together with a vectorfield $u$ normal to $\beta$, such that

a) $L \cap \beta = \partial \beta$ is contained in $K_1$ and $K_2$.

b) $u | \partial \beta$ is also normal to $L$,

c) the orientations of $S^{n+2}$ induced by $L$, $u$ and some orientation of $\beta$ are opposite to each other in the two endpoints $\partial \beta$.

Following [3] we define:

DEFINITION 1.1. The data $(L, b)$ determine a 1-handle, i.e. a submanifold of $S^{n+2}$ diffeomorphic to $I \times D^n$ (unique up to isotopy moving $\partial I \times D^n$ only in in $L$), which is used to form the connected sum $K_1 \#_b K_2$ of $K_1$ and $K_2$. The resulting $(r-1)$-component link $L(b) = (K_1 \#_b K_2, K_3, \cdots, K_r)$ is called the *fusion of L along the band b*.

For $n=1$ any two choices of $u$ for some arc $\beta$ differ by some element of $\pi_1(SO_2) \cong \boldsymbol{Z}$ twisting the vectorfield. But for $n \geq 2$ any two vectorfields are homotopic rel boundary. So the isotopy class rel boundary of $\beta$ determines $b$. This was pointed out by J. Levine.

We choose an orientation preserving embedding $\bar{b}: I \times D^{n+1} \to S^{n+2}$ (which is unique up to isotopy), such that $\bar{b}(I \times (1/2) D^n)$ is the 1-handle corresponding to $b$. Then $C_b := \bar{b}((1/2) \times S^n)$ is the *n-sphere linking b*, which is contained in $S^{n+2} \backslash L$ and in $S^{n+2} \backslash L(b)$ by suitable choice of $\bar{b}$. We shall orient $\beta$ from $K_1$ to $K_2$. Then $C_b$ inherits a canonical orientation.

DEFINIITON 1.2. The link $\hat{L}(b) = (K_1 \#_b K_2, C_b, K_3, \cdots, K_r)$ is called the *strong fusion of L along b*.

DEFINITION 1.3. Pairs $(L, b)$ and $(L', b')$ are called *isotopic*, if there is a diffeomorphism $h$ of $S^{n+2}$, isotopic to the identity, such that $h(L)=L'$ and $h(b)=b'$, i.e. $h(\beta)=\beta'$ and $u'_{h(x)}=dh_x(u_x)$ for all $x\in\beta$.

Then fusion (resp. strong fusion) is a construction from isotopy classes of pairs $(L, b)$, where $L$ is an $r$-component link, to isotopy classes of $(r-1)$ (resp. $r$)-component links.

The following two results are essential to the understanding of (strong) fusion, the second one being an easy improvement of [2, 2.3].

**Proposition 1.4.** *The diffeomorphism type of the* $(n+2)$*-manifold, which is the result of surgery on the longitudes of* $\hat{L}(b)$ (0*-framing on* $C_b$ *for* $n=1$) *does not depend on the band* $b$.

This is folklore (see for example [18]). ■

**Proposition 1.5.** *Each fusion or strong fusion of a SHB-link is a SHB-link.*

Proof. Recall that the *HBL* condition only depends on the $(n+2)$-manifold, which is the result of (0-framed for $n=1$) surgery on the longitudes of the link in consideration. Let $L$ be a sublink of the *HBL* $L'$, $b$ a band for $L$. We can arrange that $\bar{b}(I\times D^{n+1})$ does not intersect $L'\backslash L$. The link $L'\cup C_b$ is also a *HBL*. We fuse a parallel copy of $K_2$ to $K_1$ to get the link $L''$. $L''$ is a *HBL*, since the fusion corresponds to a handle-slide on the surgery manifold corresponding to $L'\cup C_b$. But $L(b)$ and $\hat{L}(b)$ are both sublinks of $L''$. ■

REMARK 1.6. By 1.4 all properties of $\hat{L}(b)$, which only depend on the surgery-manifold, do not depend on the bands. This applies for instance in those cases, where the existence of a homomorphism from the link group into a certain group is considered, which *kills longitudes*. Examples: *HBLs*, *E*-links, $\hat{F}$-links, vanishing $\bar{\mu}$-invariants (compare [2], [3]).

Now we study the way in which basings can be used as a coordinate system for bands. This will be more general than actually needed for the purposes of this paper, but can be applied in many situations.

For a given link $L$ choose basepoints $*_i\in\partial N(L)$ in the components $T_i$ of $\partial N(L)$, $1\le i\le r$, $*\in X(L)$. A *basing* of the $i$-th component is a choice of proper arc $\gamma_i\subset X(L)$ with $\gamma_i(0)=*$, $\gamma_i(1)=*_i$. Note that each basing of the $i$-th component determines a *meridian* $\mu_i$ and a *longitude* $\lambda_i$ in the usual way, i.e. $\mu_i=[\gamma_i\, m_i\, \gamma_i^{-1}]$ and $\lambda_i=[\gamma_i\, l_i\, \gamma_i^{-1}]$ in $\pi(L)$, where $m_i$ resp. $l_i$ are meridional resp. longitudinal curves on $T_i$.

There is an obvious notion of (ambient) isotopy of triples $(L, \gamma, b)$, where $\gamma=(\gamma_1, \gamma_2)$ is a basing of the first two components of $L$ and $b$ is a band joining these components.

Let $\pi_1(T_1)$ and $\pi_1(T_2)$ be considered as subgroups of $\pi(L)$ via $\gamma$. The arc $\tilde{\beta}:=\beta\cap X(L)$ has its two boundary points in $T_1\cup T_2$.

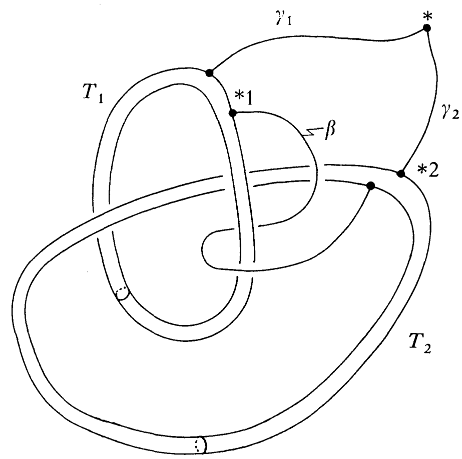


Figure 1

Up to isotopy fixing $(L, \gamma)$ these can be assumed to be $*_1$, $*_2$. Let $\bar{\beta}$ be the double coset $\pi_1(T_1)\,[\gamma_1\,\tilde{\beta}\,\gamma_2^{-1}]\,\pi_1(T_2)$. We define $\rho(L, \gamma, b)=(\pi(L), \bar{\beta})$. Then $\rho$ is well defined from isotopy classes of triples $(L, \gamma, b)$ to isomorphism classes of pairs $(\pi, \bar{\beta})$. Here an isomorphism of $(\pi, \bar{\beta})$ and $(\pi', \bar{\beta}')$ is an isomorphism of groups $\bar{h}$: $\pi\to\pi'$, such that $\bar{h}(\bar{\beta})=\bar{\beta}'$.

In higher dimensions we can reconstruct $b$ from $\bar{\beta}$ as follows:

**Proposition 1.7.** *If $n\geq 2$ and for a given pair $(L, \gamma)$ and bands $b, b'$ we know that $\bar{\beta}=\bar{\beta}'\subset\pi(L)$, then there exists an isotopy $h$ of $S^{n+2}$ fixing $L$, which maps $b$ to $b'$.*

Proof. Because of Levine's observation we don't have to worry about normal vectorfields. We know that $\bar{\beta}=\bar{\beta}'$, so $a_1[\gamma_1\,\tilde{\beta}\,\gamma_2^{-1}]\,a_2=[\gamma_1\,\tilde{\beta}'\,\gamma_2^{-1}]$ for $a_1\in\pi_1(T_1)$, $a_2\in\pi_1(T_2)$. Then it is easy to see that $a_1'\,\tilde{\beta}\,a_2'\simeq\tilde{\beta}'$ rel boundary for certain loops $a_i'\in\pi_1(T_i, *_i)$, $i=1, 2$ ($\simeq$means homotopic). Now isotope $\beta$ fixing $L$ by moving $\tilde{\beta}(0)$ in $T_1$ and $\tilde{\beta}(1)$ in $T_2$ until $\tilde{\beta}\simeq\tilde{\beta}'$ rel boundary. Since homotopy implies isotopy for $n\geq 2$ the result follows. ■

If $n=1$, then bands may link and a single band may knot. In this case $\bar{\beta}$ does not measure "band information" completely, even if we forget twisting of bands.

EXAMPLE 1.8. The Whitehead double of a knot $K\subset S^3$ is the fusion of the Hopf-link $H$ in $S^3$ along a band, which follows the knot. The $\bar{\beta}$ for the corresponding pairs $(H, b)$ are all the same.

## 2. Bands for Boundary Links

We shall introduce an invariant of $(L, b)$ for boundary links $L$, which is derived from $\rho$ in section 1.

Throughout $F$ will denote the free group of rank $r$ with basis $\{x_1, \cdots, x_r\}$, where $r$ is the number of components of $L$. Recall that (via Pontryagin Thom construction) a choice $\{V_1, \cdots, V_r\}$ of bordism class rel boundaries of Seifert surfaces for $L$ corresponds to a pair $(\phi, \mu)$, where $\mu$: $F\to\pi(L)$ is a meridian map (homotopy basing), and $\phi$: $\pi(L)\to F$ is a splitting homomorphism [5], [9]. Such a pair $(\phi, \mu)$ will be called *special.*

Let $\gamma=(\gamma_1, \cdots, \gamma_r)$ be a basing, $\mu_i$ be associated to $\gamma_i$, and $(\phi, \mu)$ be a special pair with $\mu(x_i)=\mu_i$.

Definition 2.1. The *band word* corresponding to $(\phi, \mu)$ is the double-coset mod $\langle x_1\rangle$ from the left and $\langle x_2\rangle$ from the right of the word $w:=\phi[\gamma_1 \tilde{\beta} \gamma_2^{-1}]$ $\in F$ ($\langle\ \rangle$ denotes the subgroup generated by).

This is of course just $\phi(\bar{\beta})$ with $\bar{\beta}$ from section 1.

There is the following geometric way to derive the *band word* $w$ of $(L, b)$ from the Seifert surfaces $\{V_i\}$ corresponding to $(\phi, \mu)$: Go along $\beta$ (from $K_1$ to $K_2$), which we assume transversal to $\cup V_i$. Each time $\beta$ intersects $V_i$ with positive resp. negative orientation write down $x_i$ resp. $x_i^{-1}$. Note that each occurence $x_j x_j^{-1}$ in the not reduced word $w$ can be eliminated geometrically by attaching a 1-handle to $V_j$, thus performing surgery rel boundaries. If $w=1$, then $L(b)$ and $\hat{L}(b)$ are boundary links in the obvious way.

**Lemma 2.2.** *A bordism class rel boundaries of Seifert surfaces for $L$ determines $w\in F$ up to the $\boldsymbol{Z}\times\boldsymbol{Z}$-action above. Moreover, each such change of $w$ by this action is induced by a bordism rel boundaries.*

Proof. Let $(\tilde{V}_i)$ be a bordism rel boundaries between $(V_i)$ and $(V_i')$ in $X(L)\times I$. By transversality $\beta\times I$ intersects $\cup\tilde{V}_i$ in a collection of arcs. The boundary points of these arcs are contained in $V_1, \cdots, V_r, V_1', \cdots, V_r'$ or in $\partial V_i\times I$ for $i=1, 2$. This proves the first assertion. Since we can isotope $\beta$ to change $w$ by multiplication with powers of $x_1$ from the left and $x_2$ from the right arbitrarily, the result is proved. ■

Now we consider a variation of the special pair $(\phi, \mu)$. Let $CA$ denote the group of special automorphisms of $F$, i.e. automorphisms which map each generator to a conjugate of itself ($F$ free on $\{x_1, \cdots, x_r\}$). It is known that $CA$ acts transitively on special pairs $(\phi, \mu)$ by $\alpha\cdot(\phi, \mu)=(\alpha\circ\phi, \phi\circ\mu^{-1})$ for $\alpha\in CA$ [5]. Moreover, $CA$ is generated by special automorphisms $\overline{\alpha_{ij}}(i\neq j)$, where $\overline{\alpha_{ij}}(x_j)$ $=x_i x_j x_i^{-1}$ and $\overline{\alpha_{ij}}(x_l)=x_l$ for $l\neq j$ [9].

It is easy to see how $w$ changes, when $(\phi, \mu)$ is replaced by $\overline{\alpha_{ij}}\cdot(\phi, \mu)$. Let

$\alpha_{ij}\cdot w$ denote the word corresponding to Seifert surfaces determined by $\overline{\alpha_{ij}}\cdot(\phi, \mu)$:

$$(*)\qquad \begin{cases} \alpha_{ij}\cdot w = \overline{\alpha_{ij}}(w)\,, & j\neq 1, 2 \\ \alpha_{i1}\cdot w = x_i^{-1}\,\overline{\alpha_{i1}}(w)\,, & i\neq 1 \\ \alpha_{i2}\cdot w = \overline{\alpha_{i2}}(w)\,x_i\,, & i\neq 2 \end{cases}$$

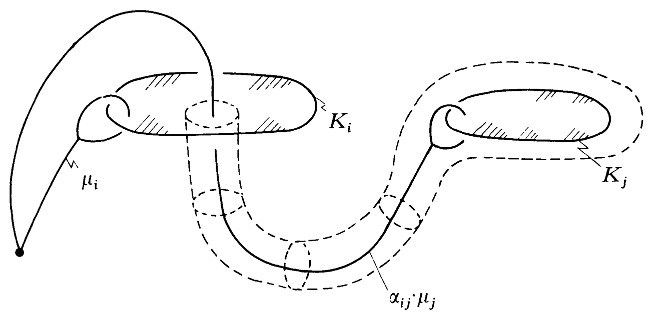


Figure 2

This suggests to introduce an extension of $CA$ as follows:

Let $F^r = F\times F\times\cdots\times F$, and for each $\alpha=(g_1, \cdots, g_r)\in F^r$ let $\bar{\alpha}$ denote the endomorphism of $F$ given by $\bar{\alpha}(x_i)=g_i\,x_i\,g_i^{-1}$ for $1\leq i\leq r$. Let $\widetilde{CA}$ denote the group of all $\alpha\in F^r$, such that $\bar{\alpha}$ is an automorphism of $F$, i.e. $\bar{\alpha}\in CA$. Then there is the exact sequence:

$$1\to \boldsymbol{Z}^r \to CA \to \widetilde{CA} \to 1$$

where $(n_1, \cdots, n_r)\in \boldsymbol{Z}^r$ is mapped to $(x_1^{n_1}, \cdots, x_r^{n_r})\in\widetilde{CA}$.

The group structure on $\widetilde{CA}$ is explicitly given by

$$(g_1, \cdots, g_r)\,(h_1, \cdots, h_r) = (\bar{\alpha}\,(h_1)\,g_1, \cdots, \bar{\alpha}(h_r)\,g_r)$$
$$(g_1, \cdots, g_r)^{-1} = (\bar{\alpha}^{-1}(g_1^{-1}), \cdots, \bar{\alpha}^{-1}(g_r^{-1}))\,.$$

Note that $\widetilde{CA}$ is generated by elements

$$\alpha_{ij} = (1, \cdots, 1, \underset{\substack{\uparrow \\ j}}{x_i}, 1, \cdots, 1)$$

for $1\leq i, j\leq r$. From $(*)$ we deduce:

**Lemma 2.3.** *$\widetilde{CA}$ acts on $F$ by*

$$\alpha\cdot w = g_1^{-1}\,\bar{\alpha}(w)\,g_2$$

*for* $\alpha=(g_1, \cdots, g_r)\in\widetilde{CA}$.

**Proposition 2.4.** *Let $L$ be a boundary link with band $b$. Then the $\widetilde{CA}$-orbit of a band word $w$ is an isotopy invariant of $(L, b)$. Moreover, each (even not reduced) word in this orbit of $w$ is realized as band word for some choice of Seifert surfaces.*

Proof. This follows from 2.2 and 2.3, [9] and [5]. If we change $(L, b)$ in its isotopy class, then Seifert surfaces can be chosen, such that the band word is unchanged. Finally note that any occurrence $x_j x_j^{-1}$ can also be generated geometrically by an obvious finger-move. ■

**Corollary 2.5.** *Let $w$ be a band word for $(L, b)$, $L$ be a boundary link. Then there are Seifert surfaces for $L$, which intersect $\beta$ only in $\partial\beta$, if and only if 1 is in the $\widetilde{CA}$-orbit of $w$.*

DEFINITION 2.6. A band for $L$ is called *essential* if 1 is not in the $\widetilde{CA}$-orbit of $w$.

EXAMPLES 2.7. Let $(L, b)$ be a given pair.
a) If $w$ does not involve $x_1$ or $x_2$, then $b$ is not essential, since $\overline{\alpha_{i1}}(w)=w$ or $\overline{\alpha_{i2}}(w)=w$ (compare (*)).
b) If $r=2$, then the usual action by automorphisms of $CA$ on $F$ is by inner automorphisms, i.e. $\alpha_{12}(w)=x_1 w x_1^{-1}$ and $\alpha_{21}(w)=x_2 w x_2^{-1}$, so $\alpha_{12}\cdot w=x_1 w$ and $\alpha_{21}\cdot w=w x_2^{-1}$. Thus the $\widetilde{CA}$-orbit is the $\boldsymbol{Z}^2$-orbit.

It is in general difficult to decide, when 1 is in the $\widetilde{CA}$-orbit of a given word $w$. The orbit of 1 is the set of all $g_1 g_2^{-1}$, for which there are $g_3, \cdots, g_r \in F$, such that $\alpha=(g_1, \cdots, g_r)\in\widetilde{CA}$. Of course $\alpha\in\widetilde{CA}$ if and only if $\{g_i x_i g_i^{-1}\}$ is a basis of $F$. This is equivalent to the existence of $h_3, \cdots, h_r\in F$, such that

$$(x_1, w x_2 w^{-1}, h_3 x_3 h_3^{-1}, \cdots, h_r x_r h_r^{-1})$$

is a basis of $F$. This suggests the following notion:

DEFINITION 2.8. Let $\{x_i\}$ be a fixed set of generators of $F$. Let $Y\subset F$ be a set of conjugates of $x_i$'s. $Y$ is called *special primitive* if there is a supplementary set $Z$ of conjugates of $x_i$'s, such that $Y\cup Z$ is a basis of $F$.

Of course special primitive implies primitive, but the converse does not hold:

EXAMPLE 2.9. Let $w=x_3 x_2 x_1$, $r=3$. Then $\{x_1, w x_2 w^{-1}\}$ is primitive, since $\{x_1, w x_2 w^{-1}\}$ is a basis of $F$. Assume that $(x_1, w x_2 w^{-1}, h x_3 h^{-1})$ is a basis for some $h\in F$. In particular $x_2$ is a word in these three elements. For $x\in F(x_1, x_2, x_3)$ let $\bar{x}$ denote the image in $F(x_1, x_2)$ under the canonical projection. Then $x_2$ is a word in $x_1$ and $\bar{w} x_2 \bar{w}^{-1}$ since $\overline{h x_3 h^{-1}}=1$. But it is easy to see that this is not possible either by a length argument or by Whitehead's algorithm [19].

For $w \in F$ let $|w|$ denote the length of the reduced word $w$.

QUESTION 2.10. Given $w \in F$ of minimal length under the $\boldsymbol{Z}^2$-action. Assume that there exists an $\alpha \in \widetilde{CA}$, such that $|\alpha \cdot w| < |w|$. Does there exist an elementary generator $\alpha_{ij}$, such that $|\alpha_{ij} \cdot w| < |w|$ ?

## 3. Detection of (Homology) Boundary Links

We recall results from [3]: Let $L$ be a *HBL* with epimorphism $\phi \colon \pi(L) \to F$ and meridians $\{\mu_i\}$. Let $\mathrm{Out}(F)$ be the group of outer automorphisms of $F$ and $\theta \colon \mathrm{Out}(F) \to \mathrm{Aut}(F^r)$ be defined by $\theta(\alpha)(g_1, \cdots, g_r) = (\alpha(g_1), \cdots, \alpha(g_r))$. Let the group structure on the semidirect product $F^r \times_\theta \mathrm{Out}(F)$ be given by

$$((g_1, \cdots, g_r), \alpha)((g_1', \cdots, g_r'), \alpha') = ((g_1 \alpha(g_1'), \cdots, g_r \alpha(g_r'), \alpha\alpha') .$$

Then $F^r \times_\theta \mathrm{Out}(F)$ acts on $F^r$ by

$$((g_1, \cdots, g_r), \alpha) \cdot (h_1, \cdots, h_r) = (g_1 \alpha(h_1) g_1^{-1}, \cdots, g_r \alpha(h_r) g_r^{-1})$$

The following results are proved in [3]:

i) The orbit of $(\phi(\mu_1), \cdots, \phi(\mu_r))$ under this action, denoted by $P(L) = [\phi(\mu_i)]$, is called the *pattern* of $L$, and is an invariant of $L$.

ii) Each $r$-tuple $(u_1, \cdots, u_r)$ of elements, which normally generate $F$, is realized as $(\phi(\mu_i))$ for some ribbon link.

**Theorem 3.1.** (Smythe [15], Cochran-Levine [3]). *A HBL L is a boundary link if and only if $(x_1, \cdots, x_r) \in P(L)$ for generators $x_1, \cdots, x_r$ of $F$.*

Note that $\phi$ can always be changed, such that $\phi(\mu_i) \equiv x_i \bmod [F, F]$. But this condition is *not* preserved by the $(F^r \times_\theta \mathrm{Out}(F))$-action.

The next result provides, similar to 3.1, a method to decide, when a *SHB*-link is a *HBL*.

It is convenient to introduce the following notion:

DEFINITION 3.2. Let $G$ be a group which abelianizes to $\boldsymbol{Z}^r$. Then $G$ is called *onto-free* if there exists an epimorphism from $G$ onto the free group of rank $r$.

**Proposition 3.3.** *Let $L'$ be an $(r+l)$-component HBL with meridians $\{\mu_i'\}$, and let $\phi' \colon \pi(L') \to F' = F(x_1, \cdots, x_{r+l})$ be an epimorphism. Let $L' = (L, L'')$ for some r-component link L. Then L is a HBL if and only if the group $G = G(r, \phi', \{\mu_i'\}) = (x_1, \cdots, x_{r+l} \mid \phi'(\mu_{r+1}'), \cdots, \phi'(\mu_{r+l}'))$ is onto-free.*

Proof. We have the commutative diagram:

$$\begin{array}{ccc} \pi(L') & \xrightarrow{\phi'} & F' \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle pr'} \\ \pi(L) & \xrightarrow{\tilde{\phi}'} & G \end{array}$$

where $\tilde{\phi}'$ is induced by $\phi'$.

Assume that $\psi: G \to F$ is an epimorphism. Then $\psi\tilde{\phi}': \pi(L) \to F$ is an epimorphism, and $L$ is a *HBL*. Conversely suppose that $\vartheta: \pi(L) \to F$ is onto. Let $\mu': F' \to \pi(L')$ be a splitting homomorphism for $\phi'$. We want to show that $\vartheta \circ pr \circ \mu': F' \to F$ factors through a homomorphism $G \to F$ (which will be onto, since $F' \cong \pi(L')/\pi(L')_\omega \to \pi(L)/\pi(L)_\omega \cong F$ is onto). But $\mu' \phi'(\mu'_j) = \mu'_j c_j$ for some $c_j \in \pi(L')_\omega$ for $1 \leq j \leq r+l$, since the kernel of $\phi'$ is $\pi(L')_\omega$. If $j > r$ then $pr(\mu'_j) = 1 \in \pi(L)$ and $pr(c_j) \in \pi(L)_\omega$, so $\vartheta \circ pr(c_j) = 1 \in F$. ■

**Theorem 3.4.** *Let $(L, b)$ be a pair, $L$ a HBL with meridians $\{\mu_i\}$ and epimorphism $\phi$. Then $L(b)$ resp. $\hat{L}(b)$ are HBLs if and only if the groups*

$$(x_1, \cdots, x_r \mid \phi(\mu_2)^{-1} \phi(B)^{-1} \phi(\mu_1) \phi(B))$$

*resp.*

$$(x_1, \cdots, x_r, z \mid \phi(\mu_2)^{-1} z^{-1} \phi(\mu_1) z))$$

*are onto-free. Here $B = [\gamma_1 \tilde{\beta} \gamma_2^{-1}] \in \pi(L)$ for basings $\gamma_1, \gamma_2$ of $K_1, K_2$, which induce the meridians $\mu_1, \mu_2$, and $\tilde{\beta} = X(L) \cap \beta$.*

Proof. We know that $L(b)$ (resp. $\hat{L}(b)$) are sublinks of the *HBL* $L(b)' := (L(b), K'_2)$ (resp. $(\hat{L}(b)' := (\hat{L}(b), K'_2))$, where $K'_2$ is a parallel pushout of $K_2$. It follows from the proof of [3, 3.1] that the pair $(\phi, \{\mu_i\})$ induces a pair $(\phi', \{\mu'_i\})$ for $L(b)'$, such that $\phi'(\mu'_2) = \phi(B^{-1} \mu_1^{-1} B \mu_2)$. This proves the assertion about $L(b)$. In case of strong fusion we first add $C_b$ to $L$ and extend $\phi$ to the epimorphism $\pi(L \cup C_b) = \pi(L) * \boldsymbol{Z} \to F(x_1, \cdots, x_r, z)$ in the obvious way ($*$ means free product). We choose a meridian of $C_b$, which maps to $z$ by this extension. Then we follow [3, 3.1] again. It turns out that always $G(r, \hat{\phi}', \{\hat{\mu}'_j\}) \cong (x_1, \cdots, x_r, z \mid z^{-1} \phi(\mu_1)^{-1} z \phi(\mu_2))$ for each $\hat{\phi}': \pi(\hat{L}(b)') \to F(x_1, \cdots, x_r, z)$ and meridians $\{\hat{\mu}'_i\}$ of $\hat{L}(b)'$. ■

REMARK 3.5. Since the isomorphism classes of $G(r-1, \phi', \{\mu'_i\})$ and $G(r, \hat{\phi}', \{\hat{\mu}'_i\})$ do not depend on input data $(\phi, \{\mu_i\})$, we will write $G_{L,b} \cong G(r-1, \phi', \{\mu'_i\})$ resp. $G_L \cong G(r, \hat{\phi}', \{\hat{\mu}'_i\})$.

The results 3.4 and 3.1 translate the geometric questions a) and b) of the introduction into algebra. For the rest of this paragraph we shall deduce several sufficient conditions from this and carry out computations, which are needed later on. We recall the following result characterizing freeness of 1-

relator groups:

**Lemma 3.6.** (Rappaport [14], Whitehead [19]). *The 1-relator group $(x_1, \cdots, x_r \mid R)$ is free if and only if the relator $R$ is a primitive element of $F = F(x_1, \cdots, x_r)$, i.e. $R$ is member of a basis of $F$.*

**Corollary 3.7.** *Let $L$ be a HBL admitting an epimorphism $\phi: \pi(L) \to F$, such that $\phi(\mu_1) = x_1$ and $\phi(\mu_2) = x_2$ (we call such links partial boundary). Let $b$ be a band for $L$. Then $\hat{L}(b)$ is a HBL. Moreover*

$$G_{L,b} \cong G_w := (x_1, \cdots, x_r \mid x_2 = w^{-1} x_1 w),$$

*where $w = \phi(B)$, $B = [\gamma_1 \tilde{\beta} \gamma_2^{-1}]$. Here $(\phi, \{\mu_i\})$ is chosen such that $\phi(\mu_1) = x_1$, $\phi(\mu_2) = x_2$, and $\gamma_1, \gamma_2$ are basings of $K_1, K_2$ inducing $\mu_1, \mu_2$. If $L$ is a boundary link, then $w$ is a band word for $b$.*

Proof. This follows from 3.4 and 3.6. ■

Of course each boundary link is partial boundary. Moreover, if $L = (K_1, \cdots, K_r)$ is partial boundary, then the sublink $(K_1, K_2)$ is a boundary link. But the following example shows that this is not sufficinet.

EXAMPLE 3.8. Let $L = (K_1, K_2, K_3)$ be a *HBL* with $(\phi, \{\mu_i\})$, such that

$$(\phi(\mu_1), \phi(\mu_2), \phi(\mu_3)) = (x_1, x_3^{-1} x_2^{-1} x_1 x_2 x_1^{-1} x_3 x_2, x_3)$$

If $L$ would be partial boundary, then there would exist an automorphism $\psi$ of $F$, such that $\psi(x_1) = x_1$ and $\psi(x_2) = g\phi(\mu_2) g^{-1}$ for some $g \in F$. In particular $\{x_1, g\phi(\mu_2) g^{-1}\}$ would be primitive for some $g \in F$. By [19] it is easily proved that this is not true. Thus $L$ is *not* partial boundary. On the other hand $[x_1, x_2^{-1} x_1 x_2 x_1 x_2]$ is the pattern of a 2-component boundary link.

From 3.7 we deduce:

**Corollary 3.9.** *Let $L$ be a partial boundary link and $w = \phi(B)$ as in 3.7. Then $L(b)$ is a HBL, if*
(i) *$x_2^{-1} w^{-1} x_1 w$ is primitive (this is equivalent to $G_w$ free) or*
(ii) *$w$ maps to $\langle t \rangle \subset F' := F(t, x_3, \cdots, x_r)$ under the epimorphism $F \to F'$ given by $(x_1 \mapsto t, x_2 \mapsto t, x_i \mapsto x_i$ for $3 \leq i \leq r)$.*

Proof. (i) follows from 3.7 and 3.6. The epimorphism in (ii) induces an epimorphism $\chi_w: G_w \to F'$, so $G_w$ is onto-free. ■

**Corollary 3.10.** *Let $L$ be a boundary link with band $b$ and band word $w$ satisfying* (ii) *in* 3.10. *Then $L(b)$ is a boundary link.*

Proof. We choose a special pair $(\phi, \mu)$ for $L$. Then there is a special pair

$(\phi', \{\mu_i'\})$ for $L(b)'$ (compare [CL, 3.1]), such that the diagram:

$$\begin{array}{ccc} \pi(L(b)') & \xrightarrow{\phi'} & F \\ {\scriptstyle pr}\big\downarrow & & \big\downarrow{\scriptstyle pr'} \\ \pi(L(b)) & \xrightarrow{\tilde{\phi}'} & G_w \\ & & \big\downarrow{\scriptstyle \chi_w} \\ & & F(t, x_3, \cdots, x_r) \end{array}$$

commutes. Thus we see that $[t, x_3, \cdots, x_r]$ is the pattern of $L(b)$. ■

The following question is related to the notion of partial boundary links: *For which $r$ there are HBLs $L$, which are not boundary links, but for which $\hat{L}(b)$ is a HBL?*

A positive answer can be given for $r \geq 3$:

EXAMPLE 3.11. Let $L$ be partial boundary with pattern $[x_1, x_2, x_3 x_2 x_3^{-1} x_1 x_2 x_3]$. Using 3.1 and [19] it is easy to prove that $L$ is not a boundary link. But $\hat{L}(b)$ is a *HBL* by 3.7.

For $r=2$ the following two conjectures are equivalent by 3.4, stating the problem in the geometric and a purely algebraic way:

CONJECTURE (H). Let $L$ be 2-component *HBL*. If $\hat{L}(b)$ is a *HBL* for some (and hence any) band $b$, then $L$ is a boundary link.

CONJECTURE (H'). Let $u_1, u_2 \in F(x_1, x_2) = F$ be elements, which normally generate $F$, but $(x_1, x_2) \notin [u_1, u_2]$. Then the HNN-extension $G_u := (x_1, x_2, z \mid u_2 = z^{-1} u_1 z)$ is not onto-free.

Now we give a convenient sufficient condition for when a fusion of a boundary link is a *HBL* and compute the resulting pattern. We begin with an example, which shows that it is not necessary that the band is inessential to $L$.

EXAMPLE 2.9. (continued). Let $L$ be a 3-component boundary link and $b$ a band for $L$ with band-word $w = x_3 x_2 x_1$. We have seen in 2.9 that $\{x_1, w w_2 w^{-1}\}$ is primitive, but not special primitive. So $L(b)$ is a *HBL* for an essential band $b$.

DEFINITION 3.12. Let $r \geq 2$ and $L$ be an $(r+1)$-component boundary link. A band $b$ for $L$ is called *tame* if for some choice of Seifert surfaces the band word $w$ has the form $w = w_1 x_{r+1}^{-1} w_2 \in F(x_1, \cdots, x_{r+1})$ (or $w = w_1 x_{r+1} w_2$) for words $w_1, w_2 \in F(x_1, \cdots, x_r) = F$. A pair $(L, b)$ as above is called *tame*. □

REMARKS 3.13.

a) A strong fusion of an $r$-component link $L$ along a band $b$ is a fusion of the $(r+1)$-component link $(L, C_b)$ along a tame band $b'$ with band-word $w'=x_{r+1}^{-1} w$, where $w$ is a band word corresponding to $b$. Fusion of a tame pair is a more general "algebraic" version of strong fusion. Figure 3 shows a tame band for the trivial 3-component link.

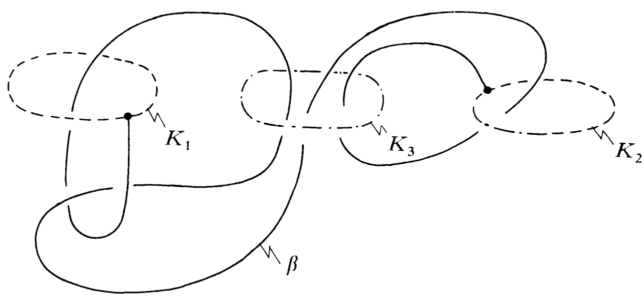


Figure 3

b) Tameness of $b$ is equivalent to the existence of Seifert surfaces, such that the band word has the form $x_{r+1}^{-1} w$. This follows from 1.4.

**Proposition 3.14.** *Let $(L, b)$ be a tame pair. Then $L(b)$ is a HBL.*

Proof. We know that

$$G_w = (x_1, \cdots, x_r, x_{r+1} \,|\, x_2 = w^{-1} x_1 w)$$

with $w=w_1 x_{r+1}^{-1} w_2$.
Then

$$(x_1, wx_2 w^{-1}, x_3, \cdots, x_r, w)$$

is a basis of $F(x_1, \cdots, x_{r+1})$ and 3.6 applies. ■

**Theorem 3.15.** *If $(L, b)$ is tame, then the pattern of $L(b)$ is given by*

$$P = [x_1, x_3, \cdots, x_r, \bar{w}_2 x_2 \bar{w}_1]\,,$$

*where for each $u \in F$ let $\bar{u}$ denote the image of $u$ under the homomorphism $x_i \mapsto x_i$ for $i \neq 2$ and $x_2 \mapsto x_2 x_1 x_2^{-1}$.*

Proof.
For a tame band $(L, b)$ we shall construct an epimorphism

$$\varepsilon\colon \pi(L(b)) \to F(x_1, \cdots, x_r)$$

as follows:
There is the canonical map

$$\tilde{\phi}' \colon \pi(L(b)) \to G_w$$

induced by

$$\phi' \colon \pi(L(b)') \to F(x_1, \cdots, x_{r+1})$$

(compare [3, 3.1]). Recall that $L(b)' = (L(b), K_2')$, where $K_2'$ is a parallel copy of $K_2$. So for a choice of meridians $\{\mu_i'\}$ of $L(b) = (K_1 \#_b K_2, K_3, \cdots, K_{r+1})$ we can assume that $\tilde{\phi}'(\mu_i') = x_i$, $i = 1, 3, \cdots, (r+1)$. Here $\mu_1'$ is a meridian of $K_1 \#_b K_2$ and $\mu_i'$ is a meridian of $K_i$ for $3 \leq i \leq r+1$.

Our assumptions imply the existence of a canonical epimorphism

$$\bar{\varepsilon} \colon G_w \to F(x_1, \cdots, x_r) = F$$

given by

$$\begin{cases} \bar{\varepsilon}(x_1) = x_1 \\ \bar{\varepsilon}(w x_2 w^{-1}) = x_1 \\ \bar{\varepsilon}(w) = x_2^{-1} \\ \bar{\varepsilon}(x_i) = x_i\,, \quad \text{for} \quad 3 \leq i \leq r\,. \end{cases}$$

So $\bar{\varepsilon}(x_2) = x_2 x_1 x_2^{-1}$, and $\bar{\varepsilon}(w_1 x_{r+1}^{-1} w_2) = x_2^{-1}$ implies $\bar{\varepsilon}(x_{r+1}) = \bar{\varepsilon}(w_2) x_2 \bar{\varepsilon}(w_1)$. Now we define $\varepsilon := \bar{\varepsilon} \circ \tilde{\phi}' \colon \pi(L(b)) \to F$. ■

## 4. More Examples

Now we construct concrete families of links to distinguish the classes (I)-(III). We need the following result:

**Lemma 4.1.** (Hillman [7, V, Thm.1]). *Let $G$ be a group which abelianizes to $\mathbf{Z}^r$ and with a presentation of deficiency $r$. If $G$ is onto-free, then the Alexander ideal $E_r(G)$ is principal.*

For definitions and further details we refer to [7].

EXAMPLE 4.2. Let $L$ be a 3-component boundary link and $b$ a band for $L$ with band-word $w = x_3^{-1} x_1 x_2 x_3$. We compute $E_2(G)$ for $G = (x_1, x_2, x_3 \mid w^{-1} x_1 w x_2^{-1})$. The Jacobi-matrix of this presentation is

$$\| t^{-2} + t^{-1} s^{-1} - t^{-2} s^{-1},\ s^{-1} - 1 - s^{-1} t^{-1},\ s^{-1}(t - t^{-1} + t^{-2} - 1) \|\,,$$

where $x_1, x_2 \mapsto t$ and $x_3 \mapsto s$ under abelianization.

So $E_2(G)$ is the ideal

$$(1-t-s,\ 1-t+ts,\ 1-t-t^2+t^3) = (1+t,\ 1-t-s,\ (1-t)^2(1+t)) = (1+t,\ 2-s)\,,$$

which is certainly not principal. It follows by 3.4. that the 2-component link $L(b)$ is not a *HBL*.

For $n=1$ and $L$ the trivial link this is the example given by Hillman [7, $V$, Figure 1].

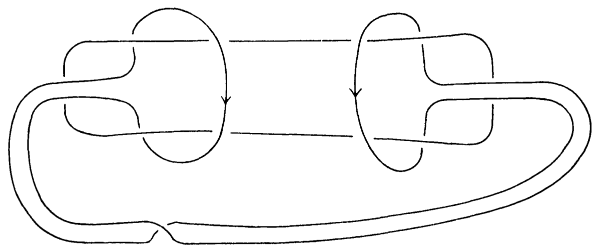

Figure 4

The next two examples show how to construct *SHB*-links, which are not *HBLs*.

EXAMPLE 4.3. Let $L$ be a 2-component boundary link and $b_i$(resp. $b_i'$) be bands with words $w_i=x_2^i\,x_1$(resp. $x_2\,x_1^i$), $i\in\mathbf{Z}\setminus\{0\}$. Then the pattern of $L_i=L(b_i)$ (resp. $L_i'$) is given by $(x_1, x_2(x_2\,x_1^i\,x_2^{-1})\,x_1)$ (resp. $(x_1, x_1(x_2^{-1}\,x_1^i\,x_2)\,x_2)$).

It is easy to prove that

$$G_{L_i}\cong G_{-i,1}\cong G_{L_i'}\,,$$

where $G_{i,j}$ is the Baumslag-group

$$(a, b, u\,|\,u[a^i, u]\,[a^j, b])\,,$$

which is known to be parafree not free (compare [1]).

Thus $L_i$ resp. $L_i'$ are *SHB*-links, which are not *HBL*.

For $n=1$ and $i=1$ this is Example VI, Figure 1 in [7].

EXAMPLE 4.4. Let $r=3$ and $V$, $W$ be elements of the commutator subgroup of $F(x_1, x_3)\subset F(x_1, x_2, x_3)=F$, $V\neq 1$. Let $L$ be a *HBL* with $\phi(\mu_1)=W\,x_1$ and $\phi(\mu_2)=W\,x_2[V, x_1^{-1}\,x_2]$. For given $(V, W)$ we can construct a (ribbon) link, such that $(w_1, w_2, w_3)=(W\,x_1, W\,x_2[V, x_1^{-1}\,x_2], x_3)$ are the images of the meridians. In fact, by [3] we only have to check that the normal closure of $\{w_1, w_2, w_3\}$ is $F$. But $x_1=W^{-1}\,w_1$, and $W$ is a commutator in $F(x_1, x_3)$, thus is contained in the normal closure of $x_3$ ($x_3=1$ implies $W=1$). Also: $x_2=W^{-1}\,w_2[x_1^{-1}\,x_2, V]=W^{-1}\,w_2(x_1^{-1}\,x_2\,V\,x_2^{-1}\,x_1)\,V^{-1}$ is in the normal closure, since $V\in F(x_1, x_3)$ and $x_1$ is in the normal closure. By 3.4. we know that

$$\begin{aligned} G_L &\cong (x_1, x_2, x_3, z\,|\,W\,x_2[V, x_1^{-1}\,x_2]=z^{-1}\,W\,x_1\,z) \\ &\cong (x_1, x_2, x_3, z\,|\,x_2[V, x_2]=[(W\,x_1)^{-1}, z])\,. \end{aligned}$$

This group is parafree not free by [17, 5.1, Thm. C]. So $G_L$ is not onto-free and $\hat{L}(b)$ is *not* a *HBL*. For $W=1$ we can construct the link $L$ and thus

$\hat{L}(b)$ explicitly. Note that the pattern $(x_1, x_2[V, x_1^{-1}x_2], x_3)$ is equivalent to $(x_1, wx_2^{-1}w^{-1}x_1, x_3)$, where $w=x_2^{-1}x_1 V x_1^{-1}x_2$. This follows from

$$\begin{aligned}\phi(\mu_2) &= (x_2 V x_2^{-1})\, x_2 x_1^{-1} (x_2 V^{-1} x_2^{-1})\, x_1 \\ &= (x_2 x_1^{-1})(x_1 V x_1^{-1})(x_1 x_2^{-1})(x_2 x_1^{-1})(x_2 x_1^{-1})(x_1 V^{-1} x_1^{-1})(x_1 x_2^{-1})\, x_1\end{aligned}$$

and the substitution $X_2^{-1}=x_2 x_1^{-1}$, i.e. $X_2=x_1 x_2^{-1}$, so

$$\phi(\mu_2) = (X_2^{-1} x_1 V x_1^{-1} X_2)\, X_2^{-1}(X_2^{-1} x_1 V^{-1} x_1^{-1} X_2)\, x_1$$

which corresponds to a change of $\phi$ induced by composition with the automorphism of $F$: $(x_1 \mapsto x_1, x_2 \mapsto x_2^{-1} x_1, x_3 \mapsto x_3)$.

The resulting pattern is the pattern of a strong fusion of a boundary link along a band with band word $w$. The resulting link $\hat{L}(b)$ for $V=[x_1, x_3]$ is shown in Figure 5 (where the the boundary link is chosen to be trivial).

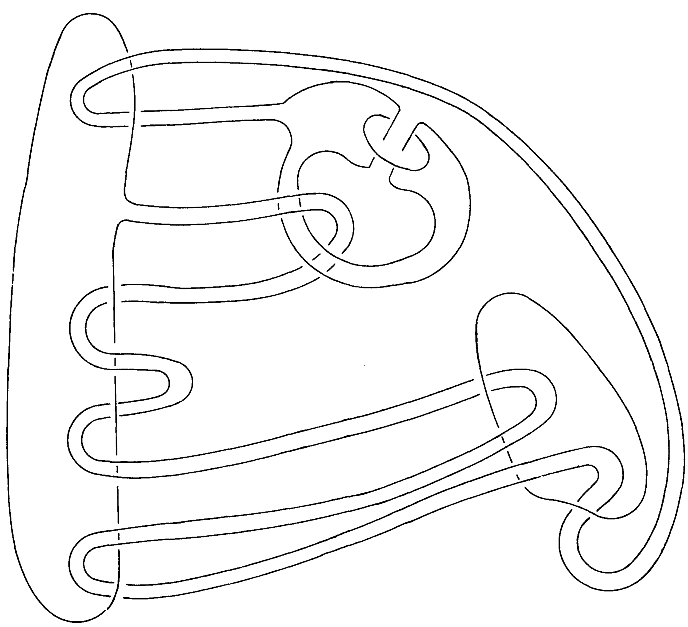

Figure 5

**Question 4.5.** For $n=1$, what is the relation between $G_w$ (or $G_L$ or $G_{L,b}$) and the ribbon group $H(R)$ of $L(b)$ (or $\hat{L}(b)$) defined by Hillman [7]?

## 5. Proof of Theorems 1 and 2

In this paragraph we prove Theorems 1 and 2 by solving the equivalent algebraic problem. By 2.5, 3.1 and 3.15 we know that Theorem 1 is equivalent to

**Theorem 5.1.** *For $w \in F$ the following two assertions are equivalent:*
i) *The trivial word 1 is in the $\widetilde{CA}$-orbit of $w$.*
ii) *The pattern $[x_1, \bar{w}x_2, x_3, \cdots, x_r]$ contains $(x_1, \cdots, x_r)$*

Then Theorem 2 follows from Theorem 1 and 2.7.b.

Proof of 5.1. The claim that i) implies ii) is obvious from the geometric picture (compare the remark following 2.1)

For the converse we have to recall Whitehead's algorithm [19] to decide when a pattern contains $(x_1, x_2, \cdots, x_r)$. To each collection of words $(w_1, w_2, \cdots, w_r)$ in $F$ there is associated a graph $\Gamma = \Gamma_w$ as follows: The graph has $2r$ vertices labelled by the generators $x_1, \cdots, x_r$ and their inverses $x_1^{-1}, \cdots, x_r^{-1}$. The edges of $\Gamma$ are given by the cyclic words $w_k$ (compare [12]): For each occurence $x_i\, x_j$ in some word $w_k$, we have an edge joining $x_i$ to $x_j^{-1}$ in $\Gamma$. Moreover, if some $w_k$ begins with $x_j$ and ends with $x_i$, we also have an edge joining $x_i$ to $x_j^{-1}$.

A vertex $v$ of $\Gamma$ is called a cut vertex, if $\Gamma = \Gamma_1 \cup v \cup \Gamma_2$, where $\Gamma_1 \cup v \neq \Gamma_1$ and $\Gamma_2 \cup v \neq \Gamma_2$ are subgraphs, $v$ is not isolated in $\Gamma_1 \cup v$ or $\Gamma_2 \cup v$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $\Gamma_1$ be chosen, such that $v^{-1} \notin \Gamma_1$.

**Lemma 5.2.** (Whitehead [W]). *Given $\{w_1, \cdots, w_r\} \neq \{x_1, \cdots, x_r\} \subset F$. If $[w_1, \cdots, w_r]$ contains $(x_1, \cdots, x_r)$, then the graph $\Gamma_w$ contains a cut vertex.*

In fact, for each cut vertex of $\Gamma = \Gamma_w$ Whitehead describes an automorphism of $F$ (called a *simple* automorphism), which reduces the length of $(w_1, \cdots, w_r)$ (which is defined by the sum of the lengths of the cyclic words $w_i$) as follows: Each generating element $w \in \{x_i, x_i^{-1}\} \cap \Gamma_1$, such that $w^{-1} \notin \Gamma_1$, is mapped to $wv$ (and $w^{-1}$ is mapped to $v^{-1} w^{-1}$). If $w$ and $w^{-1}$ are vertices of $\Gamma_1$, then $w$ is mapped to $v^{-1} wv$.

Now we assume that $[x_1, \bar{w}x_2, x_3, \cdots, x_r]$ contains a basis, but 1 is not in the $\widetilde{CA}$-orbit of $w$. We can choose $w$ of minimal length $|w| > 1$ under the $\widetilde{CA}$-action. In particular, by 2.7.a there is an occurence of $x_1^{\pm 1}$ and an occurence of $x_2^{\pm 1}$ in $w$. Also, $w$ does not begin with $x_1^{\pm 1}$ and does not end with $x_2^{\pm 1}$. So the graph $\Gamma$ associated to $(x_1, \bar{w}x_2, x_3, \cdots, x_r)$ has to contain at least the following edges:

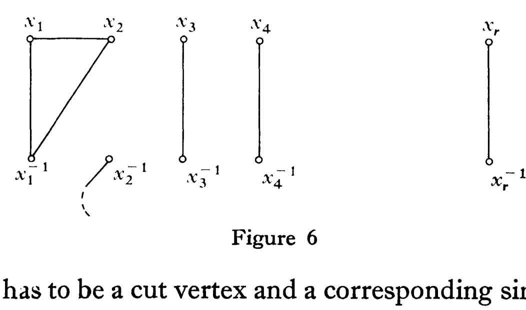


Figure 6

By 5.2 there has to be a cut vertex and a corresponding simple automorphism determined by that vertex. There are essentially five different possible cases for $v$, namely $v=x_1$, $v=x_1^{-1}$, $v=x_2$, $v=x_2^{-1}$ or $v=x_j^{\pm 1}$ for some $j\geq 2$. In each case we will find a sequence of special automorphisms, whose action will strictly reduce the length of $w$ in contradiction to the minimal choice.

The important observation is that for each $j>2$: $x_j\in\Gamma_1$ is equivalent to $x_j^{-1}\in\Gamma_1$. Assume that $v=x_1$. Then $x_1^{-1}\notin\Gamma_1$, $x_2\notin\Gamma_1$. Assume that $x_2^{-1}\notin\Gamma_1$. Then the resulting simple automorphism conjugates a set of $x_j$'s for $j>2$ by $x_1$ and leaves fixed $x_2$. Since the total length of $(w_1, \cdots, w_r)$ has to be reduced, the length of $\bar{w}x_2$ has to be reduced. But the corresponding sequence of special automorphisms applied to $w$ would have to reduce its length, which is a contradiction. Now we assume that $x_2^{-1}\in\Gamma_1$, so in addition we have to map $x_2$ to $x_1^{-1}x_2$. The corresponding simple automorphism applied to $\bar{w}x_2$ leads to

$$w(x_1, x_1^{-1}x_2x_1x_2^{-1}x_1, \text{conjugates of } x_3, \cdots, x_r)\, x_1^{-1}x_2$$

which has smaller length than $\bar{w}x_2$. Here we write $w(\alpha(x_1, \cdots, x_r))$ for the image of $w\in F$ under the endomorphism $\alpha$ of $F$. The substitution above can be realized by the $\widetilde{CA}$-action as follows: We have

$$\alpha_{12}^{-1}\cdot w = \overline{\alpha_{12}}^{-1}(w)\, x_1^{-1}\,,$$

and then we can apply a sequence of $\alpha_{ij}$'s with $i\neq j>2$ as before conjugating some of the $x_j$'s for $j>2$. The case $v=x_1^{-1}$ is similar and will be omitted.

Assume that $v=x_2$, so $x_2^{-1}\notin\Gamma_1$. Since $w$ has minimal length there has to be an occurrence of $x_1$ in $w$. So there are segments in $w$ as follows:

$$x_1^{\varepsilon}\, x_{j_1}^{\varepsilon_{j_1}}\, x_{j_2}^{\varepsilon_{j_2}}\cdots x_{j_k}^{\varepsilon_{j_k}}\, x_2^{\delta}$$

with all $j_l>2$, $\varepsilon$, $\delta\neq 0$, or

$$x_1^{\varepsilon}\, x_{j_1}^{\varepsilon_{j_1}}\, x_{j_2}^{\varepsilon_{j_2}}\cdots x_{j_k}^{\varepsilon_{j_k}}\,.$$

This gives rise to segments

$$x_1^{\varepsilon}\, x_{j_1}^{\varepsilon_{j_1}}\cdots x_{j_k}^{\varepsilon_{j_k}}\, x_2\, x_1^{\delta}$$

resp.

$$x_1^{\varepsilon}\, x_{j_1}^{\varepsilon_{j_1}}\cdots x_{j_k}^{\varepsilon_{j_k}}\, x_2$$

in $\bar{w}$.

This shows in both cases that $x_1, x_1^{-1}\notin\Gamma_1$. So the corresponding automorphism just conjugates certain $x_j$ for $j>2$, what we know can be realized by special automorphisms on $w$. From the graph follows that, if $v=x_2^{-1}$, then $x_2, x_1, x_1^{-1}\notin\Gamma_1$, so again just conjugation appears.

It remains to discuss the case $v=x_j$ for $j>2$.
There are four cases to consider:

$$\text{(i)}\quad x_2, x_2^{-1}\notin\Gamma_1 \text{ (this is easy)} \qquad \text{(ii)}\quad x_2\in\Gamma_1, x_2^{-1}\notin\Gamma_1,$$
$$\text{(iii)}\quad x_2\notin\Gamma_1, x_2^{-1}\in\Gamma_1 \qquad \text{(iv)}\quad x_2, x_2^{-1}\in\Gamma_1.$$

We only discuss (ii), the other cases are similar to prove. We know that $x_1, x_1^{-1}\in\Gamma_1$, so the corresponding simple automorphism maps $x_2$ to $x_2 x_j$ and $x_1$ to $x_j^{-1} x_1 x_j$, thus maps $\overline{w}x_2$ to

$$w(x_j^{-1} x_1 x_j, x_2 x_1 x_2^{-1}, \text{conjugates of } x_3, \cdots, x_r)\, x_2 x_j$$

Now $\alpha_{1j}^{-1}\cdot w=x_j\,\overline{\alpha_{1j}}^{-1}(w)$. So, after having applied a sequence of $\alpha_{ij}$, $i>2$, we have changed $w$ to

$$w' = x_j\, w(x_j^{-1} x_1 x_j, x_2, \text{conjugates of } x_3, \cdots, x_r)$$

But $\overline{w'}x_2$ is the same cyclic word as the image of $\overline{w}x_2$ under the simple automorphism above. Note that the simple automorphism applied to $\overline{w}x_2$ reduces the length by the number of occurences of the form $x_j x_1 x_j^{-1}$. Even when $w$ begins with $x_j^{-1}$, this would reduce the length of $\overline{w}x_2$. But then also $w$ begins with $x_j$. So we can reduce the length of $w$ by the $\widetilde{CA}$-action, in contradiction to our assumption. This proves 5.1. ■

We conclude with a discussion of the following question, which is "parallel" to Theorem 1:

*Let $L$ be an $r$-component unlink (or completely split link) and $b$ a band for $L$. When is $\hat{L}(b)$ the unlink (or completely split)?*

Recall that $L=(K_1, \cdots, K_r)$ is *completely split* if there are disjoint $(n+2)$-balls $B_i\subset S^{n+2}$, such that $B_i\supset K_i$ for $1\leq i\leq r$.

Of course, if $\hat{L}(b)$ is not a boundary link, then $\hat{L}(b)$ cannot be completely split.

**Corollary 5.3.** *Let $L$ be an $r$-component completely split link. If $b$ is essential to $L$, then $\hat{L}(b)$ is not completely split.*

For $r=2$ and $n>2$ more can be said:

**Proposition 5.4.** *Let $L$ be the 2-component unlink in $S^{n+2}$, $n>2$, and $b$ a band for $L$. Then $\hat{L}(b)$ is the unlink if and only if $b$ is inessential.*

Proof. It is sufficient to prove that $\hat{L}(b)$ is the unlink, when $b$ is inessential. Let $L=(K_1, K_2)$ and $B_1, B_2$ be the canonical disks bounding $K_1, K_2$. The band word with respect to this choice of Seifert surfaces is $x_1^r x_2^s$ for integers $r, s$. We can isotope the band to make the band word actually trivial. This means that the intersections of $\beta$ with $B_1$ resp. $B_2$ cancel algebraically taking account the order of the intersections with respect to $B_1, B_2$. Let $(P, Q)$ be a pair of points in $B_1$(or $B_2$), such that $\beta(t_1)=P$, $\beta(t_2)=Q$ and $\beta(t_1, t_2)\cap(B_1\cup B_2)=\emptyset$. We join $P$ and $Q$ in $B_1$ by some arc $\alpha$. The embedded loop $\alpha\cup\beta[t_1, t_2]$ bounds a disk $D\subset S^{n+2}$, such that $D\cap(B_1\cup B_2)=\alpha$. So the Whitney-trick can be applied to eliminate $\{P, Q\}$ and by induction all intersections $\beta\cap int\,(B_1\cup B_2)$. Finally, we can isotope $\beta$ into standard position. In particular $\beta\cap S$ is a single point for a seperating $(n+1)$-sphere $S$ for $K_1$ and $K_2$. Thus $\hat{L}(b)$ is the unlink. ■

For $n=1$ self-knotting and section-linking of $\beta$ leads to additional phenomena as was pointed out in section 1. The easiest example of a nontrivial strong fusion of a 2-component unlink with inessential band is shown in Figure 7. In fact, a tedious but straightfoward computation of the HOMFLY-polynomial shows that both $\hat{L}(b)$ and $L(b)$ are nontrivial (compare [8]).

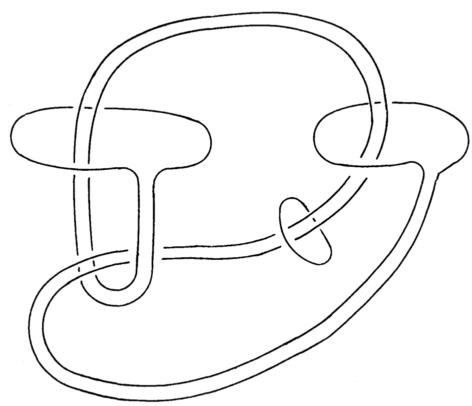

Figure 7

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Universität Siegen
Hölderlinstr. 3
5900 Siegen
Germany