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A BOUNDARY LINK IS TRIVIAL IF THE LUSTERNIK-SCHNIRELMANN CATEGORY OF ITS COMPLEMENT IS ONE

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1. Introduction

The Lusternik-Schnirelmann category cat(X) of a space X is the least integer n such that X can be covered by $n+1$ open subsets each of which is contractible to a point in X. In particular, cat(X) is a homotpoy type invariant and cat$(S^n)=1$. We know that $\pi_1(X)$ is a free group if X is a manifold and cat$(X)\leqq 1$ (cf. [2], [3] and [5]).

A locally flat knot (S^{n+2}, S^n) is topologically unknotted if and only if the category of its complement is one [10]. In fact, $S^{n+2}-S^n\simeq S^1$ if and only if cat$(S^{n+2}-S^n)=1$. We see also that a smooth knot (S^{n+2}, S^n) is unknotted if and only if cat$(S^{n+2}-S^n)=1$ when $n\neq 2$ ([7] for $n\geqq 4$, [15] for $n=3$ and [12] for $n=1$).

We will generalize this result to the smooth m-component links. A smooth (or locally flat) m-component link L stands for m smoothly (or locally flatly) embedded disjoint n-spheres $L_1\cup\cdots\cup L_m$ in S^{n+2}. A smooth (or locally flat) m-component link is called trivial if it bounds m smoothly (or locally flatly) embedded disjoint $(n+1)$-disks; boundary if it bounds a Seifert man ifold which consists of m disjoint compact smooth (or locally flat) $(n+1)$-submanifolds with connected boundary. Let $N_i=N(L_i)$ $(i=1, \cdots, m)$ be tubular neighborhoods of L_i which do not intersect each other. The $(n+2)$-dimensional compact manifold $E=S^{n+2}$- $\cup$ Int $N(L_i)$ with boundary $\partial E=\cup\,\partial N_i$ is called a link exterior and is homotopy equivalent to the link complement $S^{n+2}-L$.

In this paper we will show the following theorem by applying the unlinking criterion of boundary links due to Gutiérrez [6].

Theorem 1. *Let L be a smooth m-component boundary link in S^{n+2}. Suppose that $n\neq 2$. Then L is trivial if and only if* cat$(S^{n+2}-L)=1$.

If L is trivial, $S^{n+2}-L\simeq(\vee_m S^1)\vee(\vee_{m-1} S^{n+1})$. We have only to prove the if-part. On the other hand a classical link L is trivial if $\pi_1(S^3-L)$ is free by the loop theorem [12]. Since cat$(S^{n+2}-L)=1$ implies that $\pi_1(S^{n+2}-L)$ is free, Theorem 1 is already known for $n=1$.

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2. Unknotting of each component

Using the techniques of [10], we will show the topological unknotting of each component of L in Theorem 1 without any dimensional restriction.

Proposition 2. *Let L be a locally flat m-component boundary link such that* $\mathrm{cat}(S^{n+2}-L)=1$. *Then, for any i the knot (S^{n+2}, L_i) is topologically unknotted.*

Proof. First we need the following two lemmas.

Lemma 2.1. *Let (S^{n+2}, L) be a m-component boundary link. Suppose that $\pi_1(S^{n+2}-L)$ is a free group of rank m. Then $\pi_1(S^{n+2}-L)$ is generated by some m meridians.*

In fact, using a Seifert manifold we have an isomorphism $F_m\to\pi_1(S^{n+2}-L)/(\pi_1(S^{n+2}-L))_\omega$, where F_m is a free group of rank m generated by some m meridians by [6, p. 493, Prop. 3] and $(\pi_1(S^{n+2}-L))_\omega$ denotes the intersection $\cap_k(\pi_1(S^{n+2}-L))_k$ of the lower central series $\{(\pi_1(S^{n+2}-L))_k\}$ in $\pi_1(S^{n+2}-L)$. Since $\pi_1(S^{n+2}-L)$ is a free group, $(\pi_1(S^{n+2}-L))_\omega=\{1\}$ by [9, p. 109, Cor. 2.12] and we obtain Lemma 2.1.

Note that $\pi_1(S^{n+2}-L)$ is a free group if $\mathrm{cat}(S^{n+2}-L)=1$. If L is a link in Proposition 2, we see that $\pi_1(S^{n+2}-L_i)$ is an infinite cyclic group generated by the meridian for any i by Lemma 2.1.

Lemma 2.2. *Let L be a link in Proposition 2, Let $E(i)$ be an infinite cyclic covering of E induced from the universal covering $\tilde{E}_i$ of $E_i=S^{n+2}-\mathrm{Int}\ N(L_i)$. Then, $H_*(E(i); \boldsymbol{Z})=0$ for any i and $2\leqq *\leqq n$.*

Assuming Lemma 2.2 we will prove that $H_*(\tilde{E}_i; \boldsymbol{Z})=0$ for any $*$. The Mayer-Vietoris sequence of $(\cup(S^n\times D^2), E(i), \tilde{E}_i)$ together with Lemma 2.2 implies that $H_*(\tilde{E}_i; \boldsymbol{Z})=0$ for $2\leqq *\leqq n$, because the natural map: $H_1(\cup(S^n\times S^1); \boldsymbol{Z})\to H_1(E(i); \boldsymbol{Z})$ is an isomorphism. Since $\tilde{E}_i$ is a universal covering of E_i, $H_1(\tilde{E}_i; \boldsymbol{Z})=0$. Also $H_*(\tilde{E}_i; \boldsymbol{Z})=0$ for $*>n+1$ because $\tilde{E}_i$ is a connected $(n+2)$-manifold with non-empty boundary.

To prove the remaining case $*=n+1$, we fix a generator $t: \tilde{E}_i\to\tilde{E}_i$ of the covering transfomations. Take also a finite cyclic intermediate k-sheet covering $\tilde{E}_i\xrightarrow{j'_k}(E_i)_k\to E_i$. We fix a prime p. Let $C_*((E_i)_k)$ and $C_*(\tilde{E}_i)$ denote the complexes of $(E_i)_k$ and $\tilde{E}_i$ respectively with coefficients in $\boldsymbol{Z}_p$, and use the abbreviation $H_*(X)$ instead of $H_*(X; Z_p)$ We have an exact sequence:

$$0 \to C_*(\tilde{E}_i) \xrightarrow{t^k-1} C_*(\tilde{E}_i) \to C_*((E_i)_k) \to 0$$

which induces short exact sequences:

$$0 \to \mathrm{Coker}(t^k-1)_* | H_*(\tilde{E}_i) \to H_*((E_i)_k) \to \mathrm{Ker}(t^k-1)_* | H_{*-1}(\tilde{E}_i) \to 0 .$$

Since $H_{n+2}((E_i)_k)=0$, we get $\mathrm{Ker}(t^k-1)_* | H_{n+1}(\tilde{E}_i)=0$ for any k. On the other hand $H_{n+1}(\tilde{E}_i)$ is a finite demensional vector space over $\boldsymbol{Z}_p$, because E_i is a homology circle [11, Ass. 5]. t_* is of finite order, that is, there is an l such that $(t^l-1)_* | H_{n+1}(\tilde{E}_i)=0$. Since we have proved that $(t^l-1)_* | H_{n+1}(\tilde{E}_i)$ is injective, we see that $H_{n+1}(\tilde{E}_i)=0$ for any fixed prime p. Therefore, $H_{n+1}(\tilde{E}_i; Z)=0$. We have $E_i \simeq S^1$ and hence (S^{n+2}, L_i) is unknotted as in [10].

In order to complete a proof of Proposition 2 it suffices to prove Lemma 2 2.

Proof of Lemma 2.2. The restriction of t on $E(i)$ gives a generator of the covering transformations of $E(i)$. We may denote it also by t. Take a finite cyclic intermediate k-sheet covering $E(i) \xrightarrow{j_k} E(i)_k \to E$ induced from $\tilde{E}_i \xrightarrow{j'_k} (E_i)_k \to E_i$. We fix a prime p and take the chain complexes of $E(i)_k$ and $E(i)$ with coefficients in $\boldsymbol{Z}_p$. We have short exact sequences as before:

$$0 \to \mathrm{Coker}(t^k-1)_* | H_*(E(i)) \xrightarrow{(j_k)_*} H_*(E(i)_k)$$
$$\xrightarrow{\partial_*} \mathrm{Ker}(t^k-1)_* | H_{*-1}(E(i)) \to 0 .$$

Assertion 2.3. *Let $2 \leqq * \leqq n$, and φ be the composition of the natural maps*

$$\mathrm{Ker}(t^k-1)_* | H_*(E(i)) \to H_*(E(i)) \to \mathrm{Coker}(t^k-1)_* | H_*(E(i)) \xrightarrow{(j_k)_*} H_*(E(i)_k)$$

There exists an element $[\gamma_k]$ of $H^1(E(i)_k)$ such that the image $\varphi(\partial_*[\sigma])$ of $\partial_*[\sigma]$ by φ coincides with $[\sigma] \cap [\gamma_k]$ for any element $[\sigma] \in H_{*+1}(E(i)_k)$.

To prove Assertion 2.3 note that the maps $i_*: H_*(E(i)) \to H_*(\tilde{E}_i)$ and $(i_k)_*: H_*(E(i)_k) \to H_*((E_i)_k)$ induced by the inclusions are isomorphisms for $2 \leqq * \leqq n$ because the ambient spaces have only additional $(n+1)$- and $(n+2)$-handles. We define φ' to be the composition of the natural maps

$$\mathrm{Ker}(t^k-1)_* | H_*(\tilde{E}_i) \to H_*(\tilde{E}_i) \to \mathrm{Coker}(t^k-1)_* | H_*(\tilde{E}_i) \xrightarrow{(j'_k)_*} H_*((E_i)_k) .$$

Choose a generator $[\gamma'_k]$ of $H^1((E_i)_k)$ so that $\varphi'(\partial_* \circ (i_k)_*[\sigma]) = (i_k)_*[\sigma] \cap [\gamma'_k]$ according to Assertion 1 ot [10] for $\tilde{E}_i$. We have $(i_k)_* \circ \varphi(\partial_*[\sigma]) = \varphi' \circ i_*(\partial_*[\sigma]) = \varphi' \circ \partial_* \circ (i_k)_*[\sigma] = (i_k)_*[\sigma] \cap [\gamma'_k] = (i_k)_*([\sigma] \cap (i_k)^*[\gamma'_k])$. We obtain Assertion 2.3 by taking $[\gamma_k] = (i_k)^*[\gamma'_k]$.

Since $\mathrm{cat}(E(i)_k)\leqq\mathrm{cat}(E)=1$, the cap product with $[\gamma_k]$ vanishes, that is, $\varphi\circ\partial_*[\sigma]=0$. We have $\mathrm{Ker}(t^k-1)_*\subset\mathrm{Im}(t^k-1)_*$ on $H_*(E(i)_k)$ for any k and $2\leqq*\leqq n$ as in [10]. The following Assertion 2.4 implies Lemma 2.2 and hence Proposition 2.

Assertion 2.4. *Let $2\leqq*\leqq n$. For any prime p there is an $l=l(p)$ such that $(t^l-1)_*|H_*(E(i))=0$.*

To prove Assertion 2.4 we have only to know that $H_*(E(i))$ is a finite dimensional vector space over $\boldsymbol{Z}_p$. As Milnor pointed out in the proof of his Assertion 5 of [11], a finite generated $\boldsymbol{Z}_p[\langle t\rangle]$-module M is finite if $t-1\colon M\to M$ is a surjection. It suffices to notice that there is an exact sequence $H_*(E(i))\xrightarrow{t-1} H_*(E(i))\to 0$ of $\boldsymbol{Z}_p[\langle t\rangle]$-modules for $2\leqq*\leqq n$. q.e.d.

3. Splitting of the link exterior and residual nilpotency of the augmentation ideal

We will show that the link exterior is splittable in our case and that the augmentation ideal of the free group ring is residual nilpotent.

Lemma 3.1. *Let $L\subset S^{n+2}$ be a m-component boundary link with exterior $E=S^{n+2}-\cup\mathrm{Int}\ N(L_i)$, Suppose that $\pi_1(E)$ is a free group of rank m and that $n\geqq3$. Then, E is splittable, that is, for any j $(0\leqq j\leqq m)$ there exist submanifolds A and B of E which satisfy the following properties:*

(1) *$E=A\cup B$ and the intersection $D=A\cap B$ is a 1-connected submanifold of E satisfying $\partial E\cap D=\varnothing$.*

(2) *$E-D$ consists of two connected components $A-D$ and $B-D$,*

(3) *$\pi_1(A)$ is a free group of rank $m-j$ generated by some $m-j$ meridians and $\pi_1(B)$ is a free group of rank j generated by some j meridians.*

Proof. Take two points e_1, e_2 in E and simple paths $p_i\colon[0,1]\to E(0\leqq i\leqq m)$ such that $\mathrm{Im}\,p_i\cap N_i$ consists of one point $p_i(1)$ for each $i\,(1\leqq i\leqq m)$, $\mathrm{Im}\,p_0\cap\mathrm{Im}\,p_i=e_1=p_0(0)=p_i(0)$ if $1\leqq i\leqq m-j$ and $\mathrm{Im}\,p_0\cap\mathrm{Im}\,p_i=e_2=p_0(1)=p_i(0)$ if $m-j+1\leqq i\leqq m$. Let $x_i=p_i(1)$ $(1\leqq i\leqq m)$ be the base points of meridian circle γ_i on $\partial N(L_i)$ and put $Y_0=\cup_{i=0}^m\mathrm{Im}\,p_i$. We may suppose that $p_i(1\leqq i\leqq m)$ are chosen so that $p_i\,\gamma_i\,p_i^{-1}(1\leqq i\leqq m)$ generate $\pi_1(E, p_0(1/2))$.

We define Y be by the union $(\cup\,\mathrm{Im}\,\gamma_i)\cup Y_0$. Let X denote the middle point $p_0(1/2)$ of p_0 in Y. We define $f'\colon\partial E\cup(\cup\,\mathrm{Im}\,p_i)\to Y$ by the condition that $f'|\cup\mathrm{Im}\,p_i$ is the identity map from $\cup\,\mathrm{Im}\,p_k$ to Y_0 and $f'|\partial N_i$ is a natural projection of ∂N_i onto $\mathrm{Im}\,\gamma_i$. Since $f'_*\colon\pi_1(E)\to\pi_1(Y)$ is an isomorphism, f' extends on the 2-skeleton of E. f' extends to $f\colon E\to Y$ by the obstruction theory because Y is 1-complex. Note that $f_*\colon\pi_1(E)\to\pi_1(Y)$ is also an isomorphism and $f^{-1}(X)\cap\partial E=\varnothing$. The rest of the proof can be carried out by a standard argument in the surgery theory as in Cappell [1] using the handle exchanging lemma [1, Lemma 1.3] and we omit it. q.e.d.

Lemma 3.2. *Let F_m be a free group of rank m with generators $t_1, \cdots, t_m$, and I the augmentation ideal generated by $t_1-1, \cdots, t_m-1$ of the group ring $\boldsymbol{Z}_p[F_m]$. Denoting the k-th power of I by I^k, we have $\cap_k I^k=0$.*

This lemma can be proved elementarily and is proved by Fox [4, Cor. 4.4] as an application of the free differential calculus. A characterization of the group whose group ring has a residual nilpotent augementation ideal is given in [8, p. 282, Theorem 2.1] and [13, p. 98, Theorem 2.26].

4. Proof of Theorem 1

Boundary links in S^{n+2} are trivial if $\pi_*(E)=0$ for $*\leqq n-1$ by the unlinking criterion due to Gutiérrez [6]. It suffices to show that $H_*(\tilde{E}; \boldsymbol{Z})=0$ for $2\leqq*\leqq n-1$ for the universal covering $\tilde{E}$ of the link complement $E=S^{n+2}-L$.

We fix a prime p and let $H_*(X)$ be the homology of X with coefficients in $\boldsymbol{Z}_p$. Let $E(1, 2, \cdots, l)$ be a covering of E induced from the universal covering of $S^{n+2}-L_1\cup L_2\cup\cdots\cup L_l$ for an integer l with $1\leqq l\leqq m$. We will prove that $H_*(E(1, 2, \cdots, l))=0$ for $2\leqq*\leqq n-1$ by induction on l. The case that $l=1$ is proved in Lemma 2.2. In fact we showed more strongly $H_*(E(1); \boldsymbol{Z})=0$ for $2\leqq*\leqq n$.

By Lemma 3.1 we can split E into $E=A\cup B$ with $\pi_1(A)=\boldsymbol{Z}$ and $\pi_1(B)=F_{m-1}$ so that the intersection $D=A\cap B$ is 1-connected and two components of $E-D$ are $A-D$ and $B-D$. We may assume that $L_l\subset A-D$ and $L_i(i\neq l)\subset B-D$. Let $F(l)$ be the group of covering transformations of the covering $p_l: E(1,2,\cdots,l)\to E$ defined above which is generated by $t_1, \cdots, t_l$. Let $\tilde{A}$ be the universal covering of A and B' be the coveing of B induced from the covering $p_{l-1}: E(1, 2, \cdots, l-1)\to E$. Then we have:

(1) $p_l^{-1}(D)$ is $F(l)$-equivariantly homeomorphic to $F(l)\times D$.
(2) $p_l^{-1}(A)$ is $F(l)$-equivariantly homeomorphic to $F(l)/\langle t_l\rangle\times\tilde{A}$.
(3) $p_l^{-1}(B)$ is $F(l)$-equivariantly homeomorphic to $F(l)/F(l-1)\times B'$.

Now we may assume that $H_*(E(1, 2, \cdots, l-1))=0$ for $2\leqq*\leqq n-1$ as the inductive hypothesis. We consider the following part of the Mayer-Vietoris exact sequence of $(p_l^{-1}(A), p_l^{-1}(B), E(1, 2, \cdots, l))$,

$$H_*(p_l^{-1}(D))\xrightarrow{(j_a,\ -j_b)}H_*(p_l^{-1}(A))\oplus H_*(p_l^{-1}(B))\xrightarrow{i_a+i_b}H_*(E(1, 2, \cdots, l))$$
$$\xrightarrow{\partial_*}H_{*-1}(p_l^{-1}(D)).$$

Assertion 4.1. *$j=(j_a, -j_b)$ is surjective and hence ∂_* is injective for $2\leqq*\leqq n-1$.*

Proof. First we will take appropriate generators of $\boldsymbol{Z}_p[F(l)]$-module $H_*(p_l^{-1}(A))$. By the above properties (1)-(3) it is natural to denote the components of $p_l^{-1}(A)$ and $p_l^{-1}(B)$ by $\tilde{A}_\alpha$ and B'_β with $\alpha \in F(l)/\langle t_l \rangle$ and $\beta \in F(l)/F(l-1)$ respectively. We have $H_*(p_l^{-1}(A)) = \oplus_\alpha H_*(\tilde{A}_\alpha)$ and $H_*(p_l^{-1}(B)) = \oplus_\beta H_*(B'_\beta)$, Note that $w \cdot \tilde{A}_\alpha = \tilde{A}_{w \cdot \alpha}$ and $w \cdot B'_\beta = B'_{w \cdot \beta}$ for $w \in F(l)$. The intersection $\tilde{A}_\alpha \cap B'_\beta$ coincides with one component of $p_l^{-1}(D)$ which will be denoted by $D_{\alpha,\beta}$. Let $p_a : E(l) \to E$ be a covering induced from the universal covering of $S^{n+2} - L_l$ and $\tilde{p}_a : E(1, \cdots. l) \to E(l)$ be a natural covering. The component $\tilde{p}_a(B'_\beta)$ of $p_a^{-1}(B)$ which will be denoted by $B_{\bar{\beta}}$ depends only on the class $\bar{\beta}$ of $\beta \in F(l)/F(l-1)$ in $\langle t_l \rangle = F(l)/NF(l-1)$ where $NF(l-1)$ is the normal closure of $F(l-1)$. $\tilde{A} \cap B_{\bar{\beta}}$ is one of the components of $p_a^{-1}(D)$. We denote this component by $D_{\bar{\beta}}$. We can remark that $\tilde{A}$ is homeomorphic to $\tilde{A}_\alpha$ for any α, and that $B_{\bar{\beta}}$ is homeomorphic to B for any $\bar{\beta}$. We consider the following commutative diagram for $2 \leqq * \leqq n-1$:

$$
(4.2) \qquad
\begin{array}{ccccc}
H_*(D_{\bar{\beta}}) & \longrightarrow & H_*(p_a^{-1}(D)) & \xrightarrow[\cong]{j'} & H_*(p_a^{-1}(A)) \oplus H_*(p_a^{-1}(B)) \\
 & {\scriptstyle (p_a | D_{\bar{\beta}})_*} \searrow \cong & \downarrow {\scriptstyle (p_a | p_a^{-1}(D))_*} & & \downarrow {\scriptstyle p} \\
 & & H_*(D) & \xrightarrow[\cong]{j''} & H_*(A) \oplus H_*(B)
\end{array}
$$

where $j' = (j'_a, -j'_b)$ and $j'' = (j''_a, -j''_b)$ are isomorphisms in the Mayer-Vietoris exact sequence of $(p_a^{-1}(A), p_a^{-1}(B), E(l))$ and (A, B, E) respectively, and $p = ((p_a | p_a^{-1}(A))_*, (p_a | p_a^{-1}(B))_*)$. Fix $\alpha_0 \in F(l)/\langle t_l \rangle$ and $\beta_0 \in F(l)/F(l-1)$. We consider the following composition of the homomorphisms:

$$
H_*(D_{\alpha_0,\beta_0}) \xrightarrow{(\tilde{p}_a | D_{\alpha_0,\beta_0})_*} H_*(D_{\bar{\beta}_0}) \xrightarrow{(p_a | D_{\bar{\beta}_0})_*} H_*(D) \xrightarrow{j''} H_*(A) \oplus H_*(B) .
$$

Note that $(\tilde{p}_a | D_{\alpha_0,\beta_0})_*$ and $(p_a | D_{\bar{\beta}_0})_*$ are isomorphisms for any $*$ and that j'' is an isomorphism for $2 \leqq * \leqq n-1$. Hereafter, we assume that $2 \leqq * \leqq n-1$ in this paragraph. Let $\{a''_k\}$ be a system of generators of $H_*(A)$. Then we put $a'_k = j'_a \circ (p_a | D_{\bar{\beta}_0})_*^{-1} \circ (j'')^{-1}(a''_k, 0) \in H_*(p_a^{-1}(A))$ and $a_k = j_a \circ (p_l | D_{\alpha_0,\beta_0})_*^{-1} \circ (j'')^{-1}(a''_k, 0) \in H_*(p_l^{-1}(A))$. Note that $p_l | D_{\alpha_0,\beta_0} = (p_a | D_{\bar{\beta}_0}) \circ (\tilde{p}_a | D_{\alpha_0,\beta_0})$. For any element $(x, 0) \in H_*(p_a^{-1}(A)) \oplus H_*(p_a^{-1}(B))$ there is an element $\bar{z}$ such that $j'(\bar{z}) = (x, 0)$ because j' is surjective. The element $\bar{z}$ is decomposed into $\bar{z} = \sum_{\bar{\beta}} \bar{z}_{\bar{\beta}} (\bar{z}_{\bar{\beta}} \in H_*(D_{\bar{\beta}}))$ uniquely. As $H_*(p_a^{-1}(B)) = \oplus_{\bar{\beta}} H_*(B_{\bar{\beta}})$ and $j'_b(\bar{z}_{\bar{\beta}}) \in H_*(B_{\bar{\beta}})$, we obtain that $j'_b(\bar{z}_{\bar{\beta}}) = 0$ for every $\bar{\beta}$. We obtain $j''_b \circ (p_a | D_{\bar{\beta}})_*(\bar{z}_{\bar{\beta}}) = (p_a | p_a^{-1}(B))_* \circ j'_b(\bar{z}_{\bar{\beta}}) = 0$. We take $w' \in \langle t_l \rangle$ such that $w' \cdot \bar{\beta}_0 = \bar{\beta}$ for each $\bar{\beta}$. Since we have $w' \cdot (p_a | D_{\bar{\beta}_0})_*^{-1} \circ (j'')^{-1}(a''_k, 0) = (p_a | D_{\bar{\beta}})_*^{-1} \circ (j'')^{-1}(a''_k, 0)$, the elements $\{w' \cdot (p_a | D_{\bar{\beta}_0})_*^{-1} \circ (j'')^{-1} (a''_k, 0)\}$ generate $(p_a | D_{\bar{\beta}})_*^{-1} \circ (j'')^{-1}(H_*(A))$ as an abelian group. Because $j'_a \circ w' \cdot (p_a | D_{\bar{\beta}_0})_*^{-1} \circ (j'')^{-1}(a''_k, 0) = w' \cdot a'_k$, each element $j'_a(\bar{z}_{\bar{\beta}})$ can be described as a linear combination of $\{w'\cdot a_k'\}$. Hence $\{a_k'\}$ generate $H_*(p_a^{-1}(A))$ as $\boldsymbol{Z}_p[\langle t_l\rangle]$-module. We take $w\in F(l)$ such that $D_{\alpha,\beta}=w\cdot D_{\alpha_0,\beta_0}$. We have $w\cdot a_k=j_a(w\cdot(p_l|D_{\alpha_0,\beta_0})_*^{-1}\circ(j'')^{-1}(a_k'',0))=j_a\circ(p_l|D_{\alpha,\beta})_*^{-1}\circ(j'')^{-1}(a_k'',0)$. Since $\tilde{A}_\alpha$ is homeomorphic to $p_a^{-1}(A)$, the elements $\{w\cdot a_k\}$ generate $H_*(\tilde{A}_\alpha)$ as $\boldsymbol{Z}_p[\langle t_l\rangle]$-modlue. Since $H_*(p_l^{-1}(A))=\oplus_\alpha H_*(\tilde{A}_\alpha)$, the elements $\{a_k\}$ generate $H_*(p_l^{-1}(A))$ as $\boldsymbol{Z}_p[F(l)]$-module. We write down the result obtained above as the following lemma.

Lemma 4.3. *Let $2\leqq *\leqq n-1$ and fix the elements $\alpha_0\in F(l)/\langle t_l\rangle$ and $\beta_0\in F(l)/F(l-1)$. The elements $\{a_k\}$ defined above generate $H_*(p_l^{-1}(A))$ as $\boldsymbol{Z}_p[F(l)]$-module. If put $z_k=(p_l|D_{\alpha_0,\beta_0})_*^{-1}\circ(j'')^{-1}(a_k'',0)\in H_*(p_l^{-1}(D))$, we have $j(z_k)=(a_k,j_b(z_k)),j_b(z_k)\in H_*(B_{\beta_0})$ and $(p_l)_*j_b(z_k)=j_b''(p_l)_*(z_k)=0$.*

Secondly we note that j_b is surjective. In fact, it follows from the inductive hypothesis. Let $\tilde{p}_{l-1}\colon E(1,\cdots,l)\to E(1,\cdots,l-1)$ denote the natural covering. The component $\tilde{p}_{l-1}(\tilde{A}_\alpha)$ of $p_{l-1}^{-1}(A)$ which will be denoted by $A_{\bar{\alpha}}$ depends only on the class $\bar{\alpha}$ of $\alpha\in F(l)/\langle t_l\rangle$ in $F(l-1)=F(l)/N(\langle t_l\rangle)$ where $N(\langle t_l\rangle)$ is the normal closure of $\langle t_l\rangle$. $A_{\bar{\alpha}}\cap B'$ is one of the components of $p_{l-1}^{-1}(D)$. We denote this component by $D_{\bar{\alpha}}$. By the inductive hypothesis on l that $H_*(E(1,2,\cdots,l-1))=0$ for $2\leqq *\leqq n-1$, the natural map:

$$j_{l-1}\colon H_*(p_{l-1}^{-1}(D))\xrightarrow{((j_{l-1})_a,\,-(j_{l-1})_b)}\oplus_{\bar{\alpha}}H_*(A_{\bar{\alpha}})\oplus H_*(p_{l-1}^{-1}(B))$$

in the Mayer-Vietoris exact sequence of $(p_{l-1}^{-1}(A),p_{l-1}^{-1}(B),E(1,\cdots,l-1))$ is surjective. Hence for any $y\in H_*(B_\beta')$ there exists $\bar{x}\in H_*(p_{l-1}^{-1}(D))$ such that $j_{l-1}(\bar{x})=(0,(\tilde{p}_{l-1}|p_l^{-1}(B))_*(y))$. The element $\bar{x}$ is decomposed into $\bar{x}=\sum\bar{x}_{\bar{\alpha}}$ $(\bar{x}_{\bar{\alpha}}\in H_*(D_{\bar{\alpha}}))$ uniquely. We have $(i_{\bar{\alpha}})_*(\bar{x}_{\bar{\alpha}})=0$ in $H_*(A_{\bar{\alpha}})$ where $i_{\bar{\alpha}}\colon D_{\bar{\alpha}}\to A_{\bar{\alpha}}$ is the natural inclusion. Note that $A_{\bar{\alpha}}$ is homeomorphic to A for any α and $p_{l-1}^{-1}(B)$ is homeomorphic to B_β' for any β. For each $\beta\in F(l)/F(l-1)$ we have a unique section $\bar{s}_\beta\colon p_{l-1}^{-1}(B)\to B_\beta'$ of the covering $(\tilde{p}_{l-1}|p_l^{-1}(B))\colon p_l^{-1}(B)\to p_{l-1}^{-1}(B)$. $\bar{s}_\beta|D_{\bar{\alpha}}\colon D_{\bar{\alpha}}\to D_{\alpha,\beta}$ is a section of the covering $(\tilde{p}_{l-1}|(p_l^{-1}(D))\colon p_l^{-1}(D)\to p_{l-1}^{-1}(D)$. Let x be an element of $H_*(p_l^{-1}(D))$ corresponding to $\bar{x}$, that is, $x=(\bar{s}_\beta|p_{l-1}^{-1}(D))_*(\bar{x})$. Then, $j_b(x)=\sum_{\bar{\alpha}}j_b(\bar{s}_\beta|D_{\bar{\alpha}})_*(\bar{x}_{\bar{\alpha}})=\sum_{\bar{\alpha}}(\bar{s}_\beta)_*(j_{l-1})_b(\bar{x}_{\bar{\alpha}})=(\bar{s}_\beta)_*(\tilde{p}_{l-1}|p_l^{-1}(B))_*(y)=y$. We obtain that $j_b\colon H_*(p_l^{-1}(D))\to H_*(p_l^{-1}(B))$ is surjective for $2\leqq *\leqq n-1$.

Finally we will prove that j is surjective. We assume that j is not surjective. If every element $(a_k,0)$ in Lemma 4.3 is contained in $\operatorname{Im} j$, then j becomes surjective because j_b is surjective. Hence there exists an element $(a_{k_0},0)$ which is not in $\operatorname{Im} j$. We may assume $k_0=0$ by reordering the indices. For a $\boldsymbol{Z}_p[F_l]$-module M we define $\omega(l)\colon M^l\to M$ by $\omega(l)(x_1,\cdots,x_l)=\sum_j(t_j-1)x_j$, where M^r is the r times direct sum of M. We denote the homomorphism $\omega(l)$ defined for $M=H_*(E(1,2,\cdots,l))$ or $M=\operatorname{Coker} j$ by $\xi(l)$ or $\eta(l)$ respectively. Due to N. Sato [14] we have a homology long exact sequence:

$$(4.4)\quad \cdots\to H_{*+1}(E)\to H_*(E(1,\cdots,l))^l\xrightarrow{\xi(l)}H_*(E(1,\cdots,l))\to H_*(E)\to\cdots.$$

Since $H_*(E)=0$ for $2\leqq *\leqq n$, we have $\xi(l)\colon H_*(E(1, 2, \cdots, l))^l\to H_*(E(1, 2, \cdots, l))$ is injective for $2\leqq *\leqq n-1$. So is the restriction $\xi(l)\,|\,\mathrm{Im}\,(i_a+i_b)\colon (\mathrm{Im}\,(i_a+i_b))^l\to \mathrm{Im}\,(i_a+i_b)$. Since $\mathrm{Im}\,(i_a+i_b)$ and $\mathrm{Coker}\,j$ are isomorphic as $\boldsymbol{Z}_p[F(l)]$-module, $\eta(l)$ is injective for $2\leqq *\leqq n-1$.

Now it suffices to construct an element z such that $j(z)=((t_l-1)\,a_0, \sum_{i=1}^{l-1}(t_i-1)\,w_i)$ for some w_i by assuming $2\leqq *\leqq n-1$. This would imply $\eta(l)(([0, w_1], \cdots, [0, w_{l-1}], [a_0, 0]))=[j(z)]=0$ in $\mathrm{Coker}\,j$. Because $\eta(l)$ is injective, we have $([0, w_1], \cdots, [0, w_{l-1}], [a_0, 0])=0$. In paticular, $[a_0, 0]=0$ in $\mathrm{Coker}\,j$. On the other hand, by the definition of a_0 we have $(a_0, 0)\notin \mathrm{Im}\,j$ and we would get a contradiction and complete a proof of Assertion 4.1.

Since $(a_0, j_b(z_0))\in \mathrm{Im}\,j$ by Lemma 4.3 and $(a_0, 0)\notin \mathrm{Im}\,j$, we obtain that $j_b(z_0)\neq 0$. By [14] again we have a homology long exact sequence:

$$(4.5)\qquad \cdots\to H_{*+1}(B)\to H_*(B'_\beta)^{l-1}\xrightarrow{\xi_\beta(l-1)} H_*(B'_\beta)\xrightarrow{(p_l|B'_\beta)_*} H_*(B)\to\cdots.$$

where $\xi_\beta(l-1)\colon H_*(B'_\beta)^{l-1}\to H_*(B'_\beta)$ is defined by $\xi_\beta(l-1)\,(x_1, \cdots, x_{l-1})=\sum_{j=1}^{l-1}(t_j-1)\,x_j$ for the $Z_p[F(l-1)]$-module $H_*(B'_\beta)$. Since $j_b(z_0)\in H_*(B'_{\beta_0})$ and $(p_l)_*j_b(z_0)=0$ by Lemma 4.3, there exists $u_i\in H_*(B'_{\beta_0})$ such that $j_b(z_0)=\sum_{i=1}^{l-1}(t_i-1)\,u_i$ by using the exact sequence (4.5) with $\beta=\beta_0$. Note that j_a and j_b are $\boldsymbol{Z}_p[F(l)]$-homomorphisms. We have $j_a(t_l\,z_0)=t_l\,a_0$ and $(p_l)_*\circ j_b(t_l\,z_0)=0$. There are elements $v_i\in H_*(t_l\,B'_{\beta_0})$ such that $j_b(t_l\,z_0)=\sum_{i=1}^{l-1}(t_i-1)\,v_i$ by (4.5) with $\beta=t_l\,\beta_0$. If we set $z=(t_l-1)\,z_0$, we have $j(z)=((t_l-1)\,a_0, \sum_{i=1}^{l-1}(t_i-1)\,(v_i-u_i))$ by the above facts and Lemma 4.3. This z is a desired element and the proof of Assertion 4.1 is completed.

Now we are in a position to prove Theorem 1. For a $\boldsymbol{Z}_p[F_l]$-module M we define $\omega_k\colon M^{l^k}\to M$ by $\omega_1(x_1, \cdots, x_l)=\omega(l)\,(x_1, \cdots, x_l)=\sum_j(t_j-1)\,x_j$ and $\omega_j=\omega_{j-1}\circ(\omega_1^{l^{k-1}})$ inductively, where M^r is r times direct sum of M and ω_1^r is r times direct sum of ω_1. Hereafter, we denote the $\boldsymbol{Z}$-homomorphism ω_k defined above for $M=H_*(E(1, 2, \cdots, l))$ or $M=H_*(p_l^{-1}(D))$ by ξ_k or η_k respectively. Since $H_*(E)=0$ for $2\leqq *\leqq n$, $\xi_1\colon H_*(E(1, 2, \cdots, l))^l\to H_*(E(1, 2, \cdots, l))$ is surjective for $2\leqq *\leqq n$ by (4.4). Recall the definition $\xi_k=\xi_{k-1}\circ\xi_1^{l^{k-1}}$ and we see that ξ_k is surjective for $2\leqq *\leqq n$ and any k by induction on k. Since $\partial_*\colon H_*(E(1, 2, \cdots, l))\to H_{*-1}(p_l^{-1}(D))$ is a $\boldsymbol{Z}_p[F(l)]$-homomorphism, we have $\partial_*\circ\xi_k=\eta_k\circ(\partial_*)^{l^k}$. We obtain that $\partial_*(H_*(E(1, 2, \cdots, l)))\subset \mathrm{Im}\,\eta_k$ for any k and $2\leqq *\leqq n$. Since $H_{*-1}(p_l^{-1}(D))$ is isomorphic to $\boldsymbol{Z}_p[F(l)]\otimes H_{*-1}(D)$ as $\boldsymbol{Z}_p[F(l)]$-module, we can take a system of basis $b_1, \cdots, b_s$ of $H_{*-1}(p_l^{-1}(D))$. We define $(\eta_k)_i$ by the restriction $\eta_k\,|\,\boldsymbol{Z}_p[F(l)]\,\langle b_i\rangle$, and $I_i\langle b_i\rangle$ by $\mathrm{Im}\,(\eta_1)_i$. We see that I_i is the augmentation ideal I of $\boldsymbol{Z}_p[F(l)]$ for each i and $\mathrm{Im}\,(\eta_k)_i=I^k\langle b_i\rangle$. We have $\bigcap_k \mathrm{Im}\,\eta_k=\bigcap_k(\bigoplus_i I^k\langle b_i\rangle)=\bigoplus_i\bigcap_k I^k\langle b_i\rangle$. Since $\bigcap_k I^k\langle b_i\rangle=\{0\}$ by Lemma 3.2, we obtain that $\bigcap_k \mathrm{Im}\,\eta_k=\{0\}$. For any prime p and $2\leqq *\leqq n-1$ we know that ∂_* is injective by Assertionn 4.1 and that $\partial_*(H_*(E(1, 2, \cdots, l)))\subset \mathrm{Im}\,\eta_k$ for any k. We have $H_*(E(1, 2, \cdots, l))=0$ for any prime p and $2\leqq * \leqq n-1$. We obtain $H_*(E(1, 2, \cdots, l); \boldsymbol{Z})=0$ for $2\leqq * \leqq n-1$. Since $E(1, 2, \cdots, l)=\tilde{E}$ for $l=m$, the proof is completed.

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