Tamura, H.
Osaka J. Math.
29 (1992) 135–159

SPECTRAL AND SCATTERING THEORY FOR 3-PARTICLE HAMILTONIAN WITH STARK EFFECT: ASYMPTOTIC COMPLETENESS

Dedicated to Professor Shige Toshi Kuroda on his 60th birthday

HIDEO TAMURA

(Received May 7, 1991)

1. Introduction

In the present work we prove the asymptotic completeness for 3-particle Schrödinger operators with Stark effect. Consider a system of three particles in a uniform electric field $\mathcal{E}\in \boldsymbol{R}^3$. The energy Hamiltonian for such a system takes the form

$$-\sum_{j=1}^{3}(\Delta/2m_j+e_j\langle\mathcal{E}, r_j\rangle)+\sum_{1\leq j<k\leq 3}V_{jk}(r_j-r_k),$$

where m_j, e_j and $r_j\in \boldsymbol{R}^3$, $1\leq j\leq 3$, denote the mass, charge and position vector of the j-th particle, while $-e_j\langle\mathcal{E}, r_j\rangle$, $\langle\ ,\ \rangle$ being the usual scalar product in the Euclidean space, is the energy of interaction with the electric field and the real function V_{jk} is the potential interaction between the j-th and k-th particles.

During the last decade, the spectral and scattering theory of many particles in the absence of electric field has made major progress by many works [2,9, 11,13]. Among these works, Sigal-Soffer [13] first proved the asymptotic completeness of wave operators for N-particle scattering systems with a large class of short-range potentials (see also Graf [4] and Tamura [14]). The spectral and scattering theory of one(two)-particle systems in the presence of electric field has been also studied by many authors [1,5,6,10,15], but there seems to be only a few works on the scattering problem of many-particle systems. Korotyaev [8] has proved the asymptotic completeness of 3-particle systms by making use of the Faddeev equation method. We here prove the asymptotic completeness of wave operators by a different method. The idea of proof, which is, in principle, similar to that in [13], is based on the Mourre commutator method and on the propagation estimate showing that the relative motion of particles is asymptotically concentrated on classical trajectories. In particular, it is not necessarily assumed that a 2-particle subsystem Hamiltonian with zero reduced charge does not have a zero energy resonance. This improves slightly the results obtained by [8], although we have to impose a somewhat restrictive smoothness assumption on the pair potentials V_{jk}.

We proceed to the precise formulation of the obtained results. We begin by removing the center-of-mass motion. For notational brevity, we take the mass m_j as

$$m_j = 1, \qquad 1 \leq j \leq 3,$$

and, for such a 3-particle system, we define the configuration space X in the center-or-mass frame as

$$X = \{r = (r_1, r_2, r_3) \in \boldsymbol{R}^{3\times 3}: \sum_{j=1}^{3} r_j = 0\}.$$

Let

$$E = (e_1\mathcal{E}, e_2\mathcal{E}, e_3\mathcal{E}) \in \boldsymbol{R}^{3\times 3}$$

and define $E_X \in X$ as

$$E_X = \text{projection of } E \text{ onto } X.$$

We also assume that

$$E_X \neq 0 .$$

Then the energy operator H of the 3-particle system under consideration in the center-of-mass frame has the form

$$H = -\Delta/2 - \langle E_X, r\rangle + V \qquad \text{on } L^2(X),$$

where $V(r)$ is defined by the sum of pair potentials

$$V(r) = \sum_{1\leq j<k\leq 3} V_{jk}(r_j - r_k).$$

We here make the following assumption on the pair potential V_{jk}.

$(V)_\rho$ $V_{jk}(y)$, $y \in \boldsymbol{R}^3$, is a C^2-smooth real function and has the following decaying property as $|y| \to \infty$:

(V.0) $V_{jk}(y) = O(|y|^{-\rho})$ for some $\rho > 1/2$,
(V.1) $\partial_y^\alpha V_{jk}(y) = o(1)$, $|\alpha| = 1$,
(V.2) $\partial_y^\alpha V_{jk}(y) = O(1)$, $|\alpha| = 2$.

The constant ρ is used with the meaning ascribed above throughout the entire discussion. Under assumption $(V)_\rho$, the operator H formally defined above admits a unique self-adjoint realization in the space $L^2(X)$. We denote by the same notation H this self-adjoint realization.

Let

$$P_H: L^2(X) \to L^2(\mathrm{X}) = \text{eigenprojection associated with } H.$$

Roughly speaking, the problem of asymptotic completeness which we consider here is to determine completely the asymptotic states as $t \to \pm\infty$ of the solution $\psi(t)=\exp(-itH)\psi_0$ to the Schrödinger equation

$$i\partial_t \psi = H\psi\,, \quad \psi_0 \in \mathrm{Range}(Id-P_H)\,.$$

Such asymptotic states are characterized by the ranges of wave operators.

To describe precisely the obtained results, we have to introduce several basic notations and definitions in many-body scattering theory.

We use the letter a or c to denote a cluster decomposition and denote by $\#(a)$ the number of clusters in a. Throughout the discussion, we consider only a cluster decomposition a with $\#(a)=2$ or 3.

For a given 2-cluster decomposition $a=\{(j, k), l\}$ with $j<k$, we define the subspaces X^a and X_a of X as follows:

$$X^a = \{r \in X : r_j + r_k = 0\}\,,$$
$$X_a = \{r \in X : r_j = r_k\}\,.$$

As is easily seen, the two subspaces X^a and X_a are mutually orthogonal with respect to the scalar porduct $\langle\ ,\ \rangle$ and span X

$$X = X^a \oplus X_a\,,$$

so that we may write $L^2(X)$ as

$$L^2(X) = L^2(X^a) \otimes L^2(X_a)\,.$$

Let π^a: $X \to X^a$ and π_a: $X \to X_a$ be the projections onto X^a and X_a, respectively. For a generic vector $x \in X$, we write the projection of x onto X^a and X_a as $x^a = \pi^a x$ and $x_a = \pi_a x$, respectively, and also we define E^a and E_a as

$$E^a = \pi^a E_X\,, \quad E_a = \pi_a E_X$$

for E_X defined above.

Next we introduce the cluster Hamiltonian H_a with $2 \le \#(a) \le 3$. For a 2-cluster decomposition $a=\{(j, k), l\}$, we often use the notation V_a to denote

$$V_a(r_j - r_k) = V_{jk}(r_j - r_k)\,.$$

According to this notation, we define H_a by

$$H_a = -\frac{1}{2}\Delta - \langle E_X, r\rangle + V_a \qquad \text{on } L^2(X)\,.$$

On the space $L^2(X)=L^2(X^a)\otimes L^2(X_a)$, the operator H_a has the following decomposition:

$$H_a = H^a \otimes Id + Id \otimes T_a\,,$$

where

$$H^a = -\Delta/2 - \langle E^a, x^a \rangle + V_a \qquad \text{on } L^2(X^a),$$
$$T_a = -\Delta/2 - \langle E_a, x_a \rangle \qquad \text{on } L^2(X_a).$$

If $a = \{(1), (2), (3)\}$ is a 3-cluster decomposition, we define the cluster Hamiltonian H_a as the unperturbed Hamiltonian

$$H_0 = -\frac{1}{2}\Delta - \langle E_X, r \rangle .$$

Now assume that $E^c = 0$ for some 2-cluster decomposition c. Then the 2-particle subsystem Hamiltonian H^c associated with c has in general bound states and hence scattering channels corresponding to such bound states may arise even in a 3-particle system in an electric field as in the case of the absence of field. Thus the consideration is divided into the following two cases:

Case (i) $E^a \neq 0$ for any 2-cluster decomposition a,
Case (ii) $E^c = 0$ for some 2-cluster decomposition c.

By the asumption $E_X \neq 0$, one can easily see that there exists at most one 2-cluster decomposition c with $E^c = 0$.

Finally we introduce the wave operators. We define $W_0^{\pm}: L^2(X) \to L^2(X)$ by

$$W_0^{\pm} = s - \lim_{t \to \pm\infty} \exp(itH) \exp(-itH_0) .$$

Let c be as in case (ii) and let $P^c: L^2(X^c) \to L^2(X^c)$ be the eigenprojection associated with H^c. Then we further define $W_c^{\pm}: L^2(X) \to L^2(X)$ by

$$W_c^{\pm} = s - \lim_{t \to \pm\infty} \exp(itH) \exp(-itH_c) P^c \otimes Id .$$

If the wave operators $W_0^{\pm}$ and $W_c^{\pm}$ exist, then these operators can be easily shown to have the following properties: (i) their ranges are closed in $L^2(X)$ and are contained in Range $(Id - P_H)$; (ii) their ranges are mutually orthogonal.

We are now in a position to formulate the main theorem.

Theorem 1.1. (ASYMPTOTIC COMPLETENESS). *Let the notaitons be as above. Then one has the following statements.*

(i) *Consider the case* (i). *Assume* $(\mathrm{V})_\rho$ *with* $\rho > 1/2$. *Then the wave operators* $W_0^{\pm}$ *exist and are asymptotically complete*

$$\text{Range } W_0^{\pm} = \text{Range}(Id - P_H) .$$

(ii) *Consider the case* (ii). *Let the 2-cluster decomposiiion c be as in case* (ii). *Assume* $(\mathrm{V})_\rho$ *with* $\rho > 1$. *Then the wave operators* $W_0^{\pm}$ *and* $W_c^{\pm}$ *exist and are asymptotically complete*

$$\text{Range } W_0^{\pm} \oplus \text{Range } W_c^{\pm} = \text{Range}(Id - P_H) .$$

We conclude this section by making a brief comment on the above theorem.

REMARK 1.2. In statement (ii), the decay condition $V_a(y)=O(|y|^{-\rho})$, $\rho>1$, for $a\neq c$, c being as in case (ii), is used to prove only the existence of the wave operators $W_c^{\pm}$. If we assume that the eigenstate $\varphi(x^c)\in L^2(X^c)$ associated with the zero eigenvalue of H^c has the decaying property

$$(1+|x^c|)^\nu \varphi(x^c)\in L^2(X^c)$$

for some $\nu>1/2$, then statement (ii) can be proved to remain true under the weaker decay assumption $V_a(y)=O(|y|^{-\rho})$, $\rho>1/2$, for $a\neq c$.

REMARK 1.3. As stated above, the asymptotic completeness of 3-particle systems in an electric field has been proved by Korotyaev [8]. Roughly speaking, it is assumed in the work [8] that the 2-particle subsystem Hamiltonian H^c with zero reduced charge, c being again as in case (ii), has no zero energy resonance and that the pair potential V_c has the decaying property $V_c(y)=O(|y|^{-\nu})$ for some $\nu>2$. Thus statement (ii) improves slightly the results obtained by [8], although in the present work, the additional restrictive C^2-smoothness assumption is imposed on the pair potentials V_{jk}.

2. Two-particle Stark Hamiltonian

In this section, we make a brief review on the known facts about the spectral properties of 2-particle Stark Hamiltonian.

We consider the 2-particle unperturbed Stark Hamiltonian

$$T_1 = T_0+V_1 \qquad \text{on } L^2(\boldsymbol{R}_y^3)$$

in the center-of-mass frame, where

$$T_0 = -\Delta/2 \qquad \text{on } L^2(\boldsymbol{R}_y^3)$$

and

$$V_1 = -e_0\langle \mathcal{E}_0, y\rangle\,, \quad \mathcal{E}_0\in \boldsymbol{R}^3\,.$$

We write the coordinates $y=(y_1, y_\perp)\in \boldsymbol{R}^1\times\boldsymbol{R}^2$ and, for notational brevity, we take the electric field $\mathcal{E}_0$, as $\mathcal{E}_0=(1, 0, 0)$, so that $V_1=-e_0 y_1$, $e_0\neq 0$.

We denote by $p=(p_1, p_\perp)\in\boldsymbol{R}^1\times\boldsymbol{R}^2$ the coordinates dual to $y=(y_1, y_\perp)$ and by $\hat{u}(p)$ the Fourier transform of $u(y)$

$$\hat{u}(p) = (2\pi)^{-3/2}\int \exp(-i\langle y, p\rangle)u(y)\, dy\,,$$

where the integration with no domain attached is taken over the whole space. This abbreviation is used throughout. We also denote by $\varphi(D)$, $D=(D_1, D_\perp)=(-i\partial/\partial y_1, -i\partial/\partial y_\perp)$, the pseudodifferential operator with symbol $\varphi(p)$. Similar notations $\varphi(D_1)$ and $\varphi(D_\perp)$ are used for pseudodifferential operators with symbols $\varphi(p_1)$ and $\varphi(p_\perp)$, respectively.

Let $\varphi_0(D_1)$ be defined by

$$\varphi_0(D_1) = \exp(ie_0^{-1} D_1^3/6)\,.$$

Then it follows that

$$\varphi_0(D_1)(D_1^2/2+V_1) = V_1\varphi_0(D_1)$$

and hence we have

$$T_1 = \varphi_0(D_1)^*(|D_\perp|^2/2+V_1)\varphi_0(D_1)\,. \tag{2.1}$$

where $|D_\perp|^2$ denotes the negative Laplacian with respect to the variables $y_\perp \in \boldsymbol{R}^2$. This relation implies the essential self-adjointness of the operator T_1 on the Schwartz space $\mathcal{S}(\boldsymbol{R}_y^3)$ and also we see after a simple calculation (Perry [10]) that

$$\exp(-itT_1) = \exp(-ie_0^2t^3/6)\exp(-itV_1)\exp(-ie_0t^2D_1/2)\exp(-itT_0)\,. \tag{2.2}$$

We now consider the 2-particle perturbed Stark Hamiltonian

$$T_2 = T_1+V_2 \qquad \text{on } L^2(\boldsymbol{R}_y^3)\,,$$

where the potential $V_2(y)$ is assumed to satisfy the assumption $(\mathrm{V})_\rho$ with $\rho > 1/2$. The operator T_2 is also essentially self-adjoint on $\mathcal{S}(\boldsymbol{R}_y^3)$. We denote by the same notation T_2 this self-adjoint realization in $L^2(\boldsymbol{R}_y^3)$. We here summarize the spectral properies of T_2 obtained by the works [1, 5, 10, 15]. These spectral properties have been verified under much weaker assumptions on V_2.

Proposition 2.1. *Assume $V_2(y)$ to satisfy the assumption* $(\mathrm{V})_\rho$ *with* $\rho>1/2$. *Then the operator T_2 has following spectral properties.*

(i) *T_2 has no bound states.*

(ii) *For all $\lambda \in \boldsymbol{R}^1$, the resolvents $(T_2-(\lambda\pm i\kappa))^{-1}$ have the boundary values $(T_2-(\lambda\pm i0))^{-1}$ as $\kappa\to 0$, as an operator from the weighted L^2 space*

$$L_\gamma^2(\boldsymbol{R}_y^3) = \{u(y) \colon \langle y\rangle^\gamma u \in L^2(\boldsymbol{R}_y^3)\}, \quad \langle y\rangle = (1+|y|^2)^{1/2},$$

into $L_{-\gamma}^2(\boldsymbol{R}_y^3)$ for $\gamma>1/4$.

(iii) *The operators $(T_2-(\lambda\pm i0))^{-1}\colon L_\gamma^2(\boldsymbol{R}_y^3)\to L_{-\gamma}^2(\boldsymbol{R}_y^3)$, $\gamma>1/4$, are locally Hölder continuous in $\lambda\in\boldsymbol{R}^1$.*

(iv) *The wave operators*

$$W^\pm(T_1, T_2) = s-\lim_{t\to\pm\infty} \exp(itT_2)\exp(-itT_1)$$

exist and are asymptotically complete

$$\text{Range } W^\pm(T_1, T_2) = L^2(\boldsymbol{R}_y^3)\,.$$

3. Resolvent estimate at high energies

In this section, we keep the same notations as in the previous section and study the resolvent estimate at high energies of the 2-particle Stark Hamiltonian T_2. The obtained result plays a basic role in proving the local commutator estimate (Mourre estimate) of the 3-particle Stark Hamiltonian H under consideration.

The goal here is to prove the following result.

Proposition 3.1. *For any* $\gamma>1/4$,

$$\|\langle y\rangle^{-\gamma}(T_2-(\lambda\pm i0))^{-1}\langle y\rangle^{-\gamma}\| = o(1)\,, \quad |\lambda|\to\infty\,,$$

where $\|\cdot\|$ *denotes the operator norm when considered as an operator from* $L^2(\boldsymbol{R}^3_y)$ *into itself.*

This proposition is obtained as an immediate consequence of the following

Proposition 3.2. *For any* $\gamma>1/4$,

$$\|\langle y\rangle^{-\gamma}(T_1-(\lambda\pm i0))^{-1}\langle y\rangle^{-\gamma}\| = o(1)\,, \quad |\lambda|\to\infty\,.$$

In fact, by assumption $(\mathrm{V})_\rho$ with $\rho>1/2$, the operator

$$V_2(T_1-(\lambda\pm i0))^{-1}\colon L^2_\gamma(\boldsymbol{R}^3_y)\to L^2_\gamma(\boldsymbol{R}^3_y)$$

is well-defined for γ, $1/4<\gamma\le\rho/2$, and also, by Proposition 3.2,

$$Id+V_2(T_1-(\lambda\pm i0))^{-1}\colon L^2_\gamma(\boldsymbol{R}^3_y)\to L^2_\gamma(\boldsymbol{R}^3_y)$$

is invertible for $|\lambda|\gg1$ large enough. The inverse is bounded uniformly in $|\lambda|\gg1$. Hence Proposition 3.1 follows immediately from the resolvent equation.

We now prove Proposition 3.2. The proof is rather long and is divided into several steps.

Proof of Proposition 3.2. For brevity, we take the charge e_0 as $e_0=1$ and consider the $+$ case only.

(i) We start with the spectral representation for T_1 with $e_0=1$. Let $\varphi_0(D_1)$ be again defied by

$$\varphi_0(D_1) = \exp(iD_1^3/6)$$

with $e_0=1$. This is the convolution operator with the kernel

$$\psi_0(s) = 2^{1/3}\,Ai(2^{1/3}\,s)\,, \quad s\in R^1\,,$$

where $Ai(s)$ is the Airy function

$$Ai(s) = (2\pi)^{-1} \int \exp(isq) \exp(iq^3/3)\, dq\,.$$

For later reference, we here note that the Airy function $Ai(s)$ obeys the following estimates:

$$|Ai(s)| \leq C(1+|s|)^{-1/4}, \tag{3.1}$$

$$|(d/ds)\, Ai(s)| \leq C(1+|s|)^{1/4}, \tag{3.2}$$

and hence so does $\psi_0(s)$.

Let $\mathcal{F}=\mathcal{F}_{y\perp\to p\perp}$ be the partial Fourier transformation in the variables $y_\perp \in \boldsymbol{R}^2$. We define the unitary operator $J_0: L^2(\boldsymbol{R}^3_y) \to L^2(\boldsymbol{R}^1_\theta; L^2(\boldsymbol{R}^2_{p\perp}))$ by

$$J_0 u(\theta, p_\perp) = J_0(\theta) u(p_\perp)\,,$$

where

$$J_0(\theta) u(p_\perp) = \int \psi_0(|p_\perp|^2/2-\theta-y_1)(\mathcal{F}u)(y_1, p_\perp)\, dy_1\,. \tag{3.3}$$

By relation (2.1), this unitary operator gives the spectral representation for T_1 in the sense that T_1 is transformed into the multiplication operator by θ in the space $L^2(\boldsymbol{R}^1_\theta; L^2(\boldsymbol{R}^2_{p\perp}))$;

$$J_0 T_1 u(\theta, p_\perp) = \theta \times J_0 u(\theta, p_\perp)\,.$$

(ii) Set

$$\nu_\varepsilon(y_\perp) = (1+|y_\perp|^2)^{-\varepsilon/2}, \quad \varepsilon>0\,,$$

and define the operator $A_k(\theta): L^2(\boldsymbol{R}^2_{y\perp}) \to L^2(\boldsymbol{R}^2_{y\perp})$, $\theta \in \boldsymbol{R}^1$, by

$$A_k(\theta) = (Id+(|D_\perp|^2/2-\theta)^2)^{-k}, \quad k>0\,.$$

Lemma 3.3. *Let the notations be as above. If* $0<k<1/4$ *and* $0<\varepsilon<1/2$, *then*

$$\|A_k(\theta)\nu_\varepsilon\| \leq C_d(1+|\theta|)^{-d}$$

for any d, $0<d<2k\varepsilon$, *as an operator from* $L^2(\boldsymbol{R}^2_{y\perp})$ *into itself.*

Proof. To prove the lemma, it suffices to consider only θ with $\theta \gg 1$ large enough.

Let $E(\mu)=E(\mu; |D_\perp|^2/2)$, $\mu>0$, be the spectral resolution associated with $|D_\perp|^2/2$. We know that for $\gamma>1/2$,

$$\|\nu_\gamma (d/d\mu) E(\mu) \nu_\gamma\| = O(\mu^{-1/2})\,, \quad \mu \to \infty\,, \tag{3.4}$$

as an operator from $L^2(\boldsymbol{R}^2_{y\perp})$ into itself. Let

$$\Gamma = \{\mu \in \boldsymbol{R}^1 : |\mu-\theta| < \theta^{1/2}\}$$

and let Γ^c be its complement in $\boldsymbol{R}^1$. We write

$$\nu_\gamma A_k(\theta)^2\nu_\gamma = \nu_\gamma(A_k(\theta)^2E(\Gamma^c)+A_k(\theta)^2E(\Gamma))\nu_\gamma .$$

The norm of the first operator on the right side is of order $O(\theta^{-2k})$ as $\theta\to\infty$ and also the second one is represented as the integral

$$\int_\Gamma (1+(\mu-\theta)^2)^{-2k}\nu_\gamma(d/d\mu)E(\mu)\nu_\gamma\, d\mu .$$

If $0<k<1/4$, then we see by (3.4) that the norm of this operator is also of order $O(\theta^{-2k})$. Hence

$$\|A_k(\theta)\nu_\gamma\| = O(\theta^{-k}) .$$

Thus, by interpolation, the lemma follows immediately. □

(iii) Set

$$\nu_{\delta\varepsilon}(y) = \langle y_1\rangle^{-\delta}\langle y_\perp\rangle^{-\varepsilon} .$$

Let $\psi(s)$, $s\in \boldsymbol{R}^1$, be a bounded function with

$$|\psi(s)|\le C(1+|s|)^{-k}, \quad k>0 .$$

Define the operator $J_\psi(\theta)$, $\theta\in\boldsymbol{R}^1$, as

$$J_\psi(\theta)u(p_\perp) = \int \psi(|p_\perp|^2/2-\theta-y_1)(\mathcal{F}u)(y_1, p_\perp)\, dy_1 .$$

Lemma 3.4. *Let the notations be as above. Let* $1/4<\delta<1/2$ *and* $0<\varepsilon<1/2$. *If* $1/2-\delta<k\le 1/4$, *then* $J_\psi(\theta)\nu_{\delta\varepsilon}$: $L^2(\boldsymbol{R}^3_y)\to L^2(\boldsymbol{R}^2_{p\perp})$ *is bounded and*

$$\|J_\psi(\theta)\nu_{\delta\varepsilon}\|\le C_d(1+|\theta|)^{-d}$$

for any d, $0<d<(k+\delta-1/2)\varepsilon$.

Proof. We evaluate the value $|J_\psi(\theta)\nu_{\delta\varepsilon}u(p_\perp)|^2$ by use of the Schwarz inequality. By assumption, we have

$$|J_\psi(\theta)\nu_{\delta\varepsilon}u(p_\perp)|^2\le C(1+(|p_\perp|^2/2-\theta)^2)^{-(k+\delta-1/2)}\int |(\mathcal{F}\nu_\varepsilon u)(y_1, p_\perp)|^2\, dy_1 .$$

Hence

$$\int |J_\psi(\theta)\nu_{\delta\varepsilon}u(p_\perp)|^2\, dp_\perp\le C\int\int |A_m(\theta)\nu_\varepsilon u(y_1, y_\perp)|^2\, dy_1\, dy_\perp$$

with $m=(k+\delta-1/2)/2, 0<m<1/4$, $A_m(\theta)$ being as in Lemma 3.3. Thus Lemma 3.3 with $k=m$ completes the proof. □

Let $J_0(\theta)$ be defined by (3.3). If we take δ and ε as $1/4<\delta<1/2$ and $0<\varepsilon<1/2$, then it follows from Lemma 3.4 with $k=1/4$ that

$$\|J_0(\theta)\nu_{\delta\varepsilon}\|\le C_d(1+|\theta|)^{-d} \tag{3.5}$$

for any d, $0<d<(\delta-1/4)\varepsilon$, as an operator from $L^2(\boldsymbol{R}^3_y)$ into $L^2(\boldsymbol{R}^2_{p\perp})$.

(iv) Set

$$\alpha = |p_\perp|^2/2-\theta-y_1, \quad \beta = |p_\perp|^2/2-\sigma-y_1.$$

Then, by (3.2), we have

$$|\psi_0(\alpha)-\psi_0(\beta)| \le C|\theta-\sigma|(1+|\alpha|+|\beta|)^{1/4}.$$

On the other hand, by (3.1) ,we have

$$|\psi_0(\alpha)-\psi_0(\beta)| \le C((1+|\alpha|)^{-1/4}+(1+|\beta|)^{-1/4}).$$

Hence, by interpolation, it follows that for any $\eta>0$ small enough,

$$|\psi_0(\alpha)-\psi_0(\beta)| \le C_\eta|\theta-\sigma|^\eta\{(1+|\alpha|)^{\eta/4}(1+|\beta|)^{-(1-\eta)/4} + (1+|\beta|)^{\eta/4}(1+|\alpha|)^{-(1-\eta)/4}\}.$$

Assume that

$$|\theta-\sigma| \le (1+|\theta|)/2.$$

Then we have

$$|\psi_0(\alpha)-\psi_0(\beta)| \le C_\eta|\theta-\sigma|^\eta(\psi_\eta(\alpha;\theta)+\psi_\eta(\beta;\theta))$$

with

$$\psi_\eta(s;\theta) = (1+|s|)^{-(1-2\eta)/4}+(1+|\theta|)^{\eta/4}(1+|s|)^{-(1-\eta)/4}.$$

Let δ and ε be again as above. Take η so small that

$$0<\eta<2(\delta-1/4)\varepsilon/(1+\varepsilon).$$

Then Lemma 3.4 shows that if $|\theta-\sigma|\le(1+|\theta|)/2$,

$$\|(J_0(\theta)-J_0(\sigma))\nu_{\delta\varepsilon}\| \le C_{\eta d}|\theta-\sigma|^\eta(1+|\theta|)^{-d} \tag{3.6}$$

for any d, $0<d<(\delta-1/4)\varepsilon/2$, as an operator from $L^2(\boldsymbol{R}_y^3)$ into $L^2(\boldsymbol{R}_{p\perp}^2)$.

(v) The proof of the proposition is completed in this step.

Let $\gamma>1/4$ be as in the proposition. We assume without loss of generality that $1/4<\gamma<1$ and decompose γ into $\gamma=\delta+\varepsilon$ with $1/4<\delta<1/2$ and $0<\varepsilon<1/2$. For such a pair (δ, ε), we take η as above and define the operator $I_0(\theta)$: $L^2(\boldsymbol{R}_y^3)\to L^2(\boldsymbol{R}_y^3)$, $\theta\in\boldsymbol{R}^1$, by

$$I_0(\theta) = \nu_{\delta\varepsilon}J_0(\theta)^* J_0(\theta)\nu_{\delta\varepsilon}.$$

By (3.5) and (3.6), this operator has the following properties:

$$\|I_0(\theta)\| \le C_d(1+|\theta|)^{-2d},$$
$$\|I_0(\theta)-I_0(\sigma)\| \le C_{\eta d}|\theta-\sigma|^\eta(1+|\theta|)^{-2d},$$

for any d, $0<d<(\delta-1/4)\varepsilon/2$, σ being as above.

Recall the spectral representation for T_1 in step (i). Then we have

$$\nu_{\delta\varepsilon}(T_1-\lambda-i\kappa)^{-1}\nu_{\delta\varepsilon}=\int\frac{1}{\theta-\lambda-i\kappa}I_0(\theta)\,d\theta\,.$$

Hence the Privalov lemma proves that for $d>0$ as aobve,

$$\|\nu_{\delta\varepsilon}(T_1-\lambda-i\kappa)^{-1}\nu_{\delta\varepsilon}\|=O(|\lambda|^{-2d}),\quad |\lambda|\to\infty\,,$$

uniformly in κ, $0<\kappa\leq 1$. This completes the proof of the proposition. □

For later reference, we heie mention one application of Proposition 3.1.

Let $\lambda\in\boldsymbol{R}^1$ be fixed. We take a non-negative smooth function $f_\delta\in C_0^\infty(\boldsymbol{R}^1)$, $0\leq f_\delta\leq 1$, with a small support around λ

$$\operatorname{supp} f_\delta\subset\{s\in\boldsymbol{R}^1\colon |s-\lambda|<\delta\}$$

for $\delta>0$ small enough.

Corollary 3.5. *Let the notations be as above. Let $V_3(y)$, $y\in\boldsymbol{R}^3_y$, be a real-valued bounded functıon vanıshing at infinity. Then, as an operator from $L^2(\boldsymbol{R}^3_y)$ into ıtself,*

$$\|V_3 f_\delta(\theta+T_2)\|=o(1)\,,\quad \delta\to 0\,,$$

uniformly in $\theta\in\boldsymbol{R}^1$.

Proof. It suffices to prove the corollary for V_3 such that $V_3(y)=O(|y|^{-\gamma})$, $\gamma>1/4$, as $|y|\to\infty$.

Let $E(\mu)=E(\mu;T_2)$, $\mu\in\boldsymbol{R}^1$, be the spectral resolution associated with T_2. Consideı the operator $V_3 f_\delta(\theta+T_2)^2 V_3$. This operatoı is represented as

$$V_3 f_\delta(\theta+T_2)^2 V_3=\int f_\delta(\theta+\mu)^2 V_3(d/d\mu)E(\mu)V_3\,d\mu\,.$$

By Proposition 3.1,

$$\|V_3(d/d\mu)E(\mu)V_3\|\leq C$$

for C independent of μ. This proves that

$$\|V_3 f_\delta(\theta+T_2)\|^2=\|V_3 f_\delta(\theta+T_2)^2 V_3\|=o(1)\,,\quad \delta\to 0\,.$$

Hence the pıoof is complete. □

4. Commutator calculus

In this seation, we develope a commutator calculus, which is also used as a basic tool to prove the local commutator estimate of the 3-particle Stark Hamiltonian H.

We begin by fixing some notations. First we denote by $\mathscr{B}(X)$ the class of bounded operators acting on $L^2(X)$

$$\mathscr{B}(X)=\{B\colon L^2(X)\to L^2(X) \text{ is a bounded operator}\}.$$

Next we introduce the function class $Q_k(X)$, $k\in R^1$, by

$$Q_k(X)=\{f(x)\in C^\infty(X)\colon |\partial_x^\alpha f(x)|\le C_\alpha\langle x\rangle^{k-|\alpha|}\}.$$

For $q_k(x)\in Q_k(X)$, we denote by q_k the miltiplication operator by $q_k(x)$. In later application, we often use q_k as the multiplication operator by $\langle x\rangle^k\in Q_k(X)$.

The next lemma is easy to prove. In fact, foı the proof, we have only to note that

$$q_{-1/2}\nabla_x(H+i)^{-1}\in\mathscr{B}(X). \tag{4.1}$$

Lemma 4.1. *Assume that* $0\le k\le 1/2$. *Then one has:*

(i) $q_{-k}(H+i)^{-1}q_k\in\mathscr{B}(X)$.

(ii) *The commutator* $[(H+i)^{-1}, q_k]q_{1/2-k}\in\mathscr{B}(X)$.

Lemma 4.2. *Assume that* $0\le k\le 1/2$. *Then one has:*

(i) $[\exp(itH), q_k](H+i)^{-1}\in\mathscr{B}(X)$
for all $t\in \boldsymbol{R}^1$ *and the operator norm obeys the bound* $O(|t|)$ *as* $|t|\to\infty$.

(ii) $(H+i)^{-1}[\exp(itH), q_k]q_{1/2-k}(H+i)^{-1}\in\mathscr{B}(X)$
for all $t\in \boldsymbol{R}^1$ *and the operator norm obeys the bound* $O(|t|^2)$ *as* $|t|\to\infty$.

Proof. (i) Let

$$T_0=-\Delta/2 \qquad \text{on } L^2(X).$$

The operator under consideration is written as

$$\int_0^t \exp(i(t-s)H)i[T_0, q_k](H+i)^{-1}\exp(isH)\,ds.$$

Hence, this, together with (4.1), proves (i).

(ii) (ii) is also proved similarly. We can write the operator under consideration as

$$\int_0^t \exp(i(t-s)H)(H+i)^{-1}i[T_0, q_k]q_{1/2-k}(H+i)^{-1}\exp(isH)\,ds$$
$$+\int_0^t \exp(i(t-s)H)(H+i)^{-1}i[T_0, q_k][\exp(isH), q_{1/2-k}](H+i)^{-1}\,ds.$$

The first operator is bounded by (4.1) and the second one is also bounded by (4.1) and (i). This proves (ii). □

Lemma 4.3. *Let* $g\in C_0^\infty(\boldsymbol{R}^1)$. *Assume that* $0\le k\le 1/2$. *Then*

$$q_{-k}g(H)q_k\in\mathscr{B}(X).$$

Proof. Let $\hat{g}(t)$, $t\in\boldsymbol{R}^1$, be the Fourier transformation of g. Then we have

$$q_{-k}g(H)q_k=(2\pi)^{-1/2}\int\hat{g}(t)\{q_0\exp(itH)+q_{-k}[\exp(itH),q_k]\}\,dt.$$

Hence, by Lemma 4.2,

$$q_{-k}g(H)q_k(H+i)^{-1}\in\mathcal{B}(X),$$

which, together with Lemma 4.1, yields

$$q_{-k}g(H)(H+i)^{-1}q_k\in\mathcal{B}(X).$$

This completes the proof. □

Lemma 4.4. *Let* $g\in C_0^\infty(\boldsymbol{R}^1)$. *Assume that* $0\le k\le 1/2$. *Then*

$$[g(H),q_k]q_{1/2-k}\in\mathcal{B}(X).$$

Proof. We write

$$[g(H),q_k]=(2\pi)^{-1/2}\int\hat{g}(t)[\exp(itH),q_k]\,dt.$$

Hence, by Lemma 4.2,

$$(H+i)^{-1}[g(H),q_k]q_{1/2-k}(H+i)^{-1}\in\mathcal{B}(X). \tag{4.2}$$

We assert that

$$(H+i)^{-1}[g(H),q_k](H+i)^{-1}q_{1/2-k}\in\mathcal{B}(X). \tag{4.3}$$

To prove this we write the above operator as

$$(H+i)^{-1}[g(H),q_k]q_{1/2-k}(H+i)^{-1}$$
$$+(H+i)^{-1}(g(H)q_k-q_kg(H))[(H+i)^{-1},q_{1/2-k}].$$

The first operator is bounded by (4.2). Since

$$q_k[(H+i)^{-1}\ q_{1/2-k}]\in\mathcal{B}(X)$$

by Lemma 4.1 the second operator is also bounded by Lemma 4.3. Thus, assertion (4.3) is verified.

We now write

$$g(H)=(H+i)^{-1}f(H)(H+i)^{-1}$$

with $f\in C_0^\infty(R^1)$. Then, after a simple commutator calculation, we see by (4.3) and by Lemmas 4.1 and 4.3 that the operator $[g(H),q_k]q_{1/2-k}$ in question is bounded. This completes the proof. □

Let H_a be the cluster Hamiltonian associated with the 2-cluster decomposition a. We now take $q_{0a}\in\mathcal{Q}_0(X)$ to satisfy that on the support of q_{0a}, the intercluster potential $I_a(x)$ behaves like

$$I_a(x) = H - H_a = V - V_a = O(|x|^{-\rho}), \quad |x| \to \infty . \tag{4.4}$$

Lemma 4.5. *Assume $q_{0a}(x) \in Q_0(X)$ to satisfy the above condition. Let $g \in C_0^\infty(\boldsymbol{R}^1)$. Then*

$$(g(H) - g(H_a)) q_{0a} q_{1/2} \in \mathscr{B}(X) .$$

Proof. First, by Lemma 4.1, we can easily see that

$$((H+i)^{-1} - (H_a+i)^{-1}) q_{0a} q_{1/2} \in \mathscr{B}(X) . \tag{4.5}$$

We write the difference $g(H) - g(H_a)$ as the integral

$$(2\pi)^{-1/2} \int \hat{g}(t) \{ \int_0^t \exp(isH) i I_a \exp(i(t-s)H_a)\, ds \}\, dt .$$

Hence, we have by Lemma 4.2 and (4.4) that

$$(g(H) - g(H_a)) q_{0a} q_{1/2} (H_a + i)^{-1} \in \mathscr{B}(X) ,$$

which implies that

$$(g(H) - g(H_a)) (H_a + i)^{-1} q_{0a} q_{1/2} \in \mathscr{B}(X) . \tag{4.6}$$

We now write $g(H) = f(H)(H+i)^{-1}$ with $f \in C_0^\infty(\boldsymbol{R}^1)$; similarly for $g(H_a)$. Then the lemma follows from (4.5) and (4.6). □

5. Local commutator estimate

The aim of this section is to prove the local commutator estimate for the 3-particle Stark Hamiltonian H. Such an estimate is called the Mourre estimate, which has been played a basic role in the spectral and scattering theory for many-particle systems in the absence of electric fields ([3,9,11,13]).

Let $E_X \neq 0$ be as in section 1. We write E_X as

$$E_X = E_0 \omega , \quad E_0 = |E_X| > 0 ,$$

for $\omega \in S_X$, S_X being the unit sphere in X. We also write $x \in X$ as

$$x = z\omega + z_\perp$$

with $z \in \boldsymbol{R}^1$ and $z_\perp \in \Pi_\omega$, Π_ω being the hyperplane orthogonal to ω. With these notations, the Hamiltonian H under consideration is represented as

$$H = -\Delta/2 - E_0 z + V.$$

We further define the operator A_1 as

$$A_1 = \langle \omega, (1/i) \nabla_x \rangle . \tag{5.1}$$

Proposition 5.1. *Assume* $(\mathrm{V})_\rho$ *with* $\rho>1/2$. *Let the notations be as above. Fix* $\lambda\in\boldsymbol{R}^1$ *arbitrarily. Let* $f\in C_0^\infty(\boldsymbol{R}^1)$ *be a non-negative smooth function supported in a small neighborhood around* λ. *Then, for any* $\delta>0$ *small enough, one can take the support of* f *around* λ *so small that for a compact operator* $K=K_\delta$ *acting on* $L^2(X)$,

$$f(H)i[H, A_1]f(H)\geq(E_0-\delta)f(H)^2+K$$

in the form sense.

REMARK 5.2. As is seen from the proof below, this proposition remains true without assuming (V.2).

Before proving the proposition, we here introduce a smooth non-negative partition of unity, $\{j_a(x)\}$, $0\leq j_a\leq 1$, over X such that: $(j.1)$

$$\sum_a j_a(x)^2 = 1 \qquad \text{on } X,$$

where a ranges over all the cluster decompositions with $2\leq\#(a)\leq 3$. $(j.2)$ $j_a\in Q_0(X)$. $(j.3)$ On the support of j_a,

$$I_a = V-V_a = O|(x|^{-\rho}), \qquad |x|\to\infty .$$

If, in particular, $\#(a)=3$, then $V=O(|x|^{-\rho})$ on the support of j_a.

Proof of Proposition 5.1. Let $\mathscr{B}_0(X)$ stand for the class of compact operators acting on $L^2(X)$. Throughout the proof, we denote by the same notation K compact operators.

(i) By a simple calculation, we have

$$i[H, A_1] = E_0+i[V, A_1] .$$

Let $\{j_a\}$ be the partition of unity introduced above. Then it follows that

$$f(H)i[V, A_1]f(H) = \sum_{\#(a)=2} f(H)j_a i[V_a, A_1]j_a f(H)+K .$$

Thus it suffices to prove that for a 2-cluster decomposition a,

$$f(H)j_a i[V_a, A_1]j_a f(H)\geq -\delta f(H)^2+K . \tag{5.2}$$

(ii) The next lemma follows immediately from Lemmas 4.4 and 4.5.

Lemma 5.3. *Let* $g\in C_0^\infty(\boldsymbol{R}^1)$. *Then one has:*
(i) $[g(H), j_a]\in\mathscr{B}_0(X)$,
(ii) $j_a(g(H)-g(H_a))\in\mathscr{B}_0(X)$.

Recall the notation $\omega^a=\pi^a\omega$. Let $A_1^a=\langle\omega^a, (1/i)\nabla_x\rangle$. Then we have $[V_a, A_1]=[V_a, A_1^a]$. We now write $f(H)=f(H)g(H)$ with $g\in C_0^\infty(\boldsymbol{R}^1)$. We may assume that g is also supported in a small neighborhood around $\lambda\in\boldsymbol{R}^1$. By

Lemma 5.3, we have

$$f(H)j_a i[V_a, A_1]j_a f(H) = f(H)j_a g(H_a)i[V_a, A_1^a]g(H_a)j_a f(H)+K\,.$$

(iii) Suppose case (i). If $E_a=\pi_a E_X\neq 0$, then we make use of the spectral representation for the 2-particle Stark Hamiltonian

$$T_a = -\Delta/2-\langle E_a, x_a\rangle \qquad \text{on } L^2(X_a)$$

to obtain that the operator $g(H_a)i[V_a, A_1^a]g(H_a)$ is represented as the direct integral

$$\int \oplus g(\theta+H^a)i[V_a, A_1^a]g(\theta+H^a)\, d\theta\,. \tag{5.3}$$

We apply Corollary 3.5 to the above direct integral. By assumption (V.1), $[V_a, A_1^a]=o(1)$, $|x^a|\to\infty$, as a function of the x^a variables. Hence we can take the support of g so small that

$$g(H_a)i[V_a, A_1^a]g(H_a)\geq -\delta. \tag{5.4}$$

This proves (5.2) in the case that $E_a\neq 0$. If $E_a=0$, then we make use of the Fourier transformation in the x_a variables to obtain a direct integral representation similar to (5.3). By Corollary 3.5 again, we can prove (5.4). Thus (5.2) is verified if case (i) is assumed.

(iv) Next we deal with case (ii). Let the 2-cluster decomposition c be as in case (ii). Then $\omega^c=0$ and hence $[V_c, A_1]=[V_c, A_1^c]=0$. This proves (5.2) for the 2-cluster decomposition c. If $a\neq c$, then the same argument as in step (iii) applies and (5.2) can be also verified in this case. At any case, we can prove (5.2) and the proof is completed. □

6. Spectral properties of 3-particle Stark Hamiltonian

We can prove the folloiwng spectral properties of H.

Proposition 6.1. *Assume* $(\mathrm{V})_\rho$ *with* $\rho>1/2$. *Then one has:*

(i) *The set* $\sigma_p(H)$ *of point spectrum of* H *is discrete with possible accumulating points* $\pm\infty$.

(ii) *For* $\lambda\in \boldsymbol{R}^1\backslash\sigma_p(H)$,

$$\sup_{0<\kappa\leq 1} \|\langle x\rangle^{-\gamma}(H-(\lambda\pm i\kappa)^{-1}\langle x\rangle^{-\gamma}\|\leq C$$

for $\gamma>1/4$, *as an operator from* $L^2(X)$ *into itself, where the constant* C *can be taken locally uniformly in* $\lambda\in\boldsymbol{R}^1\backslash\sigma_p(H)$.

The above proposition can be obtained as a consequnece of Proposition 5.1 in the same way as in [9, 11]. In particular, statement (ii) follows, if we take account fo the facts that by assumption (V.2), the double commutator

$[[V, A_1], A_1]$ is bounded on $L^2(X)$ and that $q_{-1/2}A_1(H+i)^{-1}$ is also bounded on $L^2(X)$. The assumption (V.2) is ued for the first time in proving Proposition 6.1.

REMARK 6.2. The same statements as in Proposition 6.1 remain true for the cluster Hamiltonian H_a. In this case, $\sigma_p(H_a)=\emptyset$.

REMARK 6.3. We can prove the non-existence of bound states, $\sigma_p(H)=\emptyset$, under assumption $(V)_\rho$ with $\rho>3/4$. The detailed proof of this result will be discussed elsewhere.

7. Propagation estimate

Throughout this section, $(V)_\rho$ with $\rho>1/2$ is assumed to be satisfied and also we assume, without loss of generality, that $1/2<\rho\leq 1$. The next result can be derived from Proposition 6.1 as a consequence of the smoothness theorem due to Kato [7].

Proposition 7.1. *Assume'* $(V)_\rho$ *with* $\rho>1/2$. *Then one has:*

(i) *Let* $\psi\in \mathrm{Range}(Id-P_H)$, P_H *being the eigenprojection of* H. *Then* exp $(-itH)\psi$ *converges weakly to zero as* $t\to\pm\infty$.

(ii) *Let* Λ *be an open bounded interval with* $\bar{\Lambda}\cap\sigma_p(H)=\emptyset$, $\bar{\Lambda}$ *being the closure of* Λ. *Then the multiplication operator by* $\langle x\rangle^{-\gamma}$, $\gamma>1/4$, *is H-smooth on* Λ;

$$\int \|\langle x\rangle^{-\gamma}\exp(-itH)E(\Lambda)\psi\|^2_{L^2(X)}\,dt\leq C_\Lambda\|\psi\|^2_{L^2(X)},$$

where $E(\Lambda)=E(\Lambda\colon H)$ *denotes the spectral resolution onto* Λ *of* H.

The aim here is to prove the (non-) propogation estimate for the propagator $\exp(-itH)$, which shows that the relative motion of particles is asymptotically concentrated on classical trajectories. As is easily seen, charged classical particles go to infinity along the direction of electric field. The result is formulated as follows.

Proposition 7.2. *Assume* $(V)_\rho$ *with* $\rho>1/2$. *Let* $\omega\in S_X$ *be again the direction of electric field and let* Λ *be as in Proposition* 7.1. *Suppose that a non-negative function* $q_0\in Q_0(X)$ *vanishes in a small coninal neighborhood of* ω; $|x|\omega\notin \mathrm{supp}\, q_0$ *for* $|x|\geq 1$. *Then the multiplication operator by* $q_0\langle x\rangle^{-1/4}$ *is H-smooth on* Λ.

Let A_1 be defined by (5.1). We define A_2 as

$$A_2=\langle x\rangle^{-1/4}A_1\langle x\rangle^{-1/4}.$$

Then we have the following

Proposition 7.3. *Assume* $(\mathrm{V})_\rho$ *with* $\rho>1/2$. *Write again* $x\in X$ *as* $x=z\omega+z_\perp$ *with* $z_\perp\in\Pi_\omega$ *and* E_X *as* $E_X=E_0\omega$ *with* $E_0=|E_X|>0$. *Fix* $\lambda\in\boldsymbol{R}^1\backslash\sigma_p(H)$ *arbitrarily and take a non-negative function* $f\in C_0^\infty(\boldsymbol{R}^1)$ *to be supported in a small neighborhood around* λ; $\operatorname{supp} f\cap\sigma_p(H)=\emptyset$. *Then one can take the support of* f *so small that for some bounded operator* $B\in\mathscr{B}(X)$,

$$f(H)i[H, A_2]f(H)$$
$$\geq\frac{E_0}{2}f(H)\langle x\rangle^{-1/4}\tilde{q}_0\langle x\rangle^{-1/4}f(H)+f(H)\langle x\rangle^{-\rho/2}B\langle x\rangle^{-\rho/2}f(H)$$

in the form sense, where

$$\tilde{q}_0(x)=((1+|z|^2+|z_\perp|^2)^{1/2}-z)/\langle x\rangle\,.$$

The propagation estimate can be obtained as an immediate consequence of Propositions 7.1 and 7.3. In fact, since $A_2(H+i)^{-1}$: $L^2(X)\to L^2(X)$ is bounded, Proposition 7.2 follows from the relation

$$\frac{d}{dt}\|A_2\exp(-itH)f(H)\psi\|_{L^2(X)}^2$$
$$=\langle f(H)i[H, A_2]f(H)\exp(-itH)\psi, \exp(-itH)\psi\rangle_{L^2(X)}\,.$$

Proof of Proposition 7.3. The proof is divided into several steps. Throughout the proof, we denote by B bounded operators on $L^2(X)$ and by Q_k the multiplication operator by $\langle x\rangle^k$. We also again denote by q_k the multiplication operator by $q_k(x)\in Q_k(X)$.

(i) We start by calculating the commutator

$$i[H, A_2]=i[-\Delta/2, A_2]+i[-E_0z, A_2]+i[V, A_2]\,.$$

By a simple calculation, the second operator on the right side is equal to

$$i[-E_0z, A_2]=Q_{-1/4}E_0Q_{-1/4}\,. \tag{7.1}$$

(ii) Since

$$[\Delta, Q_{-1/4}]=-(i/2)Q_{-5/4}D_0+q_{-9/4}$$

with

$$D_0=\langle x/\langle x\rangle, (1/i)\nabla_x\rangle\,,$$

the first operator takes the form

$$i[-\Delta/2, A_2]=-\frac{1}{4}Q_{-3/4}(A_1D_0+D_0^*A_1)Q_{-3/4}$$
$$+q_{-5/4}(A_1+D_0+D_0^*)q_{-5/4}+q_{-14/4}\,.$$

The next lemma is easy to prove.

Lemma 7.4.

$$A_1D_0+D_0^*A_1\leq -2\Delta$$

in the form sense.

If we write

$$\Delta/2=-H-E_0z+V,$$

then the lemma above shows that

$$i[-\Delta/2, A_2]\geq -Q_{-3/4}HQ_{-3/4}-Q_{-3/4}E_0zQ_{-3/4}$$
$$+Q_{-3/4}VQ_{-3/4}+q_{-5/4}(A_1+D_0+D_0^*)q_{-5/4}+q_{-14/4}\,.$$

We analyze the first and fourth operators on the right side. If $0\leq k\leq 1/2$, then it follows from (4.1) and Lemma 4.1 that

$$q_{-k-1/2}\nabla_x(H+i)^{-1}q_k\in\mathcal{B}(X)\,. \tag{7.2}$$

Write $f(H)=f(H)g(H)(H+i)^{-1}$ with $g\in C_0^\infty(\boldsymbol{R}^1)$ and calculate

$$HQ_{-3/4}(H+i)^{-1}=Q_{-3/4}-iQ_{-3/4}(H+i)^{-1}+[-\Delta/2, Q_{-3/4}](H+i)^{-1}.$$

Then, by (7.2), we make use of the commutator calculus developed in section 4 to obtain that

$$Q_{1/2}g(H)(H+i)^{-1}Q_{-3/4}HQ_{-3/4}(H+i)^{-1}g(H)Q_{1/2}\in\mathcal{B}(X)\,.$$

Similarly we obtain that

$$Q_{1/2}g(H)(H+i)^{-1}q_{-5/4}(A_1+D_0+D_0^*)q_{-5/4}(H+i)^{-1}g(H)Q_{1/2}\in\mathcal{B}(X)\,.$$

Thus we have

$$\begin{aligned} &f(H)i[-\Delta/2, A_2]f(H)\\ &\quad\geq -f(H)Q_{-3/4}E_0zQ_{-3/4}f(H)+f(H)Q_{-1/2}BQ_{-1/2}f(H)\,. \end{aligned}\tag{7.3}$$

(iii) Let $\tilde{q}_0$ be as in the proposition. Recall the relation (7.1). We calculate the difference $E_0-Q_{-1/2}E_0zQ_{-1/2}$ as

$$E_0-Q_{-1/2}E_0zQ_{-1/2}=E_0\tilde{q}_0\,.$$

Thus, by (7.1) and (7.3), we have

$$\begin{aligned} &f(H)(i[-\Delta/2, A_2]+i[-E_0z, A_2])f(H)\\ &\quad\geq E_0f(H)Q_{-1/4}\tilde{q}_0Q_{-1/4}f(H)+f(H)Q_{-1/2}BQ_{-1/2}f(H)\,. \end{aligned}\tag{7.4}$$

(iv) We analyze the commutator $i[V, A_2]$. Let $\{j_a\}$ be the partition of unity introduced in section 5. We can construct $\{j_a\}$ to have the additional property that for all cluster decompositions a with $2\leq\#(a)\leq 3$, (7.5) $\nabla_x j_a$ vanishes in a conical neighborhood of ω.

By the definition of A_2, it follows that

$$i[V, A_2] = \sum_a Q_{-1/4} j_a i[V, A_1] j_a Q_{-1/4}.$$

We again write $f(H) = f(H)g(H)(H+i)^{-1}$. If $\#(a)=3$, then we again make use of (7.2) and of the commutator calculus to obtain that

$$Q_{\rho/2} g(H)(H+i)^{-1} Q_{-1/4} j_a [V, A_1] j_a Q_{-1/4} (H+i)^{-1} g(H) Q_{\rho/2} \in \mathcal{B}(X)$$

and also, if $\#(a)=2$, then

$$Q_{\rho/2} g(H)(H+i)^{-1} Q_{-1/4} j_a [I_a, A_1] j_a Q_{-1/4} (H+i)^{-1} g(H) Q_{\rho/2} \in \mathcal{B}(X),$$

where $I_a = V - V_a$ is the intercluster potential associated with a. Hence we have

$$f(H) i[V, A_2] f(H) = \sum_{\#(a)=2} f(H) Q_{-1/4} j_a i[V_a, A_1^a] j_a Q_{-1/4} f(H) + f(H) Q_{-\rho/2} B Q_{-\rho/2} f(H).$$

(v) We first deal with case (i). Write $f(H) = f(H)\tilde{g}(H)$ with $\tilde{g} \in C_0^\infty(\boldsymbol{R}^1)$. The function $\tilde{g}$ is also assumed to be supported in a small neigborhood around λ. By interpolation, it follows from Lemmas 4.3, 4.4 and 4.5 that

$$Q_{1/4}(\tilde{g}(H) - \tilde{g}(H_a)) j_a Q_{1/4} \in \mathcal{B}(X),\ Q_{1/4}[\tilde{g}(H_a), j_a] Q_{1/4} \in \mathcal{B}(X).$$

Hence we have

$$\tilde{g}(H) Q_{-1/4} j_a i[V_a, A_1^a] j_a Q_{-1/4} \tilde{g}(H) = Q_{-1/4} j_a \tilde{g}(H_a) i[V_a, A_1^a] \tilde{g}(H_a) j_a Q_{-1/4} + Q_{-1/2} B Q_{-1/2}.$$

We now apply the same argument as used in the proof of Proposition 5.1 to the first operator on the right side. We can take the support of $\tilde{g}$ so small that

$$f(H) Q_{-1/4} j_a i[V_a, A_1^a] j_a Q_{-1/4} f(H) \geq -\delta f(H) Q_{-1/4} j_a^2 Q_{-1/4} f(H) + f(H) Q_{-1/2} B Q_{-1/2} f(H)$$

for δ, $0<\delta \ll 1$, small enough. By property (7.5), we may assume that j_a with $\#(a)=2$ vanishes in a small conical neighborhood of ω. Hence, by (7.4), the proposition is verified in case (i).

Next we consider the case (ii). Let c be as in case (ii). Then $[V_c, A_1^c]=0$ and hence the same arguments as above prove the proposition in this case also. Thus the proof is now completed. □

8. Asymptotic clustering

Throughout this section, we again assume $(V)_\rho$ with $\rho > 1/2$ to be satisfied.

Let Λ be an open bounded interval avoiding $\sigma_p(H)$. We take g and $g_0 \in C_0^\infty(\Lambda)$ to satisfy the relation $g_0 g = g$. Let $\{j_a\}$ be again the partition of unity with the additional property (7.5). Set $\tilde{j}_a(x) = j_a(x)^2$. Then we have

$$\exp(itH)g(H)\psi$$
$$= \sum_a \exp(-itH_a)\{g_0(H_a)\exp(itH_a)\tilde{j}_a \exp(-itH)g(H)\psi\} + o(1)$$

as $t \to \pm\infty$, where $o(1)$ denotes a term converging strongly to zero in $L^2(X)$ as $t \to \pm\infty$. The relation above follows readily from the facts that $\exp(-itH)$ $g(H)\psi$ converges weakly to zero as $t \to \pm\infty$ and that $(g_0(H) - g_0(H_a))\tilde{j}_a$ and $[g_0(H_a), \tilde{j}_a]$ are both compact operators on $L^2(X)$. We rewrite the term with brackets in the integral form

$$g_0(H_a)\tilde{j}_a g(H)\psi + \int_0^t g(H_a)\exp(isH_a)i\{H_a\tilde{j}_a - \tilde{j}_a H\}\exp(-itH)g(H)\psi\, ds\,.$$

The operator $H_a\tilde{j}_a - \tilde{j}_a H$ in the above integral is expressed as

$$H_a\tilde{j}_a - \tilde{j}_a H = [-\Delta/2, \tilde{j}_a] - \tilde{j}_a I_a$$

Since the first operator on the right side is represented in the form

$$[-\Delta/2, \tilde{j}_a] = (H_a + i)\{q_0 Q_{-1/4} B Q_{-1/4} q_0 + Q_{-1/2} B Q_{-1/2}\}(H+i)$$

for some $q_0 \in Q_0(X)$ having the property as in Proposition 7.2 and since the second operator satisfies $I_a = O(|x|^{-\rho})$ as $|x| \to \infty$ on the support of $\tilde{j}_a$, it follows from Propositions 7.1 and 7.2 that the above integral converges strongly as $t \to \pm\infty$. Thus we can prove that there exist $\psi_a^\pm \in L^2(X)$ such that

$$\text{(8.1)} \qquad \exp(-itH)g(H)\psi = \sum_a \exp(-itH_a)\psi_a^\pm + o(1)\,, \quad t \to \pm\infty\,.$$

When the relation as above holds, the Hamiltonian H under consideration is said to be asymptotically clustering.

9. Existence of wave operators

In this seation we prove the existence of wave operators.

Proposition 9.1. *Consider the case* (i). *Assume that*

$$|V_a(y)| \le C(1+|y|)^{-\rho}, \quad \rho > 1/2\,.$$

for all 2-clusetr decompositions a. Then the wave operators $W_0^\pm$ *exist*

$$W_0^\pm = s - \lim_{t \to \pm\infty} \exp(itH)\exp(-itH_0)\,.$$

Proof. We consider the $+$ case only. To prove the existence of W_0^+, it suffices to show that as a function of t,

$$\|V\exp(-itH_0)\varphi\|_{L^2(X)}\in L^1(1,\infty) \tag{9.1}$$

for φ in some dense set of $L^2(X)$. As such a dense set, we take

$$\mathcal{D}=\{\varphi\in\mathcal{S}(X)\colon \hat{\varphi}\in C_0^\infty(X')\},$$

where X' stands for the space dual to X. Fix one of 2-cluster decompositions and denote it by $a=\{(j,k),l\}$. As coordinates over X, we take $x=(x^a,x_a)\in X^a\oplus X_a$ associated with a and write the pair potential V_a as $V_a(r_j-r_k)=U(x^a)$. To prove (9.1), we consider the integral

$$J_a(t)=\iint |U(x^a)\exp(-itH_0)\varphi(x^a,x_a)|^2\,dx^a\,dx_a$$

for $\varphi\in\mathcal{D}$. By relation (2.2), it follows that

$$|\exp(-itH_0)\varphi(x^a,x_a)| = |\exp(-itT_0)\varphi(x^a-E^at^2/2,\,x_a-E_at^2/2)|$$

with $T_0=-\Delta/2$. Here it should be noted that $E^a\neq 0$ by assumption. By a change of variables, we have

$$J_a(t)=\iint |U(x^a+E^at^2/2)\exp(-itT_0)\varphi(x^a,x_a)|^2\,dx^a\,dx_a\,.$$

Set

$$\varphi_t(x)=\exp(-itT_0)\varphi(x^a,x_a)\,,\quad t>1\,.$$

Then

$$|\varphi_t(x)|\le Ct^{-6/2} \tag{9.2}$$

and also, if

$$\operatorname{supp}\hat{\varphi}\subset\{p\in X'\colon |p|<R\}$$

for some R, it follows that

$$|\varphi_t(x)|\le C_N(1+t+|x|)^{-N},\quad N\gg 1\,, \tag{9.3}$$

for $|x|>2Rt$. We decompose the integral $J_a(t)$ into two parts

$$J_a(t)=\iint_{|x|<2Rt}\dots dx^a\,dx_a+\iint_{|x|>2Rt}\dots dx^a\,dx_a\,.$$

By (9.3), the second integral on the right side is of order $O(t^{-N})$, $N\gg 1$, as $t\to\infty$. Since $E^a\neq 0$ and since U obeys the bound $U(x^a)=O(|x^a|^{-\rho})$ as $|x^a|\to\infty$, we have by (9.2) that the first integral is of order $O(t^{-4\rho})$. This proves (9.1) and completes the proof. □

Proposition 9.2. *Consider the case* (ii). *Let c be as in case* (ii). *Assume that for $a\neq c$,*

$$|V_a(y)|\le C(1+|y|)^{-\rho},\quad \rho>1/2\,,$$

and that

$$|V_c(y)|\leq C(1+|y|)^{-\rho}, \quad \rho>1\,.$$

Then the wave operators $W_0^{\pm}$ *exist.*

Proof. Let $\mathscr{D}$ be as in the proof of Proposition 9.1. If $a\neq c$, we have already shown that

$$\|V_a \exp(-itH_0)\varphi\|_{L^2(X)}\in L^1(1,\infty)$$

for $\varphi\in\mathscr{D}$. If V_c has the decaying property as in the proposition, we can pvove this also for V_c in the same way as above. This completes the proof. □

Proposition 9.3. *Consider the case* (ii). *Let c be as in case* (ii). *Assume that*

$$|V_a(y)|\leq C(1+|y|)^{-\rho}, \quad \rho>1,$$

for all 2*-cluster decompositions a. Then the wave operators* $W_c^{\pm}$ *exist*

$$W_c^{\pm}=s-\lim_{t\to\pm\infty}\exp(itH)\exp(-itH_c)P^c\otimes Id\,.$$

We omit the proof. The proof is based on the L^p-L^q estimate for the free propagator $\exp(it\Delta/2)$ and relation (2.2) enables us to prove the proposition in the same way as in the case of absence of electric fields (Reed-Simon [12]). Similarly we can also prove the existence of the wave operators $W_c^{\pm}$ under the assumptions stated in Remark 1.2.

10. Asymptotic completeness

In this seation we prove the asymptotic completeness of wave operators and complete the proof of the main theorem.

For brevity, we deal with case (ii) only. A similar argument applies to case (i) also. The relation which we have to prove is

$$\text{Range } W_0^{\pm}\oplus\text{Range } W_c^{\pm}=\text{Range }(Id-P_H)\,.$$

We further prove this for the $+$ case only. The inclusion relation $\subset$ is obvious by the intertwining property of wave operators. Let

$$\mathscr{D}=\{g(H)\psi\in L^2(X)\colon g\in C_0^\infty(R^1),\ \operatorname{supp} g\cap\sigma_p(H)=\emptyset\}\,.$$

Then the set $\mathscr{D}$ is dense in Range $(Id-P_H)$. Thus we have only to show that

$$\mathscr{D}\subset\text{Range } W_0^{+}\oplus\text{Range } W_c^{+}, \tag{10.1}$$

because Range W_0^+ and Range W_0^+ are both closed. Let $g(H)\psi\in\mathscr{D}$. Then, by the asymptotic clustering property (8.1), there exist $\psi_a\in L^2(X)$ such that

$$\exp(-itH)g(H)\psi = \sum_a \exp(-itH_a)\psi_a + o(1)\,, \quad t\to\infty\,.$$

We approximate ψ_a with $\#(a)=2$ as

$$\psi_a \sim \sum_{\text{finite}} \theta^a \otimes \theta_a$$

with $\theta^a \in L^2(X^a)$ and $\theta_a \in L^2(X_a)$. If $a \neq c$, then it follows from Proposition 2.1 that

$$\exp(-itH_a)\theta^a \otimes \theta_a = \exp(-itH_0)\varphi^a \otimes \theta_a + o(1)\,, \quad t\to\infty\,,$$

for some $\varphi^a \in L^2(X^a)$. If $a=c$, then

$$\begin{aligned}&\exp(-itH_c)\theta^c \otimes \theta_c \\ &\qquad = \exp(-itH_c)P^c\theta^c \otimes \theta_c + \exp(-itH_0)\varphi^c \otimes \theta_c + o(1)\,, \quad t\to\infty\,,\end{aligned}$$

for some $\varphi^c \in L^2(X^c)$. This relation follows from the asymptotic completeness of 2-particle Schrödinger operators $-\Delta+V$ with short-range potentials V. Thus, for any $\varepsilon>0$ small enough, there exist $\psi_{0\varepsilon}$, and $\psi_{c\varepsilon} \in L^2(X)$ such that

$$\begin{aligned}&\limsup_{t\to\infty} \|\exp(-itH)g(H)\psi \\ &\qquad -\exp(-itH_0)\psi_{0\varepsilon} - \exp(-itH_c)(P^c \otimes Id)\psi_{c\varepsilon}\|_{L^2(X)} < \varepsilon\end{aligned}$$

and hence, by Propositions 9.2 and 9.3,

$$\|g(H)\psi - W_0^+\psi_{0\varepsilon} - W_c^+\psi_{c\varepsilon}\|_{L^2(X)} < \varepsilon\,.$$

This proves (10.1) and completes the proof of the main theorem.

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Department of Mathematics
Ibaraki University
Mito Ibaraki 310, Japan