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REMARKS ON SYMMETRIZATION OF 2×2 SYSTEMS AND THE CHARACTERISTIC MANIFOLDS

ENRICO BERNARDI* AND TATSUO NISHITANI**

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1. Introduction

Let L be a first order differential operator on $C^\infty(\Omega, \boldsymbol{C}^N)$ where Ω is an open set in $\boldsymbol{R}^{n+1}$ with coordinates $x=(x_0, x_1, \cdots, x_n)=(x_0, x')$. We say that L is strongly hyperbolic at $\hat{x}\in\Omega$ with respect to x_0 if the Cauchy problem for $L+Q$ is C^∞ well posed near $\hat{x}$ for every $Q\in C^\infty(\Omega, M_N(\boldsymbol{C}))$ with respect to x_0, that is there are a neighborhood $\omega\subset\Omega$ of $\hat{x}$ and a positive number ε such that $L+Q$ is an isomorphism on $\{v\in C^\infty(\omega, \boldsymbol{C}^N); v=0 \text{ in } x_0<\hat{x}_0+\tau\}$ for every $|\tau|<\varepsilon$ (for more details, see [2], [3]). Choosing a local coordinates $(x, \xi)=(x_0, x', \xi_0, \xi')$ in the cotangent bundle $T^*\Omega$ and a basis for $\boldsymbol{C}^N$ let

$$L(x, \xi) = L_1(x, \xi)+L_0(x)$$

be the complete symbol of L, L_1 being the principal symbol. Let $h(x, \xi)$ denote the determinant of $L_1(x, \xi)$ which is in $C^\infty(T^*\Omega)$.

If the Cauchy problem for $L+Q$ is C^∞ well posed near $\hat{x}$ with respect to x_0, it follows from the Lax-Mizohata theorem that $h(x, \xi+\tau dx_0)=0$ admits only real zeros τ for every $\xi\in T^*_x\Omega\backslash 0$, x close to $\hat{x}$. Therefore we are always assuming h to be hyperbolic in this sense.

If L is strongly hyperbolic at $\hat{x}\in\Omega$ with respect to x_0 then one can find a neighborhood U of $\hat{x}$ such that either h is effectively hyperbolic or the rank of L_1 is at most $N-2$ in every multiple characteristic on $T^*_x\Omega$, $x\in U$ (see [2], [3]). Since the situation for effectively hyperbolic determinants has already been elucidated (cf. [1]) it is natural to see what happens when the rank of L_1 falls to $N-2$ or less at a multiple characteristic $\rho\in T^*_{\hat{x}}\Omega\backslash 0$. In particular, this condition turns out to be $L_1(\rho)=O$ (2×2 zero matrix) when $N=2$. The hyperbolicity and h being the determinant of L_1 with $L_1(\rho)=O$ give a bound for the rank of the Hessian of h at ρ. Indeed, denoting by $\operatorname{Hess} h(\rho)$ the Hessian of h at ρ, we have

*Department of Mathematics, University of Bologna
**Department of Mathematics, College of General Education, Osaka University

Lemma 1.1. (Lemma 4.1 in [2]) *Let $N=2$ and $L_1(\rho)=O$. Then* rank Hess $h(\rho)\leqq 4$. *In particular, if L_1 is real then* rank Hess $h(\rho)\leqq 3$.

We say that Hess $h(\rho)$ has maximal rank if rank Hess $h(\rho)=4$ (resp. if rank Hess $h(\rho)=3$ when L_1 is real). Our partial converse result is then the following.

Theorem 1.1. *Let $N=2$. Assume that $L_1(\rho)=O$ and* Hess $h(\rho)$ *has maximal rank at every double characteristic $\rho\in T^*_{\hat{x}}\Omega\backslash 0$. Then L is strongly hyperbolic at $\hat{x}$ with respect to x_0.*

The assumption of Theorem 1.1 implies that the doubly characteristic set $\Sigma=\{(x, \xi); h(x, \xi)=dh(x, \xi)=0\}$ of h is a manifold. Indeed

Proposition 1.1. *Let $N=2$. Assume that $L_1(\rho)=O$ and* Hess $h(\rho)$ *is of maximal rank at a double characteristic ρ. Then Σ is a C^∞ manifold near ρ with* codim $\Sigma=$ rank Hess $h(\rho)$ *on which L_1 vanishes.*

Theorem 1.1 will be proved constructing a suitable symmetrizer for L_1, more precisely we have

Proposition 1.2. *Let $N=2$. Assume that $L_1(\rho)=O$ and Σ is a C^∞ manifold near $\rho=(\hat{x}, \hat{\xi})\in T^*_{\hat{x}}\Omega\backslash 0$ with* codim $\Sigma=$ rank Hess $h(\rho)$ *on which L_1 vanishes. Then L_1 has a symmetrizer near ρ, that is, there is a 2×2 matrix valued symbol $S(x, \xi')$ defined near $\rho'=(\hat{x}, \hat{\xi}')$, homogeneous of degree 0 in ξ' such that*

$$S^*(x, \xi') = S(x, \xi') \quad \textit{and} \quad S(x, \xi') \quad \textit{is positive definite,}$$
$$S(x, \xi')\, L_1(x, \xi) = L_1^*(x, \xi)\, S(x, \xi')\,,$$

where L_1^* denotes the adjoint matrix of L_1.

Remark 1.1. The assertion of Proposition 1.1 is equivalent to: $S=\{(x, \xi); L_1(x, \xi)=O\}$ is a C^∞ manifold with codim $S=$ rank Hess $h(\rho)$.

When $L_1(\rho)=O$ and L_1 is real, hence the maximal rank of Hess $h(\rho)$ is 3, Proposition 1.1 can be easily seen. Since Proposition 1.2 was proved in [2] when rank Hess $h(\rho)\leqq 3$ and Theorem 1.1 is an immediate consequence of Proposition 1.2, it will be enough to prove Propositions 1.1 and 1.2 assuming rank Hess $h(\rho)=4$.

Here we note that the result can be easily generalized to $N\times N$ system such that all characteristics of h are at most double. The theorems below follow easily from Propositions 1.1, 1.2 and the same arguments proving Theorem 2.3 in [2].

Theorem 1.2. *Assume that every multiple characteristic on $T^*_{\hat{x}}\Omega\backslash 0$ is at most double. Suppose that either h is effectively hyperbolic or* rank $L_1\leqq N-2$ *and*

Hess h *has maximal rank in every double characteristic on* $T^*_{\hat{x}}\Omega\backslash 0$. *Then* L *is strongly hyperbolic at* $\hat{x}$ *with respect to* x_0.

Theorem 1.3. (cf. Theorem 2.3 in [2]) *Assume that every multiple characteristic on* $T^*_{\hat{x}}\Omega\backslash 0$ *is at most double and one of the following conditions is satisfied in every double characteristic* $\rho\in T^*_{\hat{x}}\Omega\backslash 0$:

(1) h *is effectively hyperbolic at* ρ,

(2) *the doubly characteristic set* Σ *of* h *is a* C^∞ *manifold near* ρ *on which* rank $L_1\leqq N-2$.

Then L *is strongly hyperbolic at* $\hat{x}$ *with respect to* x_0.

2. Proof of Lemma and Propositions

We first note that we may assume that $L_1(0, 1, \cdots, 0)=-I_2$, the identity matrix of order 2, so that

$$L_1(x, \xi) = -\xi_0 I_2+A'(x, \xi'),\ \ A'(x, \xi') = \sum_{j=1}^{n} A_j(x)\,\xi_j,\ \ A_j(x)\in C^\infty(\Omega, M_2(\boldsymbol{C}))$$

which is also written

$$L_1(x, \xi) = -(\xi_0-\frac{1}{2}\operatorname{Tr} A'(x, \xi'))\,I_2+A(x, \xi'),\ \ \operatorname{Tr} A(x, \xi') = 0\,.$$

Here $g(x, \xi')=$the determinant of $A(x, \xi')=\det A(x, \xi')\leqq 0$ and $\operatorname{Tr} A'(x, \xi')=$ the trace of $A'(x, \xi')$ is real which follow from the hyperbolicity of h. Let us denote

$$A(x, \xi') = \begin{bmatrix} a(x, \xi') & b(x, \xi') \\ c(x, \xi') & -a(x, \xi') \end{bmatrix},\ \ \rho = (\hat{x}, \hat{\xi}),\ \ \rho' = (\hat{x}, \hat{\xi}')\,.$$

We first show Lemma 1.1. Note that

$$\text{Hess}\, h(\rho) = d\eta\circ d\eta-(da\circ da+db\circ dc)$$

where $\eta=\xi_0-2^{-1}\operatorname{Tr}A'$ and $db\circ dc$ denotes the symmetric tensor product of db and dc. Then it is enough to show that the rank of the quadratic form $Q=da\circ da+db\circ dc$ (at ρ'), which is non negative definite, is at most 3 (resp. at most 2 when a, b, c are real). Here we recall that a real quadratic form $Q(X)$ in $T_\rho(T^*\Omega)$ which is non negative definite cannot vanish on a linear subspace $V\subset T_\rho(T^*\Omega)$ unless codim $V\geqq$rank Q.

Denoting by $\Re a$ and $\Im a$ the real part and the imaginary part of a respectively we see that

$$\begin{aligned} 0\leq Q &= d\Re a\circ d\Re a-d\Im a\circ d\Im a+d\Re b\circ d\Re c-d\Im b\circ d\Im c \\ &\leq d\Re a\circ d\Re a+d\Re b\circ d\Re c-d\Im b\circ d\Im c\,. \end{aligned}$$

It is clear that Q vanishes on $\{X; d\Re a(X)=d\Re b(X)=d\Im b(X)=0\}$ which shows that rank $Q\leqq 3$. The same argument shows that rank $Q\leqq 2$ if a, b, c are real.

We turn to the proofs of Propositions 1.1 and 1.2. As noted in Introduction it is enough to prove these propositions assuming rank Hess $h(\rho)=4$. Since the hypothesis rank Hess $h(\rho)=4$ reduces to rank Hess $g(\rho')=3$ we may assume that Q is non negative definite and has rank 3.

We first remark that $d\Re a(\rho')\neq 0$. If it were not true we would have

$$0\leq Q = -d\Im a\circ d\Im a+d\Re b\circ d\Re c-d\Im b\circ d\Im c\leq d\Re b\circ d\Re c-d\Im b\circ d\Im c\,.$$

It is clear that there is a linear subspace $V(\subset T_\rho(T^*\Omega))$ with codim $V\leqq 2$ on which Q vanishes and hence rank $Q\leqq 2$. This contradicts the assumption.

Set $\varphi=\Re a$ and denote by $b|_{\varphi=0}$ the restriction of b to the surface $\{\varphi=0\}$.

Lemma 2.1. *Let $b=\beta\varphi+\tilde{b}$, $c=\gamma\varphi+\tilde{c}$ with $\tilde{b}=b|_{\varphi=0}=\tilde{b}_1+i\tilde{b}_2$, $\tilde{c}=c|_{\varphi=0}=\tilde{c}_1+i\tilde{c}_2$ where $\tilde{b}_i$, $\tilde{c}_i$ are real. Then we have*

$$d\tilde{b}_i\neq 0\,,\quad d\tilde{c}_i\neq 0 \text{ at } \rho'\,,\quad i=1,2\,.$$

Proof. Denoting $\Im a=\alpha\varphi+\tilde{a}$, $\tilde{a}=a|_{\varphi=0}$, one can write

$$A(x,\xi')=\varphi\begin{bmatrix}(1+i\alpha) & \beta\\ \gamma & -(1+i\alpha)\end{bmatrix}+\begin{bmatrix} i\tilde{a} & \tilde{b}_1+i\tilde{b}_2\\ \tilde{c}_1+i\tilde{c}_2 & -i\tilde{a}\end{bmatrix}.$$

From the non-positivity of g on $\{\varphi=0\}$ it follows that

$$\tilde{b}_1\tilde{c}_1-\tilde{b}_2\tilde{c}_2-\tilde{a}^2\geqq 0\,, \tag{2.1}$$

$$\tilde{b}_1\tilde{c}_2+\tilde{b}_2\tilde{c}_1=0 \tag{2.2}$$

near ρ'. Suppose, for instance, that $d\tilde{b}_1(\rho')=0$ and hence $d\tilde{b}_2=0$ or $d\tilde{c}_1=0$ (at ρ') by (2.2). If $d\tilde{b}_2=0$ then $d\tilde{a}=0$ by (2.1) and then Q vanishes on $\{X; d\varphi(X)=0\}$ because $da=(1+i\alpha)\,d\varphi$ at ρ'. This is a contradiction. The other cases will be proved similarly.

Lemma 2.2. *$d\tilde{b}_1$ is not proportional to $d\tilde{b}_2$ at ρ'. There is a positive function $m(x,\xi')$ defined near ρ', homogeneous of degree 0 in ξ' such that*

$$\tilde{c}_1(x,\xi')=m(x,\xi')\,\tilde{b}_1(x,\xi')\,,\quad \tilde{c}_2(x,\xi')=-m(x,\xi')\,\tilde{b}_2(x,\xi')\,.$$

Proof. Suppose that $d\tilde{b}_2=kd\tilde{b}_1$ at ρ' with some $k\in\boldsymbol{R}$ and hence $d\tilde{c}_2=-kd\tilde{c}_1$ by (2.2) at ρ'. Since (by (2.1))

$$d\tilde{b}_1\circ d\tilde{c}_1-d\tilde{b}_2\circ d\tilde{c}_2-d\tilde{a}\circ d\tilde{a}=(1+k^2)\,d\tilde{b}_1\circ d\tilde{c}_1-d\tilde{a}\circ d\tilde{a}\geq 0\,,$$

$d\tilde{b}_1$ and $d\tilde{c}_1$ must be proportional to $d\tilde{a}$ at ρ' if $d\tilde{a}\neq 0$. Then it is clear that Q vanishes on $\{X; d\tilde{a}(X)=d\varphi(X)=0\}$ which is a contradiction. If $d\tilde{a}=0$ (at ρ')

then Q vanishes on $\{X; d\varphi(X)=d\tilde{c}_1(X)=0\}$ which also gives a contradiction. This proves the first assertion. The second assertion easily follows from the first one and (2.1), (2.2).

The following lemma proves Proposition 1.1.

Lemma 2.3. *Let $\beta=\beta_1+i\beta_2$, $\gamma=\gamma_1+i\gamma_2$, β_i, γ_i real. Set $\psi_i=\tilde{b}_i+\beta_i\varphi$ $(i=1, 2)$, $B=\gamma_2+m\beta_2$, $C=\gamma_1-m\beta_1$. Then we have*

$$A=\varphi\begin{bmatrix}1 & 0\\ C+iB & -1\end{bmatrix}+\psi_1\begin{bmatrix}-iB/2 & 1\\ m & iB/2\end{bmatrix}+\psi_2\begin{bmatrix}-iC/2 & i\\ -im & iC/2\end{bmatrix}.$$

Moreover $d\varphi$, $d\psi_i$ are linearly independent at ρ' and the set $\{(x, \xi'); A(x, \xi')=O\}$ is given by

$$S=\{(x, \xi'); \varphi(x, \xi')=\psi_1(x, \xi')=\psi_2(x, \xi')=0\}\ .$$

Proof. Recall that

$$A=\varphi\begin{bmatrix}1+i\alpha & \beta\\ \gamma & -(1+i\alpha)\end{bmatrix}+\begin{bmatrix}i\tilde{a} & \tilde{b}_1+i\tilde{b}_2\\ m(\tilde{b}_1-i\tilde{b}_2) & -i\tilde{a}\end{bmatrix}.$$

We observe the imaginary part of g:

$$\Im g=2\alpha\varphi^2+2\tilde{a}\varphi+\Im(\beta\gamma)\,\varphi^2+\Im(\gamma+\beta m)\,\varphi\tilde{b}_1+\Re(\gamma-\beta m)\,\varphi\tilde{b}_2\,.$$

Since $\Im g=0$ near ρ' and $d\varphi\neq 0$ at ρ' it follows that

$$2\alpha\varphi+2\tilde{a}+\Im(\beta\gamma)\,\varphi+\Im(\gamma+\beta m)\,\tilde{b}_1+\Re(\gamma-\beta m)\,\tilde{b}_2=0 \tag{2.3}$$

near ρ'. Now we set

$$D=\Im(\beta\gamma),\ B=\Im(\gamma+\beta m)=\gamma_2+\beta_2\,m,\ C=\Re(\gamma-\beta m)=\gamma_1-\beta_1\,m\,.$$

Noticing $D=\beta_1 B+\beta_2 C$ it follows from (2.3) that

$$(\alpha\varphi+\tilde{a})=-\frac{1}{2}(\psi_1 B+\psi_2 C) \tag{2.4}$$

which shows that $a=(1+i\alpha)\,\varphi+i\tilde{a}=\varphi-i(\psi_1 B+\psi_2 C)/2$. On the other hand it is easy to see

$$m(\tilde{b}_1-i\tilde{b}_2)+\gamma\varphi=(C+iB)\,\varphi+m(\psi_1-i\psi_2),\ \ \tilde{b}_1+i\tilde{b}_2+\beta\varphi=\psi_1+i\psi_2$$

since $\gamma_1=C+m\beta_1$ and $\gamma_2=B-m\beta_2$. These prove the first part. The rest of the assertion is obvious.

Lemma 2.4.

$$4m-(B^2+C^2)>0 \ \textit{at}\ \rho'\,.$$

Proof. Let us set $\tilde{B}=B|_{\varphi=0}$, $\tilde{C}=C|_{\varphi=0}$. From (2.4) it follows that

$$\tilde{a} = -(\tilde{B}\tilde{b}_1+\tilde{C}\tilde{b}_2)/2\,.$$

On the other hand (2.1) and Lemma 2.2 give that

$$m(\tilde{b}_1^2+\tilde{b}_2^2)-\tilde{a}^2\geqq 0 \text{ near } \rho'\,.$$

Since the quadratic form $m(d\tilde{b}_1\circ d\tilde{b}_1+d\tilde{b}_2\circ d\tilde{b}_2)-(\tilde{B}\,d\tilde{b}_1+\tilde{C}\,d\tilde{b}_2)\circ(\tilde{B}\,d\tilde{b}_1+\tilde{C}\,d\tilde{b}_2)/4$ is the restriction of Q to $\{X; d\varphi(X)=0\}$, this must have rank 2 and then positive definite. This shows that $4m-(\tilde{B}^2+\tilde{C}^2)>0$ at ρ' and hence the result.

To finish the proof of Proposition 1.2 we give a required symmetrizer for L_1:

$$S(x,\xi')=\begin{bmatrix} 2m(x,\xi') & -C(x,\xi')+iB(x,\xi') \\ -(C(x,\xi')+iB(x,\xi')) & 2 \end{bmatrix}$$

which satisfies $S(x,\xi')=S^*(x,\xi')$ clearly. Using Lemma 2.3 it is easy to check that $S(x,\xi')\,A(x,\xi')=A^*(x,\xi')\,S(x,\xi')$ and hence $S(x,\xi')\,L_1(x,\xi)=L_1^*(x,\xi)\,S(x,\xi')$. The positivity of S follows from Lemma 2.4.

Added in proof: after submitted the paper we knew that L. Hörmander has obtained a strong stability property of double characteristics of maximal rank, including Proposition 1.1 in "Hyperbolic systems with double characteristics", 1990, preprint.

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