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A REMARK ON M_p-GROUPS

Dedicated to Professor Kazuhiko Hirata on his 60th birthday

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1. Introduction

Let FG be the group algebra of a finite group G over an algebraically closed field F of characteristic $p>0$. We call an FG-module V monomial if V is induced from some 1-dimensional FH-module for some subgroup H of G. An ordinary character χ of G is called monomial if χ is induced from some linear character of some subgroup of G. We call G an M_p-group if every irreducible FG-module is monomial. We call G an M-group if every irreducible ordinary character of G is monomial. For details, see a paper of Bessenrodt [1] and a book of Isaacs [4]. It is well known that M-groups are solvable (15.7 in [2]). M_p-groups are also solvable (3.8 in [6]). By Fong-Swan's theorem, M-groups are M_p-groups for any prime p. But M_p-groups need not be M-groups. For example, $SL(2, 3)$ is an M_2-group but not an M-group. So we investigate conditions for M_p-groups to be M-groups. Namely,

Theorem 3. *Let G be a p-nilpotent group. Then G is an M-group if and only if G is an M_p-group.*

Throughout this paper, groups are finite groups, F is an algebraically closed field of characteristic $p>0$, FG-modules are finitely generated right FG-modules, and characters are ordinary characters. Let χ be a character of a group G. We write χ^* for the Brauer character corresponding to χ. Let H be a subgroup of G and φ be a character of H. We write χ_H for the restriction of χ to H and φ^G for the induced character from φ. We use the same notation for modules. When M and N are FG-modules, we write $N|M$ if N is a direct summand of M. We write $\mathrm{Irr}(G)$ for the set of all irreducible characters of G. For the other notation and terminology we shall refer to books of Dornhoff [2] and Nagao and Tsushima [5].

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2. Consequences

Let H be a normal subgroup of G and φ be an irreducible character of H. We denote the inertia group of φ in G by $I_G(\varphi)$. When φ is irreducible, we put

$$\mathrm{Irr}(G|\varphi) = \{\chi\in\mathrm{Irr}(G)|(\chi_H, \varphi)\neq 0\}\ .$$

Next theorem will be a powerful tool if we consider conditions for M_p-groups to be M-groups.

Theorem 1. *Let G be a finite group. Assume that G has a normal p'-subgroup N such that G and N satisfy the followings.*

(a) *G is an M_p-group.*

(b) *G/N is an M-group.*

(c) *Every proper subgroup of G containing N is an M-group.*

(d) *Every G-invariant irreducible character of N is extendible to G.*

Then G is an M-group.

Proof. Let $\chi\in\mathrm{Irr}(G|\varphi)$ where $\varphi\in\mathrm{Irr}(N)$. If $I_G(\varphi)$ is a proper subgroup of G then there exists $\xi\in\mathrm{Irr}(I_G(\varphi)|\varphi)$ such that $\xi^G=\chi$. From (c), ξ is monomial so is χ.

Assume $I_G(\varphi)=G$. From (d), φ is extendible to G. Let χ_0 be an extention of φ. Because $(\chi_0^*)_N=\varphi^*$ and N is a p'-group, χ_0^* is an irreducible Brauer character of G. Since G is an M_p-group, there exist a subgroup H of G and a linear character λ of H such that $(\lambda^*)^G=\chi_0^*$. Since $(\lambda^G)^*=\chi_0^*$ is irreducible, λ^G is irreducible and an extention of φ. By 3.5.12 in [5],

$$\mathrm{Irr}(G|\varphi) = \{\lambda^G\eta|\eta\in\mathrm{Irr}(G/N)\}\ .$$

Now every η is monomial, so is $\lambda^G\eta$. So χ is monomial. The proof is completed.

Generally, normal subgroups of M_p-groups need not be M_p-groups. But next theorem holds.

Theorem 2. *Let G be an M_p-group and N be a normal subgroup of G such that $|G: N|=p$. Then N is an M_p-group.*

Proof. Let U be an irreducible FN-module. Since N is normal in G, there exists an irreducible FG-module V such that $U|V_N$. Since G is an M_p-group, there exist a subgroup H and a 1-dimensional FH-module W such that $V=W^G$. If the ineitia group of U in G is G then U is extendible to G. Thus we may assume $U=V_N$. By Mackey's decomposition,

$$U = V_N = (W^G)_N = \oplus_{t\in H\backslash G/N}(W^t{}_{H^t\cap N})^N\ .$$

But U is irreducible. So $G=HN$ and U is monomial. We may assume that the inertia group of U in G is N. Then by Clifford's theorem, $V_N=\oplus_{t\in G/N}U^t$. If H is contained in N then by Mackey's decomposition,

$$V_N = (W^G)_N = \oplus_{t\in H\backslash G/N}(W^t{}_{H^t\cap N})^N = \oplus_{t\in G/N}(W^t{}_{H^t})^N .$$

Since U is irreducible, $U\cong(W^t{}_{H^t})^N$ for some $t\in G/N$. So U is monomial. We may assume that H is not contained in N. So $G=$HN. Let Q be a vertex of W. Since $\dim_F W=1$, Q is a Sylow p-subgroup of H. Since $V=W^G$ and $V=U^G$, Q is in $H\cap N$. Now

$$p = |G\colon N| = |HN\colon N| = |H\colon H\cap N|\,\Big|\,|H\colon Q| .$$

But Q is a Sylow p-subgroup of H, a contradiction. Hence U is monomial. So N is an M_p-group.

Next theorem is our main result.

Theorem 3. *Let G be a p-nilpotent group. Then G is an M-group if and only if G is an M_p-group.*

Proof. We know that M-groups are M_p-groups. So we shall show that G is an M-group if G is an M_p-group by induction on $|G|$. Since G is p-nilpotent G has a normal p-complement N. We show that G and N satisfy the conditions in Theorem 1. Now (a) and (b) are satisfied. By 3.5.11 in [5], (d) is satisfied. Let H be a proper subgroup of G containing N. Since G/N is a p-group, H is an M_p-group by Theorem 2. Then H is an M-group by inductive hypothesis. So (c) is satisfied. Then G is an M-group.

Corollary 4. *Let G be an M-group and p-nilpotent. Then a subgroup H of G such that $|G\colon H|$ is p-power is an M-group.*

Proof. This is immediate from Theorem 2 and Theorem 3.

References

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