

HEEGAARD SPLITTING FOR SUTURED MANIFOLDS AND MURASUGI SUM

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1. Introduction

The concept of sutured manifolds was introduced by Gabai, and it has been playing an important role in the 3-manifold theory ([3]–[7]). On the other hand, in [2], Casson and Gordon defined Heegaard splittings of compact orientable 3-manifolds with boundaries by using compression bodies. We note that this enables us to define Heegaard splittings for sutured manifolds. In this paper, we study complementary sutured manifolds for Seifert surfaces from the viewpoint of this Heegaard splitting.

Firstly, we give the definition of Heegaard splittings for sutured manifolds following [2]. A *compression body* W is a cobordism $\text{rel } \partial$ between surfaces ∂_+W and ∂_-W such that $W \cong \partial_+W \times I \cup 2\text{-handles} \cup 3\text{-handles}$ and ∂_-W has no 2-sphere components. It is easy to see that if $\partial_-W \neq \emptyset$ and W is connected, W is obtained from $\partial_-W \times I$ by attaching a number of 1-handles along the disks on $\partial_-W \times \{1\}$ where ∂_-W corresponds to $\partial_-W \times \{0\}$. We denote the number of these 1-handles by $h(W)$. Let (M, γ) be a sutured manifold such that $R_+(\gamma) \cup R_-(\gamma)$ has no 2-sphere components and $T(\gamma) = \emptyset$. We say that (W, W') is a *Heegaard splitting* of (M, γ) if both W and W' are compression bodies, $M = W \cup W'$ with $W \cap W' = \partial_+W = \partial_+W'$, $\partial_-W = R_+(\gamma)$, and $\partial_-W' = R_-(\gamma)$. Assume that $R_+(\gamma)$ is homeomorphic to $R_-(\gamma)$. Then we define the *handle number* $h(M, \gamma)$ of (M, γ) as follows:

$$h(M, \gamma) = \min \{h(W); (W, W') \text{ is a Heegaard splitting of } (M, \gamma)\}.$$

Note that $h(M, \gamma)$ corresponds to the Heegaard genus of a closed 3-manifold.

For the definitions of a $2n$ -Murasugi sum and a complementary sutured manifold, see Section 2. Let R be a Seifert surface in S^3 obtained by a $2n$ -Murasugi sum of two Seifert surfaces R_1 and R_2 whose complementary sutured manifolds (M_i, γ_i) ($i=1, 2$) are irreducible. Let (M, γ) be the complementary sutured manifold for R . In this paper, we consider the relations between $h(M_i, \gamma_i)$ ($i=1, 2$) and $h(M, \gamma)$. In fact, we prove:

Theorem 1. *Under the above notation, we have*

$$h(M_1, \gamma_1) + h(M_2, \gamma_2) - (n-1) \leq h(M, \gamma) \leq h(M_1, \gamma_1) + h(M_2, \gamma_2).$$

We say that an oriented surface R in S^3 is a *fiber surface* if ∂R is a fibered like with R a fiber. Then we have:

Theorem 2. *If R_1 is a fiber surface, then $h(M, \gamma) = h(M_2, \gamma_2)$.*

Note that R is a fiber surface if and only if $h(M, \gamma) = 0$. Thus as a consequence of Theorem 1, we obtain the next fact.

Corollary ([3], [15]). *If R_1 and R_2 are fiber surfaces, then R is a fiber surface.*

Theorem 2 means that $h(M, \gamma) = h(M_1, \gamma_1) + h(M_2, \gamma_2)$ if R_1 is a fiber surface. Moreover in Section 5, we give examples to show that the inequality of Theorem 1 is best possible for any n . We note that Theorem 1 corresponds to the additivity of the Heegaard genus under a connected sum proved by Haken ([8], [10]). In fact, it is proved by using Haken type results for a disk properly embedded in a sutured manifold (see Section 3).

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2. Preliminaries

Throughout this paper, we work in the piecewise linear category and manifolds are oriented. For the definitions of standard terms of 3-dimensional topology, see [9]. For a topological space B , $\#B$ denotes the number of components of B . Let H be a subcomplex of a complex K . Then $N(H; K)$ denotes a regular neighborhood of H in K . Let N be a manifold embedded in a manifold M with $\dim N = \dim M$. Then $\text{Fr}_M N$ denotes the frontier of N in M .

We recall the definition of a *sutured manifold* [4]. A sutured manifold (M, γ) is a compact 3-dimensional manifold M together with a set $\gamma (\subset \partial M)$ of mutually disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. In this paper, we treat the case of $T(\gamma) = \emptyset$. The core curves of $A(\gamma)$, say $s(\gamma)$, are the *sutures*. Every component of $R(\gamma) = \partial M - \text{Int } A(\gamma)$ is oriented, and $R_+(\gamma)$ ($R_-(\gamma)$ resp.) denotes the union of the components whose normal vectors point out (into resp.) M . Moreover, the orientations of $R(\gamma)$ must be coherent with respect to the orientations of $s(\gamma)$.

We say that a sutured manifold (M, γ) is a *product sutured manifold* [6] if (M, γ) is homeomorphic to $(F \times I, \partial F \times I)$ with $R_+(\gamma) = F \times \{1\}$, $R_-(\gamma) = F \times \{0\}$, $A(\gamma) = \partial F \times I$, where F is a surface and I is the unit interval

[0, 1]. Let L be an oriented link in S^3 and R a Seifert surface for L . The exterior $E(L)$ of L is the closure of $S^3 - N(L; S^3)$. Then $R \cap E(L)$ is homeomorphic to R , and we often abbreviate $R \cap E(L)$ to R . $(N, \delta) = (N(R; E(L)), N(\partial R; \partial E(L)))$ has a product sutured manifold structure $(R \times I, \partial R \times I)$. (N, δ) is called the *product sutured manifold for R* . The sutured manifold $(N^c, \delta^c) = (\text{cl}(E(L) - N), \text{cl}(\partial E(L) - \delta))$ with $R_+(\delta^c) = R_-(\delta)$ is the *complementary sutured manifold for R* . Let R be a Seifert surface in S^3 and (M, γ) the complementary sutured manifold for R . Then R is a fiber surface if and only if (M, γ) is a product sutured manifold, that is, $h(M, \gamma) = 0$.

Sutured manifold decomposition is an operation to obtain a new sutured manifold (M', γ') from a sutured manifold (M, γ) by decomposing along an oriented proper surface S (see [4]). The notation for this operation is as follows:

$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

This paper focuses on a very special type of a sutured manifold decomposition (see Figure 2.2). A properly embedded disk D in (M, γ) is a *product disk* if $\partial D \cap A(\gamma)$ consists of two essential arcs in $A(\gamma)$. A *product decomposition* $(M, \gamma) \xrightarrow{D} (M', \gamma')$ is a sutured manifold decomposition along a product disk D . Note that each compression body W can be regarded as a sutured manifold with $A(\gamma) = \partial(\partial_- W) \times I$. In this sense, we define a product disk and a product decomposition for (W, γ) in this paper.

Next, we recall some definitions of Casson and Gordon [2]. Let W be a compression body. If W is homeomorphic to $\partial_- W \times I$, we say that W is *trivial*. A *complete disk system* \bar{D} for a compression body W is a disjoint union of disks $(D^2, \partial D^2) \subset (W, \partial_+ W)$ such that W cut along \bar{D} is homeomorphic to either a 3-ball or $\partial_- W \times I$ according to whether $\partial_- W$ is empty or not. In this paper, we treat the case of $\partial_- W \neq \emptyset$.

Let (M, γ) be a sutured manifold and (W, W') a Heegaard splitting of (M, γ) . We say that $W \cap W' = F$ is a *Heegaard surface* of (W, W') . Then $\partial F = \partial(\partial_+ W) = \partial(\partial_+ W') = s(\gamma)$.

The next theorem follows from [2].

Theorem ([2]). *Let (M, γ) be a sutured manifold such that $R_+(\gamma) \cup R_-(\gamma)$ has no 2-sphere components and $T(\gamma) = \emptyset$. Then (M, γ) has a Heegaard splitting.*

EXAMPLE 2.1. Let R be an unknotted annulus in S^3 with n -full twists and (M, γ) the complementary sutured manifold for R . Then M is homeomorphic to a solid torus and $A(\gamma)$ is the union of two annuli which wrap ∂M longitudinally n -times and meridionally once.

Case 1: $n = 1$.

In this case, ∂R is a Hopf link and it is easy to see that $(M, \gamma) \cong (R_+(\gamma) \times I, \partial R_+(\gamma) \times I)$. Hence (M, γ) is a product sutured manifold and $h(M, \gamma) = 0$.

Case 2: $n \geq 2$.

In this case, firstly we show that (M, γ) is not a product sutured manifold, so that $h(M, \gamma) > 0$. Let α be a generator of $\pi_1(R_+(\gamma))$ and β a generator of $\pi_1(M)$. If (M, γ) is a product sutured manifold, the homomorphism between fundamental groups $i_*: \pi_1(R_+(\gamma)) \rightarrow \pi_1(M)$ induced by the inclusion $i: R_+(\gamma) \rightarrow M$ is an isomorphism. However, $i_*(\alpha) = \beta^n$ or β^{-n} ($n \geq 2$), a contradiction. Hence $h(M, \gamma) > 0$. Now we split M into two components as follows. Take an essential arc a on $R_-(\gamma)$ and push a into M so that $\partial a \subset R_+(\gamma)$.

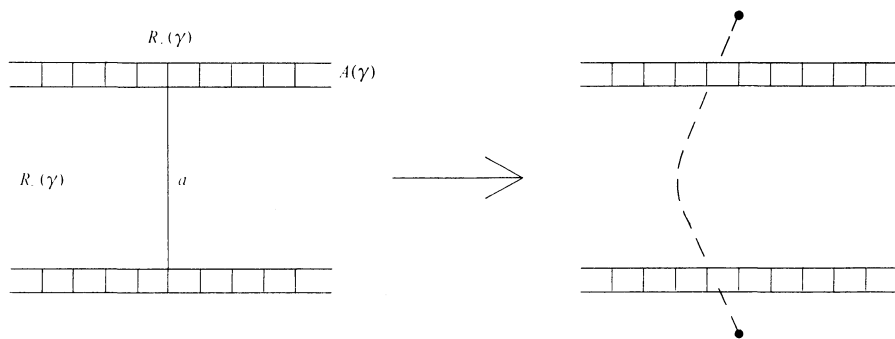


Figure 2.1

Let $X = N(R_+(\gamma) \cup a; M)$, $Y = \text{cl}(M - X)$, then $(X, A(\gamma) \cap X)$ and $(Y, A(\gamma) \cap Y)$ are homeomorphic to $(\text{Annulus} \times I \cup 1\text{-handle}, \partial(\text{Annulus}) \times I)$. Hence (X, Y) is a Heegaard splitting of (M, γ) , and we have $h(M, \gamma) = 1$.

A surface $R (\subset S^3)$ is a $2n$ -Murasugi sum of two surfaces R_1 and R_2 in S^3 if:

- (1) $R = R_1 \cup_D R_2$, where D is a $2n$ -gon, i.e., $\partial D = \mu_1 \cup \nu_1 \cup \dots \cup \mu_n \cup \nu_n$ (possibly $n=1$), where μ_j, ν_j (resp.) is an arc properly embedded in $R_1 (R_2$ resp.).
- (2) There exist 3-balls \bar{B}_1, \bar{B}_2 in S^3 such that:
 - (i) $\bar{B}_1 \cup \bar{B}_2 = S^3$, $\bar{B}_1 \cap \bar{B}_2 = \partial \bar{B}_1 = \partial \bar{B}_2 = S^2$; a 2-sphere,
 - (ii) $R_1 \subset \bar{B}_1, R_2 \subset \bar{B}_2$ and $R_1 \cap S^2 = R_2 \cap S^2 = D$.

When D is a 2-gon, the Murasugi sum is known as a connected sum. When D is a 4-gon, the Murasugi sum is known as a plumbing.

Let R be a $2n$ -Murasugi sum of R_1 and R_2 , and let (M, γ) , (M_1, γ_1) and (M_2, γ_2) be the complementary sutured manifolds for R, R_1 and R_2 respectively. By the definition of a $2n$ -Murasugi sum, there is a 2-sphere S^2 along which R is summed and the summing disk D . Let S be the $4n$ -gon $S^2 - (\text{Int} D \cup \text{Int} N(\partial R))$, then we have:

Lemma 2.2. (cf. [5]). (M_1, γ_1) and (M_2, γ_2) are obtained from (M, γ) by the sutured manifold decomposition along S with an appropriate orientation.

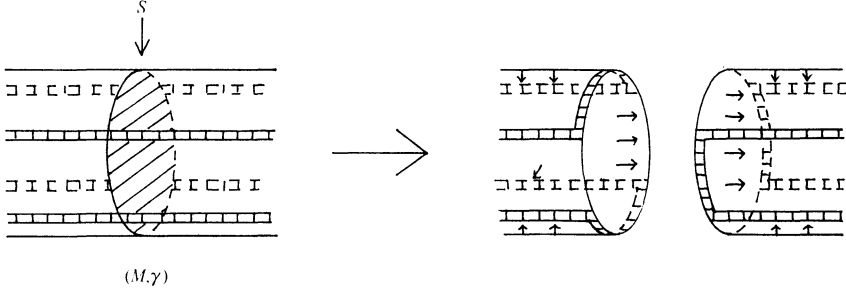


Figure 2.2

Proof. View S^2 as $R^2 \cup \infty$ where $\partial R \cap S^2$ is a set of evenly spaced points x_1, \dots, x_{2n} on the unit circle and S is the disk in $R^2 - \text{Int } N(\partial R \cap R^2; R^2)$. Let $a_i(\gamma) = s(\gamma) \cap S$ be a point for $x_i (i=1, 2, \dots, 2n)$. View $N(S^2; S^3)$ as $S^2 \times I$ with $S^2 = S^2 \times \{1/2\}$, $S^2 \times \{0\} \subset \bar{B}_1$, $N(S^2; S^3) \cap \partial R = (\partial R \cap S^2) \times I$. We can take $R_1 \times I$ so that $R_1 \times \{1\} \cap S^2 = D$ and $R_1 \times \{0\} \cap (S^2 \times I) \subset S^2 \times \{0\}$. For the sutured manifold (M_1, γ_1) , we can regard that M_1 is $\text{cl}(\bar{B}_1 - R_1 \times I) \cup_s \bar{B}_2$ and $s(\gamma_1)$ is obtained from $s(\gamma) \cap \bar{B}_1$ by attaching the subarc of the unit circle connecting $a_i(\gamma)$, $a_{i+1}(\gamma)$ for i odd by choosing a cyclic ordering of x_i 's well. Similarly, for (M_2, γ_2) , we can regard that M_2 is $\text{cl}(\bar{B}_2 - R_2 \times I) \cup_s \bar{B}_1$ and $s(\gamma_2)$ is obtained from $s(\gamma) \cap \bar{B}_2$ by attaching the subarc of the unit circle connecting $a_i(\gamma)$, $a_{i+1}(\gamma)$ for i even. Note that $\text{cl}(\bar{B}_1 - R_1 \times I) \cup_s \bar{B}_2$ is homeomorphic to $\text{cl}(\bar{B}_1 - R_1 \times I)$, and $\text{cl}(\bar{B}_2 - R_2 \times I) \cup_s \bar{B}_1$ is homeomorphic to $\text{cl}(\bar{B}_2 - R_2 \times I)$. On the other hand, we see that the sutured manifold obtained from (M, γ) by the sutured manifold decomposition along S with an appropriate orientation is equivalent to the disjoint union of $(\text{cl}(\bar{B}_1 - R_1 \times I), \gamma_1)$ and $(\text{cl}(\bar{B}_2 - R_2 \times I), \gamma_2)$. This completes the proof of Lemma 2.2. (see Figure 2.3)

In this paper, we call S in Lemma 2.2 a *cross section disk*.

Let P be a properly embedded surface in a compression body W . P is called *boundary compressible toward* $\partial_+ W$ if there exists a disk D in W such that $D \cap P = \alpha$; an arc in ∂D and $D \cap \partial_+ W = \beta$; an arc in ∂D , with $\alpha \cap \beta = \partial \alpha = \partial \beta$, $\alpha \cup \beta = \partial D$, and either α is essential in P or α is inessential in P and the boundaries of all disk components of $\text{cl}(P - \alpha)$ intersect $\partial(\partial_- W) \times I$. If P is not boundary compressible toward $\partial_+ W$, then we say that P is *boundary incompressible toward* $\partial_+ W$.

Now, let P be a connected surface properly embedded in a compression body W such that each component of $\partial P \cap \partial(\partial_- W) \times I$ is an essential arc in $\partial(\partial_- W) \times I$.

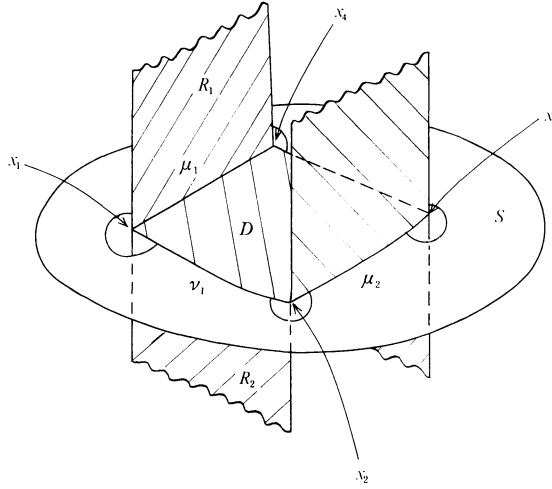


Figure 2.3

Lemma 2.3. *Assume that $\partial P \cap \partial_+ W \neq \emptyset$ and P is incompressible and boundary incompressible toward $\partial_+ W$. Then P is either*

- (1) *an annulus such that one boundary component is contained in $\partial_+ W$ and the other is contained in $\partial_- W$.*
- (2) *a disk whose boundary component is contained in $\partial_+ W$, or*
- (3) *a product disk in W .*

Proof. Firstly, we will show the special case.

Claim. *Assume that W is trivial, then P is either (1), (2) or (3).*

Proof. If ∂P is contained in only $\partial_+ W$ and $\partial_- W$, (1) or (2) occurs by the proof of Lemma 1.1 in [2]. Thus we suppose that there is an arc component in $\partial P \cap \partial_+ W$. By moving P by an isotopy, if necessary, we may suppose that each component of $\partial P \cap \partial(\partial_- W) \times I$ is a vertical arc with respect to the product structure of W . Let α_1 be a component of $P \cap \partial(\partial_- W) \times I$, and β a component of $P \cap \partial_- W$ such that $\beta \cap \alpha_1 \neq \emptyset$. Then there is another component α_2 of $P \cap \partial(\partial_- W) \times I$ such that $\alpha_2 \cap \beta \neq \emptyset$. We note that $\alpha_1 \cup \beta \cup \alpha_2$ is an arc properly embedded in $\text{cl}(\partial W - \partial_+ W)$. Let c_1 and c_2 be the components of $\text{Fr}_{\text{cl}(\partial W - \partial_+ W)} N(\alpha_1 \cup \alpha_2 \cap \beta; \text{cl}(\partial W - \partial_+ W))$, and A_i a product disk in W such that c_i is contained in ∂A_i ($i=1, 2$). Each component of $A_i \cap P$ is a circle or an arc such that the boundary is contained in $\partial_+ W$. Since W is irreducible and P is incompressible and boundary incompressible toward $\partial_+ W$, we can suppose that $A_i \cap P = \emptyset$ ($i=1, 2$) by standard innermost circle and outermost arc argument. Then we have conclusion (3) of the claim.

Let \bar{D} be a complete disk system for W . By standard innermost circle and outermost arc arguments, we see that P is isotoped so that $\bar{D} \cap P = \emptyset$ (cf. [2], [10]). Then we have the conclusion of Lemma 2.3.

Similarly, we have:

Lemma 2.4. *Let (W, γ) be a sutured manifold and (W', γ') the sutured manifold obtained from (W, γ) by a product decomposition. Then (W, γ) is a compression body if and only if (W', γ') is a compression body. Moreover, $h(W) = h(W')$.*

3. Haken type results for disks properly embedded in sutured manifolds

Let (M, γ) be an irreducible sutured manifold such that $R_+(\gamma) \cup R_-(\gamma)$ contain no closed surfaces and $T(\gamma) = \emptyset$. Let (W, W') be a Heegaard splitting for (M, γ) and E a properly embedded disk in (M, γ) such that each component of $\partial E \cap A(\gamma)$ is an essential arc in $A(\gamma)$ and F a Heegaard surface of (W, W') .

The purpose of this section is to show the next proposition.

Proposition 3.1. *We can assume that each component of $E \cap W'$ is a product disk by moving F by an isotopy rel ∂ .*

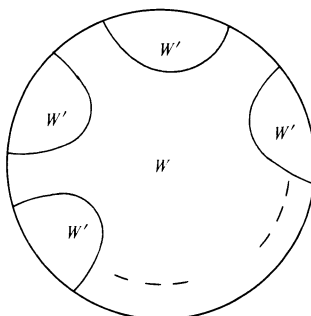


Figure 3.1

Before the proof of this proposition, we give a lemma and definitions.

Lemma 3.2. *By moving F by an isotopy rel ∂ , we may suppose that every component of $E \cap W'$ is a disk as in Figure 3.2.*

Proof. Let $\alpha_1, \dots, \alpha_p$ be arcs embedded in W' such that each α_i is obtained from a core of 1-handles of W' by extending to $\partial_- W' \times I$ vertically (hence $\partial \alpha_i \subset \partial_- W'$). F is isotopic to $\text{Fr}_{W'} N(\partial_- W' \cup (\cup \alpha_i); W')$ rel ∂ . Then we may suppose that $N(\partial_- W' \cup (\cup \alpha_i); W') \cap E = N((\partial_- W' \cup (\cup \alpha_i)) \cap E; E)$. Hence every component of $E \cap W'$ is a disk as in Figure 3.2. This completes the proof of Lemma 3.2.

Let B_1, \dots, B_m be the components of $E \cap W'$ which are contained in $\text{Int } E$, and B_{m+1}, \dots, B_{m+n} be other components of $E \cap W'$ such that $\partial B_{m+1} \cap \partial E, \dots, \partial B_{m+n} \cap \partial E$ are on ∂E in this order (see Figure 3.2) and $\sigma_i = \text{Fr}_E B_i, \Sigma = \Pi \sigma_i$.

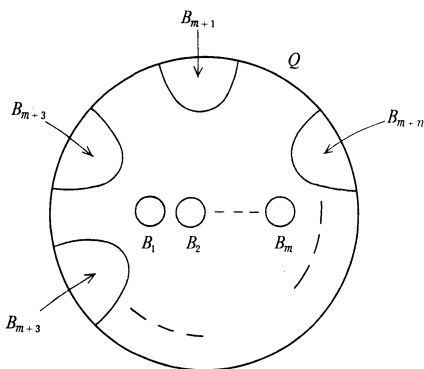


Figure 3.2

Let Q be $E \cap W$ and α an arc properly embedded in Q with $\partial\alpha \subset \Sigma$. We say that α is an *inessential* arc in Q if it is homotopic to an arc in Σ rel ∂ . If α is not inessential, then it is *essential*. Suppose that α is an essential arc in Q . We say that α is of type *I* (type *II* resp.) if α joins a component of Σ (two different components of Σ resp.). Let $\{\alpha_1, \dots, \alpha_n\}$ be a system of mutually disjoint essential arcs in Q . $\{\alpha_1, \dots, \alpha_n\}$ is a *complete system of arcs* for Q if the closure of each component of $Q - (\cup N(\alpha_i; Q))$ is a disk. Set $\bar{\alpha} = \cup \alpha_i$ and let σ be a component of Σ . We say that σ is a *distinguished circle related to $\bar{\alpha}$* if each component of $\bar{\alpha}$ meeting σ is of type *II*.

Proof of Proposition 3.1

Let \bar{D} be a complete disk system of W and \bar{D}' a union of product disks in W such that $\bar{D} \cap \bar{D}' = \emptyset$, $(\partial Q \cap \partial_- W) \cap \bar{D}' = \emptyset$ and each component of $\text{cl}(W - N(\bar{D} \cup \bar{D}'))$ is a 3-cell. By using a standard innermost circle argument, we may suppose that each component of $Q \cup (\bar{D} \cup \bar{D}')$ is an arc since M is irreducible. We denote the system of arcs $Q \cap (\bar{D} \cup \bar{D}')$ by $\bar{\alpha}$ and we suppose that $\#(E \cap W')$ is minimal.

Assume that some component of $E \cap W'$ is contained in $\text{Int}E$ (i.e., $m \geq 1$).

Claim 3.3. *We may assume that $\bar{\alpha}$ is a complete system of arcs for Q .*

Proof. By replacing $\bar{D} \cup \bar{D}'$ if necessary, we can suppose that each component of $\bar{\alpha}$ is an essential arc on Q (cf. [13]). By the irreducibility of M and the minimality of $\#(E \cap W')$, we see that Q is incompressible in W . Then we have this claim.

We can easily verify the following

Claim 3.4. *There exists at least one distinguished circle σ_i ($i=1, 2, \dots, m$) in Σ related to $\bar{\alpha}$.*

For the proof of the next claim, see [13].

Claim 3.5. *If at least one of the circles $\sigma_1, \dots, \sigma_m$ is a distinguished circle related to $\bar{\alpha}$, then there exists a disk E' in M with $\sharp(F \cap E') \leq m+n-1$ such that E' is isotopic to E and that each component of $E' \cap W'$ is a disk.*

If some component of $E \cap W'$ is contained in $\text{Int}E$, it contradicts the minimality of $\sharp(E \cap W')$ by Claims 3.3, 3.4 and 3.5. Then we have Proposition 3.1.

4. Proof of Theorems

The purpose of this section is to prove Theorems stated in Section 1.

Proof of Theorem 1

Case 1. Connected sum.

Let F be a Heegaard surface of (W, W') which realizes $h(M, \gamma)$ and S the cross section disk. By Proposition 3.1, we may assume that $S \cap F$ consists of an arc. Let $W_i (W'_i \text{ resp.}) (i=1, 2)$ be the components obtained by cutting $W (W' \text{ resp.})$ along $S \cap W (S \cap W' \text{ resp.})$. Then W_i and $W'_i (i=1, 2)$ are the compression bodies by Lemma 2.4. Therefore, we can assume that (W_1, W'_1) and (W_2, W'_2) give a Heegaard splitting of (M_1, γ_1) and (M_2, γ_2) respectively. Then, we have $h(M, \gamma) = h(W_1) + h(W_2) \geq h(M_1, \gamma_1) + h(M_2, \gamma_2)$. By a similar argument, we have $h(M, \gamma) \leq h(M_1, \gamma_1) + h(M_2, \gamma_2)$. This completes the proof in Case 1.

Case 2. Other cases.

Firstly, we will show that $h(M, \gamma) \leq h(M_1, \gamma_1) + h(M_2, \gamma_2)$. Let S be the cross section disk as in the proof of Lemma 2.2. Since (M_1, γ_1) and (M_2, γ_2) are obtained from (M, γ) by a sutured manifold decomposition along S , we call M_i in M the part of $M_i (i=1, 2)$. Let $S_1 (S_2 \text{ resp.})$ be the component of $\text{Fr}_M N(S; M)$ in the part of M_1 (the part of $M_2 \text{ resp.})$. Then we may suppose that $N(S; M) = S \times I$, $S = S \times \{1/2\}$, $S_1 = S \times \{0\}$ and $S_2 = S \times \{1\}$. Moreover we can suppose that each component of $s(\gamma) \cap N(S; M)$ is of the form $a_i \times I$, and set $a_i^1 = a_i \times \{0\}$ and $a_i^2 = a_i \times \{1\}$ ($i=1, \dots, 2n$). (For a_i , see Lemma 2.2.). In the following, we identify (M_1, γ_1) ($(M_2, \gamma_2) \text{ resp.})$ with the component of the sutured manifold decomposition from (M, γ) by decomposing along $S_2 (S_1 \text{ resp.})$ which contains the part of $M_1 (M_2 \text{ resp.})$. Further we assume that each component of $S_2 \cap s(\gamma_1) (S_1 \cap s(\gamma_2) \text{ resp.})$ joins $a_i^2, a_{i+1}^2 (a_i^1, a_{i+1}^1 \text{ resp.})$ for i odd (for i even resp.).

A Heegaard surface $F (F_j (j=1, 2) \text{ resp.})$ is said to be a *nice Heegaard surface* of $(M, \gamma) ((M_j, \gamma_j) (j=1, 2) \text{ resp.})$ if it satisfies the following conditions:

(1) $S_1 \cap F (S_1 \cap F_j \text{ resp.})$ consists of arcs joining a_i^1 and a_{i+1}^1 for i even

- (2) $S_2 \cap F(S_2 \cap F_j$ resp.) consists of arcs joining a_i^2 and a_{i+1}^2 for i odd
 (3) $F \cap (S \times I)(F_j \cap (S \times I)$ resp.) is a disk.

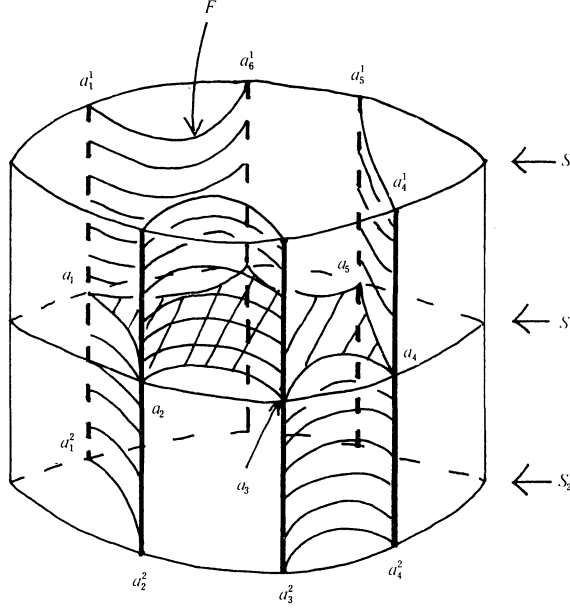


Figure 4.1

A ∂ -compressing hierarchy for a properly embedded disk E_0 in a compression body W is a finite sequence

$$E_0 \xrightarrow{D_1} E_1 \xrightarrow{D_2} \dots \xrightarrow{D_k} E_k$$

of boundary compressions toward $\partial_+ W$ for which

- (1) each D_i is a boundary compressing disk toward $\partial_+ W$ for E_{i-1} ,
- (2) E_i is obtained from E_{i-1} by boundary compression toward $\partial_+ W$ along D_i ,
- (3) each component of E_k is boundary incompressible toward $\partial_+ W$.

The length of a ∂ -compressing hierarchy is the number of boundary compressing disks D_i .

Let (W_1, W'_1) ((W_2, W'_2) resp.) be a Heegaard splitting of (M_1, γ_1) ((M_2, γ_2) resp.). We denote a Heegaard surface of (W_1, W'_1) ((W_2, W'_2) resp.) by F_1 (F_2 resp.). We may assume that each component of $S_2 \cap W_1$ ($S_1 \cap W_2$ resp.) is a "product disk", then each component of $S_1 \cap W_1$ ($S_2 \cap W_2$ resp.) is a product disk too as illustrated in Figure 4.2.

Lemma 4.1. *The Heegaard surface F_j ($j=1, 2$) is ambient isotopic rel γ_j ($j=1, 2$) to a nice Heegaard surface.*

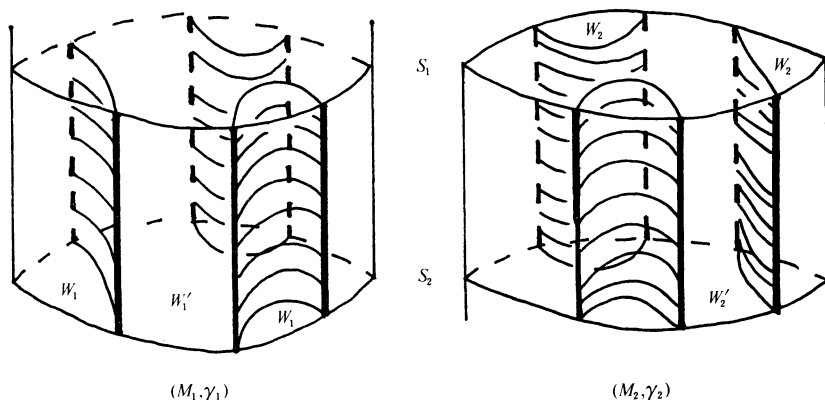


Figure 4.2

Proof. We prove this lemma for (W_1, W'_1) . Put $E_0 = S \cap W'_1$. Then E_0 is a properly embedded disk in W'_1 such that $\partial E_0 \cap \partial_- W'_1$ consists of n arcs. Hence, by Lemma 2.3, there is a ∂ -compressing hierarchy $E_0 \xrightarrow{D_1} E_1 \xrightarrow{D_2} \dots \xrightarrow{D_{n-1}} E_{n-1}$. (Note that the length of the ∂ -compressing hierarchy is $n-1$ since $\#E_{i+1} = \#E_i + 1$ for each i .) We denote $E_{i-1} \cap D_i$ by α_i and we can assume that $\alpha_i \cap \alpha_j = \emptyset$ by an isotopy. Hence we may consider that $\alpha_1, \dots, \alpha_{n-1}$ are arcs properly embedded in $S \cap W'_1$. We can assume that $D_i \cap (S \times [0, 1/2])$ is of the form $\alpha_i \times [0, 1/2]$, where S corresponds to $S \times \{1/2\}$. Let β_i be an arc $D_i \cap (S_1 \cap W'_1)$ on D_i corresponding to α_i , and D'_i be a subdisk of D_i such that $D'_i = \text{cl}(D_i - D_i \cap (S \times 0, 1/2])$. Since $\alpha_i \cap \alpha_j = \emptyset$, we may suppose that $\beta_i \cap \beta_j = \emptyset$. Then D'_1, \dots, D'_{n-1} determines a ∂ -compressing hierarchy for $S_1 \cap W'_1$. Do the ∂ -compressing hierarchy of D'_1, \dots, D'_{n-1} for $S_1 \cap W'_1$, but in this case, we consider that we move F_1 by compressing along D'_i , i.e., slide $F_1 \cap D'_i$ across D'_i and pass β_i . Then we can see that the final surface is a nice Heegaard surface.

Now suppose that (W_1, W'_1) ((W_2, W'_2) resp.) satisfies $h(W_1) = h(M_1, \gamma_1)$ ($h(W_2) = h(M_2, \gamma_2)$ resp.). By Lemma 4.1, F_1 and F_2 are nice and we may assume $F_1 \cap N(S, M) = F_2 \cap N(S, M)$. Thus $F_1 \cup F_2$ forms a surface F in M as illustrated in Figure 4.1.

We show that F is a Heegaard surface of (M, γ) . Let W and W' be the closure of the components of $M - F$ as illustrated in Figure 4.1. Then $W(W'$ resp.) can be naturally regarded as a sutured manifold, and $S_2 \cap W(S_1 \cap W'$ resp.) is a disjoint union of n product disks of $W(W'$ resp.). The sutured manifold obtained from W by decomposing along $S_2 \cap W$ is a disjoint union of W_1 and $\tilde{W}'_2 = W'_2 \cap \text{cl}(M_2 - N(S; M_2))$. Note that W_1 and \tilde{W}'_2 are compression bodies, since \tilde{W}'_2 is a component of the sutured manifold obtained from the compression body W'_2 by decomposing along the product disks $S_2 \cap W$ (see Lemma 2.4).

Hence, by Lemma 2.4, W is a compression body, and $h(W) = h(W_1) + h(\tilde{W}'_2) = h(W_1) + h(W_2)$. Similarly, W' is also a compression body and therefore (W, W') gives a Heegaard splitting of (M, γ) . Thus we have $h(M, \gamma) \leq h(W) = h(W_1) + h(W_2) = h(M_1, \gamma_1) + h(M_2, \gamma_2)$.

Next, we will show that $h(M_1, \gamma_1) + h(M_2, \gamma_2) - (n-1) \leq h(M, \gamma)$. Let (W, W') be a Heegaard splitting of (M, γ) .

Lemma 4.2. *Suppose that (W, W') be a Heegaard splitting of (M, γ) which has a nice Heegaard surface. Then (W, W') induces Heegaard splittings (W_i, W'_i) of (M_i, γ_i) ($i=1, 2$) such that $h(W) = h(W_1) + h(W_2)$.*

Proof. Since (W, W') has a nice Heegaard surface, we may suppose that each component of $W \cap S_2$ and $W' \cap S_1$ is a product disk. Let $W_1(W'_1$ resp.) be the submanifold of $W(W'$ resp.) obtained by cutting along S_2 such that the part of M_1 is contained in $W_1 \cup W'_1$. By using Lemma 2.4, we see that both W_1 and W'_1 are compression bodies, and therefore (W_1, W'_1) is a Heegaard splitting of (M_1, γ_1) . Let $W_2(W'_2$ resp.) be the submanifold of $W(W'$ resp.) obtained by cutting along S_1 such that the part of M_2 is contained in $W_2 \cup W'_2$. Similarly, (W_2, W'_2) is a Heegaard splitting of (M_2, γ_2) . Since $F \cap (S \times I)$ is a disk, $h(W) = h(W_1) + h(W_2)$. Then we have the conclusion of Lemma 4.2.

By Proposition 3.1, we may suppose that every component of $S \cap W$ and $S \cap W'$ is a disk as in Figure 3.1 and that each component of $S \cap F$ joins a_i, a_{i+1} for i even (see Figure 4.3).

Let $S \cap W$ be E_0 and $E_0 \xrightarrow{D_1} E_1 \xrightarrow{D_2} \dots \xrightarrow{D_{n-1}} E_{n-1}$ be a ∂ -compressing hierarchy.

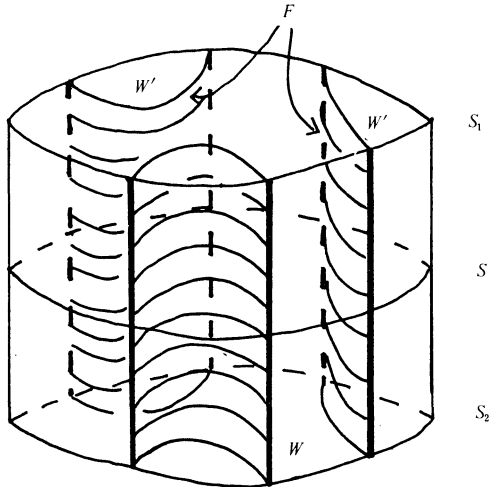


Figure 4.3

We assume that the part of M_i is defined for S before the boundary compression. If D_i is contained in the part of M_2 , we say that D_i is *good*, and if D_i is contained in the part of M_1 we say that D_i is *bad*. Let α_i be the arc $E_{i-1} \cap D_i$. If $N(\alpha_i; D_i)$ is contained in the part of M_2 and D_i is not good, we say that D_i is *semi good*. If $N(\alpha_i; D_i)$ is contained in the part of M_1 and D_i is not bad, we say that D_i is *semi bad*. If D_1, \dots, D_{n-1} are all good, we say that this ∂ -compressing hierarchy is *good*.

We denote $E_{i-1} \cap D_i$ by α_i and we can assume that $\alpha_i \cap \alpha_j = \emptyset$ by an isotopy. Hence we may consider that $\alpha_1, \dots, \alpha_{n-1}$ are arcs properly embedded in $S \cap W$.

By the proof of Lemma 4.1, we have:

Lemma 4.3. *If $S \cap W$ has a good ∂ -compressing hierarchy, (W, W') has a nice Heegaard surface.*

We suppose that D_i is bad or semi bad some i . Let \tilde{F} be a surface obtained from F by attaching 1-handles in the neighborhood of α_i contained in the part of M_2 for all bad or semi bad disks D_i as illustrated in Figure 4.4, and let \tilde{W}, \tilde{W}' be the closure of the components of $M - \tilde{F}$ corresponding to W, W' respectively.

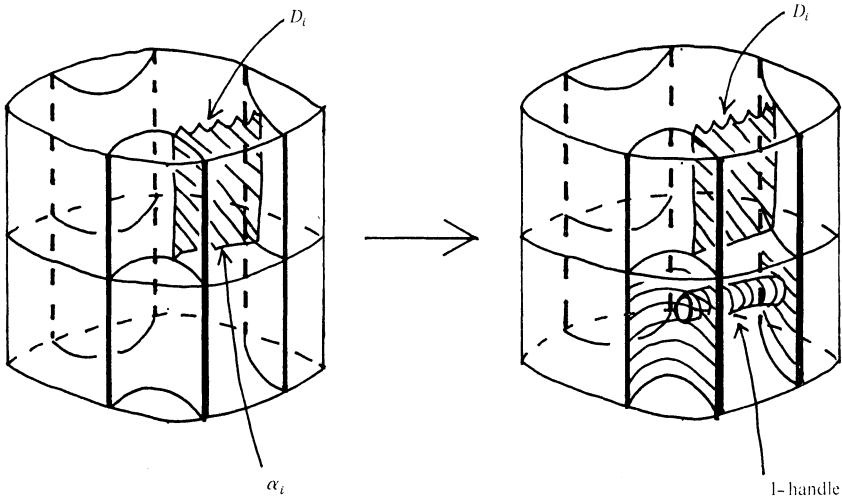


Figure 4.4

Then we have:

Claim 4.4. *(\tilde{W}, \tilde{W}') is a Heegaard splitting of (M, γ) .*

Proof. Since \tilde{W}' is obtained from a compression body W' by attaching 1-handles on $\partial_+ W'$, \tilde{W}' is a compression body. So we have only to prove that

\tilde{W} is a compression body. Let $D_{i_1}, \dots, D_{i_t} (i_1 < \dots < i_t)$ be the bad or semi bad disks and H_1, \dots, H_t by be the corresponding 1-handles. Put $F_0 = F$ and let $F_\ell (1 \leq \ell \leq t)$ be the surface obtained from $F_{\ell-1}$ by attaching the 1-handle H_ℓ . Let W_ℓ be the closure of the component of $M - F_\ell$ corresponding to W . Then we have $F_t = \tilde{F}$ and $W_t = \tilde{W}$. For each D_i , there is a 3-cell C_i such that C_i is homeomorphic to $D_i \times I$ with; D_i corresponds to $D_i \times \{1/2\}$; $\alpha_i \times I \subset S$; $\text{cl}(\partial D_i - \alpha_i) \times I \subset F$. Then we may suppose that for $j > i$, each component of $D_j \cap C_i$ (if exists) is of the form $D_i \times \{\text{a point}\}$ in C_i . In this case, we say that a subdisk of D_j is parallel with the trace of D_i . Moreover we may suppose that D_j intersects each of the attaching solid 1-handle ($\cong D^2 \times I$) in disks of the form $(\text{arcs}) \times I$. Now we show inductively that $W_\ell (1 \leq \ell \leq t)$ is a compression body. Let \tilde{D}_{i_1} be a disk obtained from D_{i_1} by extending α_{i_1} to H_1 vertically in $S \times I$. Cut W_1 along \tilde{D}_{i_1} , then we get a compression body homeomorphic to W . Hence W_1 is a compression body. Assume that $W_{\ell-1}$ is a compression body. Then we attach the ℓ -th 1-handle in a neighborhood of α_{i_ℓ} . If D_{i_ℓ} is semi bad, we have two cases, i.e., whether D_{i_ℓ} contains subdisks which are parallel with the trace of bad or semi bad disks $D_j (i_1 \leq j \leq i_\ell)$ or not. If D_{i_ℓ} is a bad disk, or a semi bad disk which does not contain subdisks which are parallel with the trace of bad or semi bad disks, W_ℓ is a compression body by the same argument in case of W_1 . If D_{i_ℓ} is a semi bad disk which contains subdisks which are parallel with the trace of bad or semi bad disks, let B_1, \dots, B_m be disk components of $D_{i_\ell} \cap (\text{solid 1-handles})$ and \tilde{D}_{i_ℓ} be a disk obtained from D_{i_ℓ} by extending to α_{i_ℓ} to H_ℓ vertically in $S \times I$ and $D'_{i_\ell} = \text{cl}(\tilde{D}_{i_\ell} - D_{i_\ell})$. Let D''_{i_ℓ} be the component of $\text{cl}(D_{i_\ell} - \cup B_h)$ which contains α_{i_ℓ} . Then $\text{Int } D''_{i_\ell} \cap (1\text{-handles}) = \emptyset$. Let D be a disk $D'_{i_\ell} \cup D''_{i_\ell}$. Cut W_ℓ along D , then we get a compression body homeomorphic to $W_{\ell-1}$. Hence W_ℓ is a compression body. Then we have the conclusion.

Claim 4.5. $S \cap \tilde{W}$ has a good ∂ -compressing hierarchy.

Proof. If D_i is bad or semi bad, let \tilde{D}_i be the disk as in the proof of Claim 4.4 and let D'_i be a disk $\text{cl}(\tilde{D}_i - D_i)$. Replace D_i by D'_i , then we get a good boundary compressing disk D'_i toward \tilde{F} . If D_i is semi good, D_i contains subdisks which are parallel with the trace of bad or semi bad disks $D_j (1 \leq j < i)$. Let B_1, \dots, B_h be disk components of $D_i \cap (1\text{-handles})$. Let D'_i denote a subdisk of D_i where D'_i is the component of $\text{cl}(D_i - \cup B_m)$ which contains α_i . Then D'_i is a good boundary compressing disk and replace D_i to D'_i . Then it is clear that these boundary compressing disks constitute good ∂ -compressing hierarchy.

Lemma 4.6. $h(M_1, \gamma_1) + h(M_2, \gamma_2) - t \leq h(M, \gamma)$, where t is the number of the bad or semi bad disks.

Proof. Suppose that (W, W') is a Heegaard splitting of (M, γ) such that $h(W)=h(M, \gamma)$. By Claims 4.4, 4.5, and Lemma 4.3, (\tilde{W}, \tilde{W}') has a nice Heegaard surface. Then (\tilde{W}, \tilde{W}') induces Heegaard splittings $(\tilde{W}_1, \tilde{W}'_1)$ of (M_1, γ_1) and $(\tilde{W}_2, \tilde{W}'_2)$ of (M_2, γ_2) such that $h(\tilde{W})=h(\tilde{W}_1)+h(\tilde{W}_2)$, by Lemma 4.2. Since $h(\tilde{W})=t+h(W)=t+h(M, \gamma)$ and $h(\tilde{W}_i) \geq h(M_i, \gamma_i)$ ($i=1, 2$), $t+h(M, \gamma) \geq h(M_1, \gamma_1)+h(M_2, \gamma_2)$. Hence we have the conclusion.

By Lemma 4.6 and the fact that $S \cap W$ has a ∂ -compressing hierarchy of length $n-1$, we have $h(M_1, \gamma_1)+h(M_2, \gamma_2)-(n-1) \leq h(M, \gamma)$. This completes the proof of in Case 2.

Proof of Theorem 2

We note that the idea of the proof of Theorem 2 is due to ([6] proof of Theorem 3.1). Let $E=S^2-\text{Int } D$, where D is an attaching disk and S^2 is a 2-sphere of a $2n$ -Murasugi sum, and T the surface obtained by summing R_1 and R_2 along E . After a tiny isotopy, we may suppose that $R \cap T=\phi$ (see Figure 4.5).

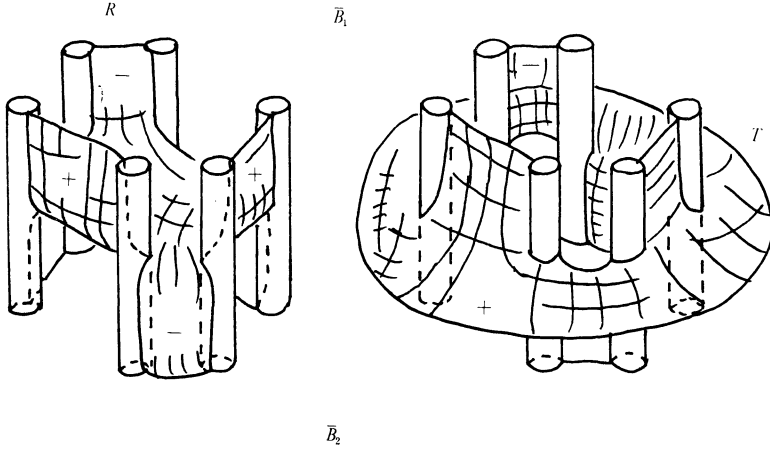


Figure 4.5

T is a properly embedded in (M, γ) and $\partial T=s(\gamma)$. We assume that + side and - side of R and T are as in Figure 4.5. R and T separates (M, γ) into two sutured manifold (M', γ') and (M'', γ'') , where M' (M'' resp.) is the closure of the component of $M-T$ between T and $R_-(\gamma)$ ($R_+(\gamma)$ resp.). Let $\gamma'=\gamma \cap M'$ ($\gamma''=\gamma \cap M''$) and D_i (D_i' resp.) be the product disks in (M', γ') ((M'', γ'') resp.) corresponding to $\mu_i \times I$ ($\nu_i \times I$ resp.) for $1 \leq i \leq n$. Then the product disks $\cup D_i'$ decompose (M'', γ'') into a product sutured manifold $((R_2-\text{Int } N(D)) \times I, \partial(R_2-\text{Int } N(D)) \times I)$ and a sutured manifold homeomor-

phic to (M_1, γ_1) . Since R_1 is a fiber surface, (M_1, γ_1) is a product sutured manifold, and hence (M'', γ'') is a product sutured manifold by Lemma 2.4. Hence $(M', \gamma') \cong (M, \gamma)$. The product disks $\cup D_i^2$ decompose (M', γ') into a product sutured manifold $((R_1 - \text{Int } N(D)) \times I, \partial(R_1 - \text{Int } N(D)) \times I)$ and a sutured manifold homeomorphic to (M_2, γ_2) . Thus by the proof of Theorem 1 for the case $n=1$, we see $h(M', \gamma') = h(M_2, \gamma_2)$. Hence we have $h(M, \gamma) = h(M_2, \gamma_2)$. This completes the proof of Theorem 2.

5. Examples

In this section, we give examples of Murasugi sums for which we have $h(M_1, \gamma_1) + h(M_2, \gamma_2) - (n-1) = h(M, \gamma)$ for any n .

Let L be the pretzel link $L = P(4, 4, \dots, 4)$ in S^3 and R a Seifert surface for L as illustrated in Figure 5.1.

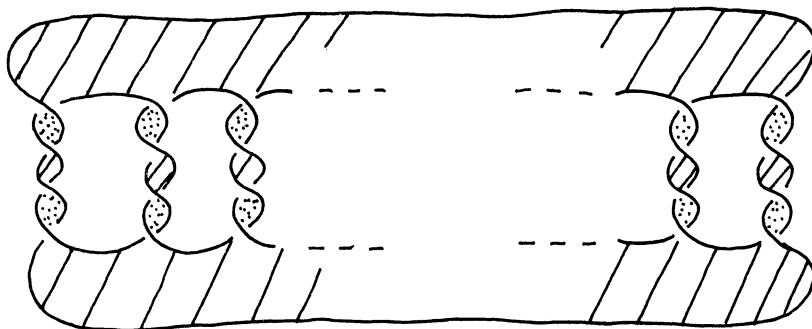


Figure 5.1

Let \tilde{R} be a Seifert surface obtained from two copies of R by a $2n$ -Murasugi sum as in Figure 5.2.

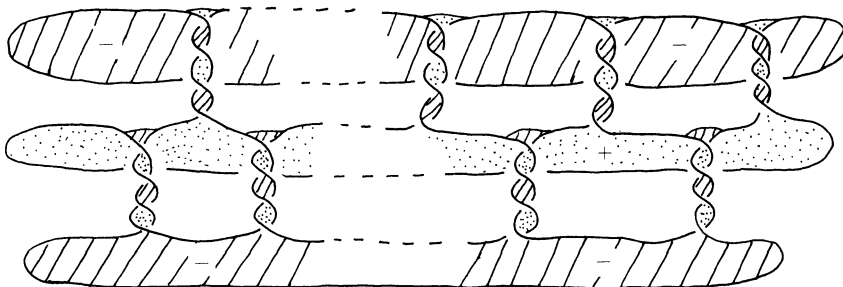


Figure 5.2

Let (M, γ) and $(\tilde{M}, \tilde{\gamma})$ be the complementary sutured manifolds for R and \tilde{R} respectively. Then we have:

Claim 5.1. $h(M, \gamma) = n - 1$.

Proof. Note that M is a cube with $(n - 1)$ handles, since the product sutured manifold for R is a cube with $(n - 1)$ handles normally embedded in S^3 . At first, we show that $h(M, \gamma) \geq n - 1$. Let (N, δ) be a product sutured manifold for R and we take the free basis $l_1, \dots, l_{n-1}, m_1, \dots, m_{n-1}$ of $H_1(\partial N)$ and the free basis a_1, \dots, a_{n-1} of $H_1(R_-(\delta))$ as illustrated in Figure 5.3.

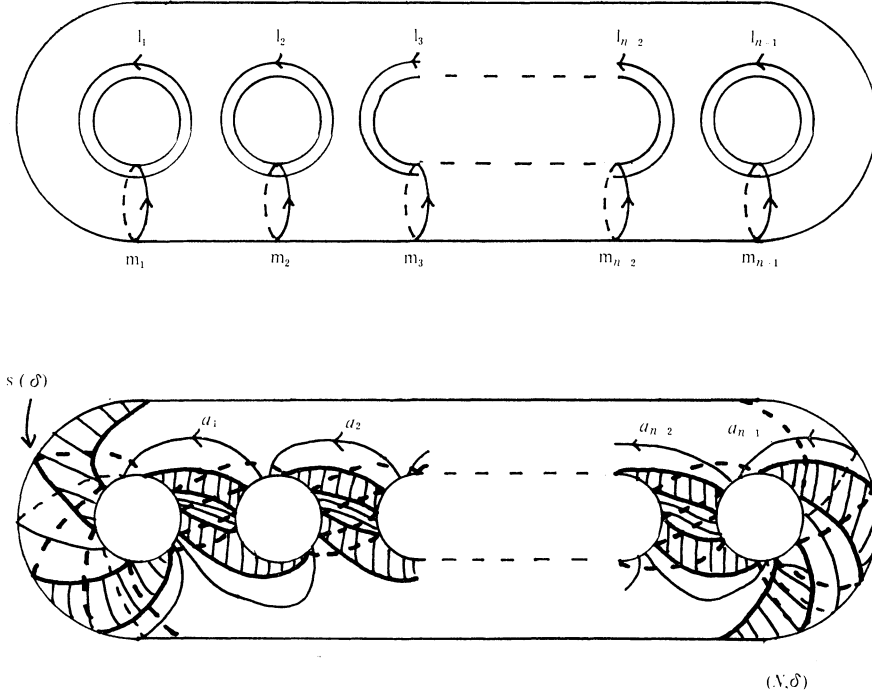


Figure 5.3

Let \bar{l}_i be the elements of $H_1(M)$ represented by $m_i (1 \leq i \leq n - 1)$. Then $\{\bar{l}_1, \dots, \bar{l}_{n-1}\}$ is the free basis of $H_1(M)$, and we have $a_1 = 4\bar{l}_1 - 2\bar{l}_2$, $a_i = -2\bar{l}_{i-1} + 4\bar{l}_i - 2\bar{l}_{i+1} (2 \leq i \leq n - 2)$, $a_{n-1} = -2\bar{l}_{n-1} + 4\bar{l}_{n-1}$. Now, assume that $h(M, \gamma) \leq n - 2$, then $M \cong R_+(\gamma) \times I \cup (n - 2)$ 1-handles $\cup (n - 2)$ 2-handles. Since $H_1(R_+(\gamma)) \approx \langle a_1, \dots, a_{n-1} \rangle$, we have $H_1(M) \approx \langle a_1, \dots, a_{n-1}, h_1, \dots, h_{n-2} | r_1, \dots, r_{n-2} \rangle$, where $h_i (1 \leq i \leq n - 2)$ is a generator corresponding to attaching 1-handle and $r_j (1 \leq j \leq n - 2)$ is a relation arising from a 2-handle. Then, $H_1(M) / i_* H_1(R_+(\gamma)) \approx \langle h_1, \dots, h_{n-2} | r_1, \dots, r_{n-2} \rangle$, and this abelian group is generated by $(n - 2)$ elements. On the other hand, $H_1(M) / \langle a_1, \dots, a_{n-1} \rangle \approx \langle \bar{l}_1, \dots, \bar{l}_{n-1} | 4\bar{l}_1 - 2\bar{l}_2, -2\bar{l}_1 + 4\bar{l}_2 - 2\bar{l}_3, \dots, -2\bar{l}_{n-3} + 4\bar{l}_{n-2} - 2\bar{l}_{n-1}, -2\bar{l}_{n-2} + 4\bar{l}_{n-1} \rangle \approx \bigoplus_{n-2}^{n-2} \mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$, and this abelian group cannot be generated by less than $(n - 1)$ elements by the fundamental theorem on the abelian group. This is a contradiction, and hence $h(M, \gamma) \geq n - 1$.

Next, we show that $h(M, \gamma) = n - 1$. We consider properly embedded arcs $\alpha_1, \dots, \alpha_{n-1}$ in (M, γ) such that $\partial\alpha_i \subset R_-(\gamma) = R_+(\delta)$ as illustrated in Figure 5.4.

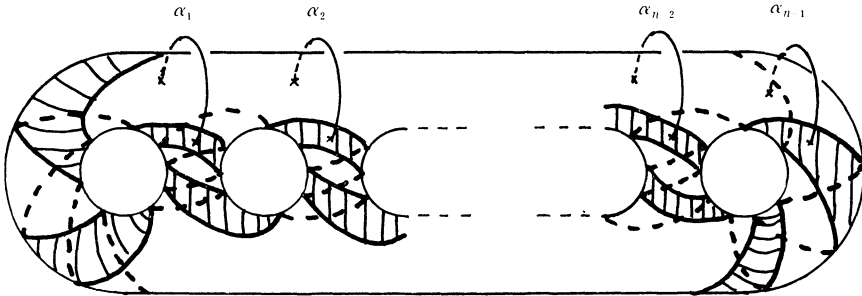


Figure 5.4

Put $X = R_-(\gamma) \times I \cup N(\alpha_i; M)$, then X is a compression body with $h(X) = n - 1$. On the other hand, Figure 5.5 shows that $\text{cl}(M - X)$ is also a compression body with $h(\text{cl}(M - X)) = n - 1$. Then $h(M, \gamma) = n - 1$.

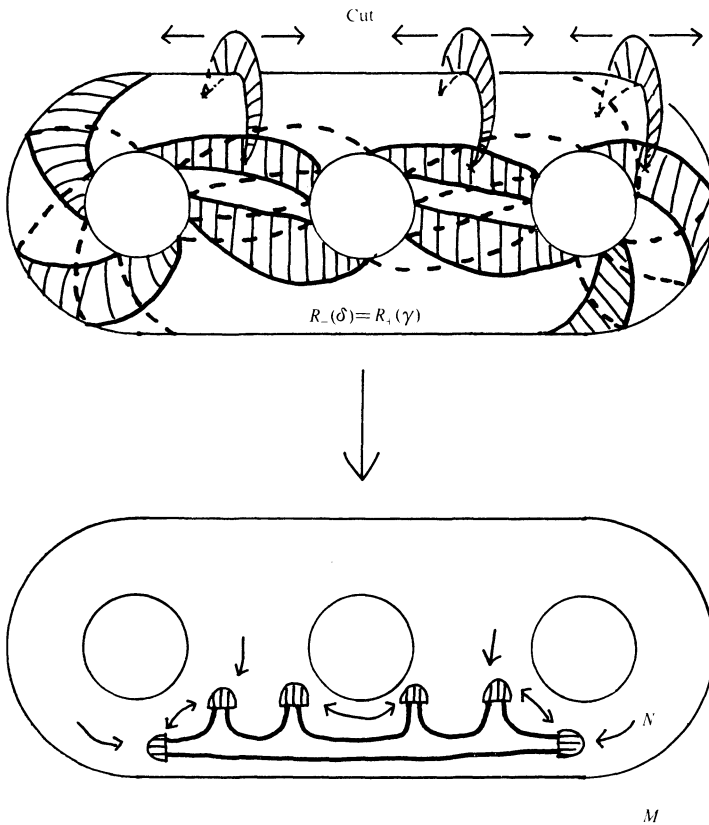


Figure 5.5

Claim 5.2. $h(\tilde{M}, \tilde{\gamma})=n-1$.

Proof. We assume that $+$ -side and $-$ -side of \tilde{R} are as in Figure 5.2. Let $(\tilde{N}, \tilde{\delta})$ be a product sutured manifold for \tilde{R} and x_1, \dots, x_{2n-2} the free basis of $H_1(R_+(\tilde{\delta}))$ as illustrated in Figure 5.6.

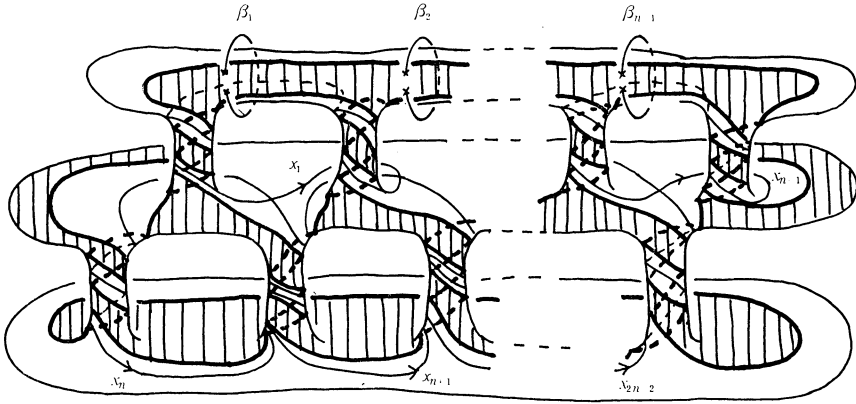


Figure 5.6

We consider properly embedded arcs $\beta_1, \dots, \beta_{n-1}$ in $(\tilde{M}, \tilde{\gamma})$ such that $\partial\beta_i \subset R_+(\tilde{\gamma})=R_-(\tilde{\delta})$ as illustrated in Figure 5.6. Put $X=R_+(\tilde{\gamma}) \times I \cup N(\beta_i; \tilde{M})$, then X is a compression body with $h(X)=n-1$. Put $Y=\text{cl}(\tilde{M}-X)$, and let d_i be a properly embedded disk in Y corresponding to the dual of $N(\beta_i; \tilde{M})$ ($1 \leq i \leq n-1$). We may choose d_i so that x_i ($1 \leq i \leq n-1$) crosses transversely d_i ($1 \leq i \leq n-1$) once. Then cutting \tilde{M} along d_i and moving by an isotopy, we can assume that x_1, \dots, x_{n-1} do not cross other meridian disks in \tilde{M} and we can see that x_i ($n \leq i \leq 2n-2$) crosses transversely a meridian disk of \tilde{M} once. By an argument similar to the proof of Claim 5.1, Y is homeomorphic to $R_-(\tilde{\gamma}) \times I \cup (n-1)$ 1-handles. Thus we have $h(\tilde{M}, \tilde{\gamma}) \leq n-1$, on the other hand, by Theorem 1, $n-1 \leq h(\tilde{M}, \tilde{\gamma}) \leq 2n-2$. Hence we have $h(\tilde{M}, \tilde{\gamma})=n-1$.

References

- [1] G. Burde, H. Zieschang: *Knots*, De Gruyter Studies in Math. 5, Walter de Gruyter, Berlin and New York, 1985.
- [2] A. Casson, C. Gordon: *Reducing Heegaard splitting*, *Topology and its Appl.* **27** (1987), 275–283.
- [3] D. Gabai: *The Murasugi sum is a natural geometric operation*, *Contemp. Math.* **20** (1983), 131–143.
- [4] D. Gabai: *Foliations and the topology of 3-manifolds*, *J. Diff. Geom.* **18** (1983), 445–503.

- [5] D. Gabai: *The Murasugi sum is a natural geometric operation II*, *Contemp. Math.* **44** (1985), 93–100.
- [6] D. Gabai: *Detecting fibered links in S^3* , *Comm. Math. Helv.* **61** (1986), 519–555.
- [7] D. Gabai: *Genus is superadditive under band connected sum*, *Topology* **26** (1987), 209–210.
- [8] W. Haken: *Some results on surfaces in 3-manifolds*, *Studies in Modern Topology*, Math. Assoc. Amer., Prentice Hall, 1968.
- [9] J. Hempel: *3-manifolds*, *Ann. of Math. Studies No.86*, Princeton University Press, Princeton N.J., 1976.
- [10] W. Jaco: *Lectures on three manifolds topology*, *CBMS Regional Conf. Ser. in Math* **43**, 1980.
- [11] T. Kobayashi: *Fibered links and unknotting operations*, *Osaka J. Math.* **26** (1989), 699–742.
- [12] T. Kobayashi: *Uniqueness of minimal genus Seifert surfaces for links*, *Topology and its appl.* **33** (1989), 265–279.
- [13] M. Ochiai: *On Haken's theorem and its extension*, *Osaka J. Math.* **20** (1983), 461–468.
- [14] M. Scharlemann: *Sutured manifolds and generalized Thurston norms*, *J. Diff. Geom.* **29** (1989), 557–614.
- [15] J. Stallings: *Constructions of fibered knots and links*, *Proc. Symp. Pure Math. AMS* **27** (1975), 315–319.

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