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HEREDITARY RINGS AND RELATIVE PROJECTIVES

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We have given some characterizations of right Nakayama rings related to almost relative projectives or almost relative injectives [12]. In this paper we shall study particularly the condition (C) (resp (C*)) in [12]. Let R be a right artinian ring and let M, N, U and V be R-modules. (C): M is almost N/N'-projective for any submodule N' of N, provided M is almost N-projective (resp. (C*): U is almost V'-injective for any submodule V' of V, provided U is almost V-injective). We shall replace the role of N (resp. V) by that of M (resp. U) in the above.

We shall give several characterizations of semi-primary rings whose Jacobson radical is square-zero in the above manner and in the similar manner for relative projectives, respectively. Further from those viewpoints we shall characterize a certain type of hereditary rings over which every submodule of any indecomposable quasi-projective module is also quasi-projective (cf. [6]), and two-sided Nakayama rings with radical square-zero, respectively.

1. Relative projectives

In this paper we always assume that R is a ring with identity. Every module M is a unitary right R-module. We shall denote *the length, the Jacobson radical* and *an injective hull of* M by $|M|$, $J(M)$ and $E(M)$, respectively. By Soc(M) and $\mathrm{Soc}_i(M)$ we denote *the socle* and *the ith lower Loewy series of* M. We follow [4] and [11] for definitions of almost relative projectives and almost relative injectives.

In this section we study some conditions below, when M is N-projective for R-modules M and N (resp. U is V-injective for R-modules U and V).

(E) *M/M' is N-projective* and

(F) *M' is N-projective*

for any submodule M' of M, provided M is N-projective.

(resp.

(E*) *U' is V-injective* and

(F*) *U/U' is V-injective*

for any submodule U' is of U, provided U is V-injective).

We first give a remark on the above conditions. Take any R-module T. Then R is always T-projective as R-modules. If we assume (E) (resp. (F)) for R, then every factor module R/I (resp. I) is T-projective, and hence R/I (resp. I) is projective, where I is a right ideal of R. Therefore R is semi-simple (resp. right hereditary). Further let M be a quasi-projective module. Then M is M-projective. If (F) holds true, N is M-projective for any submodule N of M, and hence N is N-projective (cf. [16], § 16), i.e. N is quasi-projective. Hence (F) implies

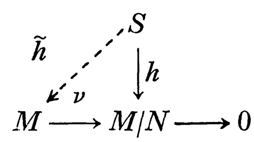
(G) *every submodule of finitely generated and quasi-projective module P is quasi-projective,*

which was studied in [6].

In the following we shall skip proofs for injectives if they are dual to ones for projectives.

Lemma 1. *Let $M\subset N$ be R-modules and S a simple R-module. Assume that S is isomorphic to a sub-factor module T/N of M. If S is M-projective, then there exists a simple submodule S' of M such that $T=S'\oplus N$.*

Proof. This is clear from the following diagram:



where h is the given isomorphism of S to T/N.

Proposition 1. *Let R be a semi-perfect ring. Then the following conditions are equivalent:*

1) *(E) holds ture when M and N are any local modules.*
1^*) *(E^*) holds true when U and V are any uniform modules.*
2) *R is semi-simple.*

Proof. 1)→2) Let e be a primitive idempotent. Since eR is eR-projective, eR/eJ is eR-projective by (E). Hence $eJ=0$ from Lemma 1. The remaining parts are similar.

Theorem 1. *Let R be a (right) artinian ring. Then the following conditions are equivalent:*

1) *(F) holds true when M and N are any local modules.*
1^*) *(F^*) holds true when U and V are any uniform modules.*

2) *R is a (right) hereditary ring with* $J^2=0$.
3) *Every proper submodule of any local module is projective.*
3$^\sharp$) *Every proper factor module of any uniform module is injective.*
4) *(F) holds true when M and N are any finitely generated R-modules.*
4$^\sharp$) *(F$^\sharp$) holds true when U and V are any finitely generated R-modules.*
5) *(F) holds true when M is finitely generated and quasi-projective.*
6) *(G) holds ture.*

Proof. 1)→2) Let e be a primitive idempotent and $eJ/eJ^2=\Sigma_i\oplus S_i$, where the S_i are simple. Assume $S_1\sim fR/fJ$ for a primitive idempotent f. eR/eJ^2 is fR/fJ^2-projective by [1], p. 22, Exercise 4. Since $S_1\subset eR/eJ^2$, S_1 is fR/fJ^2-projective by (F). Hence $fJ=0$ by Lemma 1, and S_1 is projective. Accordingly eJ/eJ^2 is projective and hence $eJ=eJ^2\oplus\Sigma_i\oplus S_i'$; $S_i'\sim S_i$ for all i. Therefore $eJ=\Sigma_i\oplus S_i'$ is projective and so $eJ^2=0$. Thus R is a (right) hereditary ring with $J^2=0$ [2].

2)→3) We assume that R is a hereditary ring with $J^2=0$. Since eJ is projective and semi-simple, every factor module of eJ is projective. Hence every proper submodule D/A of eR/A is projective.

3)→1) This is trivial.

1$^\sharp$)→2) and 2)→3$^\sharp$) They are dual to 1)→2) and 2)→3), respectively.

2)→4) Let R be a right artinian hereditary ring with radical square-zero. Then J is semisimple and projective. Let M be a finitely generated R-module and $P=e_1R\oplus e_2R\oplus\cdots\oplus e_nR$ a projective cover of M, i.e. $M\sim P/Q$. Let A be any submodule of P containing Q. Since P is a lifting module, $P=P_1\oplus P_2$, $A\supset P_1$ and $A\cap P_2$ is small in P_2. Let π_i be the projection of P onto P_i. Put $Q_i=Q\cap P_i$ and $Q^i=\pi_i(Q)$. Then h: $Q^2/Q_2\sim Q^1/Q_1$ (see [11], p. 449) and $P/(Q_1\oplus Q_2)=P_1/Q_1\oplus P_2/Q_2\supset A/(Q_1\oplus Q_2)=P_1/Q_1\oplus(A\cap P_2)/Q_2\supset Q/(Q_1\oplus Q_2)$. Since $A\cap P_2=\pi_2(A)\subset J(P_2)$, $A\cap P_2$ is semisimple and projective. Hence $(A\cap P_2)/Q_2=Q^2/Q_2\oplus Q^*/Q_2$ for some submodule Q^* of A and $P_1/Q_1\oplus(A\cap P_2)/Q_2=P_1/Q_1\oplus(Q^2/Q_2)(h)\oplus Q^*/Q_2$; $Q/(Q_1\oplus Q_2)=(Q^2/Q_2)(h)=\{q+Q_2+h(q+Q_2)\,|\,q\in Q^2\}$. Therefore $A/Q\sim(A/(Q_1\oplus Q_2))/(Q/(Q_1\oplus Q_2))\sim P_1/Q_1\oplus Q^*/Q_2$. Now P_1 is a projective cover of P_1/Q_1, since P is that of M, and we assume that M is N-projective for a finitely generated R-module N. Let θ be any homomorphism of P_1 to N. Then θ is trivially extendible to a homomorphisms θ' of P to N. Since $\theta'(Q)=0$ by [1], p. 22, Exercise 4, $\theta(Q_1)=0$. Therefore P_1/Q_1 is N-projective. Since $(A\cap P_2)/Q_2$ (and hence Q^*/Q_2) is projective, A/Q is N-projective. Therefore (F) holds true for any finitely generated R-modules.

4)→1) and 4$^\sharp$)→1$^\sharp$) Those are trivial.

2)→4$^\sharp$) Assume that R is (right) hereditary. Let $U\supset U'$ and V be finitely generated R-modules. We may assume $E=\mathrm{E}(U)=\mathrm{E}(U')\oplus E_2$ and put $\mathrm{E}(U')=E_1$. Since U' is essential in E_1, $U'\supset\mathrm{Soc}(E_1)$. Furthermore since $E_1/\mathrm{Soc}(E_1)$ is semisimple and injective by 2), so is $U^*/U,'$ for any submodule $U^*(\supset U')$ in E_1. Now $\bar{E}=E/U'=\bar{E}_1\oplus\bar{E}_2\supset U/U'=\bar{U}$, where $\bar{E}_1=E_1/U'$ and $\bar{E}_2=(E_2+U')/U'\sim E_2$ (via ρ). Let π_i be the projection of $\bar{E}$ onto $\bar{E}_i$. Put $\bar{U}^i=\pi_i(\bar{U})$ and $\bar{U}_i=\bar{E}_1\cap\bar{U}$. Then h: $\bar{U}^2/\bar{U}_2\sim\bar{U}^1/\bar{U}_1$ and $\bar{E}_1=\bar{U}_1\oplus\bar{U}^*\oplus\bar{E}^*{}_1$, since $\bar{E}_1$ is semisimple, where τ: $\bar{U}^1/\bar{U}_1\sim\bar{U}^*$. Let ν be the natural epimorphism of $\bar{U}^2$ to $\bar{U}^2/\bar{U}_2$. $\bar{U}^1/\bar{U}_1(\subset\bar{E}_1)$ being injective, $\tau h\nu$ is extended to σ: $\bar{E}_2\to\bar{E}_1$ with $\sigma(\bar{U}_2)=0$. Further $\bar{E}=\bar{E}_1\oplus\bar{E}_2(\sigma)$ and $\bar{U}=\bar{U}_1\oplus\bar{E}_2(\sigma)\cap\bar{U}$. Assume that U is V-injective and take a homomorphism θ: $V\to\bar{E}_2(\sigma)$. We have the natural isomorphism μ: $\bar{E}_2(\sigma)\to\bar{E}_2$, $(\mu(x+\sigma(x))=x$ for $x\in\bar{E}_2)$. Put $\theta^*=\rho\mu\theta$: $V\to E_2\subset E_1\oplus E_2$. Then $\theta^*(V)\subset E_2\cap U$ by [1], Proposition 4.5. Hence $\theta(V)\subset\mu^{-1}\rho^{-1}(E_2\cap U)=\mu^{-1}(\bar{E}_2\cap\bar{U})=(\bar{E}_2\cap\bar{U})(\sigma)=\bar{E}_2\cap\bar{U}\subset\bar{U}$ for $\sigma(\bar{U}_2)=0$. Accordingly $\theta(V)\subset\bar{U}\cap\bar{E}_2(\sigma)$ and hence $\bar{U}\cap\bar{E}_2(\sigma)$ is V-injective. Furthermore $\bar{U}_1$ is injective. Therefore $\bar{U}$ is V-injective.

4)→5) This is trivial.

5)→6) This is shown before Lemma 1.

6)→2) This is due to [6].

In Proposition 1 we have used a fact that (E) (resp. (E*)) holds true for local modules $M=eR/A$ and $N=eR/B$, i.e., M and N have the same projective cover eR, where e runs through over all the primitive idempotents (resp. for unifrom modules U and V in $E(S)$, where S runs through over all the simple modules). On the other hand, we have used, in Theorem 1, a fact that (F) holds true for local modules eR/A and fR/C. From this observation we restrict ourselves to a case $e\sim f$ in (F). By (H) (resp. (H*)) we denote the condition (F) (resp. (F*)) satisfied only for $M=eR/A$ and $N=eR/B$, *where A, B are submodules of eR and e is any primitive idempotent* (*resp. only for U and V in* E(S) *and S is any simple module*). Similarly we define (I) where the quasi-projective module P in (G) is indecomposable.

We note the following fact. Let T be the basic ring of R. It is well known that the category of all the right R-modules is equivalent to that of all the right T-modules. Further the local modules correspeond to each other. Hence we may assume that R is a basic ring when we study local modules.

In general we do not know a characterization of rings with (H). However we study it in a praticular case.

Lemma 2. *Assume $J^2=0$. Let A be an R-module. Consider a diagram*

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow h & & \\ eR/B & \xrightarrow{\nu} & eR/C & \longrightarrow & 0 \end{array}$$

for $B\subset C\subset eR$. If h is not an epimorphism, then there exists $\tilde{h}$: $A\to eR/B$ with $\nu\tilde{h}=h$.

Proof. The above diagram induces

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow h & & \\ \nu^{-1}(h(A)) & \longrightarrow & h(A) & \xrightarrow{\nu} & 0 \end{array}$$

If $h(A)\neq eR/C$, $h(A)\subset eJ/C$. Since eJ is semi-simple and $\nu^{-1}(h(A))\subset eJ/B$, the lemma is clear.

Proposition 2. *Let R be semi-perfect. If (H) holds true, then* 1): $eJe=0$ *for each primitive idempotent e, and* 2): *(I) holds true.*

Proof. 1): Let x be an element in eRe. Then $eR\supset xeR\sim eR/A$ for some A. Since eR is eR-projective, eR/A is also eR-projective by (H). Hence $A=0$ or $A=eR$ by [1], p. 22, Exercise 4. Therefore $eJe=0$. 2) is given before Lemma 1.

Corollary 1. *Let R be a basic and right artinian ring. Let $1=e_1+e_2+\cdots+e_n$, where $\{e_i\}$ is a set of mutually orthogonal primitive idempotenmts.* 1): *If $n=1$, (H) holds ture if and only if R is a division ring.* 2): *If $n=2$, (H) holds true if and only if $J^2=0$ and $e_iJe_i=0$ for $i=1,2$.* 3): *If $J^2=0$, (H) holds true if and only if $e_iJe_i=0$ for all i.* 4): *If (H) holds true, then $J^n=0$.*

Proof. 1) is clear from Proposition 2. Assume $1=e_1+e_2$ and (H). Then if $e_1J\neq 0$, by Proposition 2 $\mathrm{Soc}(e_1R)\sim(e_2R/e_2J)^{(t)}$; the direct sum of t-copies of e_2R/e_2J. Since e_1R is e_1R-projective, e_2R/e_2J is e_1R-projective by (H). Hence $e_1Je_2J=0$. Similarly $e_2Je_1J=0$. Therefore $J^2=\Sigma\, e_iJe_jJ=0$. In the same manner we can show that for each e_i there exists $e_j(\neq e_i)$ such that $e_iJe_jJ=0$. Hence $J^n=0$. Finally assume $J^2=0$ and $e_iJe_i=0$ for all i. Then eJ is smi-simple, and hence (H) holds true by Lemma 2.

We refer [7], [11] and [12] for definitions of Nakayama rings and co-Nakayama rings.

Corollary 2. *Let R be a right Nakayama ring. Then (H) holds true if and only if $e_iJe_i=0$ for all i.*

Proof. "only if" part is given in Proposition 2. We note that if R is right Nakayama, then $e_iJe_i=0$ if and only if any two of distinct (simple) sub-factor modules of e_iR are not isomorphic to each other for all i. We suppose $e_iJe_i=0$. Assume that eR/eJ^j is eR/eJ^i-projective. Then $i\leqq j$. Take any diagram:

$$\begin{array}{ccccc} & & eJ^{k'}/eJ^{j} & & \\ & & \big\downarrow h & & \\ eR/eJ^{i} & \xrightarrow{\nu} & eR/eJ^{k} & \longrightarrow & 0 \end{array}$$

Put $eJ^{k'}/eJ^{j} \sim fR/fJ^{q}$ for a primitive idempotent f. Since $h(eJ^{k'}/eJ^{j})=eJ^{k'}/eJ^{k}$ by the initial remark, the above diagram induces the following

$$\begin{array}{ccccc} & & fR/fJ^{q} & & \\ & & \big\downarrow h' & & \\ fR/fJ^{q-y} & \xrightarrow{\nu'} & fR/fJ^{q-x} & \longrightarrow & 0 \end{array}$$

where $y<x$, $fR/fJ^{q-x} \sim h(eJ^{k'}/eJ^{j})$ and $fR/fJ^{q-y} \sim eJ^{k'}/eJ^{i}$. Since h' is given by a unit in fRf and $y=j-i\geqq o$, we obtain $\tilde{h}'$: $fR/fJ^{q}\to fR/fJ^{q-y}$ with $\nu'\tilde{h}'=h'$. Therefore we get $\tilde{h}$: $eJ^{k'}/eJ^{j}\to eR/eJ^{i}$ with $\nu\tilde{h}=h$.

If R is right Nakayama, then (I) holds true, however (H) does not in general. Hence though (G) and (F) are equivalent over right artinian rings, (I) and (H) are not. Further we have $e_iJe_i=0$ for every hereditary ring, and we shall show in the next section that (H) holds true only on very special hereditary rings.

2. Hereditary rings with (H)

In the last part of the previous section, we consider the property (H). We shall study artinian hereditary rings with (H) in this section. Now we assume that R is a (basic and artinian) hereditary ring. Then R has the following form by [8], Theorem 1

$$(1) \qquad \begin{pmatrix} K_1 & M_{12} & M_{13} & \cdots\cdots & & M_{1n} \\ 0 & K_2 & M_{23} & \cdots\cdots & & M_{2n} \\ & & & \cdots\cdots & & \\ 0 & & & & K_{n-1} & M_{n-1n} \\ 0 & & & & 0 & K_n \end{pmatrix}$$

where the e_{ii} are matrix units, the $K_i=e_{ii}Re_{ii}$ are division rings and the $M_{ij}=e_{ii}Re_{jj}$ are K_i-K_j bimodules.

Let R be as in (1) and $e_i=e_{ii}$. We observe submodules in e_1R, Let $B\supset A$ be any submodules in e_1J. Then e_1R/A is always e_1R/K_1A-projective by [1], p. 22, Exercise 4. Since B is projective, and hence a lifting module, $B=B_1\oplus B_2$ and $A\supset B_2$, $B_1\cap A=A_1$ is small in B_1. We can assume $B_1=(e_aR)^{*(t_a)}\oplus(e_bR)^{*(t_b)}\oplus\cdots\oplus(e_mR)^{*(t_m)}$, where $a<b<\cdots<m$, f_i: $e_iR\sim(e_iR)^*$ is given by an element in M_{1i}. Let A_1^{ij} be the projection of A_1 into the jth component of $((e_iR)^*)^{(t_i)}$.

Then $A_1 \subset \Sigma \oplus A_1^{ij}$. Since B_1 is a projective cover of B_1/A_1, B/A $(=B_1/A_1)$ is e_1R/K_1A-projective if and only if

$$(*) \qquad K_1A \supset \textstyle\sum_{k=a}^{m} \sum_{j=1}^{t_k} M_{1k} f_k^{-1}(A_1^{kj}), \text{ where } K_1 = e_1Re_1.$$

Conversely if e_1R/A is e_1R/C-projective, then $K_1A \subset C$ by [1], p. 22, Exercise 4, and furthermore if B/A is e_1R/K_1A-projective, then B/A is e_1R/C-projective for $C \supset K_1A \supset \mathrm{Hom}_R(B, e_1R)A$. Therefore

(H) holds true if and only if $(*)$ holds ture, where e_1 and $A \subset B$ run through over all the primitive idempotents and the submodules in e_1R, respectively.

We shall consier the same criterion for (I). Assume $K_1A = A$ in the above. Let A_i, B_i be as above. Then B/A is quasi-projective if and only if for the same decomposition of B_1 as above

$(*')$ $\quad A_1 = \Sigma_i \Sigma_j \Sigma_{k \leq i} \Sigma_{p=1}^{t_k} f_{kp} M_{ki} f_i^{-1}(A_1^{ij})$, *where the indices i, j and k run in the decomposition of B_1 and $f_{kp} = f_k$: $e_kR \to$(the p-th component of $(e_kR)^{*(t_k)}$.*

From now on we always assume that R is a basic and hereditary ring with (I) given in (1). Since e_1J is projective,

$$(2) \qquad e_1J \overset{g}{\sim} e_kR \oplus e_sR \oplus \cdots, \quad \text{where } e_i = e_{ii} \text{ and } 1 < k, s \cdots.$$

We put $g^{-1}(e_iR) = (e_iR)' \subset e_1J$ and $g^{-1}(M_{ip}) = M_{ip}' \subset e_1J$.

Proposition 3. *Let R be a basic and hereditary ring with (I). Assume $e_1J \sim e_kR \oplus e_sR \oplus \cdots$ as in* (2), *where $k < s$. Then* 1): *either $M_{kp} = 0$ or $M_{sp} = 0$ $(s \leqq p)$, provided $M_{pq} \neq 0$ for some $q (> p)$ $(M_{ss} = K_s)$.* 2): *If (H) holds and $e_kJ \neq 0$, M_{1k} is cyclic as a K_1-K_k bimodule.*

Proof. 1) Since $\mathrm{Hom}_R(e_s'R, e_k'R) = 0$, $K_1M_{kp}'R \subset \Sigma_{b \leq k} \oplus (e_bR)' \subset e_1J$, where the b are indices in (2). Hence $K_1M_{kp}'R \cap M_{sp}'R \subset (\Sigma_{b \leq k} \oplus (e_bR)') \cap (e_sR)' = 0$. Now we may assume $M_{pq'}' = 0$ for all $q' (p < q' < q)$ and $M_{pq} \neq 0$. We note $K_1M_{kp}'R = K_1M_{kp}' \oplus K_1M_{kp}'M_{pq} \oplus \cdots \oplus K_1M_{kp}'M_{pn}$ and $M_{sp}'R = M_{sp}' \oplus M_{sp}'M_{pq} \oplus M_{sp}'M_{pq+1} \oplus \cdots \oplus M_{sp}'M_{pn}$. Put $A = K_1M_{kp}'(M_{pq} \oplus M_{pq+1} \oplus \cdots) + B$, where $B = K_1M_{sp}'(M_{pq+1} \oplus \cdots)$. Since $B \subset M_{1q+1} \oplus \cdots \oplus M_{1n}$ and $K_1M_{kp}'R \cap M_{sp}'R = 0$, $(K_1M_{kp}'R + M_{sp}'R + B)/A \sim K_1M_{kp}' \oplus M_{sp}' \oplus M_{sp}'M_{pq} (= D)$. A being characteristic in eR, eR/A is eR/A-projective, and hence D is quasi-projective by (I). Since K_1M_{kp}' and M_{sp}' are K_p-modules, we obtain a non-zero homomorphism h: $K_1M_{kp}' \to M_{sp}'$, provided $M_{kp} \neq 0$ and $M_{sp} \neq 0$. Take a diagram

$$\begin{array}{ccc} & & D \\ & \tilde{h} \swarrow & \downarrow \pi \\ & & K_1M_{kp}' \\ & & \downarrow h \\ D & \xrightarrow{\ \nu\ } & D/M_{sp}'M_{pq}=K_1M_{kp}'\oplus M_{sp}' \longrightarrow 0 \end{array}$$

where π is the projection and ν is the natural epimorphism. (Note that all the maps are R-homomorphisms.)

Then there exists $\tilde{h}: D \to D$ with $\nu\tilde{h}=\nu h$. Hnce $0 \neq \tilde{h}(K_1M_{kp}') \subset (M_{sp}'+M_{sp}'M_{pq}) \cap M_{1p}=M_{sp}'$. However $K_1M_{kp}'M_{pq} \subset A$ and the natural map $M_{sp} \otimes_{K^s} M_{pq} \to M_{sp}M_{pq}$ is an isomorphism by [8], Theorem 1, a contradiction. Therefore either $M_{kp}=0$ or $M_{sp}=0$.

2) Assume (H) and $e_kJ \neq 0$. We apply (*) to $A=m_{1k}e_kJ \subset B=(e_kR)'$, where $m_{1k} \neq 0$ in M_{1k} gives f_k. Then $K_1m_{1k}e_kJ=K_1m_{1k}K_ke_kJ=M_{1k}e_kJ$. Since the natural maps $M_{1k} \otimes_{K^k} e_kJ \to M_{1k}e_kJ$ and $K_1m_{1k}K_k \otimes_{K^k} e_kJ \to K_1m_{1k}K_ke_kJ$ are isomorphisms by [8], Theorem 1, $K_1m_{1k}K_k=M_{1k}$.

Corollary 1. *Let R be a hereditary ring as in Proposition 3 and let k and s be as above. We assume (I). If either $M_{kp'} \not\subset \mathrm{Soc}(e_kR)$ or $M_{sp'} \not\subset \mathrm{Soc}(e_sR)$ for some p', then $M_{kp'}=0$ or $M_{sp'}=0$. Hence any simple sub-factor modules of e_kR are never isomorphic to any ones of e_sR, provided they are not derived from their socles.*

Proof. From the assumption and [8], Theorem 1, there exists an integer q' such that $M_{p'q'} \neq 0$.

Corollary 2. *Let R be as in Corollary 1. We gather together isomorphic components in (2) and put $e_1J=((e_kR)')^{(n_k)} \oplus ((e_sR)')^{(n_s)} \oplus \cdots$. Then $((e_kR)')^{(n_k)}$ is characteristic in e_1R, provided $e_kJ \neq 0$.*

Proof. Let u, u' and k be indices in (2). If $k>u$, $M_{uk}=0$ from Proposition 3, and hence $\mathrm{Hom}_R(e_kR, e_{u'}R)=0$ for any $u' \neq k$. Therefore $K_1(e_kR)' \subset ((e_kR)')^{(n_k)}$.

We shall study the remaining part on Corollary 1, namely $M_{kq} \subset \mathrm{Soc}(e_kR)$. Let D_1 and D_2 be division rings and M_1, M_2 D_1-D_2 bimodules. Put $M=M_1 \oplus M_2$. Consider the following condition: for any element $m=m_1+m_2$; $m_i \in M_i$

(3) $D_1mD_2=D_1m_1D_2 \oplus D_1m_2D_2$, *i.e., for any D_1-D_2 submodule N of M, $N=M_1 \cap N \oplus M_2 \cap N$.*

If $D_1=D_2$ are fields and the M_i are usual D_1-D_1 bimodules, then M does not satisfy (3). Assume next that there exists a non-trivial automorphism σ of D_1. Let $M_1=D_1m_1=m_1D_1$ be a usual D_1-D_1 bimodule. Put $M_2=D_1m_2$ and define

$m_2d=d^{\sigma}m_2$ for $d\in D_1$. Then $M=M_1\oplus M_2$ satisfies (3) as D_1-D_1 bimodules.

Proposition 4. *Let R be a hereditary ring with (I) as in Proposition 3. 1): Let e_kR and e_sR be as in (2). Assume $0\neq M_{ip}\subset \mathrm{Soc}(e_iR)\subsetneq e_iR$ for some p and $i=k, s(k\neq s)$. Then K_1M_{kp}' and K_1M_{sp}' satisfy (3) as K_1-K_p bimodules. 2): If $n_k>1$ in Corollary 2 and $e_kJ\neq 0$, we assume $((e_kR)')^{(n_k)}=X_1\oplus X_2$; the X_i are characteristic in e_1R. If X_i contains a non-zero right K_p-module Y_i contained in $\mathrm{Soc}(e_kR')^{(n_k)}$ for $i=1,2$, then K_1Y_1 and K_1Y_2 satisfy (3) as K_1-K_p bimodules.*

Proof. Assume $k<s$. Then $K_1M_{ip}'\subset((e_iR)')^{(n_i)}$ from Corollary 2 for $i=k,s$. Let m_i be any element in K_1M_{iq}' and put $A=K_1(m_k+m_s)K_{p'}$ which is a characteristic submodule in e_1R and is contained in $((e_kR)')^{(n_k)}\oplus((e_sR)')^{(n_s)}$ $(=F)$. Then F/A is F/A-projective from (I). Hence A is also a characteristic submodule in F, since A is small in F. Accordingly $A\supset K_1m_kK_p\oplus K_1m_sK_p\supset A$. We can show 2) in the same manner.

In the above, we studied the structure of R, provided e_1J was a direct sum of distinct projective modules e_kR. We can not easily describe the structure of R, even though $e_1J\sim e_kR$. Here we shall explore several examples. It is clear, from Proposition 2, that every hereditary ring with $J^2=0$ satisfies (H). Let $K_1\supset K_2$ and K_3 be fields such that K_1 has a K_2-automorphism σ and $[K_1: K_2]=2$. Take the $M=M_1\oplus M_2$ after (3). Then M satisfies (3), if $\sigma\neq 1$, and put

$$R_o=\begin{pmatrix} K_2 & K_1 & M \\ 0 & K_1 & M \\ 0 & 0 & K_1 \end{pmatrix} \qquad \left(R_o'=\begin{pmatrix} K_1 & K_1 & M_{13} \\ 0 & K_1 & M_{23} \\ 0 & 0 & K_3 \end{pmatrix}\right),$$

where the M_{i3} are any K_i-K_3 bimodules such that R_o' is hereditary. Set $M_o=(m_1+m_2)K_1$ in M and $A=e_{22}M_oe_{33}$ in R_o. Then $e_{12}A$ is a characteristic submodule of $e_{11}R$. However $A(\subset B\sim e_{11}J)$ is not a characteristic submodule of $e_{22}R$, provided $\sigma\neq 1$. We note $e_{11}J\sim e_{22}R$. Hence (I) does not hold true from (*'). If $\sigma=1$, (H) holds true (see R_1 below). Further R_o is a K_2-algebra and satisfies all the conditions in Theorem 2 below, except the condition: $K_1=K_2$. On the other hand R_o' satisfies (H) from (*).

We can easily show

$$R_1=\begin{pmatrix} K_1 & K_1 & K_1 & M \\ 0 & K_1 & 0 & M_1 \\ 0 & 0 & K_1 & M_2 \\ 0 & 0 & 0 & K_1 \end{pmatrix} \qquad \left(R_1'=\begin{pmatrix} K_2 & K_1 & K_1 & M \\ 0 & K_1 & 0 & M_1 \\ 0 & 0 & K_1 & M_2 \\ 0 & 0 & 0 & K_1 \end{pmatrix}\right)$$

is hereditary ring with (H), provided $\sigma\neq 1$. $e_{11}J\sim e_{22}R_1\oplus e_{33}R_1$, and $(0, 0, 0, M_1)$

$=\mathrm{Soc}(e_2R_1)$, $(0,0,0,M_2)=\mathrm{Soc}(e_{33}R_1)$, and R_1 does not satisfy (I), provided $\sigma=1$. R_1' does not satisfy (I) for all σ.
Put

$$R_2=\begin{pmatrix} K_1 & K_1 & K_1 \\ 0 & K_2 & K_2 \\ 0 & 0 & K_2 \end{pmatrix} \qquad \left(R_2'=\begin{pmatrix} K_2 & K_1 & K_1 \\ 0 & K_2 & K_2 \\ 0 & 0 & K_2 \end{pmatrix}\right).$$

Then R_2 is hereditary and $e_{11}J\sim e_{22}R_2\oplus e_{22}R_2$. R_2 satisfies (H). Contrarily R_2' does not satisfy (I) by Proposition 4.
Put

$$R_3=\begin{pmatrix} K_1 & K_1 & K_1\otimes_{K_2}K_1 \\ 0 & K_2 & K_1 \\ 0 & 0 & K_2 \end{pmatrix} \qquad \text{(cf. } R_2\text{)}$$

R_3 does not satisfy (I) from (*′). In R_2 $\mathrm{Soc}(e_{11}R_2)$ does not contain proper characteristic submodules, however $\mathrm{Soc}(e_{11}R_3)$ does a characteristic submodule $K_1(0,0,1\otimes 1+v\otimes 1)$ $(=A)$ in R_3, which does not satisfy (*′) for $A\subset B=e_{11}J$, where $K_1=K_2\oplus vK_2$.

It is very hard for the author to interpret generally (*′) in terms of structures of R. Hence in the last part of this section, we shall determine the structure of a basic and hereditary algebra over a field K which satisfies (I) and assumption:

$$K_1=K_2=\cdots=K_n=K \text{ in (1).}$$

From now on we always assume that R is such an algebra. Then every submodule in e_iR is characteristic. Further (3) is never staisfied. Hence $e_iJ\sim e_{i(1)}R\oplus e_{i(2)}R\oplus\cdots$; $i(k)\neq i(s)$ for $k\neq s$, from Proposition 4, if $e_{i(j)}R$ is not simple.

Thus if R satisfies (I), then

$$\text{(4)} \qquad \begin{aligned} e_1J\sim\, & e_{i(1)}R\oplus e_{i(2)}R\oplus\cdots\oplus e_{i(p)}R \\ & \oplus(e_{j(1)}R)^{(m_1)}\oplus(e_{j(2)}R)^{(m_2)}\oplus\cdots\oplus(e_{j(q)}R)^{(m_q)}, \end{aligned}$$

where $e_{i(u)}J\neq 0$ for each u and $e_{j(v)}J=0$ for each v. We note

(5) *if $M_{kt}\neq 0$ for any k and $t=$ some $j(a)$ in the above, then $M_{kt}\subset\mathrm{Soc}(e_kR)$.*

Further from [8], Theorem 1, if a simple component in $\mathrm{Soc}(e_iR)$ is isomorphic to a submodule in $M_{jp}\subset e_jR$, then $M_{jp}\subset\mathrm{Soc}(e_jR)$ (cf. (5)).

The following lemma is well known (see [9]).

Lemma 3. *Let M and N be R-modules such that every sub-factor module* *of M is never isomorphic to any one of N. Then $P=M\cap P\oplus N\cap P$ for any submodule P of $M\oplus N$.*

In the similar manner to the proof of Proposition 3, we can obtain the following lemma.

Lemma 4. *Let R be the algebra as above. We assume (I). Then if $M_{jk}\neq 0$ for some $j, k(j\neq k)$, i.e., $M_{ij}\not\subset \mathrm{Soc}(e_iR)$, then $|M_{ij}|\leqq 1$ for all $i(<j)$.*

We assume

$$M_{ij}=u_{ij}K \text{ or } =0\,. \tag{6}$$

The following lemma is clear.

Lemma 5. *Let R be a hereditary algebra in (1) whose structure is as in (6). Then every sub-factor module of $e_{i(r)}R$ is never isomorphic to any sub-factor module of $e_{i(s)}R$, where $i(r)$ and $i(s)$ are indices in (4) $(s\neq r)$ and e_1 in (4) runs through all e_j.*

Theorem 2. *Let K be a field and R a basic and right artinian hereditary K-algebra such that $R/J\sim\Sigma\oplus K$. Then R satisfies (I) if and only if R has the following structure:*

1) *$R/\mathrm{Soc}(R)$ is an algebra as in (6).*

2) *Any simple component in $\mathrm{Soc}(e_{i(k)}R)$ is never isomorphic to any one in $\mathrm{Soc}(e_{i(k')}R)$ for $k\neq k'$, where $i(k)$, $i(k')$ run through over all the indices in (4) and e_1 in (4) runs through over all the primitive idempotents.*

In this case (H) and (I) are equivalent to each other.

Proof. Suppose (I). Then we obtain 1) from Lemma 4 and 2) from Proposition 4. Conversely we assume 1) and 2). Then from Lemma 5 e_iJ has the following direct decomposition for each i: $e_iJ\sim D\oplus\Sigma_k\oplus F_k$; i) D is semi-simple, and ii) the F_k are indecomposable and non-simple projectives $(=e_{\rho(k)}R)$ and every simple sub-factor module of F_k is never isomorphic to any one of $F_{k'}$ for all $k\neq k'$. Let R be of the form (1). By induction on n, the degree of matrix, we shall show (H). We assume that (H) holds true for $M=e_{jj}R/A'$ and $N=e_{jj}R/B'$; all $j>1$, and we shall show that (H) holds true for $e_{11}R/A$ and $e_{11}R/B$. Put $e_{11}=e$, and assume that eR/A is eR/B-projective. Then $A\subset B$. Take a proper submodule C/A of eR/A and consider a diagram

$$\begin{array}{ccccc} & & C/A & & \\ & & \downarrow h & & \\ eR/B & \xrightarrow{\nu} & eR/E & \longrightarrow & 0 \end{array}$$

Since $h(C/A)\subset eJ/E$, we can derive the diagram from the above

$$(7)\qquad \begin{array}{ccccc} & & C/A & & \\ & & \downarrow h & & \\ eJ/B & \xrightarrow{\nu} & eJ/E & \longrightarrow & 0 \end{array}$$

From $eJ\supset A$, we can easily see from i), ii), (5) and Lemma 3 that $A=(A\cap D^*)\oplus\Sigma_k\oplus(A\cap F_k)$ after a little change of a direct decomposition of eJ: $=D^*\oplus\Sigma_k\oplus F_k$, where D^* is semi-simple (cf. [9], the proof of Proposition 8). Further noting that $A\cap D^*$ is a direct summand of D^*, i.e., $D^*=D_1^*\oplus(A\cap D^*)$ and that $D_1^*\oplus\Sigma_k\oplus F_k\supset B/(A\cap D^*)\supset\Sigma_k\oplus(A\cap F_k)$, we obtain further direct decompositions

$$eJ=(A\cap D^*)\oplus D_1^*\oplus\Sigma_k\oplus F_k\supset B=(A\cap D^*)\oplus(B\cap D_1^*)\oplus\Sigma_k\oplus (B\cap F_k)\supset A=(A\cap D^*)\oplus\Sigma_k\oplus(A\cap F_k).$$

From the above observation, let $C=C'\oplus\Sigma_k\oplus(C\cap F_k)\supset A=A'\oplus\Sigma_k\oplus(A\cap F_k)$, where $C'\supset A'$ are semi-simple. Then $C/A=C'/A'\oplus\Sigma_k\oplus(C\cap F)/(A\cap F_k)$. In order to show that C/A is eR/B-projective, we may show that each simple component C_i^* of C'/A' (resp. $(C\cap F_k)/(A\cap F_k)$) is eR/B-projective. Hence we can replace C/A by C_i^* or $(C\cap F_k)/(A\cap F_k)$ in (7). We have similar decompositions

$$eJ=D'\oplus\Sigma_k\oplus F_k\supset E=E'\oplus\Sigma_k\oplus(E\cap F_k)\supset B=B'\oplus\Sigma_k\oplus(B\cap F_k)$$
and $D'\supset E'\supset B'\supset A'$.

Then we have $\nu=\nu_1+\nu_2$ and $h=h_1+h_2$, where ν_1: $D'/B'\to D'/E'$, ν_2: $\Sigma_s\oplus(F_s/(B\cap F_s))\to\Sigma_s\oplus(F_s/(E\cap F_s))$, h_1: $X\to D'/E'$ and h_2: $X\to\Sigma_s\oplus(F_s/(E\cap F_s))$, where $X=C_i^*$ or $(C\cap F_k)/(A\cap F_k)$. Since $D'\supset E'\supset B'$ are semi-simple, we obtain

$$\tilde{h}_1\colon X\to D'/B' \text{ with } \nu_1\tilde{h}_1=h_1\,.$$

Assume first $X=C_i^*(\sim e_{j(p)}R)$. If $h_2(C_i^*)\neq 0$, then there exists k such that $h_2(C_i^*)\subset(M_{\rho(k)j(p)}+(E\cap F_k))/(E\cap F_k)$ and $M_{\rho(k)j(p)}\subset\mathrm{Soc}(F_k)$ by ii) and (5). Then we can derive the following diagram:

$$\begin{array}{ccccc} & & C_i^* & & \\ & & \downarrow h_2 & & \\ (\mathrm{Soc}(F_k)+(B\cap F_k))/(B\cap F_k) & \xrightarrow{\nu_2} & (\mathrm{Soc}(F_k)+(E\cap F_k))/(E\cap F_k) & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Soc}(F_k)$ is semi-simple, we obtain also

$$\tilde{h}_2\colon C_i^*\to(\mathrm{Soc}(F_k)+(B\cap F_k))/(B\cap F_k)\subset F_k/(B\cap F_k) \text{ with } \nu_2\tilde{h}_2=h_2\,.$$

Finally assume $X=(C\cap F_k)/(A\cap F_k)$. Then $h_2(X)\subset F_k/(E\cap F_k)$ by ii). Moreover since $A\cap F_k\subset B\cap F_k$, $F_k/(A\cap F_k)$ is $F_k/(B\cap F_k)$-projective. Hence there exists

$$\tilde{h}_2\colon X \to F_k/(B\cap F_k) \text{ with } \nu_2\tilde{h}_2 = h_2$$

by induction hypothesis. Therefore C/A is eR/B-projective. Thus (H) holds true and hence (I) does.

We can completely determine the styles of hereditary algebras in Theorem 2. Let M_1, M_2 be non-zero K-vector spaces. Then there are only three styles of the above algebras, when $n=3$.

$$\begin{pmatrix} K & M_1 & M_2 \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}, \begin{pmatrix} K & 0 & M_1 \\ 0 & K & M_2 \\ 0 & 0 & K \end{pmatrix} \text{ and } \begin{pmatrix} K & K & M_1\oplus M_2 \\ 0 & K & M_1 \\ 0 & 0 & K \end{pmatrix}$$

We note that R_0 before Lemma 3 shows that Theorem 2 is not true if $K\neq K_i$ for some i and R_0' is a hereditary algebra with (H) as right R_0'-modules, but not as left R_0'-modules, if $K_1=K_2=K_3$ and $M_{ij}=K_1\oplus K_1$.

3. Almost relative projectives and almost relative injectives

In this section we shall study the same problem for almost relative projectives (resp. injectives). We consider the following conditions:

(J) *M/M' is almost N-projective and*

(K) *M' is almost N-projective*

for any submodule M' of M, provided M is almost N-projective.

(resp.

(J^*) *U' is almost V-injective* and

(K^*) *U/U' is almost V-injective*

for any submodule U' of U, provided U is almost V-injective).

Proposition 5. *Let R be a perfect ring. (J) holds true when M and N are any local modules (resp. any $M=eR/A$ and $N=eR/B$ for a fixed primitive idempotent e) if and onld if R is a right Nakayama ring with $J^2=0$ (resp. eR is a uniserial module with $(eJ)^2=0$ and $|eR|<\infty$).*

Proof. Since fR is almost eR/A-projective for any submodule A of eR, fR/B is almost eR/A-projective by (J). Hence R is a ring stated in the proposition by [11], Theorem 3. The converse is clear from the same theorem. We can use [12], Theorem 4 in case of $M=eR/A$ and $N=eR/B$.

Proposition 5*. *Let R be as above. (J^*) holds true when U and V are any uniform modules (resp. any submodules U and V in* E(S) *for a fixed simple module S) if and only if R is right co-Nakayama ring with $J^2=0$ (every simple sub-factor module of E, execpt* Soc(E) *and* E/J(E), *is not isomorphic to S).*

Similarly to Theorem 2 in [12], we have

Proposition 6. *Let R be a two-sided artinian ring. Then the following conditions are equivalent:*

1) *(J) holds true for any finitely generated (and indecomposable) modules M and N.*

1^*) *(J^*) holds true for any finitely generated (and indecomposable) modules U and V.*

2) *R is a two-sided Nakayama ring with $J^2=0$.*

3) *Any two of finitely generated R-modules are mutually almost relative projective.*

Proof. 2)→3) If R is a two-sided Nakayama ring with $J^2=0$, then every finitely generated and indecomposable R-module is local. Hence 3) holds true by [11], Theorem 3.

1)→2) Let (J) hold ture. Assume $e_1J\sim e_2J$ via f for primitive idempotents e_1 and e_2. Put $N=(e_1R\oplus e_2R)/\{x+f(x)\mid x\in e_1J\}$. Since e_1R is (almost) N-projective, e_1R/e_1J is almost N-projective by 1). Then N is decomposable by [12], Lemma 3. Hence R is left Nakayama by [15], Lemmas 2.1 and 4.3. Therefore R is two-sided Nakayama.

The remaining implications are clear.

Proposition 7. *Let R be a perfect ring. Then the following conditions are equivalent:*

1) *(K) holds ture when M and N are any local modules.*

1^*) *(K^*) holds ture when U and V are any uniform modules.*

2) *$J^2=0$.*

3) *Let M and N be any local modules. Then every proper submodule of M is almost N-projective.*

4) *Every module is almost R-projective.*

Proof. 1)→2) Let $eJ^{n-1}\neq 0$ and assume $n>2$. We put $\bar{e}\bar{R}=eR/eJ^{n-1}\supset \bar{e}\bar{J}^{n-2}=eJ^{n-2}/eJ^{n-1}$. Then $\bar{e}\bar{J}^{n-2}$ is semi-simple and let $\bar{e}\bar{J}^{n-2}=\bar{B}_1\oplus\bar{B}_2\oplus\bar{B}_3\oplus\cdots$, where the $\bar{B}_i$ are simple, and $eJ^{n-2}\supset B_i\supset eJ^{n-1}$. Since $eJeeJ^{n-1}\subset eJ^n$, eR/eJ^{n-1} is almost eR/eJ^n-projective by [5], Proposition 2. Hence B_i/eJ^{n-1} is almost eR/eJ^n-projective by 1). Take a diagram

$$\begin{array}{ccccc} & & B_i/eJ^{n-1} & & \\ & & \downarrow h & & \\ eR/eJ^n & \xrightarrow{\nu} & eR/eJ^{n-1} & \longrightarrow & 0 \end{array}$$

where h is the inclusion.
Since $n>2$, $B_i\subset eJ$ and h is not an epimorphism. Therefore there exists a simple submodule K in eR/eJ^n with $B_i/eJ^n=K\oplus eJ^{n-1}/eJ^n$ (cf. Lemma 1). Hence

$B_i/eJ^n \subset \mathrm{Soc}(eR/eJ^n)$. Since $eJ^{n-2}=\Sigma_i B_i$, $eJ^{n-2}/eJ^n=\Sigma_i B_i/eJ^n \subset \mathrm{Soc}(eR/eJ^n)$. Hence $0=\mathrm{Soc}(eR/eJ^n)J \supset (eJ^{n-2}/eJ^n)J$, and so $eJ^{n-1}=eJ^n$, a contradiction. Accordingly $n \leqq 2$.

2)→3) Since $J^2=0$, eJ is semi-simple. Let D/A be any proper submodule of eR/A. Then D/A is semi-simple. In order to show 3) we may assume that D/A is simple. Take a diagram:

$$\begin{array}{ccccc} & & D/A & & \\ & & \downarrow h & & \\ fR/B & \xrightarrow{\ \nu\ } & fR/C & \longrightarrow & 0 \end{array}$$

If h is an epimorphism, $C=fJ$, and hence putting $h^{-1}\nu=\tilde{h}$, we have $h\tilde{h}=\nu$. If h is not an epimorphism, we can find $\tilde{h}: D/A \to fJ/B \subset fR/B$ with $\nu\tilde{h}=h$ by Lemma 2.

3)→1) This is trivial.

1*)→2) and 2)→1*) Those are dual to 1)→2) and 2)→1), respectively.

4)→2) Since eR/eJ is almost fR-projective, $0=fJeeJ=fJeJ$ for any primitive idempotents e and f. Hence $J^2=\Sigma_{i,j}e_iJe_jJ=0$, where $1=\Sigma_i e_i$.

2)→4) Let M be an R-module. Take a projective cover P of M. Then $M \sim P/Q$ and $Q \subset PJ$. Let θ be any element in $\mathrm{Hom}_R(P, eJ)$. Then $\theta(Q) \subset \theta(PJ) \subset eJ^2=0$. Hence M is almost eR-projective by [13], Theorem 2, and M is almost R-projective by [10], Theorem 2.

REMARK. 1) Related to Proposition 7, we note that if every indecomposable module is R-projective, then R is semi-simple.

2) In the above 1)→2), we have used a fact that (K) holds true only for hollow modules $M=eR/A$ and $N=eR/B$. Further the property in Proposition 7 is left and right symmetric.

Finally we study (K) for any finitely generated R-modules M and N.

First we assume (K) only in case of M is an indecomposable and projective module. Then since eR is a (almost) N-projective for any finitely generated R-module N, R satisfies (17) in [14] (cf. the remark in §4 of [14]), and hence R is a right almost hereditary ring given in Theorem 3 of [14]. As a consequence in this case (K) holds true when M is a finitely generated projective module.

Proposition 8. *Let R be a (two-sided) artinian ring. Then (K) holds true for any finitely generated R-modules M and N if and only if R is a right almost hereditary ring with $J^2=0$ and (K) holds true when N is local.*

Proof. Assume (K). Then R is right almost hereditary as above and $J^2=0$ by Proposition 7. Conversely, we assume that R is a (basic) right almost hereditary ring with $J^2=0$. Let M be a finitely generated R-module and P a projective cover of M. Let T be a submodule of P containing Q ($P/Q \sim M$).

Since $J^2=0$, $T/Q \sim P_1/Q_1 \oplus Q^*/Q_2$ as in the proof of 2)→4) of Theorem 1, where $Q^* \subset \mathrm{J}(P_2)$. Hence since Q^*/Q_2 is a direct summand of $\mathrm{J}(P_2)$, Q^*/Q_2 is almost projective by assumption. Suppose that M is almost N-projective for a finitely generated R-module N. We first assume that N is indecomposable. If N is not local, M is N-projective by [10], Theorem 1. Hence P_1/Q_1 is N-projective by the proof of 2)→4) of Theorem 1. Therefore T is almost N-projective. We have the same result for a local module by assumption. Hence we obtain (K) by [10], Theorem 2.

Corollary 1. *Let R be a right Nakayama, right almost hereditary ring with $J^2=0$. Then (K) holds ture for any finitely generated R-modules M and N.*

Proof. Since R is a right Nakayama ring with $J^2=0$, the set of local modules consists of $\{eR, eR/eJ\}_e$. Hence by Proposition 7 (K) holds true when N is local.

Next we shall study (K) when M is quasi-projective. The following corollary corresponds to the equivalence 1) and 5) in Theorem 1.

Corollary 2. *Let R be a (two-sided) artinian ring. Then the following conditions are equivalent:*

1) *(K) holds true when M is an indecomposable and quasi-projective module.*
2) *(K) holds true when M is finitely generated and quasi-projective.*
3) *R is a right almost hereditary ring with $J^2=0$.*

Proof. We assume 1). Then $J^2=0$ from the proof of Proposition 7. We have shown before Proposition 8 that R is a right almost hereditary ring. Hence we obtain 3). Conversely we assume 3). Let M be a finitely generated and quasi-projective module. We shall use the same notations as in the proof of Proposition 8. Then $P=P_1 \oplus P_2$ and $Q=Q_1 \oplus Q_2$, since M is quasi-projective. Hence if P/Q is almost N-projective, so is P_1/Q_1. Therefore we obtain 2) from the proof of Proposition 8.

Let $K_1 \supsetneqq K_2$ be fields. Then

$$R=\begin{pmatrix} K_2 & K_1 & 0 \\ 0 & K_1 & K_1 \\ 0 & 0 & K_1 \end{pmatrix}$$

is a right Nakayama and right almost hereditary ring with $J^2=0$, which is neither hereditary nor two-sided Nakayama. Since R is not left almost hereditary, (K) is not left and right symmetric for finitely generated R-modules. We note that we can not replace "quasi-projective" in 5) of Theorem 1 by "indecomposable and quasi-projective" (cf. 2) in Corollary 2 above).

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