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Osaka J. Math.
27 (1990), 843-861

NON-STANDARD REPRESENTATIONS OF DISTRIBUTIONS II

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(Received November 6, 1989)

1. Introduction

This paper is a continuation of the previous one [1]. Its aim is to represent the space $\mathcal{S}'(\boldsymbol{R})$ of tempered distributions on $\boldsymbol{R}$, the space $\mathcal{S}(\boldsymbol{R})$ of rapidly decreasing functions on $\boldsymbol{R}$ and the Fourier transformation on the space $\mathcal{S}'(\boldsymbol{R})$ by using a kind of standardization of functions and transformations on *a* $*$-finite subset of a lattice with infinitesimal mesh (see Definition below).

Fix an even infinite integer in $^*\boldsymbol{N}-\boldsymbol{N}$. Let $\varepsilon=1/H$ and $\boldsymbol{L}={}^*\boldsymbol{Z}\cdot\varepsilon$. Put $\boldsymbol{X}=\{x\in\boldsymbol{L}\mid -H/2\leqq x<H/2\}$. Then, X is a $*$-finite subset of L of cardinality H^2.

We have $\boldsymbol{Z}\subsetneqq\boldsymbol{X}\subsetneqq{}^*\boldsymbol{R}$. Let

$$R(X)=\{\varphi\colon X\to{}^*\boldsymbol{C}\ \text{(internal)}\}$$

and assume that every φ in $R(X)$ is always extended to a function on $\boldsymbol{L}$ with period H. With this convention, the sum $\sum_{x\in\boldsymbol{L},x_0\leqq x<x_0+H}\varphi(x)$ does not depend on the choice of $x_0\in\boldsymbol{L}$. When $x_0=-H/2$, we write this sum as $\sum_{x\in\boldsymbol{X}}\varphi(x)$ or, in short, $\sum_{\boldsymbol{X}}\varphi$.

The following definition is due to G. Takeuti.

Definition. For $x\in\boldsymbol{X}$, let $\delta(x)=\boldsymbol{H}$ for $x=0$ and $\delta(x)=0$ for $x\neq0$.

Proposition 1. *For $x\in\boldsymbol{X}$, we have*

$$\delta(x)=\sum_{y\in\boldsymbol{L},\,0\leqq y<H}\varepsilon e^{2\pi ixy}=\sum_{y\in\boldsymbol{L},\,0\leqq y<H}\varepsilon e^{-2\pi ixy}\,.$$

The proof is trivial by the summation formula of finite geometric series.

Definition. For functions φ, ψ in $R(x)$, we define Fourier transform $F\varphi$, inverse Fourier transform $\bar{F}\varphi$ and the convolution $\varphi*\psi$ by following formulas respectiaely:

1) Posthumous manuscript translated and arranged by Norio Adachi, Toru Nakamura and Masahiko Saito.

$$F\varphi(x)=\sum_{y\to x}\varepsilon e^{-2\pi ixy}\varphi(y)\,,\quad \bar{F}\varphi(x)=\sum_{y\in x}\varepsilon e^{2\pi ixy}\varphi(y)\,,$$
$$\varphi*\psi(x)=\sum_{y\in x}\varepsilon\varphi(x-y)\psi(y)\,.$$

If we consider φ ih $R(X)$ as a vector $(\varphi(r\varepsilon))_{0\leq r<H^2}$ with H^2 rows, then Fourier transformation F is an $H^2\times H^2$ matrix

$$\frac{1}{H}(e^{-2\pi irs/H^2})_{0\leq r,s<H^1}$$

and the inverse Fourier transformation $\bar{F}$ is the complex conjugate of this matrix. We shall use same symbols φ, F and $\bar{F}$ for above vectors and matrices.

Note that the trace of $\bar{F}$ multiplied by H is a Gauss sum.

DEFINITION. 1) The external subspace $A_T(\boldsymbol{R})$ of $R(X)$ is the set of all $\varphi\in R(X)$ such that $\sum_X\varepsilon\varphi^*f=\sum_{x\in X}\varepsilon\varphi(x)^*f(x)$ is finite for every f in $\mathcal{S}(\boldsymbol{R})$.

2) The external subspace $M_1(\boldsymbol{R})$ of $R(X)$ is the set of all $\varphi\in R(X)$ such that $\sum_X\varepsilon|\varphi|$ is finite.

3) The external subspace $M(\boldsymbol{R})$ of $\boldsymbol{R}(X)$ is the set of all $\varphi\in R(X)$ such that $\sum_{*K\cap X}\varepsilon|\varphi|$ is finite for every compact subset K of $\boldsymbol{R}$.

4) Define $\Gamma_\varphi(f)={}^0\sum_X\varepsilon\varphi^*f$ for $\varphi\in A_T(\boldsymbol{R})$ and $f\in\mathcal{S}(\boldsymbol{R})$, where ${}^0\alpha$ is the standard part of a finite element in $^*\boldsymbol{C}$. Then, Γ_φ is a linear form on $\mathcal{S}(\boldsymbol{R})$, i.e. an element of the algebraic dual $\mathcal{S}(\boldsymbol{R})^*$ of $\mathcal{S}(\boldsymbol{R})$.
We have thus obtained a mapping Γ from $A_T(\boldsymbol{R})$ to $\mathcal{S}(\boldsymbol{R})^*$: $\varphi\mapsto\Gamma_\varphi$ $(\varphi\in A_T(\boldsymbol{R}))$. As in Theorem 1 of [1], we can prove that Γ is surjective.

DEFINITION. 1) Define mappings D_+ and D_- from $R(X)$ to $R(X)$ by formulas

$$D_+\varphi(x)=(\varphi(x+\varepsilon)-\varphi(x))/\varepsilon\,,\quad D_-\varphi(x)=(\varphi(x)-\varphi(x-\varepsilon)/\varepsilon\,.$$

2) Define a function λ in $R(x)$ by

$$\lambda(x)=(e^{2\pi i\varepsilon x}-1)/\varepsilon=2\pi i(\sin\pi\varepsilon x/\pi\varepsilon)e^{\pi i\varepsilon x}$$

and define mappings λ and $\bar{\lambda}$ from $R(X)$ to $R(X)$ by

$$(\lambda\varphi)(x)=\lambda(x)\varphi(x)\,,\quad (\bar{\lambda}\varphi)(x)=\overline{\lambda(x)}\varphi(x)\,.$$

3) Let $T(\boldsymbol{R})$ be the smallest (external) subspace of $R(X)$ which includes $M_1(\boldsymbol{R})$ and is stable under D_+, D_-, λ, $\bar{\lambda}$. Namely, a function is in $T(R)$ if and only if it is a finite sum of functions which are obtained from functions in $M_1(\boldsymbol{R})$ by operating D_+, D_-, λ, $\bar{\lambda}$ finitely many times succesively. □

In this paper, we shall obtain following results:

(1) $T(\boldsymbol{R}) \subseteq A_T(\boldsymbol{R})$ (a part of Theorem 2).

(2) $T(\boldsymbol{R})$ is stable under F and $\bar{F}$ (Theorem 1).

(3) If $\varphi \in T(\boldsymbol{R})$, we have

a) $\Gamma_\varphi \in \mathcal{S}'(\boldsymbol{R})$.

b) $\Gamma_{D_\pm \varphi} = (\Gamma_\varphi)'$, $\Gamma_{\varphi\lambda}(t) = (2\pi it)\Gamma_\varphi(t)$, $\Gamma_{\bar{\lambda}\varphi}(t) = (-2\pi it)\Gamma_\varphi(t)$.

c) $\Gamma_{F\varphi} = F\Gamma_\varphi$, $\Gamma_{\bar{F}\varphi} = \bar{\mathcal{F}}\Gamma_\varphi$ (Theorem 4), where $\mathcal{F}$ is Fourier transformation on the space $S'(\boldsymbol{R})$.

(4) The mapping Γ from $T(\boldsymbol{R})$ to $\mathcal{S}'(\boldsymbol{R})$: $\varphi \to \Gamma_\varphi$ is surjective (Theorem 4).

DEFINITION. 1) $U(\boldsymbol{R})$ is the set of functions φ in $R(X)$ such that $\varphi(x)$ is finite for every $x \in X$ and that $\varphi(x) \simeq \varphi(y)$ whenever $x \simeq y$.

2) $Q(\boldsymbol{R})$ is the set of functions φ in $U(\boldsymbol{R})$ suh such that iterated operations of D_+, D_-, λ, $\bar{\lambda}$ do not bring φ outside $U(\boldsymbol{R})$.

3) For a real number t, let ${}^\Delta t = \max\{x \in X \mid x \leqq t\}$. For a function φ in $Q(\boldsymbol{R})$, we can define a function ${}^\vee\varphi$: $\boldsymbol{R} \to \boldsymbol{C}$ by ${}^\vee\varphi(t) = {}^0(\varphi({}^\Delta t))$ for $t \in \boldsymbol{R}$.

We shall obtain following results:

(1) For $1 \leqq p < \infty$, the sum $\sum_X \varepsilon |\varphi|^p$ is finite for every $\varphi \in Q(\boldsymbol{R})$ (Proposition 10).

(2) $Q(\boldsymbol{R})$ is stable under D_+, D_-, λ, $\bar{\lambda}$, F, $\bar{F}$ and closed under multiplication (i.e. φ, $\psi \in Q(\boldsymbol{R})$ implies $\varphi\psi \in Q(\boldsymbol{R})$) (Theorem 6).

(3) If $\varphi \in Q(\boldsymbol{R})$, then ${}^\vee\varphi \in \mathcal{S}(\boldsymbol{R})$ and $\Gamma_\varphi = T_{{}^\vee\varphi}$, where $T_{{}^\vee\varphi}$ is the distribution on $\boldsymbol{R}$ defined by ${}^\vee\varphi$. Namely, if we denote by μ Lebesgue measure on $\boldsymbol{R}$, then $\Gamma_\varphi(t) = \int_{\boldsymbol{R}} {}^\vee\varphi f d\mu$ for $f \in \mathcal{S}(\boldsymbol{R})$.

(4) For $\varphi \in Q(\boldsymbol{R})$, we have (Theorem 7)

$$
\begin{aligned}
&{}^\vee(D_\pm\varphi) = ({}^\vee\varphi)', \quad {}^\vee(\lambda\varphi) = (2\pi it){}^\vee\varphi, \quad {}^\vee(\bar{\lambda}\varphi) = (-2\pi it){}^\vee\varphi, \\
&{}^\vee(F\varphi) = \mathcal{F}({}^\vee\varphi), \quad {}^\vee(\bar{F}\varphi) = \bar{\mathcal{F}}({}^\vee\varphi).
\end{aligned}
$$

(5) If $h \in \mathcal{S}(\boldsymbol{R})$, then ${}^*h \mid X$ belongs to $Q(\boldsymbol{R})$ and ${}^\vee({}^*h \mid X) = h$ (Theorem 8). In particular, the map: $\varphi \mapsto {}^\vee\varphi$ from $Q(\boldsymbol{R})$ to $\mathcal{S}(\boldsymbol{R})$ is surjective.

2. Fourier analysis on $\boldsymbol{R}(X)$

Fourier analysis on $R(X)$ is essentially that of a finite cyclic group interpreted in the universe of internal sets. Proposition 1 writes $\delta = \mathrm{F}1 = \bar{F}1$, where 1 is the constant function on X with value 1.

Proposition 2. *Write $1_{R(X)}$ the identity map of $R(X)$ and let φ, ψ be in $R(X)$.*

a) *F is unitary, symmetric and $F^4 = 1_{R(X)}$. We have $F\bar{F} = \bar{F}F = 1_{R(X)}$ and $\sum_X \varepsilon\varphi\bar{\psi} = \sum_X \varepsilon F\varphi \cdot \overline{F\psi}$. The eigenvalues of F are 1, -1, $-i$, and i with multiplicity $H^2/4+1$, $H^2/4$, $H^2/4$ and $H^2/4-1$ respectively.*

b) $\varphi*\delta=\delta*\varphi=\varphi$, $\varphi*\psi=\psi*\varphi$, $F(\varphi*\psi)=(F\varphi)(F\psi)$, $F(\varphi\psi)=(F\varphi)*(F\psi)$.

c) $\sum_{n\in{}^*Z,\,0\leqq n<H}\delta(x-n)=\sum_{n\in{}^*Z,\,0\leqq n<H}e^{2\pi ixn}$.

d) *Let φ be a function in $R(X)$ with period* 1. *If we put* $c_n=\sum_{x\in X,\,0\leqq x<1}\varepsilon\varphi(x)\cdot e^{-2\pi ixn}$, *then we have* $\varphi(x)=\sum_{n\in{}^*Z,0\leqq x<H}c_n e^{2\pi ixn}$.

e) *A function φ in $R(X)$ is non-negative real valued if and only if we have*

$$\sum_{x,y\in X}\varepsilon^2(F\varphi)(x-y)\psi(x)\overline{\psi(y)}\geqq 0$$

for every ψ in $R(X)$.

Proof of a). H being even, H^2 is a multiple of 4 and therefore the results on Gauss sum imply that the trace of F is $1-i$ (see [2] for example). Let N_1, N_2, N_3 and N_4 be the multiplicity of eigenvalues 1, -1, $-i$ and i respectively. Then we have $N_1-N_2-iN_3+iN_4=1-i$. Let r, $s\in{}^*\boldsymbol{Z}$ with $0\leqq r$, $s<H^2$ and let

$$a_{r,s}=\begin{cases}1, & \text{if } r+s\equiv 0 \pmod{H^2}\\ 0, & \text{otherwise.}\end{cases}$$

Then $F^2=(a_{r,s})_{0\leqq r,s<H^2}$ and the multiplicity of the eigenvalues 1 and -1 of F^2 is $H^2/2+1$ and $H^2/2-1$ respectively. So we have $N_1+N_2=H^2/2+1$ and $N_3+N_4=H^2/2-1$, and we get the result.

We omit the proof of the remaining parts, which is classical.

Proposition 3. a) *For $\varphi\in R(X)$, we have $FD_+\varphi=\lambda F\varphi$, $FD_-\varphi=-\bar{\lambda}F\varphi$, $F(\lambda\varphi)=D_-F\varphi$ and $F(\bar{\lambda}\varphi)=D_+F\varphi$.*

b) *For $x\in X$ with $|x|\leqq H/2$, we have $4|x|\leqq|\lambda x(x)|\leqq 2\pi|x|$.*

Proof. a) Direct calculation.

b) If $\alpha\in{}^*\boldsymbol{R}$ and $|\alpha|\leqq\pi/2$, then we know that $\frac{2}{\pi}|\alpha|\leqq|\sin\alpha|\leqq|\alpha|$. Hence

$$\frac{2}{\pi}\left|\frac{\pi\varepsilon x}{\pi\varepsilon}\right|\leqq\left|\frac{\sin(\pi\varepsilon x)}{\pi\varepsilon}\right|\leqq\left|\frac{\pi\varepsilon x}{\pi\varepsilon}\right|.$$

Multiplying these inequalities by 2π, we have

$$4|x|\leqq\left|2\pi i\frac{\sin(\pi\varepsilon x)}{\pi\varepsilon}e^{\pi i\varepsilon x}\right|\leqq 2\pi|x|.$$

3. Fourier transformation on the space $M_T(\boldsymbol{R})$

DEFINITION. Let $M_T(\boldsymbol{R})$ be the set of functions φ in $R(X)$ such that $\sum_X\varepsilon\frac{|\varphi|}{(1+|\lambda|^2)^l}$ is finite for some standard integer $l\in\boldsymbol{N}$.

From inequalities of Proposition 3 b), the condition on φ is equivalent to the condition that $\sum_X \varepsilon \frac{|\varphi(x)|}{(1+|x|^2)^l}$ is finite for some $l \in N$.

We have $M_1(\boldsymbol{R}) \subseteq M_T(\boldsymbol{R})$ by definition. Put $\psi = \frac{\varphi}{(1+|x|^2)^l}$ for $\varphi \in M_T(\boldsymbol{R})$. $\sum_X \varepsilon |\psi|$ being finite, we have $\psi \in M_1(\boldsymbol{R})$.

We have $M_T(\boldsymbol{R}) \subseteq M(\boldsymbol{R})$. In fact, if $\varphi \in M_T(\boldsymbol{R})$ and K is a compact subset of $\boldsymbol{R}$, then

$$\sum_{K^* \cap X} \varepsilon |\varphi| = \sum_{x \in {}^*K \cap X} \frac{\varepsilon |\varphi(x)|}{(1+|x|^2)^l} (1+|x|^2)^l$$
$$\leqq \Big(\sum_{x \in X} \varepsilon \frac{|\varphi(x)|}{(1+|x|^2)^l} \sup_{t \in K} (1+|t^2|)^l ,$$

the last quantity is finite for some $l \in \boldsymbol{N}$ by definition of $M_T(\boldsymbol{R})$.

Proposition 4. *We have $M_T(\boldsymbol{R}) \subset A_T(\boldsymbol{R})$, and if $\varphi \in M_T(\boldsymbol{R})$, then $\Gamma_\varphi \in \mathcal{S}'(\boldsymbol{R})$ and $P_\varphi = \Gamma_\varphi | \mathcal{D}(\boldsymbol{R}) \in \mathcal{D}'^{(0)}(\boldsymbol{R})$.*

Proof. Let $\varphi \in M_T(\boldsymbol{R})$ and $f \in \mathcal{S}(\boldsymbol{R})$. Then there exists an integer $l \in \boldsymbol{N}$ such that $\sum_{x \to X} \varepsilon \frac{|\varphi(x)|}{(1+|x|^2)^l}$ is finite. We have therefore

$$|\sum_X \varepsilon \varphi^* f| = \left| \sum_{x \in X} \varepsilon \frac{\varphi(x)}{(1+|x|^2)^l} (1+|x|^2)^{l*} f(x) \right|$$
$$\leqq \left(\sum_{x \in X} \varepsilon \frac{|\varphi(x)|}{(1+|x|^2)^l} \right) \sup_{t \in R} (1+|t|^2)^l |f(t)| .$$

Hence $\varphi \in A_T(\boldsymbol{R})$ and $\Gamma_\varphi \in \mathcal{S}'(\boldsymbol{R})$. $P_\varphi \in \mathcal{D}'^{(0)}(\boldsymbol{R})$ follows from $\varphi \in M(\boldsymbol{R})$.

Let μ be Lebesque measure on $\boldsymbol{R}$.

Lemma 1. *Put $\boldsymbol{R}_+ = \{t \in \boldsymbol{R} \mid t \geqq 0\}$ and let h be a continuous, integrable and decreasing (in wider sense) function on $\boldsymbol{R}_+$ with values in $\boldsymbol{R}_+$. Then we have*

i) *For $N_1, N_2 \in {}^*N$ with $N_2 \leqq N_2$*

$$\sum_{j=1}^{N_1} \varepsilon^* h(j\varepsilon) \leqq \sum_{j=1}^{N_2} \varepsilon^* h(j\varepsilon) \leqq \int_{R^+} h d\mu .$$

ii) *For $N \in {}^*\boldsymbol{N} - \boldsymbol{N}$, $\sum_{j=1}^{NH} \varepsilon^* h(j\varepsilon) \simeq \int_{R_+} h d\mu$.*

Proof. i) Obvious.

ii) Put $\alpha(n) = \sum_{j=1}^{nH} \varepsilon^* h(j)$ for $n \in {}^*\boldsymbol{N}$. Then, α: ${}^*\boldsymbol{N} \to {}^*\boldsymbol{R}$ is internal and $\alpha(n) \leqq \int_{R_+} h d\mu$. We claim that there exists an infinite natural number L such that $^{*o}\alpha(n)\simeq\alpha(n)$ for all $n\leqq L$. In fact, let A be the set of all $m\in{}^*\boldsymbol{N}$ such that $n|^{*o}\alpha(n)-\alpha(n)|\leqq 1$ for all $n\leqq m$. If n is finite, then $^{*0}\alpha(n)={}^o\alpha(n)\simeq\alpha(n)$, so $\boldsymbol{N}\subset A$. The set A being internal, it contains an infinite element L.

Write $I=\int_{\boldsymbol{R}+}hd\mu$. Then, $\alpha(n)\simeq\int_0^\infty hd\mu$ and $\lim_{n\to\infty}{}^o\alpha(n)=I$. Therefore we have $^{*o}\alpha(N)\simeq I$ for all $N\in{}^*\boldsymbol{N}-\boldsymbol{N}$. If in paticular $N\leqq L$, we have $\alpha(N)\simeq{}^{*o}\alpha(N)\simeq I$. On the other hand, if $N>L$, we have $\alpha(L)\leqq\alpha(N)\leqq I$ by i), hence $\alpha(N)\simeq I$. These two relations imply the desired result.

Proposition 5. i) *For every integer $l\in\boldsymbol{N}$, we have $(1+|\lambda|^2)^{-l}\in M_1(\boldsymbol{R})$.*
ii) *If $\varphi\in M_1(\boldsymbol{R})$, then F_φ, $\bar{F}_\varphi\in M_T(\boldsymbol{R})$.*

Proof. i) By Proposition 3, it suffices to show $(1+|x|^2)^{-1}\in M_1(\boldsymbol{R})$. Writing $h(t)=(1+|t|^2)^{-1}$, we have $^*h(x)=(1+|x|^2)^{-1}$ for $x\in X$. Lemma 1 implies $\sum_{j=1}^{(H/2)H}\varepsilon^*h(j\varepsilon)\simeq\int_{\boldsymbol{R}+}(1+|t|^2)^{-1}d\mu(t)$, so $\sum_{x\in X}\varepsilon^*h(x)$ is finite.

ii) Let $\varphi\in M_1(\boldsymbol{R})$ and $\varphi\geqq 0$. Then $F\varphi(0)=\sum_X\varepsilon\varphi$ is finite and $|F\varphi(x)|\leqq\sum_{y\in x}\varepsilon|e^{-2\pi ixy}|\varphi(y)=\sum_{y\in Y}\varepsilon\varphi(y)=F\varphi(0)$. For general $\varphi\in M_1(\boldsymbol{R})$, write $\varphi=(\varphi_1-\varphi_2)+i(\varphi_3-\varphi_4)$ where $\varphi_i\geqq 0$ and $\varphi_i\in M(\boldsymbol{R})$. Then we have $|F\varphi(x)|\leqq\sum_{i=1}^4|F\varphi_i(x)|\leqq\sum_{i=1}^4F\varphi_i(0)$, so $F\varphi(x)$ is finite. Combining with i), we have $(F\varphi)(1+|\lambda|^2)^{-l}\in M_1(\boldsymbol{R})$ and therefore $F\varphi\in M_T(\boldsymbol{R})$. Same for $\bar{F}\varphi$.

Theorem 1. *The space $T(\boldsymbol{R})$ is stable under operations D_+, D_-, λ, $\bar{\lambda}$, F and $\bar{F}$.*

Proof. By definition, $T(\boldsymbol{R})$ is stable under D_+, D_-, λ, and $\bar{\lambda}$. Using loose notations, A stands for $D_\pm$ and B stands for λ and $\bar{\lambda}$. Let $\psi\in M_1(\boldsymbol{R})$ and $\varphi=A^{m_1}B^{n_1}\cdots A^{m_k}B^{n_k}\psi$. Then $F\varphi=\pm B^{m_1}A^{n_1}\cdots B^{m}A^{n_k}F\psi$. $F\psi$ is in $M_T(\boldsymbol{R})$, so in $T(\boldsymbol{R})$. We have therefore $F\varphi\in T(\boldsymbol{R})$. By the definition of $T(\boldsymbol{R})$, we get the result.

For a function of on $\boldsymbol{R}$ and for x, h in $\boldsymbol{R}$, we put

$$(\Delta_+f)(x)=f(x+h)-f(x) \quad\text{and}\quad (\Delta_-f)(x)=f(x)-f(x-h)\,.$$

Lemma 2. *If a function f on $\boldsymbol{R}$ has bounded derivative of every degree, then we have*

$$|((\Delta_+\Delta_-)^nf)(x)-h^{2n}f^{(2n)}(x)|\leqq\frac{2n}{4!}|h|^{2n+2}\sup|f^{(2n+2)}|\,.$$

Proof. By Taylor's theorem and induction.

Lemma 3. *Let f be in $\mathcal{S}(\boldsymbol{R})$. Then,*

i) $^*f\,|\,X\in M_1(\boldsymbol{R})$ (*we shall write* $^*f$ *for* $^*f\,|\,X$ *if there is no danger of confusion*).

ii) *Fro every* $l\in\boldsymbol{N}$, *there is* $c\in\boldsymbol{R}$ *such that* $(1+|\lambda|^2)^l|F^*f|\leqq c$.

iii) $(1+|\lambda|^2)^l|F^*f-{}^*(\mathcal{F}f)|\simeq 0$ *for every* $l\in\boldsymbol{N}$.

Proof. i) By Proposition 5, $(1+|x|^2)^{-l}\in M_1(\boldsymbol{R})$ for every $l\in\boldsymbol{N}$. As f is in $\mathcal{S}(\boldsymbol{R})$, there exists $c\in\boldsymbol{R}$ such that $(1+|t|^2)^l|f(t)|\leqq c$ for all $t\in\boldsymbol{R}$. Therefore we have $(1+|x|^2)^l|{}^*f(x)|\leqq c$ for all $x\in X$, which implies $^*f\in M_1(\boldsymbol{R})$.

ii) Proposition 3 implies

$$\begin{aligned}|\lambda|^{2k}(F^*f) &= (-1)^k F\{(D_+D_-)^{k*}f\}\\ &= (-1)^k F({}^*f^{(2k)})+(-1)^{k+1}F({}^*f^{(2k)}-(D_+D_-)^{k*}f)\end{aligned}$$

for $k\in\boldsymbol{N}$ and we have

$$|\lambda|^{2k}|F^*f|\leqq|F^*f^{(2k)}|+|F({}^*f^{(2k)}-(D_+D_-)^{k*}f)|\,.$$

$$|F^*f^{(2k)}|=|\sum_{y\in X}\varepsilon e^{-2\pi ixy*}f^{(2k)}(y)|\leqq\sum_{y\in X}\varepsilon\,|{}^*f^{(2k)}(y)|\,,$$

which is finite by (i) and the fact $f^{(2k)}\in\mathcal{S}(\boldsymbol{R})$.

$$\begin{aligned}|F({}^*f^{(2k)}-(D_+D_-)^{k*}f)| &\leqq\sum_{y\to X}\varepsilon\,|{}^*f^{(2k)}(y)-(D_+D_-)^{k*}f(y)|\\ &\leqq\sum_{y\in X}\varepsilon\cdot\frac{2k}{4!}\varepsilon^2\cdot\sup|f^{(2k+2)}|\quad\text{(see Lemma 2)}\\ &=\varepsilon\cdot\frac{2k}{4!}\cdot\sup|f^{2k+2)}|\,.\end{aligned}$$

iii) For every $l\in\boldsymbol{N}$, there exists $c\in\boldsymbol{R}$ such that $(1+|\lambda(x)|^2)^{l+1}|(F^*f)(x)|\leqq c$ and $(1+|\lambda(x)|^2)^{l+1}|{}^*(\mathcal{F}f)(x)|\leqq c$ for all $x\in X$. We have therefore

$$(1+|\lambda(x)|^2)^l|(F^*f)(x)-(\mathcal{F}f)(x)|\leqq\frac{2c}{1+|\lambda(x)|^2}\leqq\frac{c}{8|x|^2}\,.$$

If $x\in X$ is infinite, then $c/8|x|^2$ is infinitesimal and we get the result.

If $x\in X$ is finite, $(1+|\lambda(x)|^2)^l$ is finite by the inequality $\lambda|(x)|\leqq 2\pi|x|$. Hence it suffices for us to show that $|(F^*f)(x)-{}^*(\mathcal{F}f)(x)|\simeq 0$.

Let $e>0$ and take $m\in N$ such that $\sum_{x\in X}\varepsilon(1+|x|^2)^{-m}$ is finite. Choose a function $g\in D(R)$ such that

$$\sup_{t\in\boldsymbol{R}}(1+|t|^2)^m|f(t)-g(t)|\leqq\frac{e}{\sum_{x\in X}\varepsilon(1+|x|^2)^{-m}}$$

and $\sup_{t\in\boldsymbol{R}}|(\mathcal{F}f)(t)-(\mathcal{F}g)(t)|\leqq e$.

Let $t={}^0x\in\boldsymbol{R}$. Then we have

$$(1+|x|^2)^m|{}^*f(x)-{}^*g(x)|\leqq\sup_{t\in R}(1+|t|^2)^m|f(t)-g(t)|$$

and therefore

$$|{}^*f(x)-{}^*g(x)|\leqq(1+|x|^2)^{-m}\sup(1+|t|^2)^m|f(t)-g(t)|.$$

We shall evaluate the right-hand side of the inequality

$$\begin{aligned}|(F^*f)(x)-{}^*(\mathcal{F}f)(x)|\leqq&|(F^*f)(x)-(F^*g)(x)|\\&+|(F^*g)(x)-{}^*(\mathcal{F}g)(x)|\\&+|{}^*(\mathcal{F}f)(x)-{}^*(\mathcal{F}g)(x)|.\end{aligned}$$

$$\begin{aligned}\text{The first term}&=|\sum_{y\in X}\varepsilon e^{-2\pi ixy}({}^*f(x)-{}^*g(x))|\\&\leqq\sum_{y\in X}\varepsilon(1+|x|^2)^{-m}\sup_{t\in R}\{(1+|t|^2)^m|f(t)-g(t)|\}\leqq e.\end{aligned}$$

$$\begin{aligned}\text{The third term}\leqq&|{}^*(\mathcal{F}f)(x)-(\mathcal{F}f)(t)|+|(\mathcal{F}f)(t)-(\mathcal{F}g)(t)|\\&+|{}^*(\mathcal{F}g)(x)-(\mathcal{F}g)(t)|.\end{aligned}$$

The first and third summands are infinitesimal and the second summand is $\leqq e$.

Put $K=\text{Supp}\,(g)$. Then,

$$\begin{aligned}\text{the second term}&=|(F^*g)(x)-{}^*(\mathcal{F}g)|\\&\leqq|\sum_{y\in{}^*K\cap X}\varepsilon e^{-2\pi ixy}{}^*g(y)-\sum_{y\in{}^*K\cap X}\varepsilon e^{-2\pi ity}{}^*g(y)|\\&\quad+|\sum_{y\in{}^*K\cap X}\varepsilon e^{-2\pi ity}{}^*g(y)-\int_K e^{-\pi its}g(s)d\mu(s)|\\&\quad+|\int_K e^{-2\pi its}g(s)d\mu(s)-{}^*(\mathcal{F}g)(x)|.\end{aligned}$$

If $k\geqq m+1$,

$$\begin{aligned}&\text{the first summand}=|\sum_{y\in{}^*K\cap X}\varepsilon(e^{-2\pi ixy}-e^{-2\pi ity})(1+|y|^2)^{-k}(1+|y|^2)^k{}^*g(y)|\\&\leqq\sum_{y\in{}^*K\cap X}\frac{\varepsilon|e^{-\pi i(x-t)y}-1|}{(1+|y|^2)^k}\sup_{t\in R}(1+|t|^2)^k|g(t)|\\&\leqq\sum_{y\in X}\frac{\varepsilon2\pi|x-t||y||\cos2\pi\sigma(x-t)y+i\sin2\pi\tau(x-t)y|}{(1+|y|^2)^k}\sup_{t\in R}(1+|t|^2)^k|g(t)|\\&\leqq\sum_{y\in X}\frac{\varepsilon|y|}{(1+|y|^2)^k}\cdot4\pi|x-t|\cdot\sup_{t\in R}(1+|t|^2)^k|g(t)|\simeq0,\end{aligned}$$

where σ, $\tau\in{}^*R$ and $0<\sigma$, $\tau<1$. The second and third summand being infinitesimal, the second term is $\leqq e$.

Combining these results, we have

$$|(F^*f)(x)-{}^*(\mathcal{F}f)(x)|\leqq e+e+2e=4e.$$

The positive number e being arbitrary, we have $|(F^*f)(x)-{}^*(\mathcal{F}f)(x)|\simeq0$.

Proposition 6. i) *If* $\varphi \in M_T(R)$, *then* $F\varphi \in A_T(\boldsymbol{R})$.

ii) *If* $\varphi \in M_T(\boldsymbol{R})$ *and* $f \in \mathcal{S}(\boldsymbol{R})$, *then* $\sum_X \varepsilon (F\varphi)^* f \simeq \sum_X \varepsilon \varphi^*(\mathcal{F} f)$. *In other words,* $\Gamma_{F\varphi}(f) = \Gamma_\varphi(\mathcal{F} f)$.

Proof. Note that $\sum_X \varepsilon (F\varphi)^* f = \sum_X \varepsilon \varphi (F^* f)$.

i) Take a standard integer l so that $\sum_X \varepsilon \frac{|\varphi|}{(1+|\lambda|^2)^l}$ is finite. For every f in $\mathcal{S}(\boldsymbol{R})$, we have

$$|\sum_X \varepsilon (F\varphi)^* f| = |\sum_X \varepsilon \varphi (F^* f)| = |\sum_X \varepsilon \frac{\varphi}{(1+|\lambda|^2)^l} (1+|\lambda|^2)^l F^* f|$$

$$\leqq \Big(\sum_X \varepsilon \frac{|\varphi|}{(1+|\lambda|^2)^l}\Big) \cdot \sup_X (1+|\lambda|^2)^l |F^* f|.$$

This is finite by Lemma 3 ii) and therefore $F\varphi \in A_T(\boldsymbol{R})$.

ii) Take $l \in \boldsymbol{N}$ so that $\sum_X \varepsilon \frac{|\varphi|}{(1+|\lambda|^2)^l}$ is finite. For every f in $\mathcal{S}(\boldsymbol{R})$, we have

$$|\sum_X \varepsilon (F\varphi)^* f - \sum_X \varepsilon \varphi^* (\mathcal{F} f)| = |\sum_X \varepsilon \varphi (F^* f - {}^*(\mathcal{F} f))|$$

$$= |\sum_X \varepsilon \frac{\varphi}{(1+|\lambda|^2)^l} (1+|\lambda|^2)^l (F^* f - {}^*(\mathcal{F} f))|$$

$$\leqq \sum_X \varepsilon \frac{|\varphi|}{(1+|\lambda|^2)^l} (1+|\lambda|^2)^l |F^* f - {}^*(\mathcal{F} f)|.$$

Lemma 3 iii) implies

$$|\sum_X \varepsilon (F\varphi)^* f - \sum_X \varepsilon \varphi^* (\mathcal{F} f)| \leqq d \sum_X \varepsilon \frac{|\varphi|}{(1+|\lambda|^2)^l}$$

for every positive $d \in \boldsymbol{R}$, which is our claim.

4. Spaces $\boldsymbol{T}(\boldsymbol{R})$ and $\mathcal{S}'(\boldsymbol{R})$

Proposition 7. *Let* φ *be a function in* $R(X)$. *Then the following two conditions on* φ *are mutually equivalent:*

i) $\sum_X \varepsilon^{m+1} \frac{|\varphi|}{(1+|\lambda|^2)^l}$ *is finite for some* m *and* l *in* $\boldsymbol{N}$.

ii) $\sum_X \varepsilon^{k+1} \frac{|\varphi|^2}{(1+|\lambda|^2)^r}$ *is finite for some* k *and* r *in* $\boldsymbol{N}$.

Proof. i) ⇒ ii) $\sum_X \varepsilon^{2m+2} \frac{|\varphi|^2}{(1+|\lambda|^2)^{2l}} \leqq \Big(\sum_X \varepsilon^{m+1} \frac{|\varphi|}{(1+|\lambda|^2)^l}\Big)^2$.

ii) ⇒ i) If $k+1 \leqq 2m$ and $r \leqq 2l$, then we have $\varepsilon^{2m} \leqq \varepsilon^{k+1}$ and $(1+|\lambda|^2)^{-2l} \leqq (1+|\lambda|^2)^{-r}$. Hence we have

$$\left(\sum_X \varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}\right)^2 \leqq H^2\sum_X \varepsilon^{2m+1}\frac{|\varphi|^2}{(1+|\lambda|^2)^{2l}}$$

$$\leqq \sum_X \varepsilon^{2m}\frac{|\varphi|^2}{(1+|\lambda|^2)^r} \leqq \sum_X \varepsilon^{k+1}\frac{|\varphi|^2}{(1+|\lambda|^2)^r}\,.$$

DEFINITION. $Z_T(\boldsymbol{R})$ is the set of functions φ in $R(X)$ which satisfy mutually equivalent conditions in Proposition 7. Clearly $M_T(\boldsymbol{R})\in Z_T(\boldsymbol{R})$.

Lemma 4. *For $n\in\boldsymbol{N}$, $n\geqq 1$, we have*

$$D_+^n\lambda(x)=\frac{\lambda(\varepsilon)^n}{\varepsilon}e^{2\pi i\varepsilon x}=\frac{\lambda(\varepsilon)^n}{\varepsilon}+\lambda(\varepsilon)^n\lambda(x)\,,$$

$$D_-^n\lambda(x)=\frac{(-\bar\lambda(\varepsilon))^n}{\varepsilon}e^{2\pi i\varepsilon x}=\frac{(-\bar\lambda(\varepsilon))^n}{\varepsilon}+\bar\lambda(\varepsilon())^n\lambda(x)$$

and $|\lambda(\varepsilon)|^n/\varepsilon\leqq(2\pi)^n\varepsilon^{n-1}$.

Proof. Direct calculation for $n=1$ and induction on n.

Proposition 8. *The space $Z_T(\boldsymbol{R})$ is stable under operations D_+, D_-, λ, $\bar\lambda$, F $\bar F$ and under the multiplication of functions.*

Proof. 1° $\displaystyle\sum\varepsilon^{m+2}\frac{|D_\pm\varphi|}{(1+|\lambda|^2)^l}=\sum_{x\in X}\varepsilon^{m+1}\frac{|\varphi(x\pm\varepsilon)-\varphi(x)|}{(1+|\lambda(x)|^2)^l}$

$$\leqq\sum_{x\in X}\varepsilon^{m+1}\frac{|\varphi(x\pm\varepsilon)|}{(1+|\lambda(x)^2)|^l}+\sum_X\varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}$$

$$=\sum_{x\in X}\varepsilon^{m+1}\frac{|\varphi(x\pm\varepsilon)|}{(1+|\lambda(x\pm\varepsilon)^2|)^l}\left(\frac{1+|\lambda(x\pm\varepsilon)|^2}{1+|\lambda(x)|^2}\right)^l+\sum_X\varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}$$

$$\leqq 2^l\sum_X\varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}+\sum_X\varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}=(2^l+1)\sum_X\varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}\,.$$

This means $D_\pm\varphi\in Z_T(\boldsymbol{R})$ for $\in Z_T(\boldsymbol{R})$. Here, we used the inequality $\dfrac{1+|\lambda(x\pm\varepsilon)|^2}{1+|\lambda(x)|^2}\simeq 1\leqq 2$.

2° $\displaystyle\sum_X\varepsilon^{m+1}\frac{|\lambda\varphi|}{(1+|\lambda|^2)^{l+1}}\leqq\sum_X\varepsilon^{m+1}\frac{|\varphi|}{(1+|\lambda|^2)^l}$,

which implies $\lambda\varphi\in Z_T(\boldsymbol{R})$ for $\varphi\in Z_T(\boldsymbol{R})$. Same for $\bar\lambda\varphi\in Z_T(\boldsymbol{R})$.

3° Take $r\in\boldsymbol{N}$ so that $\sum_X\varepsilon(1+|\lambda|^2)^{-r}$ is finite. We shall show

$$\sum_X\varepsilon^{k+2r+2}\frac{|F\varphi|^2}{(1+|\lambda|^2)^r}\leqq(\varepsilon^2+\pi^2)^r\sum_X\varepsilon\frac{1}{(1+|\lambda|_2)^r}\sum_X\frac{\varepsilon^{k+1}|\varphi|^2}{(1+|\lambda|^2)^r}\,.$$

From inequalities

$$|(F\varphi)(x)|^2=|\sum_{y\in X}\varepsilon e^{-2\pi ixy}\varphi(y)|^2\leqq(\sum_X\varepsilon|\varphi|^2)\leqq H^2\sum_X\varepsilon^2|\varphi|^2=\sum_X|\varphi|^2,$$

we have

$$\sum_X\varepsilon^{k+2r+2}\frac{|F\varphi|^2}{(1+|\lambda|^2)^r}\leqq\sum_{x\in X}\sum_{y\in X}\frac{\varepsilon^{k+2r+2}|\varphi(y)|^2}{(1+|\lambda(x)|^2)^r}$$
$$=\sum_{y\in X}\varepsilon^{k+1}\frac{|\varphi|(y)|^2}{(1+|\lambda(y)|^2)^r}\sum_{x\in X}\varepsilon^{2r+1}\left(\frac{1+|\lambda(y)|^2}{1+|\lambda(x)|^2}\right)^r.$$

On the other hand, from $|\lambda(y)|\leqq 2\pi|y|\leqq\pi H$, we have

$$\sum_{x\in X}\varepsilon^{2r+1}\left(\frac{1+|\lambda(y)|^2}{1+|\lambda(x)|^2}\right)^r\leqq\sum_X\varepsilon^{2r+1}\frac{1+\pi^2H^2}{(1+|\lambda|^2)^r}=(\varepsilon^2+\pi^2)^r\sum_X\frac{\varepsilon}{(1+|\lambda|^2)^r}.$$

Combining these, we have

$$\sum_X\varepsilon^{k+2r+2}\frac{|F\varphi|^2}{(1+|\lambda|^2)^r}\leqq\left(\sum_X\varepsilon^{k+1}\frac{|\varphi|^2}{(1+|\lambda|^2)^r}\right)(\varepsilon^2+\pi^2)^r\sum_X\frac{\varepsilon}{(1+|\lambda|^2)^r}.$$

Now, if $\varphi\in Z_T(\boldsymbol{R})$, Proposition 7 implies the existence of $k, r\in N$ such that $\sum_X\frac{\varepsilon^{k+1}|\varphi|^2}{(1+|\lambda|^2)^r}$ is finite. So, by the above inequality, $\sum_X\varepsilon^{k+2r+2}\frac{|F\varphi|^2}{(1+|\lambda|^2)^r}$ is finite, and Proposition 7 implies $F\varphi\in Z_T(\boldsymbol{R})$. Same for $\bar{F}\varphi$.

4° The inequality

$$\left(\sum_X\frac{\varepsilon^{m+1+n+1}|\varphi\psi|}{(1+|\lambda|^2)^{l+s}}\right)^2\leqq\left(\sum_X\frac{\varepsilon^{2m+2}|\varphi|^2}{(1+|\lambda|^2)^{2l}}\right)\left(\sum_X\frac{\varepsilon^{2n+2}|\psi|^2}{(1+|\lambda|^2)^{2s}}\right)$$
$$\leqq\left(\sum_X\frac{\varepsilon^{m+1}|\varphi|}{(1+|\lambda|^2)^l}\right)^2\left(\sum_X\frac{\varepsilon^{n+1}|\psi|}{(1+|\lambda|^2)^s}\right)^2$$

implies $\varphi\psi\in Z_T(\boldsymbol{R})$, if φ, $\psi\in Z_T(\boldsymbol{R})$.

Proposition 9. *If $\varphi\in A_T(\boldsymbol{R})\cap Z_T(\boldsymbol{R})$, then $D_\pm\varphi$, $\lambda\varphi$, $\bar{\lambda}\varphi\in A_T(\boldsymbol{R})\cap Z_T(\boldsymbol{R})$. Moreover, if $f\in\mathcal{S}(\boldsymbol{R})$,*

$$\Gamma_{D_\pm\varphi}(f)=-\Gamma_\varphi(f'),\quad\Gamma_{\lambda\varphi}(f)=\Gamma(2\pi itf),\quad\Gamma_{\bar{\lambda}\varphi}(f)=-\Gamma(2\pi itf).$$

Proof. (1) Let $\varphi\in A_T(\boldsymbol{R})\cap Z_T(\boldsymbol{R})$. $D_\pm\varphi\in Z_T(\boldsymbol{R})$ by Proposition 8. Let $f\in\mathcal{S}(\boldsymbol{R})$ and take $m, l\in\boldsymbol{N}$ so that $\sum_{x\in X}\varepsilon^{m+1}\frac{|\varphi(x)|}{(1+|x|^2)^l}$ is finite. Then we have

$$\sum_X\varepsilon(D_\pm\varphi)^*f=\pm\sum_{x\in X}\varphi(x\pm\varepsilon)^*f(x)\mp\sum_X\varphi^*f$$
$$\simeq\mp\sum_{x\in X}\varphi(x)^*f(x\mp\varepsilon)\mp\sum_X\varphi^*f=-\sum_{x\in X}\varepsilon\varphi(x)\frac{{}^*f(x\mp\varepsilon)-{}^*f(x)}{\mp\varepsilon}$$
$$=-\left\{\sum_{k=1}^{m+1}\frac{(\mp1)^{k-1}}{k!}\varepsilon^{k-1}\sum_X\varepsilon\varphi^*f^{(k)}\right.$$
$$\left.+\frac{(\mp1)^{m+1}}{(m+2)!}\varepsilon\sum_{x\in X}\varepsilon^{m+1}\varphi(x)(\mathrm{Re}\,{}^*f^{(m+2)}(x\mp\sigma\varepsilon)+i\,\mathrm{Im}\,{}^*f^{(m+2)}(x\mp\tau\varepsilon))\right\}$$

where $\sigma, \tau \in {}^*R$ and $0<\sigma, \tau<1$. We have, for $1 \leqq k \leqq m+1$,

$$\frac{(\mp 1)^{k-1}}{k!} \varepsilon^{k-1} \sum_X \varepsilon\varphi^* f^{(k)} \simeq \frac{(\mp 1)^{k-1}}{k!} \Gamma_\varphi(f^{(k)}) \begin{cases} = \Gamma_\varphi(f') & (k=1) \\ \simeq 0 & (1<k\leqq m+1). \end{cases}$$

On the other hand, we have

$$\begin{aligned} &\frac{\varepsilon}{(m+2)!} |\varepsilon^{m+1}\varphi(x)(\mathrm{Re}\, {}^*f^{(m+2)}(x\mp\sigma\varepsilon)+i\,\mathrm{Im}\, {}^*f^{(m+2)}(x\mp\tau\varepsilon))| \\ &= \frac{\varepsilon}{(m+2)!} \left| \sum_{x\in X} \varepsilon^{m+1} \frac{\varphi(x)}{(1+|x|^2)^l} (1+|x|^2)^l (\mathrm{Re}\, {}^*f^{(m+2)}(x\mp\sigma\varepsilon) \right. \\ &\qquad\qquad \left. +i\,\mathrm{Im}\, {}^*f^{(m+2)}(x\mp\tau\varepsilon)) \right| \\ &\leqq \frac{\varepsilon}{(m+2)!} \sum_{x\in X} \varepsilon^{m+1} \frac{|\varphi(x)|}{(1+|x|^2)^l} \cdot 2 \sup_{t\in \boldsymbol{R}} (1+|t|^2)^l |f^{(m+2)}(t)| \simeq 0. \end{aligned}$$

Hence we have $D_\pm\varphi \in A_T(\boldsymbol{R})$ and $\Gamma_{D\pm\varphi}(f) = -\Gamma_\varphi(f')$ for $f\in\mathcal{S}(\boldsymbol{R})$.

(2) Let $\varphi\in A_T(\boldsymbol{R})\cap Z_T(\boldsymbol{R})$. Proposition 8 showed that $\lambda\varphi, \bar{\lambda}\varphi \in Z_T(\boldsymbol{R})$. Let $f\in\mathcal{S}(\boldsymbol{R})$ and take $m, l\in\boldsymbol{N}$ such that $\sum_{x\in X} \varepsilon^{m+1} \frac{|\varphi(x)|}{(1+|x|^2)^l}$ is finite. We can then write

$$\begin{aligned} \lambda(x) &= \frac{e^{2\pi i\varepsilon x}-1}{\varepsilon} = \sum_{k=1}^{m+1} \frac{(2\pi i)^k \varepsilon^{k-1} x^k}{k!} \\ &\quad + \frac{(2\pi i)^{m+2}\varepsilon^{m+1}}{(m+2)!} (\cos 2\pi\varepsilon\sigma x + i \sin 2\pi\varepsilon\tau x)\, x^{m+2}, \end{aligned}$$

where $\sigma, \tau\in{}^*\boldsymbol{R}$ and $0<\sigma, \tau<1$. We have

$$\begin{aligned} \sum_X \varepsilon\lambda\varphi^* f &= \sum_{k=0}^{m+1} \frac{(2i\pi)^k\varepsilon^{k-1}}{k!} \sum_{x\in X} \varepsilon\varphi(x) x^k\, {}^*f(x) \\ &\quad + \frac{(2\pi i)^{m+2}\varepsilon^{m+1}}{(m+2)!} \sum_{x\in X} \varepsilon\varphi(x)(\cos 2\pi\varepsilon\sigma x + i\sin 2\pi\varepsilon\tau x) x^{m+2}\, {}^*f(x). \end{aligned}$$

If $1\leqq k\leqq m+1$, we have

$$\frac{(2\pi i)^k\varepsilon^{k-1}}{k!} \sum_{x\in X} \varepsilon\varphi(x)^k\, {}^*f(x) \simeq \frac{(2\pi i)^k\varepsilon^{k-1}}{k!} \Gamma_\varphi(t^k f) \begin{cases} = 2\pi i \Gamma_\varphi(tf) & (k=1), \\ \simeq 0 & (1<k\leqq m). \end{cases}$$

The absolute value of remaining terms is bounded by

$$\begin{aligned} &\frac{(2\pi)^{m+2}\varepsilon}{(m+2)!} \left| \sum_{x\in X} \varepsilon^{m+1} \frac{\varphi(x)}{(1+|x|^2)^l} (1+|x|^2)^l (\cos 2\pi\varepsilon\sigma x + i\sin 2\pi\varepsilon\tau x) x^{m+2}\, {}^*f(x) \right| \\ &\leqq \frac{(2\pi)^{m+2}\varepsilon}{(m+2)!} \left(\sum_{x\in X} \varepsilon^{m+1} \frac{|\varphi|}{(1+|x|^2)^l} \right) 2 \sup_{t\in\boldsymbol{R}} (1+|t|^2)^l |t^{m+2} f(t)| \simeq 0. \end{aligned}$$

Hence we have $\lambda\varphi \in A_T(\boldsymbol{R})$ and $\Gamma_{\lambda\varphi}(f) = \Gamma_\varphi(2\pi\, itf)$ for $f \in \mathcal{S}(\boldsymbol{R})$. Same for $\bar{\lambda}\varphi$.

Theorem 2. $T(\boldsymbol{R}) \subseteq A_T(\boldsymbol{R}) \cap Z_T(\boldsymbol{R})$.

Proof. Note that $M_1(\boldsymbol{R}) \subseteq M_T(\boldsymbol{R}) \cap A_T(\boldsymbol{R}) \cap Z_T(\boldsymbol{R})$, which follows from definitions of $M_T(\boldsymbol{R})$ and $Z_T(\boldsymbol{R})$, and from Proposition 4. On the other hand, $A_T(\boldsymbol{R}) \cap Z_T(\boldsymbol{R})$ is stable under D_+, D_-, λ and $\bar{\lambda}$ (Proposition 9). Hence the definition of $T(\boldsymbol{R})$ leads us to the result.

Results in § 3, in particular Proposition 6, suggest that divided differences and their finite sums of functions in $M_T(\boldsymbol{R})$ are easier to manipulate than general functions in $T(\boldsymbol{R})$. So we hope to "approximate" a function in $T(\boldsymbol{R})$ by a finite sum of divided differences of functions in $M_T(\boldsymbol{R})$. For this purpose, we introduce an equivalence relation $\equiv$ in the space $T(\boldsymbol{R})$. Let $\boldsymbol{N}_+ = \{n \in \boldsymbol{N} \mid n > 0\}$.

DEFINITION. Let $T_0(\boldsymbol{R})$ be the set of finite sums $\sum_{i=1}^{n} \alpha_i \varphi_i$, where $n \in \boldsymbol{N}_+$, $\alpha_i \in {}^*\boldsymbol{C}$, $\alpha_i \simeq 0$ and $\varphi_i \in T(\boldsymbol{R})$ $(1 \leqq i \leqq n)$. For $\varphi, \psi \in T(\boldsymbol{R})$, we write $\varphi \equiv \psi$ if $\varphi - \psi \in T_0(\boldsymbol{R})$.

Lemma 5. *Let $Ns({}^*\boldsymbol{C}) = \{\alpha \in {}^*\boldsymbol{C} \mid \alpha$ finite$\}$.*

i) *If $\varphi, \psi \in T(\boldsymbol{R})$ and $\varphi \equiv \psi$, then $\Gamma_\varphi = \Gamma_\psi$.*

ii) *The relation $\equiv$ is compatible with addition, subtraction, multiplication by elements of $Ns({}^*\boldsymbol{C})$, λ, $\bar{\lambda}$, $D_\pm$, F and $\bar{F}$.*

iii) *If $\alpha, \beta \in Ns({}^*\boldsymbol{C})$, $\alpha \simeq \beta$ and $\varphi, \psi \in T(\boldsymbol{R})$, $\varphi \equiv \psi$, then $\alpha\varphi \equiv \beta\psi$.*

We omit the proof.

Theorem 3. *Every function φ in $T(\boldsymbol{R})$ is equivalent $(\equiv)$ to a sum $\sum_{i=1}^{q} D_+^{m_i} D_-^{n_i} \psi_i$, where $q \in N_+$, $\psi_i \in M_T(\boldsymbol{R})$ and $m_i, n_i \in \boldsymbol{N}$ $(1 \leqq i \leqq q)$.*

Proof. The definition of $T(\boldsymbol{R})$ assures that φ is of the form $\varphi = \prod_{k=1}^{l} (D_+^{m_k} D_-^{n_k} \lambda^{r_k} \bar{\lambda}^{s_k}) \psi$ where $\psi \in M_1(R)$. We proceed by induction on l. The assertion is trivial for $l=1$. Assume the result for $l-1$. Then, we can write

$$\varphi \equiv D_+^m D_-^n \lambda^r \bar{\lambda}^s \Big(\sum_{i=1}^{u} D_+^{k_i} D_-^{l_i} \psi_i\Big),$$

where $u \in \boldsymbol{N}_+$ and $\psi_i \in M_T(\boldsymbol{R})$, $k_i, l_i \in \boldsymbol{N}$ $(1 \leqq i \leqq u)$. It suffices therefore to prove the following assertion $P(r, s, k, l)$ with parameters r, s, k, l in $\boldsymbol{N}$: if $\psi \in M_T(\boldsymbol{R})$, then we can write

$$\lambda^r \bar{\lambda}^s D_+^k D_-^l \psi \equiv \sum_{j=1}^{v} D_+^{m_j} D_-^{n_j} \chi_j,$$

where $v \in \boldsymbol{N}_+$ and $\chi_j \in M_T(\boldsymbol{R})$, $m_j, n_j \in \boldsymbol{N}$ $(1 \leqq j \leqq v)$.

First, $P(0, 0, k, l)$ is trivial. We assume $P(0, s, k, l)$ and show $P(0, s+1,$

$k, l)$. We have $\bar{\lambda}^{s+1}D_+^k D_-^l\psi\equiv\sum_{j=1}^{v}\bar{\lambda}D_+^{m_j}D_-^{n_j}\chi_j$, and by Lemma 6,

$$\bar{\lambda}D_+^{m_j}D_-^{n_j}\chi_j\equiv D_+^{m_j}D_-^{n_j}(\bar{\lambda}\chi)+2\pi i(m_jD_+^{m_j-1}D_-^{n_j}\chi+n_jD_+^{m_j}D_-^{n_j-1}),$$

and we get $P(0, s+1, k, l)$ because $\bar{\lambda}\chi\in M_T(\boldsymbol{R})$.

Next, we assume $P(r, s, k, l)$ and show $P(r+1, s, k, l)$. We have $\lambda^{r+1}\bar{\lambda}^sD_+^k D_-^l\psi\equiv\sum_{j=1}^{v}\lambda D_+^{m_j}D_-^{n_j}\chi$ and by Lemma 6,

$$\lambda D_+^{m_j}D_-^{n_j}\chi\equiv D_+^{m_j}D_-^{n_j}(\lambda\chi)-2\pi i(m_jD_+^{n_j}D_-^{n_j}\chi+n_jD_+^{n_j}D_-^{n_j-1}\chi),$$

and we get $P(r+1, s, k, l)$. We have thus proved $P(r, s, k, l)$ for all $r, s, k, l\in\boldsymbol{N}$ and so Theorem 3 is proved.

Theorem 4. 1) *If* $\varphi\in T(\boldsymbol{R})$. *then* $\Gamma_\varphi\in\mathcal{S}'(\boldsymbol{R})$ *and* $\Gamma_{D_\pm\varphi}=(\Gamma_\varphi)'$, $\Gamma_{\lambda\varphi}=(2\pi it)\Gamma_\varphi$, $\Gamma_{\bar{\lambda}\varphi}=(-2\pi it)\Gamma_\varphi$.

2) *If* $\varphi\in T(R)$, *then* $\Gamma_{F\varphi}=\mathcal{F}\Gamma_\varphi$ *and* $\Gamma_{\bar{F}\varphi}=\bar{\mathcal{F}}\Gamma_\varphi$.

3) *The map*: $\varphi\mapsto\Gamma_\varphi$ *from* $T(\boldsymbol{R})$ *to* $\mathcal{S}'(\boldsymbol{R})$ *is surjective.*

Proof. (due to T. Nakamura). 1) By Theorem 3, we can assume that $\varphi\equiv D_+^m D_-^n\psi$, where $m, n\in\boldsymbol{N}$ and $\psi\in M_T(\boldsymbol{R})$. As $D_+^m D_-^n\psi\in T(\boldsymbol{R})$, we have $\Gamma_\varphi=\Gamma_{D_+^m D_-^n\psi}$ by Lemma 5. $T(\boldsymbol{R})\subseteq A_T(\boldsymbol{R})\cap Z_T(\boldsymbol{R})$ (Theorem 2) and Proposition 9 imply that $\Gamma_{D_+^m D_-^n\psi}(f)=(-1)^{m+n}\Gamma_\psi(f^{(m+n)})$, we have $\Gamma_\psi\in\mathcal{S}'(\boldsymbol{R})$ by $\psi\in M_T(\boldsymbol{R})$ and Proposition 4. Hence we have

$$(-1)^{m+n}\Gamma_\psi(f^{(m+n)})=(\Gamma_\psi)^{(m+n)}(f),$$

where $(\Gamma_\psi)^{(m+n)}$ is $(m+n)$-th derivative of Γ_ψ in the sense of distribution. We have therefore $\Gamma_\varphi=(\Gamma_\psi)^{(m+n)}\in\mathcal{S}'(\boldsymbol{R})$. By Proposition 9, we get the result.

2) By Lemma 5 ii) and Proposition 3, we have

$$F\varphi\equiv FD_+^m D_-^n\psi=(-1)^n\lambda^m\bar{\lambda}^nF\psi$$

and therefore $\Gamma_{F\varphi}=(-1)^n\Gamma_{\lambda^m\bar{\lambda}^nF\psi}$.

By Theorem 2 and Proposition 9, we have

$$(-1)^n\Gamma_{\lambda^m\bar{\lambda}^nF\psi}(f)=\Gamma_{F\psi}((2\pi it)^{m+n}f)$$

for $f\in\mathcal{S}(\boldsymbol{R})$, and by Proposition 6

$$\begin{aligned}\Gamma_{F\psi}((2\pi it)^{m+n}f)&=\Gamma_\psi(\mathcal{F}((2\pi it)^{m+n}f))=\Gamma_\psi(-1)^{m+n}(\mathcal{F}f)^{(m+n)})\\&=(\Gamma_\psi)^{(m+n)}(\mathcal{F}f)=\Gamma_{D_+^m D_-^n\psi}(\mathcal{F}f)=\Gamma_\varphi(\mathcal{F}f)=(\mathcal{F}\Gamma_\varphi)(f),\end{aligned}$$

and hence we get $\Gamma_{F\varphi}=\mathcal{F}\Gamma_\varphi$. The same for $\bar{F}$.

3) Let $T\in\mathcal{S}'(\boldsymbol{R})$. By the structure theorem of $\mathcal{S}'(\boldsymbol{R})$, there exist a bounded complex measure S and $n, k\in\boldsymbol{N}$ such that $T=\{(1+|t|^2)^kS\}^{(n)}$ (see [3]). By our previous paper [1], there exists $\psi\in M_1(\boldsymbol{R})$ such that $S(g)={}^0\sum_X\varepsilon\psi^*g$ for

$g\in\mathcal{D}(\boldsymbol{R})$. We have therefore $S\,|\,\mathcal{D}(\boldsymbol{R})=\Gamma_\psi\,|\,\mathcal{D}(\boldsymbol{R})$ and hence $S=\Gamma_\psi$. If we put $\varphi=D_+^n(1+|\lambda|^2)^k\psi$, then $\varphi\in T(\boldsymbol{R})$ and $\Gamma_\varphi=T$.

5. Spaces $Q(\boldsymbol{R})$ and $\mathcal{S}(\boldsymbol{R})$

Recall definitions in § 1. $U(\boldsymbol{R})$ is the set of functions φ in $R(X)$ such that $\varphi(x)$ is finite for all $x\in X$ and that $\varphi(x)\simeq\varphi(y)$ whenever $x, y\in X$ and $x\simeq y$. $U(\boldsymbol{R})$ is the set of bounded and uniformly continuous $\boldsymbol{C}$-valued functions on $\boldsymbol{R}$. For a function φ in $U(\boldsymbol{R})$, ${}^\vee\varphi$ is a function $\boldsymbol{R}\to\boldsymbol{C}$ defined by ${}^\vee\varphi(t)={}^0(\varphi({}^\Delta t))$ for $t\in\boldsymbol{R}$, where ${}^\Delta t=\max\{x\in X\,|\,x\leqq t\}$ and ${}^0\alpha$ is the standard part of $\alpha\in Ns({}^*\boldsymbol{C})$. These definitions and the following theorem are due to Robinson [4].

Theorem 5. 1) *If $\varphi\in U(\boldsymbol{R})$, then ${}^\vee\varphi\in U(\boldsymbol{R})$ and $\Gamma_\varphi=T_{{}^\vee\varphi}$, where $T_{{}^\vee\varphi}$ is the distribution defined by ${}^\vee\varphi$: $\Gamma_\varphi(f)=\int_{\boldsymbol{R}}{}^\vee\varphi f\,d\mu(f\in S(R))$.*

2) *If $h\in U(\boldsymbol{R})$, then ${}^*h\,|\,X\in U(\boldsymbol{R})$ and ${}^\vee({}^*h\,|\,X)=h$.*

DEFINITION. 1) For a function φ in $R(X)$, let $Y(\varphi)$ be the set of finite sums of functions of the form $\alpha\lambda^l\bar\lambda^m D_+^n D_-^k\varphi$, where $\alpha\in Ns({}^*\boldsymbol{C})$ and $l, m, n, k\in\boldsymbol{N}$.

2) $Q(\boldsymbol{R})$ is the set of functions φ in $U(\boldsymbol{R})$ such that $Y(\varphi)\subseteq U(\boldsymbol{R})$.

Proposition 10. *If $\varphi\in Q(\boldsymbol{R})$ and $1\leqq p<\infty$, then $\sum_X\varepsilon|\varphi|^p$ is finite. In particular, $Q(\boldsymbol{R})\subseteq M_1(\boldsymbol{R})\subseteq T(\boldsymbol{R})$.*

Proof. Take $l\in\boldsymbol{N}$ such that $(1+|\lambda|^2)^{-l}\in M_1(\boldsymbol{R})$. As $(1+|\lambda|^2)^{l+1}\varphi\in U(\boldsymbol{R})$, there exists $c\in\boldsymbol{R}$ such that $(1+|\lambda|^2)^l|\varphi|\leqq c$. Hence $|\varphi|\leqq c(1+|\lambda|^2)^{-l}$ and $|\varphi|^p\leqq(1+|\lambda|^2)^{-lp}$.

Lemma 7. *For φ, $\psi\in R(X)$, we have*

$$D_\pm^n(\varphi\psi)=\sum_{r=0}^{n}(\mp\varepsilon)^r\binom{n}{r}\sum_{j=1}^{n-r}\binom{n-r}{j}D_\pm^{n-j}\varphi D_\pm^{j+r}\psi\,.$$

Proof. Induction on n.

Lemma 8. *If $\varphi\in R(X)$, then $Y(D)\varphi_\pm$, $Y(\lambda\varphi)$ and $Y(\bar\lambda\varphi)$ are included in $Y(\varphi)$.*

Proof. $Y(D_\pm\varphi)\subseteq Y(\varphi)$ follows from the definition. By Lemma 4 we have

$$D_+\lambda=\frac{\lambda(\varepsilon)}{\varepsilon}+\lambda(\varepsilon)\lambda\,,\quad D_-\lambda=-\Big(\frac{\bar\lambda(\varepsilon)}{\varepsilon}+\bar\lambda(\varepsilon)\lambda\Big),$$

and hence

$$D_+\bar\lambda=\frac{\bar\lambda(\varepsilon)}{\varepsilon}+\bar\lambda(\varepsilon)\bar\lambda\,,\quad D_-\bar\lambda=-\Big(\frac{\lambda(\varepsilon)}{\varepsilon}+\lambda(\varepsilon)\bar\lambda\Big).$$

We have therefore

$$D_+(\lambda\varphi) = \lambda D_+\varphi + \frac{\lambda(\varepsilon)}{\varepsilon}\varphi + \lambda(\varepsilon)\lambda\varphi + \lambda(\varepsilon)D_+\varphi + \varepsilon\lambda(\varepsilon)\lambda(D_+\varphi),$$

$$D_-(\lambda\varphi) = \lambda D_-\varphi - \frac{\bar\lambda(\varepsilon)}{\varepsilon}\varphi - \bar\lambda(\varepsilon)\lambda\varphi - \bar\lambda(\varepsilon)D_-\varphi - \varepsilon\bar\lambda(\varepsilon)\lambda(D_+\varphi),$$

$$D_+(\bar\lambda\varphi) = \bar\lambda D_+\varphi + \frac{\bar\lambda(\varepsilon)}{\varepsilon}\varphi + \bar\lambda(\varepsilon)\bar\lambda\varphi + \bar\lambda(\varepsilon)D_-\varphi - \varepsilon\bar\lambda(\varepsilon)\bar\lambda(D_+\varphi),$$

$$D_-(\bar\lambda\varphi) = \bar\lambda D_-\varphi - \frac{\lambda(\varepsilon)}{\varepsilon}\varphi - \lambda(\varepsilon)\bar\lambda\varphi - \lambda(\varepsilon)D_-\varphi - \varepsilon\lambda(\varepsilon)\bar\lambda(D_-\varphi).$$

For the proof of $Y(\lambda\varphi)\subseteq Y(\varphi)$ and $Y(\bar\lambda\varphi)\subseteq Y(\varphi)$, it suffices to show the following assertion $P(n, k)$ with parameters $n, k\in N$:

$$D_+^n D_-^k(\lambda\varphi)\in Y(\varphi) \quad\text{and}\quad D_+^n D_-^k(\bar\lambda\varphi)\in Y(\varphi).$$

$P(0, 0)$ is trivial. Assume $P(0, k)$ and show $P(0, k+1)$. By the second formula above, we have

$$D_-^{k+1}(\lambda\varphi) = D_-^k(D_-(\lambda\varphi)) = D_-^k(\lambda D_-\varphi) - \frac{\bar\lambda(\varepsilon)}{\varepsilon}D_-^k\varphi$$
$$-\bar\lambda(\varepsilon)D_-^k(\lambda\varphi) - \bar\lambda(\varepsilon)D_-^{k+1}\varphi - \varepsilon\lambda(\varepsilon)D_-^k(\lambda(D_-\varphi)).$$

The first and the last terms belong to $Y(D_-\varphi)$ by the induction hypothesis and so to $Y(\varphi)$. The third term belongs to $Y(\varphi)$ by the induction hypothesis. and the second and the fourth terms belong to $Y(\varphi)$ by the definition, and we get $P(0, k+1)$. Similar for $\bar\lambda\varphi$.

Next, assume $P(n, k)$. We show $P(n+1, k)$. By the first formula above, we have

$$D_-^{n+1}D_+^k(\lambda\varphi) = D_+^n D_-^k(D_+(\lambda\varphi)) = D_+^n D_-^k(\lambda D_+\varphi) + \frac{\lambda(\varepsilon)}{\varepsilon}D_+^n D_-^k\varphi$$
$$+\lambda(\varepsilon)D_+^n D_-^k(\lambda\varphi) + \lambda(\varepsilon)D_+^{n+1}D_-^k\varphi + \varepsilon\lambda(\varepsilon)D_+^n D_-^k(\lambda D_+\varphi).$$

The same argument shows that five terms belong to $Y(\varphi)$. Similar for $\bar\lambda\varphi$.

Theorem 6. *$Q(\boldsymbol{R})$ is stable under multiplication and operations D_+, D_-, λ, $\bar\lambda$, F, $\bar F$.*

Proof. 1° Let $\varphi, \psi\in Q(R)$. Lemma 7 shows $Y(\varphi\psi)\subseteq Y(\varphi)Y(\psi)$ and hence $Y(\varphi\psi)\subseteq U(\boldsymbol{R})$.

2° Let $\varphi\in Q(\boldsymbol{R})$. Lemma 8 shows $Y(D_\pm\varphi)$, $Y(\lambda\varphi)$, $Y(\bar\lambda\varphi)\subseteq Y(\varphi)$, which imply $D_\pm\varphi$, $\lambda\varphi$, $\bar\lambda\varphi\in Q(\boldsymbol{R})$.

3° Let $\varphi\in Q(\boldsymbol{R})$ and we shall first show $F\varphi\in U(\boldsymbol{R})$. $|F\varphi(x)| = |\sum_{y\in X}\varepsilon e^{-2\pi ixy}\varphi(y)| \leqq \sum_{y\in Y}\varepsilon|\varphi(y)|$, which is finite because $Q(\boldsymbol{R})\subseteq M_1(\boldsymbol{R})$ (Proposition 10). Let $x, x'\in X$ and take $l\in\boldsymbol{N}$ and $c\in\boldsymbol{R}$ such that $\sum_X \varepsilon(1+|\lambda|^2)^{-l}$ is finite (Proposition 5) and that $(1+|\lambda|^2)^{l+1}|\varphi|\leqq c$. We have

$$|F\varphi(x)-F\varphi(x')| = |\sum_{y\in X}\mathcal{E}(e^{-2\pi ixy}-e^{-2\pi ix'y})\varphi(y)|$$
$$=\sum_{y\in X}\frac{|e^{2\pi i(x-x')y}-1|}{1+|\lambda(y)|^2}\frac{(1+|\lambda(y)|^2)^{l+1}|\varphi(y)|}{(1+|\lambda(y)|^2)^l}.$$

As we can write

$$e^{2\pi i(x-x')y}-1 = (2\pi i(x-x')y)(\cos 2\pi(x-x')\tau y-i\sin 2\pi(x-x')\sigma y),$$

where $\tau,\ \sigma\in{}^*\boldsymbol{R}$and $0<\sigma,\ \tau<1$, we have

$$\frac{|e^{2\pi i(x-x')y}-1|}{1+|\lambda(y)|^2}\leqq\frac{2\pi|x-x'||y|}{1+16|y|^2}\leqq\frac{\pi}{4}|x-x'|.$$

Therefore we have

$$|F\varphi(x)-F\varphi(x')|\leqq\frac{\pi}{4}|x-x'|\sum_X\frac{c}{(1+|\lambda|^2)^l}.$$

If $x\simeq x'$, then $F\varphi(x)\simeq F\varphi(x')$, that is, $F\varphi\in U(\boldsymbol{R})$.

4° We shall show $Y(F\varphi)\subseteq U(\boldsymbol{R})$, which will complete the proof of Theorem 6. Let $\alpha\in N_5({}^*\boldsymbol{C})$ and $l,\ m,\ n,\ k\in\boldsymbol{N}$. By Proposition 3, we have

$$\alpha\lambda^l\bar{\lambda}^m D_+^n D_-^k F\varphi = (-1)^{m+k}\alpha F(D_+^l D_-^m(\bar{\lambda}^n\lambda^k\varphi)).$$

The right-hand side belongs to $U(\boldsymbol{R})$, because $D_+^l D_-^n(\bar{\lambda}^n\lambda^k\varphi)\in Q(\boldsymbol{R})$. Hence we have $\alpha\lambda^l\bar{\lambda}^m D_+^n D_-^k F\varphi\in U(\boldsymbol{R})$, which says $Y(F\varphi)\subseteq U(\boldsymbol{R})$.

Theorem 7. *Let* $\varphi\in Q(\boldsymbol{R})$.

1) ${}^\vee\varphi\in\mathcal{S}(\boldsymbol{R})$ *and* $\Gamma_\varphi=T_{{}^\vee\varphi}$, *whereis* $T_{{}^\vee\varphi}$ *is the distribution on* $\boldsymbol{R}$ *defined by* ${}^\vee\varphi$.

2) ${}^\vee(D_\pm\varphi)=({}^\vee\varphi)'$, ${}^\vee(\lambda\varphi)=(2\pi it){}^\vee\varphi$, ${}^\vee(\bar{\lambda}\varphi)=(-2\pi it){}^\vee\varphi$.

3) ${}^\vee(F\varphi)=\mathcal{F}({}^\vee\varphi)$.

Proof. Theorem 5 says that $(T_{{}^\vee\varphi})'=(\Gamma_\varphi)'=\Gamma_{D_\pm\varphi}=T_{{}^\vee(D_\pm\varphi)}$. By Theorem 7 in Schwartz [3], Ch. 2, § 6, ${}^\vee\varphi\in\mathcal{C}^1(\boldsymbol{R})$ and $({}^\vee\varphi)'=(D_\pm\varphi)$. Therefore ${}^\vee\varphi\in\mathcal{C}^\infty(\boldsymbol{R})$. On the other hand, $T_{2\pi it({}^\vee\varphi)}=2\pi it\,T_{{}^\vee\varphi}=2\pi it\Gamma_\varphi=\Gamma_{\lambda\varphi}=T_{{}^\vee(\lambda\varphi)}$, which leads to $(2\pi it){}^\vee\varphi={}^\vee(\lambda\varphi)\in U(\boldsymbol{R})$ and therefore ${}^\vee\varphi\in\mathcal{S}(\boldsymbol{R})$. Finally we have $T_{{}^\vee(F\varphi)}=\Gamma_{F\varphi}=\mathcal{F}\Gamma_\varphi=\mathcal{F}T_{{}^\vee\varphi}=T_{\mathcal{F}({}^\vee\varphi)}$, and so ${}^\vee(F\varphi)=\mathcal{F}({}^\vee\varphi)$.

Theorem 8. *If* $h\in\mathcal{S}(\boldsymbol{R})$, *then* ${}^*h|X\in Q(\boldsymbol{R})$ *and* ${}^\vee({}^*h|X)=h$. *In particular, the map*: $\varphi\mapsto{}^\vee\varphi$ *from* $Q(\boldsymbol{R})$ *to* $\mathcal{S}(\boldsymbol{R})$ *is surjective.*

Proof. Write ${}^*h$ for ${}^*h|X$. If we show $Y({}^*h)\subseteq U(\boldsymbol{R})$, then ${}^*h\in Q(\boldsymbol{R})$ by the definition of $Q(\boldsymbol{R})$. Then, ${}^*h({}^\Delta t)\simeq h(t)$ for $t\in\boldsymbol{R}$ and ${}^\vee{}^*h(t)={}^0({}^*h({}^\Delta t))=h(t)$, which will complete the proof.

For showing $Y(^*h)\subseteq U(\boldsymbol{R})$, it suffices to prove the following two assertions (1) and (2):

(1) $\lambda^l \Lambda^m D_+^n D_-^k {}^*h \simeq \lambda^l \Lambda^m {}^* h^{(n+k)}$ for $l, m, n, k \in \boldsymbol{N}$.

(2) $\lambda(x)^*h(x) \simeq 2\pi i x^*h(x)$ for $x \in X$.

In fact, we have for $x \in X$

$$\begin{aligned}\lambda^l(x)\Lambda^m(x)D_+^n D_-^k {}^* h(x) &\simeq (-1)^m (2\pi i x)^{l+m*} h^{(n+k)}(x)\\ &= (-1)^{m*}((2\pi i t)^{l+m} h^{(n+k)}(t))(x)\,.\end{aligned}$$

Hence there exists $c \in \boldsymbol{R}$ such that $|\lambda^l(x)\Lambda^m(x)D_+^n D_-^k {}^*h(x)| \leqq c$, for all $x \in X$, that is, $\lambda^l \Lambda^m D_+^n D_-^k {}^*h$ is bounded. Next, let $x, y \in X$ and $x \simeq y$. Then the function $t \mapsto t^{l+m} h^{(n+k)}(t)$ is uniformly continuous, and we have

$$x^{l+m}\ {}^*h^{(n+k)}(x) \simeq y^{l+m}\ {}^*h^{(n+k)}(y)\,.$$

We have therefore $\lambda^l \Lambda^m D_+^n D_-^k {}^*h \in U(\boldsymbol{R})$, hence $Y(^*h) \subseteq U(\boldsymbol{R})$.

To show the assertions (1) and (2), we provide two lemmas.

Lemma 9. *Let $f \in C^\infty(\boldsymbol{R})$ and put*

$$\Delta_+ f(x) = f(x+h) - f(x)\,, \quad \Delta_- f(x) = f(x) - f(x-h)$$

for $x, h \in \boldsymbol{R}$. Then, for $n, k \in \boldsymbol{N}_+$, there exist $u_l, u_l', v_l, v_l' \in \boldsymbol{R}$ $(1 \leqq l \leqq n)$ and $s_j, s_j' \in \boldsymbol{R}$ $(1 \leqq j \leqq k)$ such that $0 < u_l, u_l', v_l, v_l', s_j, s_j' < 1$ and that

$$\begin{aligned}&\Delta_+^n \Delta_-^k f(x) - h^{n+k} f^{(n+k)}(x)\\ &= \frac{h^{n+k+1}}{2}\Big\{\sum_{l=1}^n (\operatorname{Re} f^{(n+k+1)}(x+lu_l h) + i \operatorname{Im} f_m{}^{(n+k+1)}(x+lu_l' h)\\ &\qquad -\sum_{j=1}^k \operatorname{Re} f^{(n+k+1)}(x-js_j h) - i \operatorname{Im} f^{(n+k+1)}(x-js_j' h))\Big\}\\ &\quad -\frac{h^{n+k+2}}{4}\Big\{\sum_{l=1}^n \sum_{j=1}^k (\operatorname{Re} f^{(n+k+2)}(x+l(v_l-j_js)h)\\ &\qquad + i \operatorname{Im} f^{(n+k+2)}(x+l(v_l'-js_j')h))\Big\}\,.\end{aligned}$$

Proof. Taylor's theorem and induction.

Lemma 10. *Let $h \in \mathcal{S}(\boldsymbol{R})$, $\alpha \in Ns(^*\boldsymbol{R})$ and $l, m \in \boldsymbol{N}$. Then $\lambda^l(x)\Lambda^m(x)^* \times (\operatorname{Re} h)(x+\alpha\varepsilon)$ and $\lambda^l(x)\Lambda^m(x)^*(\operatorname{Im} h)(x+\alpha\varepsilon)$ are finite for $x \in X$.*

Proof is direct.

Proof of the assertions (1) and (2) in Theorem 8.

(1) Put $h_1 = \operatorname{Re} h^{(n+k)}$ and $h_2 = \operatorname{Im} h^{(n+k)}$. By Lemma 9,

$$\begin{aligned}&|\lambda^l(x)\Lambda^m(x)(D_+^n D_-^k {}^*h)(x) - \lambda^l(x)\Lambda^m(x)^* h^{(n+k)}(x)|\\ &\quad = |\lambda^l(x)\Lambda^m(x)\{D_+^n D_-^k {}^*h)(x) - {}^*h^{(n+k}(x)\}|\end{aligned}$$

$$=\Big|\lambda^l(x)\bar{\lambda}^m(x)\Big[\frac{\varepsilon}{2}\{\sum_{r=1}^{n}({}^*h_1'(x+r\rho_r\varepsilon)+i{}^*h_2'(x+r\rho_r'\varepsilon))$$

$$-\sum_{s=1}^{k}({}^*h_1'(x-s\sigma_s\varepsilon)+i{}^*h_2'(x-\sigma_s'\varepsilon))\}-\frac{\varepsilon^2}{4}\sum_{r=1}^{m}\sum_{s=1}^{k}\{{}^*h_1''(x+(r\tau_r-s\sigma_s))\varepsilon$$

$$+i{}^*h_2''(x+(r\tau_r'-s\sigma_s')\varepsilon)\}\Big]\Big|$$

$$\leqq\frac{\varepsilon}{2}\{\sum_{r=1}^{n}|\,|\lambda^l(x)\bar{\lambda}^m(x){}^*h_1'(x+r\rho_r\varepsilon)|+\sum_{r=1}^{n}|\lambda^l(x)\bar{\lambda}^m(x){}^*h_2'(x+r\rho_r'\varepsilon)|$$

$$+\frac{\varepsilon^2}{4}\{\sum_{r=1}^{n}\sum_{s=1}^{k}|\lambda^l(x)\bar{\lambda}^m(x){}^*h_1''(x+(r\tau_\rho-s\sigma_s)\varepsilon)|$$

$$+\sum_{r=1}^{n}\sum_{s=1}^{k}|\lambda^l(x)\bar{\lambda}^m(x)h_1''(x+(r\tau_r'-s\sigma_s')\varepsilon|\,,$$

where $\rho_r, \tau_r, \rho_r', \tau_r'\in{}^*\boldsymbol{R}$, $0<\rho_r, \tau_r, \rho_r', \tau_r'<1$ $(1\leqq r\leqq n)$ and $\sigma_s, \sigma_s'\in\boldsymbol{R}$, $0<\sigma_s, \sigma_s'<1$ $(1\leqq s\leqq k)$. The coefficients of $\varepsilon/2$ and $\varepsilon^2/4$ in the right-hand side are finite by Lemma 10, so the assertion (1) is proved.

(2) As we have

$$\lambda(x)-2\pi ix=\varepsilon(2\pi ix)^2(\cos 2\pi\varepsilon\sigma x+i\sin 2\pi\varepsilon\tau x)\,,$$

where $\sigma, \tau\in{}^*\boldsymbol{R}$ and $0<\sigma, \tau<1$, we have

$$\begin{aligned}|\lambda(x){}^*h(x)-2\pi ix{}^*h(x)|&\leqq\varepsilon|2\pi|^2 2|x|^2|{}^*h(x)|\\&\leqq\varepsilon 8\pi^2\sup_{t\in\boldsymbol{R}}|t^2h(t)|\simeq 0\,,\end{aligned}$$

which completes the proof of Theorem 8.

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