

LOCALLY LINEAR REPRESENTATION FORMS

IB MADSEN AND MARTIN RAUSSEN

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1. Introduction

Group actions on homotopy spheres have been studied in various categories and under different aspects for many years. Best understood are the so-called *semi-linear* actions or *representation forms*. These are actions of a group G on a (homotopy) sphere M , such that, for every subgroup H of G , the fixed point set M^H is again a (homotopy) sphere, possibly empty.

In this paper, we study representation forms in the categories of locally linear topological, piecewise-linear, and smooth manifolds. Recall that an action is called *locally linear* if for every $x \in M$ of orbit type G/H , there exists an $\mathbf{R}H$ -module V and a local coordinate map from $G \times_H V$ onto a G -invariant neighbourhood of the orbit of x . This presents no restriction in the smooth category but is an important restriction in the other two cases. For a detailed discussion the reader may consult [3] and [13]. From now on all manifolds will be assumed locally linear.

Basic invariants for representation forms are their equivariant homotopy type and in particular, their *dimension function*, cf. [6], [7]. The ring of integer-valued functions on the set of conjugacy classes of subgroups of G will be denoted by $C(G)$. The dimension function $Dim X$ of a homotopy representation X is defined by:

$$(Dim X)(H) = dim X^H + 1, \quad (H) \leq G.$$

In this paper we study the following question in the PL and smooth categories:

$$\begin{aligned} & \text{Which functions } m \in C(G) \text{ are realizable by a representation} \\ & \text{form } M \text{ with } m = Dim M? \end{aligned} \quad (1.1)$$

For various reasons, an answer to this question is at present only available for actions of *cyclic groups of odd order*. First, the vanishing of the Swan obstruction in these cases allows an easy enumeration of the possible *finite* homotopy types, see (2.10), (2.13). Secondly, we use the transversality results from [17] which in the PL case limits the groups to be of odd order.

In the rest of this paper (unless otherwise specified):

$$G \text{ denotes an odd order cyclic group.} \tag{1.2}$$

The isotropy groups of an action can be recovered from the dimension function, $m = \text{Dim } X$, namely

$$\text{Iso}(m) := \{H \mid m(H) > 0; H < K \Rightarrow m(H) > m(K)\} . \tag{1.3}$$

We will often write $\text{Iso}(X)$ for $\text{Iso}(\text{Dim } X)$.

DEFINITION 1.4. *A function $m \in C(G)$ is a CW-dimension function if*

- (i) $m(H) \geq m(K) \geq 0$ and $m(H) \equiv m(K) \pmod 2$ for all $H \leq K \leq G$.
- (ii) $\text{Iso}(m)$ is closed under intersection.

The set of CW-dimension functions will be denoted by $C^+(G)$.

Proposition 1.5. *If M is a representation form, then $\text{Dim } M \in C^+(G)$.*

Proof. The first condition is an immediate consequence of P.A. Smith theory, see ([3], Ch. III). To prove (ii), let $H, K \in \text{Iso}(M)$ and $L \leq G$ such that $H \cap K < L$. Then $M^L \subseteq M^{H \cap K} \supseteq M^H \neq \emptyset$. If M^L and $M^{H \cap K}$ had the same dimension, they had to be equal, since M is locally linear. This would imply that $M^{(L, H)} = M^L \cap M^H = M^H$, hence $(L, H) = H$ and $L \leq H$, and similarly $L \leq K$. Hence $\text{dim } M^L$ has to be less than $\text{dim } M^{H \cap K}$. □

Following [6], a $(G\text{-CW})$ homotopy representation X is a $G\text{-CW}$ complex with $X^H \simeq S^{n(H)}$ and $\text{dim } X^H = n(H)$. One may ask the question (1.1) for homotopy representations. This was examined by tom Dieck and Petrie in detail, see [6], [7]. For a cyclic group G of odd order, we extract from their work:

Theorem 1.6. *A CW-dimension function $m \in C^+(G)$ with $m(H) \neq 1, 2, 3$ for all $H \leq G$, is the dimension function of some homotopy representation.*

In the locally linear manifold categories we can collapse around a stationary point to obtain a further necessary condition, namely

Proposition 1.7. *A G -representation form M with $M^G \neq \emptyset$ is homotopy linear in the sense that $M \simeq_G S(V \oplus \mathbf{R})$ for some orthogonal $\mathbf{R}G$ -module V .*

Proof. Since M is locally linear, any point $x \in M^G$ has a G -invariant neighbourhood U which is CAT-homeomorphic to the disk DV of a linear representation V of G . The collapse map

$$M \rightarrow U/\partial U \simeq DV/SV \simeq S(V \oplus \mathbf{R})$$

is equivariant and has degree 1 on all fixed point sets. It is an equivariant

homotopy equivalence by the equivariant Whitehead theorem, cf. [2]. □

REMARK 1.8. In other words, at *isotropy* subgroups $H \leq G$, a representation form is H -homotopy equivalent to the sphere of an \mathbf{RH} -module. In particular, by (1.4), a representation form M^n of even dimension n (i.e. $\text{Dim } M$ odd) is homotopy linear.

Thus, we can (and will) restrict attention to even dimension functions. It is not hard to single out those functions, which can be realized by the spheres of orthogonal representations, see (2.14) for details. We would like to give a converse to (1.5) and (1.7), but since our constructions will be based on equivariant transversality and surgery, we are forced to impose the usual gap (or stability) conditions:

DEFINITION 1.9. *A function $m \in C(G)$ satisfies the strong gap conditions, if $m(H) > 2m(K) \geq 12$ for each pair of isotropy groups $H \leq K$. A homotopy representation X satisfies the strong gap conditions, if $\text{Dim } X$ does.*

Our main result in the PL or Top category can be stated as

Theorem A. *A function $m \in C^+(G)$ which satisfies the strong gap conditions is the dimension function of a PL-representation form M , if and only if its restriction to any isotropy group H of m is the dimension function of a linear H -sphere.*

In (4.2) we translate the condition above to a numerical property of the dimension function, namely that its restriction to isotropy subgroups have positive Möbius transform. Thus modulo gap conditions, the situation is very satisfactory in the non-smooth categories. Our results in the smooth category on the other hand are less satisfactory. We give in (4.7) a sufficient condition for a dimension function to be smoothly realizable. The condition is considerably stronger than the above PL condition.

In sect. 5 we show that a smooth representation form has stably trivial tangent bundle and identify its fibre homotopy type. In the final sect. 6 we use this to compare smooth and PL realizability of dimension functions. Rather surprisingly, we have

Theorem B. *There exists PL representation forms whose dimension function cannot be realized by any smooth representation form.*

It appears in general that the question of realizing dimension functions by smooth homotopy representations involves hard questions in equivariant homotopy theory.

EXAMPLE 1.10. Suppose $n = p_1 p_2 p_3 p_4$ to be a product of four distinct odd prime numbers. Consider the function $m \in C(G)$ with even integer values

given by

$$\begin{array}{c|c|c|c|c|c}
 L & G & \mathbf{Z}/p_i p_j p_k & \mathbf{Z}/p_i p_j & \mathbf{Z}/p_i & 1 \\
 \hline
 m(L) & 0 & b & 2b+2 & a & 2a+2
 \end{array}$$

The strong gap hypothesis (1.9) is satisfied precisely if $28 \leq 4b+4 < a$. Under this condition, there is a PL-representation form with dimension function m if and only if $a \leq 5b+8$. However, there is no smooth representation form with dimension function m in the cases where for some $p=p_i$, the following conditions are satisfied: $a \geq \left(5 - \frac{1}{p-1}\right)b+9$, $b < p(p-1)$ and $a \equiv 4b+7 \pmod p$.

The paper is divided into the following sections:

- § 1. Introduction.
- § 2. Recollections on homotopy representations.
- § 3. Stiefel manifolds.
- § 4. From homotopy representations to representation forms.
- § 5. The stable tangent bundle.
- § 6. Smooth representation forms.

In [16], we claimed a stronger version of Theorem A than the one listed above in that we there dealt with homotopy types rather than just dimension functions. However, the stronger statement is unfortunately false in general. This question will be taken up elsewhere.

2. Recollections on homotopy representations

In this section, we collect in a form which is convenient for our purposes various notions and results about CW-homotopy representations, due to tom Dieck and Petrie. The presentation is of course considerably simplified since we only consider cyclic groups of odd order.

The equivariant homotopy types of G -CW homotopy representations form a semi-group $V^+(G)$ under join. Its corresponding Grothendieck group is denoted $V(G)$. This group can be described by two invariants, the *dimension* function and the *degree* function. The first is a homomorphism $\text{Dim}: V(G) \rightarrow C(G)$ into the ring of \mathbf{Z} -valued functions on the subgroups of G . For $X \in V^+(G)$

$$\text{Dim } X(H) = \dim X^H + 1 = n(H) + 1, \quad H \leq G. \tag{2.1}$$

Before describing the second invariant, we introduce an additive and a multiplicative Möbius transformation on $C(G; R)$ and on its unit group $C^\times(G; R)$, respectively where R is any commutative ring. Let μ denote the classical Möbius function: $\mu(n)=0$ if n is divisible by a square and $\mu(p_1 \cdots p_s) = (-1)^s$ if p_1, \dots, p_s are distinct primes, and $\mu(1)=1$. Define

$$\mu: C(G; R) \rightarrow C(G; R), \tag{2.2}$$

$$\mu m(H) = \sum_{H \leq K \leq G} \mu(|K: H|) \cdot m(K), \quad H \leq G;$$

$$\mu^\times: C^\times(G; R) \rightarrow C^\times(G; R) \tag{2.3}$$

$$\mu^\times d(H) = \prod_{H \leq K \leq G} d(K)^{\mu(|K: H|)}, \quad H \leq G$$

Notice that both μ and μ^\times are bijections.

The Möbius transform μ of a dimension function will be used in the sections below. We use μ^\times to define the degree function,

$$Deg: v(G) \rightarrow Pic(A(G))$$

from $v(G) = Ker Dim$ to the Picard group of the Burnside ring $A(G)$. For cyclic G , one gets from [4] that

$$Pic(A(G)) \cong \prod_{H \leq G} [(Z/|G: H|)^\times / \{\pm 1\}] \tag{2.4}$$

If $[Y - X] \in v(G)$, obstruction theory permits us to choose an equivariant map $f: X \rightarrow Y$ with $d(H) = degree(f^H)$ prime to G for all $H \leq G$, see [7]. With the identification (2.3), the H -component of Deg is given by the formula

$$Deg([Y - X])(H) = \begin{cases} \mu^\times d(H) \in (Z/|G: H|)^\times / \{\pm 1\}, & X^H \neq \emptyset \\ 1, & X^H = \emptyset \end{cases} \tag{2.5}$$

Since Swan obstructions vanish for cyclic groups, we conclude from [7, Theorem 6.3]:

Theorem 2.6. *Deg is an isomorphism.*

Let $Irr(G, R)$ denote a complete set of isomorphism classes of irreducible RG -modules. We define homomorphisms

$$F, I: RO(G) \rightarrow C(G; RO(G)):$$

For $\alpha = \sum_{V_i \in Irr(G, R)} n_i V_i$, with $n_i \in Z$,

$$F(\alpha)(H) = \alpha^H = \sum_{H \leq ker V_i} n_i V_i \tag{2.7}$$

$$I(\alpha)(H) = \alpha_H = \sum_{H = ker V_i} n_i V_i.$$

Furthermore, define a bijection σ on $C(G; A)$ and its inverse μ by

$$\sigma(f)(H) = \sum_{H \leq K \leq G} f(K) \quad \text{and} \tag{2.8}$$

$$\mu(f)(H) = \sum_{H \leq K \leq G} \mu(K; H) f(K),$$

where $f \in C(\mathbf{G}; A)$ and μ denotes the Möbius transform on the lattice of (conjugacy classes of) subgroups of \mathbf{G} [1, chap. 4]. Obviously, $F = \sigma \circ I$, and hence $I = \mu \circ F$.

For cyclic groups of odd order, the even-dimensional part of $RO(\mathbf{G})$ is isomorphic to

$$\mathbf{Z}[T] / \{T^{|\mathbf{G}|-i} + T^i \mid 0 \leq i < |\mathbf{G}|\},$$

and if $\alpha = \sum a_j \cdot T^j$ with $0 \leq j < \frac{1}{2}(|\mathbf{G}|-1)$ and $a_j \in \mathbf{Z}$,

$$F(\alpha)(H) = \alpha^H = \sum a_j \cdot T^j, \quad |H| \text{ divides } (j, |\mathbf{G}|) \text{ and } 0 \leq j < \frac{1}{2}(|\mathbf{G}|-1)$$

$$I(\alpha)(H) = \alpha_H = \sum a_j \cdot T^j, \quad |H| = (j, |\mathbf{G}|) \text{ and } 0 \leq j < \frac{1}{2}(|\mathbf{G}|-1) \quad (2.9)$$

We shall now compare $V^+(\mathbf{G})$ and $V(\mathbf{G})$ with the orthogonal representation ring and consider the forgetful homomorphism

$$i_h: RO(\mathbf{G}) \rightarrow V(\mathbf{G}).$$

which to W associates SW , the unit sphere in some \mathbf{G} -invariant inner product. Its kernel is the subgroup

$$RO_h(\mathbf{G}) = \{[W - V] \mid SW \simeq_{\mathbf{G}} SV\}.$$

Proposition 2.10. *The map i_h induces an isomorphism of $RO(\mathbf{G})/RO_h(\mathbf{G})$ with $V(\mathbf{G})$.*

Proof. Let $RO_0(\mathbf{G})$ be the kernel of $Dim \circ i_h$, and consider the diagram

$$\begin{array}{ccccc}
 & & RO_0(\mathbf{G})/RO_h(\mathbf{G}) \subseteq RO(\mathbf{G})/RO_h(\mathbf{G}) & & \\
 & \swarrow \text{Deg} & \downarrow i_h & \searrow \text{Dim} & \\
 Pic(\mathbf{G}) & & v(\mathbf{G}) & \subseteq & V(\mathbf{G}) & \xrightarrow{\text{Dim}} & C(\mathbf{G}) & (2.11) \\
 & \swarrow \text{Deg} & & \downarrow i_h & \swarrow \text{Dim} & & & \\
 & & & & V(\mathbf{G}) & & &
 \end{array}$$

From [6, Satz 2.7], we know that $Dim(RO(\mathbf{G})/RO_0(\mathbf{G})) = Dim(V(\mathbf{G}))$, even for nilpotent groups \mathbf{G} . In our case of a cyclic, odd order group,

$$Dim V(\mathbf{G}) = \{m \in C(\mathbf{G}) \mid m(H) \equiv m(\mathbf{G}) \pmod{2}, H \leq \mathbf{G}\}.$$

Hence by (2.6), it is enough to show that

$$Deg: RO_0(\mathbf{G})/RO_h(\mathbf{G}) \rightarrow Pic(A(\mathbf{G}))$$

is onto. But for each integer i prime to $|\mathbf{G}:K|$, the i 'th power map defines an equivariant map

$$f: S(T^{|\mathbf{K}|}) \rightarrow S(T^{i|\mathbf{K}|})$$

with

$$\mu^\times(\text{Deg } f)(H) = \begin{cases} i \in (\mathbf{Z}/|\mathbf{G}: K|)^\times / \{\pm 1\} & H = K \\ 1 & H \neq K. \end{cases} \quad \square$$

Geometrically, (2.10) asserts that each homotopy representation is *stably linear* in the sense that there exist $\mathbf{R}\mathbf{G}$ -modules V, W such that

$$X * SV \cong_{\mathbf{G}} SW. \tag{2.12}$$

It is known [21] that the only nilpotent groups for which all homotopy representations are stably linear, are the cyclic groups and the dihedral 2-groups.

When (2.12) is satisfied, we say that $\alpha = [W - V] \in RO(\mathbf{G})$ is *realized* by X as a homotopy representation. Not all representations can be realized, but whether or not a given representation can, is detected by its dimension function alone. Indeed:

Proposition 2.13. *Let $\alpha \in RO(\mathbf{G})$ be such that $\text{Dim } \alpha \in C^+(\mathbf{G})$. Then α is realized by some homotopy representation X .*

Proof. From (1.6), we have a homotopy representation Y with $\text{Dim } Y = \text{Dim } \alpha$. It is stably linear, so there is a representation $\beta \in RO(\mathbf{G})$ with $[Y - S\beta] = 0 \in v(\mathbf{G})$ and $\alpha - \beta \in RO_0(\mathbf{G})$. According to [7, Theorem 6.3], there is a (finite) homotopy representation X such that $\text{Deg}(S\alpha - S\beta) = \text{Deg}(X - Y) \in \text{Pic}(A(\mathbf{G}))$. Since Deg is an isomorphism, $i_k(\alpha) = X \in V(\mathbf{G})$. \square

The following proposition answers the question as to which homotopy representations can be represented *linearly*.

Proposition 2.14. *A function $m \in C^+(\mathbf{G})$ is the dimension function of an $\mathbf{R}\mathbf{G}$ -module if and only if*

$$\mu m(K) \geq 0 \quad \text{for all } K \leq \mathbf{G}.$$

Proof. With the notation from (2.2) and (2.7), note that

$$\mu \text{Dim } \alpha(K) = \dim_{\mathbf{R}}(\alpha_K) \quad \text{for any } \alpha \in RO(\mathbf{G}).$$

Choosing an integer $i(K)$ for each $K \leq \mathbf{G}$ such that $(i(K), |\mathbf{G}: K|) = 1$, we define the representation:

$$V = \mu m(\mathbf{G}) \cdot \mathbf{R} + \sum_{1 \leq K < \mathbf{G}} \frac{1}{2} \mu m(K) \cdot T^{i(K) \cdot |\mathbf{K}|}.$$

It has $\text{Dim } SV = m$. \square

3. Stiefel manifolds

Our results about realizing homotopy representations by smooth or PL-manifolds are based upon equivariant surgery techniques and in particular on transversality. This requires certain connectivity results for appropriate Stiefel spaces which we now discuss.

Given an $\mathbf{R}G$ -module W with G -invariant inner product. Let $O_G(W)$ be the group of G -isometries, and let $PL_G(W)$ be the (realization of the semi-simplicial) group of equivariant PL-homeomorphisms of W which keep the origin fixed. If U is a sub $\mathbf{R}G$ -module of W , there are fibrations

$$\begin{aligned} O_G(W)/O_G(U) &\rightarrow BO_G(U) \rightarrow BO_G(W) \\ PL_G(W)/PL_G(U) &\rightarrow BPL_G(U) \rightarrow BPL_G(W). \end{aligned}$$

We are interested in the connectivity of the fibres. In [18], we defined $PL_G(W)$ without the requirement that the origin be fixed, but the two spaces are homotopy equivalent. Thus we have

Theorem 3.1. [18]. *Suppose the dimension functions $Dim SU$ and $Dim SW$ both satisfy the strong gap conditions (1.9). Then*

$$\pi_i(PL_G(W)/PL_G(U)) = 0 \quad \text{for } i \leq \dim U^G - 1.$$

The PL result is stronger than the corresponding result where PL_G is replaced by the group O_G of G -isometries. In the latter case, the individual eigenvalues of V and W play a role, not only their order. Let $T: G \rightarrow S^1$ be a fixed (but arbitrary) faithful character.

DEFINITION 3.2. *A representation $\alpha \in RO(G)$ is called isogeneous if for every $K < G$,*

$$\alpha_K = \frac{1}{2} \mu \text{Dim } \alpha(K) \cdot T^{i(K) \cdot |K|}$$

for some integer $i(K)$ with $(i(K), |G: K|) = 1$.

For example, the representation V in the proof of (2.14) is isogeneous. If $U = \sum a_i T^i$ with $a_i \geq 0, 0 \leq i \leq \frac{1}{2}(|G| - 1)$, then by Schur's lemma,

$$O_G(U) = O(a_0) \times \prod U(a_i), \quad 1 \leq i \leq \frac{1}{2}(|G| - 1),$$

and $\dim U^G = a_0, \mu \text{Dim } SU(K) = 2 \sum a_i, (i, |G|) = |K|$. Suppose $U \subseteq W$ are both $\mathbf{R}G$ -modules; we define

$$c(U, W) = \min \{ \dim SU^G, \mu \text{Dim } SU(K) \mid K \in Iso(SU), W_K \neq U_K \}.$$

The well-known connectivity results for ordinary real and complex Stiefel manifolds give:

Proposition 3.3. *The Stiefel manifold $O_c(W)/O_c(U)$ is $c(U, W)$ -connected if U and W are isogeneous; otherwise, it is less than $c(U, W)$ -connected.*

Every CW dimension function $m \in C^+(G)$ can be realized as the dimension function of a virtual representation $\alpha \in RO(G)$; it can be chosen in a particularly nice way:

Proposition 3.4. *For every $m \in C^+(G)$, there is an $\alpha \in RO(G)$ with $Dim \alpha = m$ and such that $Res_H(\alpha)$ is isogeneous for all $H \leq G$.*

Proof. Any isogeneous representation $\alpha \in RO(G)$ with $Dim \alpha = m$ will be of the form

$$\alpha = m(G) \cdot \mathbf{R} + \sum_{K < G} \frac{1}{2} \mu m(K) \cdot T^{i(K) \cdot |K|}$$

for certain integers $i(K)$ prime to $|G:K|$. For subgroups $L < H \leq G$, the L -isotropic part of $Res_H \alpha$ is given by

$$(Res_H \alpha)_L = \sum_{K \cap H = L} \frac{1}{2} \mu m(K) \cdot T^{i(K) \cdot |K|}.$$

Hence, $Res_H \alpha$ is isogeneous if and only if the integers $i(K)$ satisfy the following system of congruences:

$$i(K) \cdot |K| \equiv i(K \cap H) \cdot |K \cap H| \pmod{|H|}, \quad K, H < G.$$

Equivalently, $i(K)$ should satisfy:

$$i(K) \cdot |K:K \cap H| \equiv i(K \cap H) \pmod{|H:K \cap H|}, \quad K, H < G.$$

Given $L \leq K \leq G$, a maximal subgroup H of the cyclic group G with $K \cap H = L$ is the following product of Sylow subgroups

$$H = \prod_{K_p \neq L_p} L_p \times \prod_{K_p = L_p} G_p$$

Hence, an application of the Chinese Remainder Theorem allows to reformulate the congruence conditions as follows:

$$L \leq K \leq G \Rightarrow i(K) \cdot |K:L| \equiv i(L) \pmod{|G_p:K_p|}$$

for those prime divisors p of G with $K_p = L_p$. Hence, the following congruences are necessary and sufficient to obtain the required α :

For primes $p \neq q$ and a q -subgroup K of G ,

$$i(K) \cdot |K| \equiv i(1) \pmod{|G_p|} \tag{3.5}$$

For an arbitrary subgroup K of G ,

$$i(K) \cdot |K: K_p| \equiv i(K_p) \pmod{|G_p: K_p|}. \tag{3.6}$$

Thus, in order to construct a representation α as in (3.4), first choose an arbitrary integer $i(1)$ prime to $|G|$. Then, for every q -subgroup K of G , $i(K)$ can be determined mod $|G: G_q|$ according to (3.5). This can be done with the aid of the Chinese Remainder Theorem, since $|K|$ is a unit mod $|G_p|$, $p \neq q$. Finally, the same type of argument determines for every subgroup K of G an integer $i(K) \pmod{|G: K|}$ satisfying (3.6). \square

Corollary 3.7. *For $K \leq H \leq G$ and $\mu \text{Res}_H m(K) \geq 0$, $(\text{Res}_H \alpha)_K$ is an \mathbf{RH} -module.*

One might think that (3.1) and (3.3) yield the same connectivity for \mathbf{RG} -modules which are isogeneous (and satisfy the strong gap conditions). But this is not the case, as the following example shows.

EXAMPLE 3.8. Suppose $G = \mathbf{Z}/pqr$ with p, q, r , distinct prime numbers. Consider the CW dimension function $m \in C^+(G)$ with values given by

H	1	\mathbf{Z}/p	\mathbf{Z}/q	\mathbf{Z}/r	\mathbf{Z}/pq	\mathbf{Z}/pr	\mathbf{Z}/qr	\mathbf{Z}/pqr
$m(H)$	$2a+2$	a	a	a	$2b+2$	$2b+2$	$2b+2$	b

where a, b are even numbers. It satisfies the strong gap condition and $\mu m(H) \geq 0$ for all $H \leq G$ if and only if $5b+8 > a > 4b+4 \geq 28$. By (3.4), there is an \mathbf{RG} -module U with $\text{Dim } SU = m$ and $\text{Res}_H U$ isogeneous for all $H \leq G$. Choose a second \mathbf{RG} -module $W \supset U$ with the same property. Then $PL_c(W)/PL_c(U)$ is $(b-1)$ -connected, and the connectivity of $O_c(W)/O_c(U)$ is

$$c(U, W) = \min\{b-1, \mu m(1) = 5b-a+8\}.$$

For $a > 4b+9$, this connectivity is less than $b-1$.

4. From homotopy representations to representation forms

Given a G -homotopy representation X , we ask if there is a smooth or PL -representation form M with the same dimension function $\text{Dim } M = \text{Dim } X$. We begin with the (locally linear) PL -case where the results are more satisfactory. We want to prove Theorem A of section 1: a dimension function $m \in C^+(G)$ satisfying the strong gap conditions is the dimension function of a PL -representation form M if and only if the restriction $\text{Res}_H(m)$ of m is the dimension func-

tion of a linear H -sphere at every isotropy group $H \in Iso(m)$. The necessity of the condition was pointed out in (1.7). Sufficiency follows from (4.1) and (4.2) below:

Theorem 4.1. *Let G be an arbitrary group of odd order and X a finite G -homotopy representation satisfying the strong gap conditions (1.9). If there are $\mathbf{R}G$ -modules V and W such that*

- (i) $X * SV \simeq_G SW$;
- (ii) $Res_H(V) \subseteq Res_H(W)$ for all $H \in Iso(X)$,

then there is a PL G -representation form M with homotopy type X .

Corollary 4.2. *Let G be a cyclic group of odd order and $m \in C^+(G)$ a function satisfying the strong gap conditions (1.9). Then, there is a PL G -representation form M with $Dim M = m$ if and only if $\mu Res_H m(K) \geq 0$ for all $K \leq H \leq G$ and $H \in Iso(m)$.*

Proof of (4.2). The necessity of (4.2) follows from (1.7) and (2.14). To prove sufficiency, let $m \in C^+(G)$ satisfy the conditions in (4.2). By (3.4) and (3.7), there is a representation $\alpha \in RO(G)$ with $Dim(S\alpha) = m$ such that $Res_H \alpha$ is isogeneous for all $H \leq G$ and an $\mathbf{R}H$ -module for all $H \in Iso(m)$. Again by (3.4) and (3.7), we can add an $\mathbf{R}G$ -module V such that $W = \alpha \oplus V$ is an $\mathbf{R}G$ -module, $Res_H V$ and $Res_H W$ are isogeneous for all $H \leq G$ and $Res_H V \subseteq Res_H W$ for all $H \in Iso(m)$. For large V , we conclude by (2.12), that there is a G -homotopy representation with $X * SV \simeq_G SW$, and hence $Dim X = Dim(S\alpha) = m$ and (4.1) can be applied. □

The rest of this section is devoted to a proof of (4.1). Given X , the first step is to construct a (locally linear) $PL G - \mathbf{R}^n$ bundle TX over X , which can play the role of the tangent bundle. This seems to be a very difficult problem in general, but in our case we are aided by the fact that X is stably linear. The equivalence (4.1.i) suggests to construct TX such that $TX \oplus \mathbf{R} \oplus \underline{V} \cong \underline{W}$ as $G - \mathbf{R}^n$ -bundles. Section 5 gives more motivation for this choice. Note that (4.1.ii) is certainly necessary for obtaining such a bundle TX .

Our $G - \mathbf{R}^n$ bundles have a distinguished zero section, cf. the definition of $PL_G(W)$ in sect. 3. Given two such bundles ξ and η over Y with the property that the fibre η_y is a sub $\mathbf{R}G_y$ -module of ξ_y , for each $y \in Y$. In [17], the concept of a G -epimorphism $\lambda: \xi \rightarrow \eta$ was defined. We add to the definition [17, (1.5)] the requirement that λ preserves the zero section. The obstruction theory of [17, sect. 2] remains valid.

Proposition 4.3. *Given ξ, η as above and suppose further:*

- (i) $Dim S(\xi_y), Dim S(\xi_y, -\eta_y) \in C^+(G_y)$ satisfy the gap conditions (1.9),

(ii) $\dim Y^{G_y} \leq \dim_{\mathbf{R}}(\xi_y^{G_y} - \eta_y^{G_y})$ for each $y \in Y$.
 Then there is a G -bundle epimorphism $\varphi: \xi \rightarrow \eta$ (preserving the 0-section). If in (ii) one requires strict inequality, then φ is unique up to regular G -homotopy.

Proof. By [17, sect. 2] we must construct a section of the G -bundle functor $\text{Epi}(\xi, \eta)$. Its fibre at $y \in Y$ is the space

$$\text{Epi}(\xi_y, \eta_y) = \text{PL}_{G_y}(\xi_y) / \text{PL}_{G_y}(\alpha_y)$$

where $\alpha_y \oplus \eta_y = \xi_y$. The obstructions to a section are classes in the Bredon cohomology groups (see [2])

$$H_G^{k+1}(Y, \omega_k); \omega_k(G/H, y) = \pi_k(\text{PL}_H(\xi_y) / \text{PL}_H(\alpha_y)).$$

Under the given assumptions these groups all vanish by (3.1). Similarly, given two bundle epimorphisms, the obstructions to a regular G -homotopy between them lie in $H_G^k(Y; \omega_k)$. Again the groups vanish by (3.1). □

We apply (4.3) with $\xi = \underline{W}$, $\eta = \underline{V} \oplus \underline{\mathbf{R}}$ to construct TX , where the underlining indicates the product bundle. There is a G -bundle epimorphism $\varphi: \underline{W} \rightarrow \underline{V} \oplus \underline{\mathbf{R}}$ over X , and we define

$$TX = \text{Ker } \varphi = \{(x, w) \in \underline{W} \mid \varphi(w) = 0_x\}. \tag{4.4}$$

It follows directly from the definition of a G -bundle epimorphism that TX is a $\text{PL } G - \mathbf{R}^n$ bundle where $n = \text{Dim } X(1) - 1$ is the ambient dimension of X . Moreover, the uniqueness statement in (4.2) shows that the isomorphism class of $TX \oplus \underline{\mathbf{R}}$ is independent of the choice of epimorphism φ .

Next, we construct a normal map over $(X, TX \oplus \underline{\mathbf{R}})$, cf. [15], [18].

Proposition 4.5. *There is a degree 1 normal map*

$$(f, \hat{f}): (M, TM \oplus \underline{\mathbf{R}}) \rightarrow (X, TX \oplus \underline{\mathbf{R}}).$$

Proof. We want to homotop the map

$$h: SW \simeq_G X * SV \rightarrow X * SV / SV \simeq_G (X \times V)_+$$

to one which is G -transverse to X . This can be achieved by the obstruction theory of [17, sect. 3]. In fact, it is the same obstruction groups, which occur for this problem and for the problem considered in (4.3); they vanish. Assuming h is already transversal, we let $M = h^{-1}(X)$ and set $f = h|_M$.

It is easily checked that f has degree 1 on each fixed set and hence, since G has odd order, has degree 1 in the sense of [15]. Transversality includes the statement that M has normal bundle \underline{V} in SW . It follows that $TM \oplus \underline{\mathbf{R}}$ is the kernel bundle of an epimorphism $\lambda: \underline{W} \rightarrow \underline{V}$. By definition $f^*(TX \oplus \underline{\mathbf{R}})$ is also

the kernel of an epimorphism $\varphi: \underline{W} \rightarrow \underline{V}$. The uniqueness statement of (4.3) applied to $\xi = \underline{W}$, $\eta = \underline{V}$ yields an isomorphism $TM \oplus \underline{R} \cong f^*(TX \oplus \underline{R})$, hence the map \hat{f} . □

Proof of (4.1). It suffices to show that the G -normal map (f, \hat{f}) of the previous proposition is normally cobordant to a G -homotopy equivalence. The obstructions to this lie in the equivariant L -group $\mathcal{L}_n^h(\pi^c X, \{T_x X\})$, and by [15],

$$\mathcal{L}_n^h(\pi^c X, \{T_x X\}) = \sum_{(H)}^{\oplus} L_{m(H)-1}^h(\mathbf{Z}[G/H]),$$

where $m(H) = \text{Dim } X(H)$ and H varies over the isotropy groups of X . Since $m(H) - 1$ is odd and $|G|$ is odd, the L -groups all vanish, [24]. □

EXAMPLE 4.6. The dimension function of (1.10) with $a > 4b + 4$ has $\mu \text{Res}_H(m)(K) \geq 0$ for all $K \leq H < G$ except possibly for $H = \mathbf{Z}/p_i p_j p_k$ and $K = 1$, where $\mu \text{Res}_H(m)(1) = 5b + 8 - a$. Thus, there is a PL - G representation form with dimension function m for $4b + 4 < a \leq 5b + 8$, but not for $a > 5b + 8$.

Our smooth realization results are weaker than the PL ones. This reflects the observation in sect. 3 that orthogonal Stiefel manifolds are less connected than the PL ones.

Theorem 4.7. *Let G be a cyclic group of odd order. Suppose $m \in C^+(G)$ satisfies the strong gap conditions (1.9). If $\mu \text{Res}_H m(K) \geq m(H) - 2$ for all $K \leq H \leq G$ and $H \in \text{Iso}(m)$, then there is a smooth representation form with dimension function m .*

Proof. The proof is similar to that of Theorem 4.1, resp. Corollary 4.2, so we indicate only the necessary modifications. We choose $\mathbf{R}G$ -modules V and W with $\text{Dim}(SW - SV) = m$, such that $\text{Res}_H W$, $\text{Res}_H V$ and $\text{Res}_H W - \text{Res}_H V$ are isogeneous for all $H \leq G$, and obtain a homotopy representation $X \in V^+(G)$ with $X * SV \simeq_c SW$ by (2.10). The stronger assumptions $\mu \text{Res}_H(m)(K) \geq m(H) - 2$ for all $K \leq H$, guarantee that the obstruction theory used in the proofs of (4.1) and (4.2) works also smoothly; the obstructions vanish. Thus we obtain a smooth normal map over X , which will be normally cobordant to a homotopy equivalence by smooth surgeries, again since $L_{2k+1}^h(\mathbf{Z}[G/H]) = 0$. □

EXAMPLE 4.8. Let $G = \mathbf{Z}/p_1 p_2 p_3$ with p_1, p_2 and p_3 distinct primes. Consider the dimension function $m \in C^+(G)$ with

$$m(\mathbf{Z}/p_i) = 2a, \quad m(1) = 4a + 2, \quad m(H) = 0 \text{ otherwise.}$$

For $a \geq 3$, m satisfies the required gap conditions (1.9), and $\mu \text{Res}_H(m)(K) \geq m(H) - 2$ for all $K \leq H \leq G$ and $H \in \text{Iso}(m)$, i.e., there is a smooth representa-

tion form M with $\dim M = m$. However $\mu m(1) = 2 - 2a < 0$, so by (2.14), there is no linear G -sphere with dimension function m .

5. The stable tangent bundle

From [11] we know that smooth homotopy spheres in the G -trivial case are stably parallelizable. We prove here a similar result for smooth G -representation forms under our standard assumption that G is cyclic of odd order (see [14] for prime order groups). Furthermore, we determine the stable fibre homotopy type of the tangent bundle TM of a smooth representation form M .

Theorem 5.1. *Let M be a smooth representation form (of odd dimension). There exist representations V and W such that $TM \oplus V \oplus R = W$.*

It should be a problem of some interest to attempt a generalization of (5.1) to more general groups G . One may wonder if smooth representation forms are always stably G -parallelizable.

Recall that $KO_G^*(\)$ has Thom isomorphism for symplectic G -bundles, and in particular for bundles of the form $V \oplus \psi^{-1}(V)$ where $\psi^{-1}(V)$ is the complex conjugate representation. Moreover, $KO_G^*(\) \otimes \mathbf{Z} \left[\frac{1}{2} \right]$ has Thom isomorphism for all complex bundles, because the functor is a direct summand of $K_G^*(\) \otimes \mathbf{Z} \left[\frac{1}{2} \right]$. The basic lemma is the following

Lemma 5.2. *Let G be any odd order group and W any RG -module without stationary lines, i.e. $W^G = 0$. Then*

$$KO_G^{2i-1}(DW, SW) = \begin{cases} 0 & \text{if } 2i \not\equiv \dim W \pmod{8}, \\ \mathbf{Z}/2 & \text{if } 2i \equiv \dim W \pmod{8}. \end{cases}$$

In the second case, Res: $KO_G^{-1}(DW, SW) \rightarrow KO_{\Gamma}^{-1}(DW, SW)$ is an isomorphism.

In the proof we find it convenient to use localization techniques and a few words about how this works are in order. The functor $KO_G^*(X)$ is a module over $RO(G)$, and hence a module over the Burnside ring $A(G)$ via the natural mapping $A(G) \rightarrow RO(G)$.

A 2-elementary subgroup of G is cyclic, so Brauer's induction theorem shows that

$$\text{Ind: } \Sigma RO(C)_{(2)} \rightarrow RO(G)_{(2)}, \quad C \leq G \text{ cyclic}$$

is surjective. The general induction theorem, see e.g. [8], gives

$$KO_C^i(X)_{(2)} \cong \varprojlim KO_C^i(X)_{(2)}, \quad C \text{ cyclic.} \tag{5.3}$$

Since G has odd order, $A(G)_{(2)}$ decomposes into a product of its localizations. The prime ideals of $A(G)_{(2)}$ are indexed by the subgroups $\Gamma \subseteq G$; call them $q(\Gamma)$ following [5]. Then

$$A(G)_{(2)} = \prod_{\Gamma \subseteq G} A(G)_{q(\Gamma)}; A(G)_{q(\Gamma)} = \mathbf{Z}_{(2)}.$$

The representation ring is a module over $A(G)$, so decomposes accordingly. From [23],

$$RO(G)_{(2)} = \prod RO(G)_{q(\Gamma)}; RO(G)_{q(\Gamma)} \cong \mathbf{Z}_{(2)}[\zeta_\Gamma + \zeta_\Gamma^{-1}]$$

where ζ_Γ is a primitive $|\Gamma|$ 'th root of 1.

Finally, we shall use (for cyclic G):

$$KO_G(X)_{q(\Gamma)} = 0 \quad \text{if } X^\Gamma = \emptyset \tag{5.4}$$

$$\text{Res: } KO_G(X)_{q(\Gamma)} \xrightarrow{\cong} KO_\Gamma(X)_{q(\Gamma)}^{\mathcal{G}/\Gamma}$$

The first claim follows from the skeleton spectral sequence, the second from induction theory, cf. [5], [19].

Proof of (5.2). By Thom isomorphism,

$$KO_G^{2i-1}(DW, SW) \otimes \mathbf{Z} \left[\frac{1}{2} \right] \cong KO_G^{2i-|W|-1}(pt) \otimes \mathbf{Z} \left[\frac{1}{2} \right] = 0$$

since $|W|$ is even. Hence we can localize at 2. According to (5.3) there is no harm in assuming G is cyclic, and to localize further at $q(\Gamma)$, $\Gamma \subseteq G$.

Localization is an exact functor, so by (5.4)

$$KO_G^{2i-1}(DW, SW)_{q(\Gamma)} \xrightarrow{\cong} KO_\Gamma^{2i-1}(DW, SW)_{q(\Gamma)}^{\mathcal{G}/\Gamma}.$$

Write $\text{Res}_\Gamma(W) = \mathbf{R}^{2k} \oplus W_\Gamma$. Then

$$KO_\Gamma^{2i-1}(DW, SW) \cong KO_\Gamma^{2i-2k-1}(DW_\Gamma, SW_\Gamma).$$

Since localization is an exact functor and since SW_Γ has empty Γ -fixed set, (5.4) gives

$$KO_\Gamma^{2i-2k-1}(DW_\Gamma, SW_\Gamma)_{q(\Gamma)} \cong KO_\Gamma^{2i-2k-1}(DW_\Gamma)_{q(\Gamma)} \cong KO_\Gamma^{2i-2k-1}(pt)_{q(\Gamma)}.$$

The groups $KO_\Gamma^{2i-1}(pt)$ are known to be zero, except if $i \equiv 0 \pmod{4}$ where

$$KO_\Gamma^{-1}(pt) \cong RO(\Gamma)/R(\Gamma),$$

Since Γ is odd, $RO(\Gamma)/R(\Gamma) \cong \mathbf{Z}/2$ and the restriction

$$\text{Res: } KO_\Gamma^{-1}(pt) \xrightarrow{\cong} KO^{-1}(pt)$$

gives the isomorphism. It follows that $KO_{\Gamma}^{-1}(pt)_{q(\Gamma)}=0$ unless $\Gamma=1$, since $q(\Gamma)$ maps to zero in $A(1)$ for $\Gamma \neq 1$. \square

Proof of (5.1). Choose representations V and W with $X*SV \simeq_c SW$ and with V symplectic. Let cX be the cone of X . By Thom isomorphism

$$KO_{\mathbb{Z}}^*(cX, X) \cong KO_{\mathbb{Z}}^{*+|V|}(DW, SW).$$

We will show that the tangent bundle TX extends over cX , hence is stably trivial. Consider the exact sequence

$$KO_G(cX) \rightarrow KO_G(X) \rightarrow KO_G^1(cX, X)$$

Since $\dim X^G = |W^G| - |V^G| - 1$ is of odd dimension (cf. (1.8)) and

$$KO_G^{1+|V|}(DW, SW) \cong KO_G^{1+|V|-|W^G|}(DW_G, SW_G)$$

where $W^G \oplus W_G = W$, we can apply (5.2). Indeed, $|W| - |W^G| \equiv |V| - |V^G| \equiv 0 \pmod{2}$. Thus

$$\text{Res: } KO_G^1(cX, X) \xrightarrow{\cong} KO^1(cX, X).$$

On the other hand, TX restricts to a stably trivial bundle in $KO_1(X)$ by the Kervaire-Milnor result, so the composite

$$KO_G(X) \rightarrow KO_G^1(cX, X) \xrightarrow{\text{Res}} KO^1(cX, X)$$

must be trivial by naturality. \square

Finally, we determine the stable fibre homotopy type of the tangent bundle of a smooth stably linear G -representation form. Let G be an arbitrary finite group and M be a smooth G -representation form.

Theorem 5.5. *There is an equivariant fibre homotopy equivalence of G -spherical fibrations*

$$\varphi: M \times M \rightarrow S(TM \oplus \mathbf{R}) \text{ over } M.$$

If M is stably linear, i.e., if there are $\mathbf{R}G$ -modules V and W such that $M*SV \simeq_c SW$, this allows us to conclude:

Corollary 5.6. *The G -sphere bundle $S(TM \oplus \mathbf{R} \oplus V)$ and SW over M are G -fibre homotopy equivalent.*

Proof of (5.5). The exponential map w.r.t. some equivariant Riemannian metric defines a fibrewise G -diffeomorphism $id \times exp: D(TM) \rightarrow M \times M$ from a suitable disk bundle of TM onto a neighbourhood of the diagonal $\Delta M \subset M \times M$. A fibrewise ‘‘inverse’’

$$c: M \times M \rightarrow \bigcup_{x \in M} D(T_x M) / S(T_x M) \cong S(TM \oplus \underline{R})$$

into the *fibrewise* quotient with the sphere bundle is given by

$$c(x, y) = \begin{cases} v \in \text{int } D(T_x M), & \text{if } \exp_x(v) = y; \\ \star = [S(T_x M)], & \text{if } y \notin \exp(\text{int } D(T_x M)). \end{cases}$$

The restriction of c to fibres over any point $x \in M$ is a G_x -homotopy equivalence. Hence, (5.5) follows from the equivariant Dold theorem [10]. \square

Proof of (5.6). Let $\bar{\varphi}$ denote a fibre homotopy inverse to the homotopy equivalence φ from (5.5) and $\psi: M * SV \rightarrow SW$ denote a G -homotopy equivalence. Then, the composite fibre map over M

$$S(TM \oplus \underline{R} \oplus \underline{V}) \xrightarrow{\bar{\varphi} * id} M \times (M * SV) \xrightarrow{id \times \psi} M \times SW$$

is a G -fiber homotopy equivalence. \square

6. Smooth representation forms

In this section, we prove Theorem B of the introduction. Supposing gap conditions, we identified in (4.2) the dimension functions of G -PL representation forms. We show here that the dimension functions of *smooth* G -representation forms have to satisfy further numerical conditions.

We assume that $m = \text{Dim } M$ for some *smooth* G -representation form M . By (2.12), there are $\mathbf{R}G$ -modules V, W such that $M * SV \cong_G SW$; in particular, $m = \text{Dim } W - \text{Dim } V$. Moreover, by (5.6)

$$[TM \oplus \underline{R} \oplus \underline{V}] = [W] \in JO_G(M). \tag{6.1}$$

We will now fix a subgroup H of G and restrict (6.1) to the fixed manifolds M^H to get our basic equation

$$[TM | M^H \oplus \underline{R} \oplus \underline{V}] = [W] \in JO_G(M^H). \tag{6.2}$$

We then proceed as follows: Any G -equivariant bundle over M^H , $H \leq G$ decomposes into a direct sum of its “ K -isotropic” pieces, $K \leq H$, and each summand is a G/H -bundle, at least if G is a direct product $G = H \times \Gamma$.

Then, (6.2) will decompose into G/H -equivariant fibre homotopy trivializations (6.6) of certain subbundles of $TM | M^H$. If H is a maximal isotropy group, calculations in KO - and JO -theory give upper bounds for the “fibre homotopy geometric dimension” of associated bundles over lens spaces.

Taking Thom spaces, we obtain in (6.11) that certain stunted lens spaces are suspensions and then use cohomology operations to obtain necessary conditions in the special case where $\Gamma = \mathbf{Z}/p$ and $|H|$ is prime to p .

Let G be a finite group, $H \leq G$ and let X be a G - CW -complex. Given a G - \mathbf{R}^n bundle E over X^H and any subgroup $K \leq H$, one may define the *virtual \mathbf{R}^n bundle*

$$I(Res_H E)(K) = \sum_{K \leq L \leq H} \mu(L; K)E^L.$$

In general, $I(Res_H E)(K)$ does not support a G/H -action. However, if $G = H \times \Gamma$, then, $\Gamma \cong G/H$ acts naturally on E^L , $L \leq H$, and hence on $I(Res_H E)(K)$.

Suppose next E is a G -vector bundle. Let $Irr(H, \mathbf{R})$ denote the set of irreducible $\mathbf{R}H$ -modules. For $V \in Irr(H, \mathbf{R})$, the “ H -isotypical part” of E is the vector bundle $Hom_H(V, E)$ of fibrewise linear homomorphisms. Define

$$\Phi^H: KO_G(X^H) \rightarrow RO(H) \otimes KO(X^H),$$

by

$$\Phi^H([E]) = [\sum_{V \in Irr(H, \mathbf{R})} V \otimes_{Hom_H(V, E)} [Hom_H(V, Res_H(E))]].$$

We use the Möbius transformation (2.7), and define

$$\begin{aligned} \mu\Phi^H(K): KO_G(X^H) &\xrightarrow{\Phi^H} RO(H) \otimes KO(X^H) \xrightarrow{I(K) \otimes id} \\ &RO(H) \otimes KO(X^H) \xrightarrow{\varepsilon \otimes id} KO(X^H). \end{aligned}$$

It is of importance to note that the element $\mu\Phi^H([E])(K)$ is represented by an honest (not just virtual) vector bundle, namely by

$$\mu\Phi^H([E])(K) = [\sum_{\substack{V \in Irr(H, \mathbf{R}) \\ ker V = K}} [Hom_H(V, Res_H(E))]].$$

The homomorphisms Φ^H and $\mu\Phi^H(K)$ can be refined to yield G/H -vector bundles, when the restriction map $Res_H: RO(G) \rightarrow RO(H)$ admits a section $\sigma_H: RO(H) \rightarrow RO(G)$. In general, Res_H fails to be onto. But when $G \cong H \times \Gamma$, the projection map $p_1: G \rightarrow H$ induces a canonical section

$$\sigma_H = p_1^*: RO(H) \rightarrow RO(G).$$

The vector bundle $Hom_H(V, E)$ then has a G/H -structure by letting G act on $Hom_H(\sigma_H V, E)$ via conjugation, and we get for each $K \leq H$,

$$\begin{aligned} \Phi^H: KO_G(X^H) &\rightarrow RO(H) \otimes KO_{G/H}(X^H), \text{ and} \\ \mu\Phi^H(K): KO_G(X^H) &\rightarrow RO(H) \otimes KO_{G/H}(X^H) \xrightarrow{\varepsilon \otimes id} KO_{G/H}(X^H). \end{aligned}$$

Lemma 6.3. *For a finite group G , the homomorphisms $[I(Res_H -)](K)$ and $\mu\Phi^H(K): KO_G(X^H) \rightarrow KO(X^H)$ coincide. When $G \cong H \times \Gamma$, the refined versions into $KO_{G/H}(X^H)$ coincide as well.*

Proof. The first statement is a combination of the canonical decomposition of real G -vector bundles with the properties of the Möbius transformation. The second statement follows from the commutative diagram

$$\begin{CD}
 KO_G(X^H) @>\Phi^H>> RO(H) \otimes KO_\Gamma(X^H) \\
 @V Res_\Gamma VV @VV \varepsilon \otimes id V \\
 @. @. KO_\Gamma(X^H)
 \end{CD} \tag{6.4}$$

Indeed, for an $\mathbf{R}G$ -module W and an $\mathbf{R}H$ -module V , Γ acts trivially on p_1^*V , so the conjugation action on $\text{Hom}_G(p_1^*V, W)$ is precisely the Γ -action on W . \square

From now on, we suppose $G=H \times \Gamma$.

Proposition 6.5. *For stably G -fibre homotopy equivalent bundles E_1, E_2 over X^H , $\mu\Phi^H(E_1)(K)$ and $\mu\Phi^H(E_2)(K)$ are stably Γ -fibre homotopy equivalent.*

Proof. According to (6.3), we may represent $\mu\Phi^H([E_i])(K)$ stably by $I(Res_H E_i)(K) = \sum_{K \leq L \leq H} \mu(L; K)E_i^L$ $i = 1, 2$. A (stable) G -fibre homotopy equivalence between E_1 and E_2 induces stable Γ -fibre homotopy equivalences between E_1^L and E_2^L for all $L \leq H$. \square

Proposition 6.6. *Let $G=H \times \Gamma$ and let M be a smooth G -representation form with $\text{Dim } M=m$, as in (6.1). For $K \leq H$,*

$$\mu\Phi^H(TM \oplus \mathbf{R})(K) \simeq_\Gamma Res_\Gamma(\sum_{S \leq \Gamma} \mu(W-V)(K \times S))$$

as elements in $JO_\Gamma(M^H)$. The real dimensions of the fibres at $x \in M^H$ are given by the non-negative numbers

$$\begin{aligned}
 \dim_x \mu\Phi^H(TM \oplus \mathbf{R})(K) &= \sum_{S \leq \Gamma} \dim \mu(W-V)(K \times S) \\
 &= \sum_{S \leq \Gamma} \mu m(K \times S) = \mu Res_H m(K).
 \end{aligned}$$

Proof. By (5.6) and (6.5) above, the bundles $\mu\Phi^H(TM \oplus \mathbf{R})(K)$ and $\mu\Phi^H(W-V)(K)$ are stably Γ -fibre homotopy equivalent for all $K \leq H$. For $\alpha = W - V \in RO(G)$,

$$\mu\Phi^H(\alpha)(K) = Res_\Gamma(\sum_{S \leq \Gamma} \mu\alpha(K \times S)) \in KO_{G/H}(M^H)$$

by (6.4), since the irreducible G -modules that restrict to an H -module with kernel K are precisely the ones with kernel $K \times S, S \leq \Gamma$. \square

The rest of this section analyses (6.6). To simplify notation, let us write it:

$$E \simeq_{\Gamma} \sum_{s \leq \tau}^{\oplus} \underline{\alpha}(S), \tag{6.7}$$

where E is a Γ -vector bundle over the Γ -representation form Y and $\alpha(S) \in RO(\Gamma)$ is a Γ -representation consisting of irreducible representations with isotropy group S . Define $n(S) \in \mathbf{Z}$ by $\dim_{\mathbf{R}} \alpha(S) = 2n(S)$. In our case (6.6), $m = \text{Dim } M$, $2n(S) = \mu m(K \times S)$ and $Y = M^H$.

To simplify further, assume Γ is a cyclic group of odd order acting freely on Y . Comments on more general situations are given in [22].

We assume that Y is the sphere SV of a free $\mathbf{R}\Gamma$ -module V with $\text{Dim}(SV) = \text{Dim } Y$. This is no essential restriction, since the equivalence (6.7) can be pulled back to SV via a Γ -equivariant map, cf. the obstruction theory in [5]. The KO_{Γ} - and JO_{Γ} -theory of free linear Γ -spheres are well-known ([9], [12], [20]): In particular, $\widetilde{KO}_{\Gamma}(Y)$ is a finite group. Furthermore, if $T(S)$ denotes the realification of an arbitrary 1-dimensional $C\Gamma$ -module with kernel S , then (cf. [12])

$$\underline{\alpha}(S) \simeq n(S) \cdot \underline{T}(S) \text{ over } Y.$$

Hence (6.7) can be written as

$$E \simeq_{\Gamma} \sum_{s < \Gamma}^{\oplus} n(S) \cdot \underline{T}(S) \oplus 2n(\Gamma) \cdot \underline{\mathbf{R}}. \tag{6.8}$$

Let $o(S)$ denote the order of $\underline{T}(S) - 2 \cdot \underline{\mathbf{R}}$ in $\widetilde{JO}(Y)$, and choose $\varepsilon(S) \in \mathbf{Z}$ such that

$$0 \leq r(S) = n(S) + \varepsilon(S)o(S) < o(S).$$

With $n = \sum_{s < \Gamma} \varepsilon(S) \cdot o(S) - n(\Gamma)$, we rewrite (6.8) stably as

$$E \oplus 2n \cdot \underline{\mathbf{R}} \simeq_{\Gamma} \sum_{s < \Gamma}^{\oplus} r(S) \cdot \underline{T}(S) \text{ over } Y. \tag{6.9}$$

If $n \geq 0$, the fibre homotopy equivalence (6.9) yields an upper bound for the ‘‘fibre homotopy geometric dimension’’ of the bundle on the right hand side. Taking Thom spaces (which we denote by a subscript $+$) on both sides of (6.9) provides the following *desuspension* result for certain *stunted* ‘‘ Γ -spheres’’.

Proposition 6.10. For $2m = \text{dim } Y + 1$,

$$\sum_{\mathbb{S}^{2n}} E_+ \simeq_{\Gamma} S(\sum_{\mathbb{S}^3 < \Gamma}^{\oplus} (r(S) \cdot \underline{T}(S)) \oplus m \cdot T(1)) / S(\sum_{\mathbb{S}^3 < \Gamma}^{\oplus} r(S) \cdot \underline{T}(S)).$$

Proof. For arbitrary $\mathbf{R}\Gamma$ -representations U and V , there is a homotopy equivalence

$$(SV \times U)_+ = SV \times DU / (SV \times SU) \simeq SV * SU / SU = S(U \oplus V) / SU.$$

Without restrictions assume $Y = SV = S(m \cdot T(1))$, and use the homotopy equivalence above for $U = \sum_{\mathbb{S}^3 < \Gamma}^{\oplus} r(S) \cdot \underline{T}(S)$. □

REMARK 6.11. Passing to Thom spaces as above seems to be essential. The usual K-theoretic obstructions to a linear desuspension similar to (6.9) are difficult to grasp, since there are many different vector bundles that are fibre homotopy equivalent to the right hand side of (6.9). This was the reason for the very limited results in § 6 in our first version [16]. It seems to be an interesting question whether there are other methods to find upper bounds for the geometric dimensions of Γ -bundles over Γ -spheres with given dimension function in the fibre, in particular, whether such a bound could be sharper than that for the “fibre homotopy geometric dimension”.

Finally we specialize to $\Gamma = \mathbf{Z}/p$, p an odd prime number, acting freely on Y , where KO - and JO -theory of a free Γ -sphere SV are known explicitly.

Proposition 6.12 ([9], [12], [20]).

Let V be a free m -dimensional Γ -module, and let $s \geq 0$, $0 \leq r < p-1$, be given by

$$m-2 = 2 \cdot (s \cdot (p-1) + r).$$

Let $X = \underline{T}(1) - \underline{1}_c \in \underline{K}_\Gamma(SV)$, and $\delta_m = \begin{cases} 1 & m \equiv 2 \pmod 8 \\ 0 & \text{otherwise} \end{cases}$

Then,

- (i) $K_\Gamma(SV) \cong \mathbf{Z}[X]/(p^{s+1} \cdot X, p^s \cdot X^{r+1}, (X+1)^p - 1)$
- (ii) $KO_\Gamma(SV) \cong \text{re}K_\Gamma(SV) \oplus \delta_m \cdot \mathbf{Z}/2$
- (iii) $JO_\Gamma(SV) \cong \mathbf{Z} \oplus \mathbf{Z}/p^s$, and $\widetilde{JO}_\Gamma(SV)$ is generated by X .

With $T = \underline{T}(1)$, (6.8) becomes the following equation over $S(m \cdot T)$:

$$E \simeq_\Gamma n(1) \cdot T \oplus 2 \cdot n(\Gamma) \cdot \underline{\mathbf{R}}$$

Of course, $2(n(1) + n(\Gamma)) = \dim E \geq 0$, but either $n(1)$ or $n(\Gamma)$ might be negative. In these cases define $a, n \in \mathbf{Z}$, and $0 \leq r < p^s$ by

$$n(1) + a \cdot p^s = r, \quad \text{and} \quad n = a \cdot p^s - n(\Gamma). \tag{6.13}$$

Then, (6.9) becomes

$$E \oplus 2n \cdot \underline{\mathbf{R}} \simeq_\Gamma r \cdot \underline{T} \text{ over } S(m \cdot T). \tag{6.14}$$

Proposition 6.15. Let $\Gamma = \mathbf{Z}/p$. If there is a Γ -homotopy equivalence (6.14) for some e -dimensional Γ -bundle over $S(m \cdot T)$ such that $n \geq 0$, then the binomial coefficients $\binom{kr+j}{i} \equiv 0 \pmod p$ for all $k \geq 1, i > j \geq 0$ such that $\frac{ke}{2} + j < i < \frac{m}{p-1}$.

Proof. Notice first, that (6.14) gives rise to a bunch of Γ -homotopy equivalences

$$k\bar{E} \oplus jT \oplus 2kn \cdot \underline{R} \simeq_{\Gamma} (kr+j) \cdot T \text{ over } S(m \cdot T).$$

The group Γ acts freely on both sides. We define $\bar{E} := E/\Gamma$, and $\mathcal{X} = S(m \cdot T) \times_{\Gamma} T$ as bundles over the lens space $L(m \cdot T) = S(m \cdot T)/\Gamma$. Dividing out the Γ -action yields homotopy equivalences

$$k\bar{E} \oplus j \cdot \mathcal{X} \oplus 2kn \cdot \underline{R} \simeq (kr+j) \cdot \mathcal{X} \text{ over } L(m \cdot T).$$

Taking Thom spaces as in (6.10) yields the desuspension homotopy equivalences

$$\sum^{2kn} (k\bar{E} \oplus j \cdot \mathcal{X})_+ \simeq L((kr+j+m) \cdot T) / L((kr+j) \cdot T).$$

The mod p -cohomology $H^*(-; \mathbf{Z}/p)$ of both sides is given by

$$\begin{cases} \mathbf{Z}/p & 2(kr+j) \leq * \leq 2(kr+j+m-1); \\ 0 & \text{otherwise} \end{cases}$$

A generator u in dimension $2(kr+j)$ projects to $t^{kr+j} \in H^{2kr+2j}(L((kr+j+m) \cdot T))$, $t \in H^2(L((kr+j+m) \cdot T))$. The proof is by calculation of Steenrod p -th powers P^i on $H^{2kr+2j}(-; \mathbf{Z}/p)$. By stability $P^i(H^{2kr+2j}(\sum^{2kn} (k\bar{E} \oplus j \cdot \mathcal{X}))) = 0$ whenever $2i > k \cdot \dim \bar{E} + 2j$. On the other hand, $P^i u$ projects to

$$P^i(t^{kr+j}) = \binom{kr+j}{i} \cdot t^{kr+j+i(p-1)} \text{ in } H^{2(kr+j+i(p-1))}(L((kr+j+m) \cdot T)).$$

Hence, the binomial coefficient has to be zero whenever the latter dimension is less than or equal to the dimension of the lens space. □

REMARK 6.16. In some cases, (6.15) implies (use $k=1, j=0$):

$$\frac{e}{2} \geq \frac{m}{p-1} - 1 \tag{6.17}$$

This occurs whenever $m \leq p^l(p-1)$ and there is no zero among the last l digits of the p -adic expansion of r . As a particular case consider $m \leq p(p-1)$ and $r \not\equiv 0 \pmod p$. One wonders whether (6.17) always follows from (6.14).

Corollary 6.18. *Let $G = \Gamma \times H$ with $\Gamma = \mathbf{Z}/p$ and H cyclic of odd order q , prime to p . Let M be a smooth fixed point free homotopy representation with even dimension function m . Define s and r by*

$$m(H) - 2 = 2 \cdot (s(p-1) + r),$$

and for every $K \leq H$ let $a(K), m(K) \in \mathbf{Z}$ with $0 \leq r(K) < p^s$ be given by

$$\frac{1}{2} \cdot \mu m(K) + a(K) \cdot p^s = r(K),$$

$$n(K) = a(K) \cdot p^s - \frac{1}{2} \cdot \mu m(K \times \Gamma).$$

If $n(K) \geq 0$, then

$$\binom{kr(K)+j}{i} \equiv 0 \pmod{p} \tag{6.19}$$

for all $k \geq 1, i > j \geq 0$ such that

$$k \cdot \mu \operatorname{Res}_H m(K) + 2j < 2i < \frac{m(H)}{p-1}.$$

If, in addition, $m(H) \leq 2p^l(p-1)$ and there is no zero among the last l digits of the p -adic expansion of $\mu m(K)$, then

$$\mu \operatorname{Res}_H m(K) \geq \frac{m(H)}{p-1} - 1. \tag{6.20}$$

EXAMPLE 6.21. Consider Example 1.10 from the introduction with $H = \mathbb{Z}/p_1 p_2 p_3 \leq \mathbb{Z}/p_1 p_2 p_3 p_4$, and four distinct odd prime numbers p_i . Whith m as in (1.10), one gets

L	1	\mathbb{Z}/p_i	$\mathbb{Z}/p_i p_j$	$\mathbb{Z}/p_i p_j p_k$	G
$m(L)$	$2a+2$	a	$2b+2$	b	0
$\mu m(L)$	$8b-2a+14$	$a-3b-6$	2	b	0
$\mu \operatorname{Res}_H m(L)$	$5b-a+8$	$a-3b-4$	$b+2$	b	—

The strong gap hypothesis is satisfied if $a > 4b + 4 \geq 28$. By (4.2), there is a PL -representation form with dimension function m , if and only if, in addition, $a \leq 5b + 8$. By (6.20), there is no smooth representation form with that dimension function if, for some $p = p_i, b \leq 2p(p-1), \mu m(1) = 8b - 2a + 14 \not\equiv 0 \pmod{p}$ and $5b - a + 8 = \mu \operatorname{Res}_H m(1) < \frac{m(H)}{p-1} - 1 = \frac{b}{p-1} - 1$.

References

[1] M. Aigner: Combinatorial theory, Springer, Berlin Heidelberg New York, 1979.
 [2] G.E. Bredon: Equivariant Cohomology Theories, Lect. Notes Math. vol. 34., Springer, Berlin Heidelberg New York, 1967.
 [3] G.E. Bredon: Introduction to compact transformation groups, Academic Press, New York, 1972.
 [4] T. tom Dieck: Homotopy equivalent group representations and Picard groups of the Burnside ring and the character ring, Manuscripta Math. 26 (1978), 179–200.

- [5] T. tom Dieck: Transformation groups and representation theory, Lect. Notes Math., vol. 766, Springer, Berlin Heidelberg New York, 1979.
- [6] T. tom Dieck: *Homotopiedarstellungen endlicher Gruppen: Dimensionsfunktionen*, Invent. Math. **67** (1978), 231–252.
- [7] T. tom Dieck, T. Petrie: *Homotopy representations of finite groups*, Publ. Math. IHES **56** (1982), 337–377.
- [8] A. Dress: *Induction and structure theorems for orthogonal representations of finite groups*, Ann. of Math, **102** (1975), 291–325.
- [9] K. Fujii, M. Sugawara: *The order of the canonical elements in the J-group of the lens space*, Hiroshima Math. J. **10** (1980), 369–374.
- [10] H. Hauschild, S. Waner: *The equivariant Dold theorem mod k and the Adams conjecture*, Illinois J. of Math. **27** (1983), 52–66.
- [11] M. Kervaire, J. Milnor: *Groups of homotopy spheres I*, Ann. of Math. **77** (1963), 504–537.
- [12] T. Kobayashi, S. Murakami, M. Sugawara: *Note on J-groups of lens spaces*, Hiroshima Math. J. **7** (1977), 387–409.
- [13] R. Lashof, M. Rothenberg: *G-smoothing theory*, Proc. Pure Math. AMS **32**, (1978), 211–266.
- [14] P. Löffler: *Über die G-Rahmbarkeit von G-Homotopiesphären*, Arch. Math. **29** (1977), 628–634.
- [15] W. Lück, I. Madsen: *Equivariant L-theory I, II*, Math. Z. **203** (1990), 503–526, Math. Z. **204** (1990), 253–268.
- [16] I. Madsen, M. Raussen: *Smooth and locally linear G-homotopy representations*, Algebraic Topology, Proceedings, Göttingen 1984, ed. by L. Smith, Lect. Notes Math. vol. 1172, Berlin Heidelberg New York, Springer, 1985, 130–156.
- [17] I. Madsen, M. Rothenberg: *On the classification of G-spheres I: equivariant transversality*, Acta Mathematica **160** (1988), 65–104.
- [18] I. Madsen, M. Rothenberg: *On the classification of G-spheres II: PL automorphism groups*, Math. Scand. **64** (1989), 161–218.
- [19] I. Madsen, J.A. Svensson: *Induction in unstable equivariant homotopy theory and non-invariance of Whitehead torsion*, Comtemp. Math. **37** (1985), 99–113.
- [20] N. Mahammed: *A propos de la K-théorie des espaces lenticulaires; K-théorie des espaces lenticulaires*, C.R. Acad. Sc. Paris **A271** (1970), 639–642; **A272** (1971), 1363–1365.
- [21] I. Nagasaki: *Homotopy representations and spheres of representations*, Osaka J. Math. **22** (1985), 895–905.
- [22] M. Raussen: *Non-smoothable homotopy representations*, Preprint, Aalborg Universitetscenter 1988.
- [23] G. Segal: *The representation ring of a compact Lie group*, Publ. Math. IHES **34** (1968), 113–128.
- [24] C.T.C. Wall: *The classification of hermitian forms VI*, Ann. of Math. **103** (1976), 1–80.

Ib Madsen
Aarhus Universitet
Matematisk Institut
Ny Munkegade
DK-8000 Aarhus C
Denmark

Martin Raussen
Aalborg Universitetscenter
Institut for Elektroniske Systemer
Afdeling for Matematik og Datalogi
Frederik Bajersvej 7
DK-9220 Aalborg ϕ
Denmark

