Nayatani, S.
Osaka J. Math.
27 (1990), 453–464

LOWER BOUNDS FOR THE MORSE INDEX OF COMPLETE MINIMAL SURFACES IN EUCLIDEAN 3-SPACE

SHIN NAYATANI

(Received July 14, 1989)

Introduction

The study of the index of minimal surfaces in Euclidean space has been quite active in these several years. Fischer-Colbrie [7], Gulliver and Lawson [8], [9] have proved independently that a complete oriented minimal surface in $\boldsymbol{R}^3$ has finite index if and only if it has finite total curvature. More recently Tysk [16] has proved that the index of a complete oriented minimal surface in $\boldsymbol{R}^3$ is bounded from above by an explicit constant times the total curvature. For the situation in higher codimensions, see [2], [6] and [13].

In this paper we study the lower bound for the index of complete oriented minimal surfaces in $\boldsymbol{R}^3$. In view of the above mentioned result due to Fischer-Colbrie et al., we may restrict our attention to the surfaces with finite total curvature. It is well known that such a surface is conformally equivalent to a compact Riemann surface with finitely many punctures and the Gauss map of the surface extends to the compactified surface as a holomorphic map. We then give a lower bound for the index in terms of an invariant of the extended Gauss map and the genus of the surface. As a corollary of this result, we give a lower bound for the index in terms of the total curvature of the surface, when all the critical values of the extended Gauss map are contained in some great circle of the target unit sphere. By applying these results we show that the index of the k-end catenoid and the Costa's surface are not less than $2k$-3 and 3 respectively. We also prove that if M is a complete oriented minimal surface of genus zero and is not one of the plane, the Enneper's surface and the catenoid, then the index of M is not less than three. Finally we prove that the index of the k-end catenoid is actually equal to $2k$-3 by explicitly solving the eigenvalue problem associated to the Jacobi operator.

The author wishes to thank Professors H. Ozeki and A. Kasue for their constant encouragement and advice. He also thanks H. Naito for useful conversations.

1. Preliminaries

Let M be a two-dimensional Riemannian manifold. For a function q on M we consider the operator $L=-\Delta+q$, acting on functions on M, where Δ is the Laplace-Beltrami operator. We denote by Q the quadratic form associated to L. Thus for a function u with compact support

$$Q(u, u)=\int_M(|du|^2+qu^2)dA\,,$$

where dA is the area element of M. For a relatively compact domain Ω in M, we define $\mathrm{Ind}(L, \Omega)$, the index of L in Ω, as the number of negative eigenvalues (counted with multiplicities) of the Dirichlet eigenvalue problem

$$Lu=\lambda u \text{ in } \Omega\,, \quad u=0 \text{ on } \partial\Omega\,.$$

It can also be defined as the maximal dimension of a subspace of $C_0^\infty(\Omega)$ on which Q is negative definite. We now define $\mathrm{Ind}(L, M)$, the index of L in M, as the supremum of the numbers $\mathrm{Ind}(L, \Omega)$ over all relatively compact domains in M.

If M is an oriented minimal surface in $\boldsymbol{R}^3$, the associated Jacobi operator L is given by $L=-\Delta-|B|^2$, where B is the second fundamental form of M. The equation $Lu=0$ is called the Jacobi equation. We define $\mathrm{Ind}(M)$, the index of M, as the index of L in M.

At this point we recall the well-known fact that isothermal coordinates for the induced metric together with the orientation give rise to a complex structure on M.

Let $G\colon M\to S^2\subset\boldsymbol{R}^3$ be the Gauss map of M, where S^2 is the unit sphere in $\boldsymbol{R}^3$. Then G is a holomorphic map with respect to the complex structure on M just mentioned and that on S^2 induced by the stereographic projection form the north pole (see [14]). We note that the formula $|B|^2=|dG|^2$ holds.

We now let M be a complete oriented minimal surface in $\boldsymbol{R}^3$ with finite total curvature. By a theorem of Osserman [14], M is conformally equivalent to a compact Riemann surface with finitely many punctures and the Gauss map G extends to the compactified surface as a holomorphic map. We denote by $\tilde{M}$ and $\tilde{G}$ the compactified surface and the extended Gauss map respectively. We fix a conformal metric on M and consider the operator $\tilde{L}=-\tilde{\Delta}-|d\tilde{G}|^2$. Then we have

$$\mathrm{Ind}(M)=\mathrm{Ind}(\tilde{L}, \tilde{M}) \tag{1.1}$$

(see Fischer-Colbrie [7, Corollary 2, p. 131]).

2. The number of nodal domains of solutions of the Jacobi equation and a lower bound for the index

In this section we shall consider a slightly general situation as follows. Let Σ be a compact Riemann surface and $G: \Sigma \to S^2$ a nonconstant holomorphic map. We fix a conformal metric on Σ and consider the operator $L_G = -\Delta - |dG|^2$. We note that $\mathrm{Ind}(L_G, \Sigma)$ and the kernel of L_G are independent of the particular choice of a metric on Σ. The following proposition is an immediate consequence of the celebrated Courant's nodal domain theorem (see [4]). If u is a solution of an elliptic equation on a surface, the set $u^{-1}(0)$ is called the nodal line of u and each connected component of $\Sigma - u^{-1}(0)$ is calied a nodal domain of u.

Proposition 2.1. *Let u be a nontrivial solution of the equation $L_G u = 0$ and N the number of nodal domains of u. Then*

$$\mathrm{Ind}(L_G, \Sigma) \geq N - 1 . \tag{2.1}$$

Proof. Let $\mathrm{Ind}(L_G, \Sigma) = i$. Then u is an $(i+1)$-th eigenfucntion (that is, an eigenfunction belonging to the $(i+1)$-th eigenvalue) of L_G. The nodal domain theorem then says that the number of nodal domains of u is not greater than $i+1$, that is, $N \leq i+1$, or $i \geq N-1$, getting the desired result.

The next lemma assures the existence of nontrivial solutions of the equation $L_G u = 0$.

Lemma 2.2. *For a fixed vector $a \in \boldsymbol{R}^3$, the function $u = a \cdot G$ satisfies the equation $L_G u = 0$. Moreover, if $a \neq 0$ then $u \neq 0$.*

Proof. The first assertion follows from the fact that the holomorphic map G satisfies the equation $\Delta G + |dG|^2 G = 0$. To show the second assertion assume $a \neq 0$ and $a \cdot G = 0$. Then the image of G is contained in a great circle of S^2. By the holomorphicity, G is a constant map, a contradiction.

REMARK. The lemma shows that $V = \{a \cdot G \mid a \in \boldsymbol{R}^3\}$ is a three-dimensional subspace of the kernel of L_G.

Let $u = a \cdot G$ for $a \in \boldsymbol{R}^3 - \{0\}$. We note that the nodal line $u^{-1}(0)$ is nothing but the inverse image by G of the great circle S^1, which is determined as the intersection of the plane $a \cdot X = 0$ and S^2, where $X = (X^1, X^2, X^3)$ is the standard coordinates on $\boldsymbol{R}^3$.

We now study the inverse image by G of a great circle of S^2. Let S^1 be a great circle of S^2. We denote by $\{q_1, \cdots, q_s\}$ the set of all the critical values of G contained in S^1, which we assume to be nonempty. Let $G^{-1}(q_i) = \{p_1^{(i)}, \cdots, p_{t_i}^{(i)}\}$ for $i = 1, \cdots, s$. We denote by $b_j^{(i)}$ the branching order of G at $p_j^{(i)}$. Set

$t=\sum_{i=1}^{s} t_i$ and $b=\sum_{i,j} b_j^{(i)}$. In the following two lemmas and their proofs we shall use some terminologies from graph theory, for which we refer the reader to [1].

Lemma 2.3. *Under the above situation, $G^{-1}(S^1)$ is an embedded pseudograph on Σ, consisting of t vertices $\{p_j^{(i)}\}$ and $b+t$ edges.*

Proof. We may assume $q_1, \cdots, q_s$ lie on S^1 in this order. By the definition of branching order, G can be expressed relative to local coordinates z (resp. w) around $p_j^{(i)}$ (resp. q_i) as

$$w = G(z) = z^{b_j^{(i)}+1}.$$

Hence there exist precisely $b_j^{(i)}+1$ lifts of the arc $q_i q_{i+1}$, starting from $p_j^{(i)}$ (here and in the following we interpret as $q_{s+1}=q_1$, etc.). The terminal point of each lift is among $p_1^{(i+1)}, \cdots, p_{t_{i+1}}^{(i+1)}$. We consider these lifts of the arcs $q_i q_{i+1}, i=1, \cdots, s$, as edges. Since G is a local homeomrophism away from the branch points, each edge has no self-intersections and any two edges do not intersect at their interiors. Thus, $G^{-1}(S^1)$ is an embedded pseudograph on Σ. It is easy to verify that the pseudograph so obtained has the required number of edges.

For a great circle S^1 of S^2, we denote by $N(\Sigma, G, S^1)$ the number of connected components of $\Sigma-G^{-1}(S^1)$. To estimate $N(\Sigma, G, S^1)$, we need the following topological lemma.

Lemma 2.4. *Let Γ be an embedded pseudograph on the compact orientable surface S of genus g. Suppose that Γ has v vertices and e edges, and $S-\Gamma$ has N components. Then*

$$v-e+N\geq 2-2g. \tag{2.2}$$

Proof. The proof is by induction on g and is a slight modification of that of Theorem 4.20 in [1]. For $g=0$, by the Euler's formula for an embedded pseudograph on the sphere, we have $v-e+N=1+k$, where k is the number of connected components of Γ. Since $k\geq 1$, (2.2) holds for $g=0$.

We now let $g>0$. Then the surface S has g handles. Draw a curve C around a handle of S so that C contains no vertices of Γ. We must consider the following two cases:

Case (i). C intersects with no edges of Γ.
We cut the handle along C and cap the two resulting holes. Then we obtain a new embedded pseudograph Γ_1 ($=\Gamma$ as abstract pseudographs) on the compact surface S_1 of genus $g-1$. Suppose that $S_1-\Gamma_1$ has N_1 components. Then it is easy to see that $N_1\leq N+1$. By the inductive hypothesis, we have $v-e+N_1\geq 2-2(g-1)$. Hence $v-e+N\geq 3-2g$ and (2.2) holds in this case.

Case (ii). C intersects with edges of Γ.

By perturbing C if necessary, we may assume that the total number of intersections of C with edges of Γ is finite, say k. Suppose that C intersects with m edges of Γ. At each of the k intersections of C with the edges of Γ we add a new vertex. And each subset of C lying between consecutive new vertices is identified as a new edge. Moreover, m edges of Γ which intersect with C is subdivided into $m+k$ new edges. Let the new embedded pseudograph so formed be denoted by Γ_1. Suppose that Γ_1 has v_1 vertices and e_1 edges, and $S-\Gamma_1$ has N_1 components. Then $v_1=v+k$ and $e_1=e+2k$. Each portion of C that became an edge of Γ_1 is in a component of $S-\Gamma$. It is easy to see that the addition of such an edge divides that component into at most two components. Since there are k such edges, we have $N_1 \leq N+k$. We now cut the handle along C and cap the two resulting holes. Then we obtain a new embedded pseudograph Γ_2 on the compact surface S_2 of genus $g-1$. Suppose that Γ_2 has v_2 vertices and e_2 edges, and $S_2-\Gamma_2$ has N_2 components. Since the vertices and edges resulting from the curve C have been divided into two copies, we have $v_2=v_1+k=v+2k$ and $e_2=e_1+k=e+3k$. Moreover, $N_2=N_1+2 \leq N+k+2$. By the inductive hypothesis, we have $v_2-e_2+N_2 \geq 2-2(g-1)$. Hence $(v+2k)-(e+3k)+(N+k+2) \geq 2-2(g-1)$. Thus, $v-e+N \geq 2-2g$, getting the desired result.

In the next proposition we adopt the following notation for simplicity. For a subset $A \subset \Sigma$, we set

$$b(G, A) = \sum_{p \in A} b(G, p)\,,$$

where $b(G, p)$ is the branching order of G at p.

Proposition 2.5. *Let S^1 be a great circle of S^2. Then*

$$N(\Sigma, G, S^1) \geq b(G, G^{-1}(S^1))+2-2g\,, \tag{2.3}$$

where g is the genus of Σ.

Proof. Consider $G^{-1}(S^1)$ as an embedded pseudograph on Σ as in Lemma 2.3. (2.3) follows by applying Lemma 2.4.

Theorem 2.6. *Let Σ be a compact Riemann surface of genus g and $G\colon \Sigma \to S^2$ a nonconstant holomorphic map. Then the inequality*

$$\mathrm{Ind}\,(L_G, \Sigma) \geq b(G, G^{-1}(S^1))+1-2g \tag{2.4}$$

holds for any great circle S^1 of S^2.

Proof. Let a be a nonzero vector in $\boldsymbol{R}^3$ orthogonal to the plane spanned by S^1 and set $u=a \cdot G$. By Lemma 2.2, u is a nontrivial solution of the equation $L_G u=0$. The number of nodal domains of u is nothing but $N(\Sigma, G, S^1)$.

(2.4) now follows by combining (2.1) and (2.3).

3. Lower bounds for the index of minimal surfaces in $\boldsymbol{R}^3$ and examples

The following theorem is a rephrasement of Theorem 2.6 (see the last paragraph in § 1, in particular, (1.1)).

Theorem 3.1. *Let M be a complete oriented nonplanar minimal surface in $\boldsymbol{R}^3$ of genus g with finite total curvature and $\tilde{G}\colon \tilde{M}\to S^2$ the extended Gauss map, where $\tilde{M}$ is the compactified surface. Then the inequality*

$$\operatorname{Ind}(M)\geq b(\tilde{G}, \tilde{G}^{-1}(S^1))+1-2g$$

holds for any great circle S^1 of S^2.

Corollary 3.2. *Let M and $\tilde{G}$ be as in Theorem* 3.1. *Suppose that all the critical values of $\tilde{G}$ are contained in some great circle of S^2. Then*

$$\operatorname{Ind}(M)\geq\frac{1}{2\pi}\int_M(-K)dA-1\,,$$

where K is the Gauss curvature of M.

Proof. By the Riemann-Hurwitz formula, the total branching order of $\tilde{G}$ is $2(n-1+g)$, where n is the degree of $\tilde{G}$. Apply Theorem 3.1 and substitute $n=(1/4\pi)\int_M(-K)dA$.

Corollary 3.3. *Let M be a complete oriented minimal surface in $\boldsymbol{R}^3$ of genus zero. Suppose that M is not one of the plane, the Enneper's surface and the catenoid. Then*

$$\operatorname{Ind}(M)\geq 3\,.$$

Proof. We may assume that the total curvature of M is finite. By the assumption the total curvature of M is not more than -8π, or equivalently the degree of the extended Gauss map, say $\tilde{G}$, is not less than two. Then it is easy to verify, using the Riemann-Hurwitz formula, that $\tilde{G}$ has at least two critical values. Take any two of them and let S^1 be the great circle of S^2 passing through these two values. Then $\tilde{G}^{-1}(S^1)$ include at least two branch points of $\tilde{G}$ and thus $b(\tilde{G}, \tilde{G}^{-1}(S^1))\geq 2$. Apply Theorem 3.1.

REMARK. Lopez and Ros [12] have proved that the Enneper's surface and the catenoid are the only complete oriented minimal surfaces in $\boldsymbol{R}^3$ with index one. The above corollary improves their result when the genus of the surface is zero.

EXAMPLE 3.4 (k-end catenoid). Let M_k be the k-end catenoid, $k\geq 2$, which was discovered by Jorge and Meeks [11]. M_k is a complete oriented minimal surface of genus zero with total curvature $-4\pi(k-1)$ and k embedded ends. Let Π: $S^2\to \boldsymbol{C}\cup\{\infty\}$ be the stereographic projection from the north pole. Then the extended Gauss map $\tilde{G}$: $\tilde{M}_k=\boldsymbol{C}\cup\{\infty\}\to S^2$ is given by

$$\Pi\cdot\tilde{G}(z)=z^{k-1}\,.$$

The critical values of $\tilde{G}$ are precisely $(0, 0, \pm 1)$ (i.e., the north and south poles), which are clearly contained in a great circle of S^2. Applying Corollary 3.2, we obtain

$$\operatorname{Ind}(M_k)\geq 2k-3\,.$$

We note that the inequality (2.3) becomes equation in this case.

Actually we can show that

$$\operatorname{Ind}(M_k)=2k-3$$

by explicitly solving the eigenvalue problem associated to the Jacobi operator of M_k (see § 4).

EXAMPLE 3.5 (Costa's surface). Let M be the Costa's surface [3]. M is a complete oriented minimal surface in $\boldsymbol{R}^3$ of genus one with total curvature -12π and three ends. It was proved by Hoffman and Meeks [10] that M was embedded. Let L be the square lattice in $\boldsymbol{C}$ generated by 1 and i and P the Weierstrass $\mathcal{P}$-function for L. Then the extended Gauss map $\tilde{G}$: $\tilde{M}=\boldsymbol{C}/L\to S^2$ is given by

$$\Pi\cdot\tilde{G}([z])=\frac{a}{P'(z)}\,,$$

where $[z]$ is the point in $\tilde{M}$ corresponding to $z\in\boldsymbol{C}$ and $a=2\sqrt{2\pi}P(1/2)$. We note that a is a positive real constant. The branching of P' is shown in the following table, where z_j $(j=1, 2, 3, 4)$ are as in Figure 1 and b is some positive real constant.

Table 1.

branch point	branching order	value
0	2	∞
z_1	1	b
z_2	1	$-b$
z_3	1	ib
z_4	1	$-ib$

Let S^1 be the great circle of S^2, corresponding to the real axis of the complex plane via Π. It passes through the critical values $\Pi^{-1}(0)$, $\Pi^{-1}(a/b)$ and $\Pi^{-1}(-a/b)$ of $\tilde{G}$. The corresponding branch points are $[0]$, $[z_1]$ and $[z_2]$ with branching order 2, 1 and 1 respectively. Applying Theorem 3.1, we obtain

$$\mathrm{Ind}(M) \geq 3 .$$

We note that the set $\tilde{G}^{-1}(S^1)=[(P')^{-1}$ (real axis $\cup \{\infty\}$)] is actually as in Figure 2 (see [5, p. 38]) and the inequality (2.3) becomes equation in this case also.

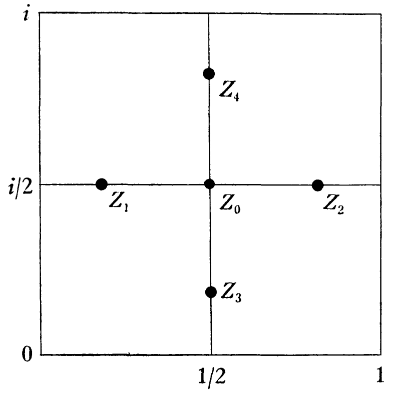

Fig. 1. The segments $\overline{z_0 z_j}$ $(j=1, 2, 3, 4)$ have the same length.

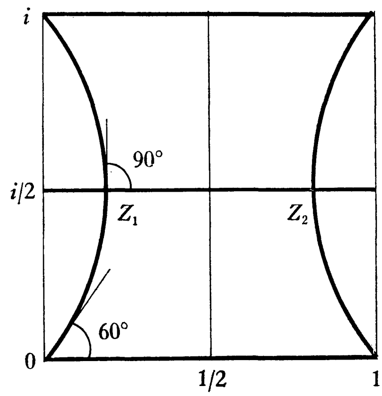

Fig. 2. The loci of real values of P' are shown as the thick lines.

4. The eigenvalue problem associated to the Jacobi operator of the k-end catenoid

Let G: $\Sigma_0=\boldsymbol{C} \cup \{\infty\} \rightarrow S^2$ be the holomorphic map of degree n defined by $\Pi \cdot G(z)=z^n$, where n is a positive integer. Then

$$G(z)=\left(\frac{2\,\mathrm{Re}(z^n)}{|z|^{2n}+1}, \frac{2\,\mathrm{Im}(z^n)}{|z|^{2n}+1}, \frac{|z|^{2n}-1}{|z|^{2n}+1}\right).$$

We fix an arbitrary conformal metric $ds^2=\mu|dz|^2$ on Σ_0 and consider the eigenvalue problem

$$-\Delta u=\lambda|dG|^2 u \qquad \text{on } \Sigma_0 . \tag{4.1}$$

Let (r, θ) be the polar coordinates on $\boldsymbol{C}$. Then the Laplace-Beltrami operator is expressed as

$$\Delta=\frac{1}{\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right),$$

and by a direct computation, we have

$$|dG|^2 = \frac{1}{\mu} \frac{8n^2 r^{2n-2}}{(r^{2n}+1)^2}.$$

Hence in these coordinates (4.1) is equivalent to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda \frac{8n^2 r^{2n-2}}{(r^{2n}+1^2)} u = 0 . \tag{4.2}$$

For real number α and nonnegative integer i, we define

$$(\alpha)_i = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+i-1) & \text{if } i \geq 1, \\ 1 & \text{if } i = 0. \end{cases}$$

The hypergeometric function $F(a, b; c; x)$ is a real analytic function of x, $|x|<1$, defined by

$$F(a, b; c; x) = \textstyle\sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{i!(c)_i} x^i , \qquad |x|<1 ,$$

where c is not a nonpositive integer. $F(a, b; c; x)$ satisfies the hypergeometric differential equation

$$x(1-x)\frac{d^2 y}{dx^2} + \{c-(a+b+1)x\}\frac{dy}{dx} - aby = 0 .$$

For nonnegative integers p and q, we set

$$\varphi_{p,q}(t) = (1-t^2)^{q/2n} F\left(p+2\frac{q}{n}+1,\ -p;\ \frac{q}{n}+1; \frac{1}{2}(1-t)\right), \qquad -1<t<1 ,$$

and

$$v_{p,q}(r) = \varphi_{p,q}\left(\frac{r^{2n}-1}{r^{2n}+1}\right), \qquad 0<r<\infty .$$

We note that $F\left(p+2\frac{q}{n}+1,\ -p;\ \frac{q}{n}+1; \frac{1}{2}(1-t)\right)$ is a polynomial of t of degree p. By a direct computation, we obtain

Lemma 4.1. *$\varphi_{p,q}(t)$ and $v_{p,q}(r)$ satisfy the ordinary differential equations*

$$(1-t^2)\frac{\partial^2 \varphi}{\partial t^2} - 2t\frac{\partial \varphi}{\partial t} + \left(2\lambda - \left(\frac{q}{n}\right)^2 \frac{1}{1-t^2}\right)\varphi = 0, \tag{4.3}$$

and

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \left(\lambda \frac{8n^2 r^{2n-2}}{(r^{2n}+1)^2} - \frac{q^2}{r^2}\right) v = 0 , \tag{4.4}$$

respectively, where $\lambda = \frac{1}{2}\left(p+\frac{q}{n}\right)\left(p+\frac{q}{n}+1\right)$.

Lemma 4.2. *$v_{p,q}(r)\cos q\theta$ and $v_{p,q}(r)\sin q\theta$ satisfy* (4.1) *with*

$$\lambda=\frac{1}{2}\left(p+\frac{q}{n}\right)\left(p+\frac{q}{n}+1\right).$$

Proof. Let u be one of the two functions above. By (4.4),

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\lambda\frac{8n^2r^{2n-2}}{(r^{2n}+1)^2}u=\frac{q^2}{r^2}u=-\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

Therefore u satisfies (4.2) (or (4.1)) in $\boldsymbol{C}-\{0\}$ with $\lambda=\frac{1}{2}\left(p+\frac{q}{n}\right)\left(p+\frac{q}{n}+1\right)$. It is easy to see that u is smooth at $z=0$ and ∞. Hence u satisfies (4.1) on the whole Σ_0.

Lemma 4.3. *$\{v_{p,q}(r)\}_{p=0,1,2,\cdots}$ forms a complete orthogonal system in $L^2\left((0,\infty),\frac{8n^2r^{2n-1}}{(r^{2n}+1)^2}dr\right)$.*

Proof. Setting $t=\frac{r^{2n}-1}{r^{2n}+1}$, we have $dt=\frac{4n^2r^{2n-1}}{(r^{2n}+1)^2}dr$. Hence it suffices to prove that $\{\varphi_{p,q}(t)\}_{p=0,1,2,\cdots}$ forms a complete orthogonal system in $L^2((-1,1),dt)$. The orthogonality follows easily from (4.3). The vertification of the completeness is also standard, so we omit it.

The next lemma follows easily from Lemma 4.3.

Lemma 4.4. *$\{v_{p,q}(r)\cos q\theta,\ v_{p,q}(r)\sin q\theta\}_{p,q=0,1,2,\cdots}$ forms a complete orthogonal system in $L^2(\Sigma_0,|dG|^2dA)$, where dA is the area element of the metric ds^2.*

In summary we have proved the following proposition.

Proposition 4.5. *The eigenvalues of* (4.1) *are exhausted by*

$$\lambda_i=\frac{i}{2n}\left(\frac{i}{n}+1\right),\qquad i=0,1,2,\cdots.$$

The eigenspace belonging to λ_i is spanned by

$$\{v_{p,q}(r)\cos q\theta,\ v_{p,q}(r)\sin q\theta\}_{pn+q=i}.$$

In particular, the multiplicity of λ_i is given by

$$m_i=\begin{cases}2p+2 & \text{if } i=pn+q,\ q\neq 0,\\ 2p+1 & \text{if } i=pn.\end{cases}$$

Theorem 4.6. *Let Σ_0 and G be as in the beginning of this section. Then*

$$\operatorname{Ind}(L_G,\Sigma_0)=2n-1.$$

Proof. We denote by Q the quadratic form associated to L_G (see § 1) and by R the Rayleigh quotient associated to the problem (4.1). Thus for a function $u \neq 0$

$$R(u) = \int_{\Sigma_0} |du|^2 dA \Big/ \int_{\Sigma_0} u^2 |dG|^2 dA .$$

We note that $Q(u,u)<0$ if and only if $R(u)<1$ for any $u \neq 0$. Therefore, by a variational characterization of the eigenvalues of (4.1), $\mathrm{Ind}(L_G, \Sigma_0)$ is equal to the number of the eigenvalues (counted with multiplicities) of (4.1) which is less than one. Hence $\mathrm{Ind}(L_G, \Sigma_0) = \sum_{i=1}^{n-1} m_i = 2n-1$, getting the desired result.

REMARK. It will be of some interest to observe that the dimension of the kernel of L_G is three for all n.

Corollary 4.7. *Let M_k be the k-end catenoid* (*see Example* 3.4). *Then*

$$\mathrm{Ind}(M_k) = 2k-3 .$$

After the completion of this paper we learned that Choe [17] obtained independently some of the results in this paper.

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan