

LINEAR PROGRAMMING RELATED TO HOMOGENEOUS PROGRAMMING

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(Received April 13, 1989)

1. Introduction with problem setting

Homogeneous programming problems (=HP) were first studied by Eisenberg [1]. His duality theorem for HP has been generalized by Schechter [8], Fujimoto [3], Gwinner [4,5] and the author [7]. The result in [5] seems to be most general among these results.

In the present paper, we introduce linear programming problems related to HP by means of "ray" and the axiom of choice. By our method, we obtain a new sufficient condition for duality in HP.

More precisely, let X and Y be convex cones with vertices at the origins in real linear spaces E_X and E_Y respectively. For simplicity, we assume that X , Y are pointed and hence X , Y contain the origins of E_X , E_Y respectively. Let f , g and h be real valued functions on X , Y and $X \times Y$ respectively. Assume that f is sublinear, that is, f is positively homogeneous and convex on Y , g is superlinear, that is, $-g$ is sublinear, $h(x, \cdot)$ is sublinear on Y and $h(\cdot, y)$ is superlinear on X for each $x \in X$ and $y \in Y$.

We call the quintuple $\{X, Y, h, f, g\}$ the primal homogeneous programming (=PHP). The value of PHP is defined by

$$(PHP) \quad M = \inf \{f(x); x \in V\},$$

where V is the set of feasible solutions of PHP, i.e.,

$$V = \{x \in X; h(x, y) \geq g(y) \text{ for all } y \in Y\}.$$

We call the quintuple $\{Y, X, -h, -g, -f\}$ the dual homogeneous programming (=DHP). The value of DHP is defined by

$$(DHP) \quad M^* = \sup \{g(y); y \in W\},$$

where W is the set of all feasible solutions of DHP and given by

$$W = \{y \in Y; h(x, y) \leq f(x) \text{ for all } x \in X\}.$$

In this paper, we use the convention that the infimum and supremum on the

empty set are equal to ∞ and $-\infty$ respectively. Obviously, $M^* \leq M$. A result which assures the equality $M=M^*$ is called a duality theorem for HP.

To state Gwinner's result in [5], we introduce some notation. For any nonempty set S , denote by R^S the set of all real functions on S . We assume that R^S is assigned the canonical product topology unless otherwise stated. Let C be the key set in the duality theorem due to Gwinner defined by

$$C = \bigcup_{x \in X} \{u \in R^Y; u \geq f(x) - h(x, \cdot) \text{ on } Y\}.$$

Gwinner gave the following duality theorem in [5; §8]:

Theorem 1.1. *Assume that V and W are nonempty, or equivalently, M and M^* are finite. If the set C is closed, then $M=M^*$ holds and PHP has an optimal solution.*

Gwinner stated this theorem as an application of [5; Theorem 2.1] which is a result of Farkas type. He noted that the closedness of C follows from any one of conditions given in [1], [3], [4], [7] and [8].

In the next section, we shall introduce the set \tilde{X} of rays of a convex cone X and define two linear programming problems related to PHP and DHP.

The author wishes to thank Professor M. Yamasaki for many valuable suggestions.

2. Linear programming problems related to HP

We say that two elements x_1 and x_2 are equivalent and denote it by $x_1 \sim_x x_2$ if there exists a positive number t such that $x_1 = tx_2$. It is clear that this is an equivalence relation. Denote by \tilde{X} the set of all equivalence classes, i.e., $\tilde{X} = X / \sim_x$ (the quotient space) and call it the set of rays of X . For $x \in X$, denote by \tilde{x} the equivalence class containing x . Note that $\{0\}$ is an element of \tilde{X} , i.e., $\tilde{0} = \{0\}$. In this paper, we assume the axiom of choice. Namely, there exists a mapping r_x from \tilde{X} to X such that $r_x(\tilde{x}) \in \tilde{x} \subset X$. We fix such a mapping.

Similarly we define an equivalence relation \sim_y on Y , and the set \tilde{Y} of rays of Y and a mapping r_y from \tilde{Y} to Y .

Denote by $L(\tilde{X}, R)$ the set of all real functions on \tilde{X} such that $u(\tilde{0})=0$ and by $L_0(\tilde{X}, R)$ the set of all $u \in L(\tilde{X}, R)$ such that $u(\tilde{x}) \neq 0$ only for finitely many $\tilde{x} \in \tilde{X}$. It is clear that $L_0(\tilde{X}, R)$ and $L(\tilde{X}, R)$ are linear spaces which are in duality with respect to the bilinear form:

$$\langle u, v \rangle_x = \sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) v(\tilde{x}) \quad \text{for } u \in L_0(\tilde{X}, R) \text{ and } v \in L(\tilde{X}, R).$$

Similarly, $L(\tilde{Y}, R)$ and $L_0(\tilde{Y}, R)$ are linear spaces which are in duality with respect to the bilinear form:

$$\langle u, v \rangle_y = \sum_{\tilde{y} \in \tilde{Y}} u(\tilde{y}) v(\tilde{y}) \quad \text{for } u \in L(\tilde{Y}, R) \text{ and } v \in L_0(\tilde{Y}, R).$$

Let us put

$$\tilde{E}_X = L_0(\tilde{X}, R), \tilde{E}_X^* = L(\tilde{X}, R); \tilde{E}_Y = L(\tilde{Y}, R), \tilde{E}_Y^* = L_0(\tilde{Y}, R).$$

Let P_X and P_Y be convex cones with vertices at the origin in \tilde{E}_X and \tilde{E}_Y defined by

$$P_X = \{u \in \tilde{E}_X; u(\tilde{x}) \geq 0 \text{ for all } \tilde{x} \in \tilde{X}\},$$

$$P_Y = \{v \in \tilde{E}_Y; v(\tilde{y}) \geq 0 \text{ for all } \tilde{y} \in \tilde{Y}\}.$$

Related to the given functions f, g and h in PHP, we define elements $\tilde{f} \in \tilde{E}_X^*$, $\tilde{g} \in \tilde{E}_Y^*$ and a linear mapping A from \tilde{E}_X to \tilde{E}_Y by

$$\tilde{f}(\tilde{x}) = f(r_X(\tilde{x})) \text{ for } \tilde{x} \in \tilde{X},$$

$$\tilde{g}(\tilde{y}) = g(r_Y(\tilde{y})) \text{ for } \tilde{y} \in \tilde{Y},$$

$$Au(\tilde{y}) = \sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) h(r_X(\tilde{x}), r_Y(\tilde{y})) \text{ for } u \in \tilde{E}_X \text{ and } \tilde{y} \in \tilde{Y}.$$

Denote by $w(\tilde{E}_X, \tilde{E}_X^*)$ the weak topology which is compatible with the duality. Then P_X and P_Y are $w(\tilde{E}_X, \tilde{E}_X^*)$ - and $w(\tilde{E}_Y, \tilde{E}_Y^*)$ -closed respectively. Furthermore A is $w(\tilde{E}_X, \tilde{E}_X^*)$ - $w(\tilde{E}_Y, \tilde{E}_Y^*)$ continuous. Thus the quintuple $\{A, P_X, P_Y, \tilde{f}, \tilde{g}\}$ is a linear programming problem in Kretschmer's sense in [6]. We call this the linearized homogeneous programming (=LHP). The value of LHP is given by

$$(LHP) \quad M_L = \inf \{\langle u, \tilde{f} \rangle_X; u \in S\},$$

where S is the set of all feasible solutions of LHP, i.e.,

$$S = \{u \in P_X; Au - \tilde{g} \in P_Y\}.$$

To obtain a dual problem for LHP along the theory due to [6], we calculate the dual cones P_X^\dagger and P_Y^\dagger of P_X and P_Y respectively and the adjoint linear mapping A^* of A . We have

$$P_X^\dagger = \{u^* \in \tilde{E}_X^*; u^*(\tilde{x}) \geq 0 \text{ for all } \tilde{x} \in \tilde{X}\},$$

$$P_Y^\dagger = \{v^* \in \tilde{E}_Y^*; v^*(\tilde{y}) \geq 0 \text{ for all } \tilde{y} \in \tilde{Y}\},$$

$$A^* v^*(\tilde{x}) = \sum_{\tilde{y} \in \tilde{Y}} v^*(\tilde{y}) h(r_X(\tilde{x}), r_Y(\tilde{y})) \text{ for } \tilde{x} \in \tilde{X} \text{ and } v^* \in \tilde{E}_Y^*.$$

The quintuple $\{A^*, P_Y^\dagger, -P_X^\dagger, -\tilde{g}, \tilde{f}\}$ is the dual problem of LHP. We call this the dual linearized homogeneous programming problem (=DLHP). The value of DLHP is given by

$$(DLHP) \quad M_L^* = \sup \{\langle \tilde{g}, v^* \rangle_Y; v^* \in S^*\},$$

where $S^* = \{v^* \in P_Y^\dagger; \tilde{f} - A^* v^* \in P_X^\dagger\}$.

To apply Kretschmer's duality theorem in this case, we define a key set G

in $\tilde{E}_Y \times R$ by Kretschmer [6; Theorem 3].

$$G = \{(Au - z, r + \langle u, \tilde{f} \rangle_X); u \in P_X, z \in P_Y, r \in R^+\},$$

where R^+ is the set of nonnegative real numbers.

Kretschmer [6; Theorem 3] yields

Theorem 2.1. *Assume that S and S^* are nonempty. If the set G is $w(\tilde{E}_Y \times R, \tilde{E}_Y^* \times R)$ -closed, then $M_L = M_L^*$ and LHP has an optimal solution.*

3. Relation between LHP and PHP

In order to study the relation between LHP (resp. DLHP) and PHP (resp. DHP), we prepare

DEFINITION 3.1. For $a \in X$, there exists a positive number s such that $a = sr_X(\tilde{a})$. We define $u_a \in P_X$ by setting $u_a(\tilde{a}) = s$ if $a \neq 0$, and $u_a(\tilde{x}) = 0$ if $\tilde{a} \neq \tilde{x}$ or $a = 0$.

Lemma 3.1. *Let $v \in \tilde{E}_Y$ and $x \in X$ satisfy $h(x, r_Y(\tilde{y})) \geq v(\tilde{y})$ for all $\tilde{y} \in \tilde{Y}$. Then $Au_x - v \in P_Y$ and $\langle u_x, \tilde{f} \rangle = f(x)$.*

Proof. Let $s > 0$ satisfy $x = sr_X(\tilde{x})$. By definition, we have

$$\begin{aligned} Au_x(\tilde{y}) - v(\tilde{y}) &= sh(r_X(\tilde{x}), r_Y(\tilde{y})) - v(\tilde{y}) \\ &= h(x, r_Y(\tilde{y})) - v(\tilde{y}) \geq 0 \end{aligned}$$

for all $\tilde{y} \in \tilde{Y}$ so that $Au_x - v \in P_Y$. Similarly we see that

$$\langle u_x, \tilde{f} \rangle_X = sf(r_X(\tilde{x})) = f(x).$$

Taking \tilde{g} as v in Lemma 3.1 we obtain

Corollary. *Let V and S be the sets of feasible solutions of PHP and LHP. Then $\{u_x; x \in V\} \subset S$ and $M_L \leq M$.*

Lemma 3.2. *Let $v \in \tilde{E}_Y$ and $u \in P_X$ satisfy $Au - v \in P_Y$ and set $a = \sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) r_X(\tilde{x})$. Then $a \in X$, $\langle u, \tilde{f} \rangle_X \geq f(a)$ and $h(a, r_Y(\tilde{y})) \geq v(\tilde{y})$ for all $\tilde{y} \in \tilde{Y}$.*

Proof. Since $\{\tilde{x} \in \tilde{X}; u(\tilde{x}) \neq 0\}$ is a finite set, $u(\tilde{x}) \geq 0$ and $r_X(\tilde{x}) \in X$ for all $\tilde{x} \in \tilde{X}$, we see that $a \in X$. Furthermore since f is sublinear, we have

$$\begin{aligned} f(a) &= f(\sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) r_X(\tilde{x})) \leq \sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) f(r_X(\tilde{x})) \\ &= \sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) \tilde{f}(\tilde{x}) = \langle u, \tilde{f} \rangle_X. \end{aligned}$$

The superlinearity of $h(\cdot, y)$ yields

$$\begin{aligned} h(a, r_Y(\tilde{y})) &= h(\sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) r_X(\tilde{x}), r_Y(\tilde{y})) \\ &\geq \sum_{\tilde{x} \in \tilde{X}} u(\tilde{x}) h(r_X(\tilde{x}), r_Y(\tilde{y})) = Au(\tilde{y}) \geq v(\tilde{y}) \end{aligned}$$

for all $\mathfrak{y} \in \tilde{Y}$.

Corollary. *Let V and S be the same as in Corollary of Lemma 3.1. Then $\{\Sigma_{\tilde{x} \in \tilde{X}} u(\tilde{x}) r_X(\tilde{x}); u \in S\} \subset V$ and $M \leq M_L$.*

By Corollaries of Lemmas 3.1 and 3.2, we obtain

Theorem 3.1. *PHP and LHP have the same value, i.e., $M = M_L$. If one of PHP and LHP has an optimal solution, then the other also has an optimal solution.*

Proof. By the above observation, we see that if $x \in V$ is an optimal solution of PHP, then u_x is an optimal solution of LHP and that if $u \in S$ is an optimal solution of LHP, then $a = \Sigma_{\tilde{x} \in \tilde{X}} u(\tilde{x}) r_X(\tilde{x})$ is an optimal solution of PHP.

Similarly we can prove

Theorem 3.2. *DHP and DHLP have the same value, i.e., $M^* = M_L^*$. If one of DHP and DLHP has an optimal solution, then the other also has an optimal solution.*

We recall the definition of the key set G in Section 2 and express it in the following form:

Lemma 3.3. *For each $x \in X$, put*

$$N_x = \{(v, q) \in \tilde{E}_Y \times R; q \geq f(x) \text{ and } h(x, r_Y(\mathfrak{y})) \geq v(\mathfrak{y}) \text{ for all } \mathfrak{y} \in \tilde{Y}\}.$$

Then $G = \cup_{x \in X} N_x$.

Proof. If $(v, q) \in G$, then there exist $u \in P_X$, $z \in P_Y$ and $r \in R^+$ such that $v = Au - z$ and $q = r + \langle u, \tilde{f} \rangle_X$. We set $a = \Sigma_{\tilde{x} \in \tilde{X}} u(\tilde{x}) r_X(\tilde{x})$. We see by Lemma 3.2 that $(v, q) \in N_a$. On the other hand, let $(v, q) \in N_x$ for some $x \in X$. Then $q \geq f(x)$ and $h(x, r_Y(\mathfrak{y})) \geq v(\mathfrak{y})$ for all $\mathfrak{y} \in \tilde{Y}$. We see by Lemma 3.1 that $q \geq f(x) = \langle u_x, \tilde{f} \rangle_X$ and $Au_x(\mathfrak{y}) \geq v(\mathfrak{y})$ for all $\mathfrak{y} \in \tilde{Y}$, so that $Au_x - v \in P_Y$ and $q - \langle u_x, \tilde{f} \rangle_X \geq 0$. Taking $z = Au_x - v$ and $r = q - \langle u_x, \tilde{f} \rangle_X$, we obtain that $(v, q) = (Au_x - z, r + \langle u_x, \tilde{f} \rangle_X) \in G$.

Theorems 2.1 and 3.2 yield the following duality theorem for HP:

Theorem 3.3. *Assume that V and W are nonempty. If the set G is $w(\tilde{E}_Y \times R, \tilde{E}_Y^* \times R)$ -closed, then $M = M^*$ holds and PHP has an optimal solution.*

4. Comparison of the closedness of C and G

Related to the key set C in Gwinner's theorem, let us put

$$C_x^- = \{u \in R^Y; u(y) \leq h(x, y) - f(x) \text{ for all } y \in Y\}$$

for $x \in X$ and

$$C^- = \bigcup_{x \in X} C_x^-.$$

Then $C^- = \{u \in R^Y; -u \in C\}$. Hence it is clear that C is closed if and only if C^- is closed. Furthermore let H_Y be the set of all positively homogeneous functions on Y and set

$$H_Y + R = \{u + r; u \in H_Y, r \in R\}.$$

Then $H_Y + R$ is a closed subspace of R^Y .

We shall prove

Theorem 4.1. $C \cap (H_Y + R)$ is closed in R^Y if and only if G is $w(\tilde{E}_Y \times R, \tilde{E}_Y^* \times R)$ -closed.

Proof. Assume that $C \cap (H_Y + R)$ is closed in R^Y . Let $\{(v_i, q_i)\}$ be a net in G which converges to $(v, q) \in \tilde{E}_Y \times R$ with respect to $w(\tilde{E}_Y \times R, \tilde{E}_Y^* \times R)$ -topology. By Lemma 3.3, there exists $x_i \in X$ such that $q_i \geq f(x_i)$ and $h(x_i, r_Y(\mathfrak{Y})) \geq v_i(\mathfrak{Y})$ for all $\mathfrak{Y} \in \tilde{Y}$. Define $v'_i, v' \in R^Y$ as follows: If y is a nonzero element in Y , then using $s > 0$ such that $y = sr_Y(\mathfrak{Y})$ we set

$$v'_i(y) = sv_i(\mathfrak{Y}) \quad \text{and} \quad v'(y) = sv(\mathfrak{Y}).$$

If $y=0$, then we set $v'_i(y) = v'(y) = 0$. We have

$$v'_i(y) = sv_i(\mathfrak{Y}) \leq sh(x_i, r_Y(\mathfrak{Y})) = h(x_i, sr_Y(\mathfrak{Y})) = h(x_i, y).$$

Put $u_i = v'_i - q_i$ and $u = v' - q$. Then $u_i \in C^- \cap (H_Y + R)$ and $\{u_i\}$ converges to u . Since $C \cap (H_Y + R)$ is closed, $C^- \cap (H_Y + R)$ is also closed, so that $u \in C^- \cap (H_Y + R)$. Thus there exists $x \in X$ such that $u \in C_x^-$, that is,

$$u(y) \leq h(x, y) - f(x) \quad \text{for all } y \in Y.$$

Since $v'(0) = h(x, 0) = 0$, we obtain $q \geq f(x)$. We prove that $v'(y) \leq h(x, y)$ for all $y \in Y$. Since $v', h(x, \cdot)$ are positively homogeneous and

$$u(ty) = v'(ty) - q \leq h(x, ty) - f(x)$$

for all $y \in Y$ and $t > 0$, we have $tv'(y) - q \leq th(x, y) - f(x)$ for all $y \in Y$ and $t > 0$. Dividing both sides by t and letting $t \rightarrow \infty$, we obtain $v'(y) \leq h(x, y)$ for all $y \in Y$. It follows that $v(\mathfrak{Y}) \leq h(x, r_Y(\mathfrak{Y}))$ for all $\mathfrak{Y} \in \tilde{Y}$. Namely, $(v, q) \in N_x \subset G$ by Lemma 3.3 and the closedness of G is proved.

Conversely assume that G is $w(\tilde{E}_Y \times R, \tilde{E}_Y^* \times R)$ -closed and let $\{u_i\}$ be a net in $C^- \cap (H_Y + R)$ which converges to $u \in R^Y$. Since $H_Y + R$ is closed, $u \in H_Y + R$. Thus to prove that $C^- \cap (H_Y + R)$ is closed, it suffices to show that $u \in C^-$.

Let $\{x_i\}$ be a net in X such that $u_i \in C_{x_i}^-$. We set $q_i = -u_i(0)$, $q = -u(0)$, $v_i(\mathfrak{Y}) = u_i(r_Y(\mathfrak{Y})) + q_i$ and $v(\mathfrak{Y}) = u(r_Y(\mathfrak{Y})) + q$ for all i and $\mathfrak{Y} \in \tilde{Y}$. The relation $u_i \in C_{x_i}^-$ yields

$$q_i \geq f(x_i)$$

and

$$u_i(ty) + q_i \leq h(x_i, ty) - f(x_i) + q_i$$

for all $y \in Y$ and $t > 0$. Since $u_i + q_i$ is a positively homogeneous function on Y , dividing the both sides of the latter inequality by t and letting $t \rightarrow \infty$, we obtain

$$u_i(y) + q_i \leq h(x_i, y)$$

for all $y \in Y$. In particular $v_i(\mathfrak{Y}) \leq h(x_i, r_Y(\mathfrak{Y}))$ for all $\mathfrak{Y} \in \tilde{Y}$. It follows that $(v_i, q_i) \in N_{x_i} \subset G$. Since $(v_i, q_i) \rightarrow (v, q)$ and G is closed, $(v, q) \in G$ and hence by Lemma 3.3 there exists $x \in X$ such that $(v, q) \in N_x$. Then $q \geq f(x)$ and $v(\mathfrak{Y}) \leq h(x, r_Y(\mathfrak{Y}))$ for all $\mathfrak{Y} \in \tilde{Y}$. The latter inequality implies that $u(y) + q \leq h(x, y)$ for all $y \in Y$. It follows that $u(y) \leq h(x, y) - q \leq h(x, y) - f(x)$ for all $y \in Y$. This means that $u \in C_x^- \subset C^-$. This completes the proof.

Corollary. *If C is closed in R^Y , then G is $w(\tilde{E}_Y \times R, \tilde{E}_Y^* \times R)$ -closed.*

It should be noted that the closedness of G does not imply that of C in general. This is shown by

EXAMPLE. Let E_X and E_Y be the Euclidean space R and $X = Y = [0, \infty)$. Define f and h by

$$f(x) = 0$$

for all $x \in X$,

$$h(x, y) = xy$$

for all $x, y \in [0, \infty)$. First we show that the set

$$C = \cup_{x \in X} \{u \in R^Y; u(y) \geq -xy \text{ for all } y \in [0, \infty)\}$$

is not closed. In fact, consider a sequence $\{u_n\}$ in R^Y defined by $u_n(y) = 0$ if $0 \leq y \leq 1/n$ and $u_n(y) = -1$ if $1/n < y < \infty$. Then $u_n \in C$ and $\{u_n\}$ converges to the function u defined by $u(0) = 0$ and $u(y) = -1$ if $0 < y < \infty$. Clearly $u \notin C$, and hence C is not closed. To prove the closedness of G , we note that $\tilde{X} = \tilde{Y} = \{\tilde{0}, \tilde{1}\}$ and \tilde{E}_X and \tilde{E}_Y can be identified with R . Let us take $r_X(\tilde{1}) = 1$ and $r_Y(\tilde{1}) = 1$. Then $\tilde{f} = 0$ on \tilde{X} , $Au(\tilde{0}) = 0$, $Au(\tilde{1}) = u(\tilde{1})$, $h(r_X(\tilde{1}), r_Y(\tilde{1})) = u(\tilde{1})$ for every $u \in \tilde{E}_X$. If $z \in P_Y$, then $z(\tilde{0}) = 0$ and $z(\tilde{1}) \geq 0$. Thus we have

$$G = \{(u(\tilde{1}) - z(\tilde{1}), r); u \in P_X, z \in P_Y, r \in R^+\} = R \times R^+$$

and hence G is closed.

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