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NOTE ON ALMOST M-INJECTIVES

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Recently, in [2], Harada and Tozaki defined 'almost M-projectives' which are generalized from the concept 'M-projectives' due to Azumaya. In this paper we shall define a dual concept 'almost M-injectives'. In the forthcoming paper [1], we will show several results dual to Harada and Tozaki's ones above. The purpose of this paper is to generalize the following Azumaya's theorem concerning to M-injectives: N is M_1- and M_2-injective if and only if N is $M_1\oplus M_2$-injective for modules N, M_1 and M_2, to a case of 'almost M-injectives'. An easy example shows that the theorem can not be modified as the same form.

Throughout this paper, R is an associative ring with identity. Every module is a unitary right R-module. We always use i, i_k and i^k ($k=1, 2, \cdots$ or $*$) to denote the inclusion maps. For modules M and N with $N\subseteq M$, we denote by $N\underset{e}{\subset} M$ and by $N\langle\oplus M$ to mean that M is an essential extension of N and that N is a direct summand of M, respectively. For modules M, N and a homomorphism f: $M\to N$, $M(f)$ denotes $\{m+f(m)\,|\,m\in M\}$. For a module M, unif. dim (M) and $||M||$ denote its uinform dimension and composition length, respectively. If for each simple submodule S of M there is a direct summand M' of M such that $S\underset{e}{\subset} M'$, we say that M is extending for simple modules. For a sst T, $|T|$ denotes its cardinal number.

Our main result is the following.

Theorem. *Let U_k be a uniform module of finite composition length for $k=0, 1, 2, \cdots, n$. Then the following two conditions are equivalent*;

(1) *U_0 is almost $\sum_{k=1}^{n}\oplus U_k$-injective.*

(2) *U_0 is almost U_k-injective for every $k=1, 2, \cdots, n$ and if* Soc $(U_0)\approx$ Soc$(U_k)\approx$Soc(U_l) *(any k, $l\in\{1, 2, \cdots, n\}$, $k\neq l$) then (i) U_0 is U_k-and U_l-injective or (ii) $U_k\oplus U_l$ is extending for simple modules.*

DEFINITION. Let M and N be R-modules. We say that N is *almost M-injective* if at least one of the following conditions holds for each submodule L of M and each homomorphism f: $L\to N$:

(1) There exists a homomorphism $\bar{f}$: $M\to N$ such that $\bar{f}\cdot i=f$,

(2) There exists a non-zero direct summand M_1 of M and a homomorphism $\hat{f}: N \to M_1$ such that $\hat{f} \cdot f = \pi \cdot i$, where $\pi: M \to M_1$ is a projection of M onto M_1.

In this definition, for a given diagram:

$$(*) \qquad \begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\;i\;} & M \\ & & {\scriptstyle f}\downarrow & & \\ & & N & & \end{array}$$

we call that *the first* (*respectively, second*) *case occurs in the diagram* (*) if the condition (1) (respectively, (2)) holds in the diagram.

Lemma A. *Let U be a uniform module and X an indecomposable module. If U is almost X-injective and $||U|| \geq ||X||$, U is X-injective.*

Proof. Consider a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\;i\;} & X \\ & & {\scriptstyle f}\downarrow & & \\ & & U & & \end{array}$$

Assume that the second case occurs in this diagram. Let $\hat{f}: U \to X$ be a homomorphism such that $i = \hat{f} \cdot f$. (Note that X is indecomposable.) Then $\hat{f}$ is a monomorphism since U is a uniform module, and so $||U|| \leqq ||X||$. We have $||U|| = ||X||$ from the assumption $||U|| \geqq ||X||$. Therefore $\hat{f}$ is an isomorphism. Then $f = \hat{f}^{-1} \cdot i$.

Lemma B. *Let M and N be R-modules. Consider a diagram:*

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\;i\;} & M \\ & & {\scriptstyle f}\downarrow & & \\ & & N & & \end{array}$$

and put $K := \mathrm{Ker}(f)$. Then if the second case occurs in this diagram, there is a proper direct summand M' of M which contains K.

In particular, if $K \subset_e M$, then the first case occurs.

Proof. Since the second case happens, we have a direct decomposition $M = M' \oplus M''$ and $\hat{f}: N \to M'$ for which the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\ i\ } & M=M'\oplus M'' \\ & & f\downarrow & & \downarrow \pi' \text{ : projection} \\ & & N & \xrightarrow{\ \bar{f}\ } & M' \end{array}$$

is commutative. Then $\pi'(K)=\pi'\cdot i(K)=\bar{f}\cdot f(K)=0$, and $K\subseteq \mathrm{Ker}(\pi')=M''\langle\oplus M$. Since $M'\neq 0$, M'' is a proper direct summand.

Now we prepare for Lemma C below. Let N, M_1 and M_2 be modules, and put $M:=M_1\oplus M_2$. Consider a diagram:

$$\text{(1)} \qquad \begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\ i\ } & M \\ & & f\downarrow & & \\ & & N & & \end{array}$$

From this diagram we induce the following for $k=1, 2$:

$$\text{(2-}k\text{)} \qquad \begin{array}{ccccc} 0 & \longrightarrow & L_k & \xrightarrow{\ i_k\ } & M_k \\ & & f|_{L_k}\downarrow & & \\ & & N & & \end{array}$$

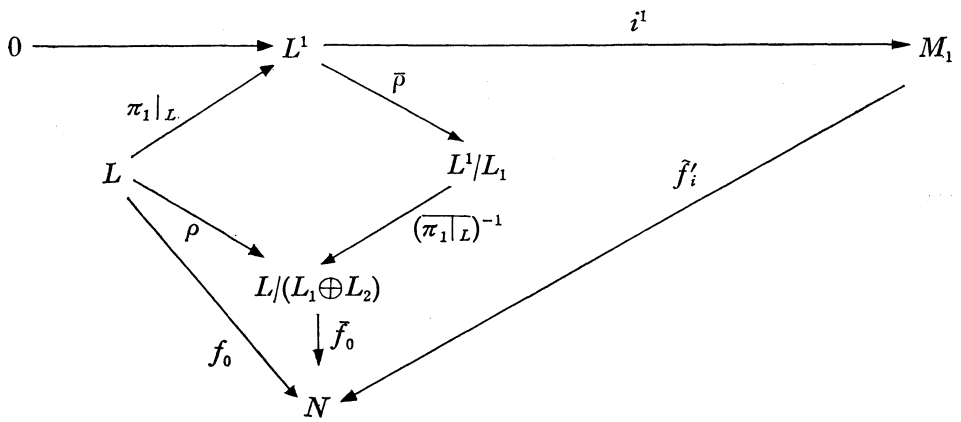
where $L_k:=L\cap M_k$. Moreover when the first case occurs in both diagrams (2–1) and (2–2) (let $\bar{f}_k\colon M_k\to N$ be homomorphisms such that $f|_{L_k}=\bar{f}_k\cdot i_k$ for $k=1, 2$), we shall consider the following for $k=1, 2$:

$$\text{(3-}k\text{)} \qquad \begin{array}{ccccc} 0 & \longrightarrow & L^k & \xrightarrow{\ i^k\ } & M_k \\ & & f'_k\downarrow & & \\ & & N & & \end{array}$$

where, letting $\pi_k\colon M(=M_1\oplus M_2)\to M_k$ be the projection, $L^k:=\pi_k(L)$ and the homomorphisms f'_k is defined as follows: Put $f_0:=f-(\sum_{k=1}^{2}\bar{f}_k\cdot\pi_k|_L)\colon L\to N$. Since $f_0(L_1\oplus L_2)=0$ (from the definition of $\bar{f}_k$), the canonical map $\bar{f}_0\colon L/(L_1\oplus L_2)\to N$ is induced. We let $f'_k\colon L^k\to N$ be the composite map: $L^k\xrightarrow{\text{natural epi.}}L^k/L_k\xrightarrow[\text{natural iso.}]{\approx}L/(L_1\oplus L_2)\xrightarrow{\ \bar{f}_0\ }N$.

Lemma C. *Assume that N be almost M_1- and M_2-injective. Consider a* *diagram* (1) *and induce the above diagrams. If the first case occurs in both diagrams* (2–1) *and* (2–2) *and does in either* (3–1) *or* (3–2), *then so does in the diagram* (1).

Proof. We say that the first case occurs in the diagram (3–1). Let $\tilde{f}'_1$: $M_1 \to N$ be a homomorphism such that $f'_1 = \tilde{f}'_1 \cdot i^1$. The diagram (3–1) induces the following commutative diagram:



where ρ is the canonical epimorphism.
Then, note that $\pi_k|_L = \pi_k \cdot i$,

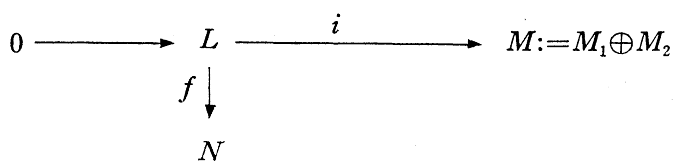
$$\begin{aligned} f &= f_0 + \left(\sum_{k=1}^{2} \tilde{f}_k \cdot \pi_k|_L\right) \\ &= \tilde{f}'_1 \cdot i^1 \cdot (\pi_1|_L) + \left(\sum_{k=1}^{2} \tilde{f}_k \cdot \pi_k|_L\right) \\ &= (\tilde{f}'_1 \cdot i^1 + \tilde{f}_1) \cdot (\pi_1|_L) + \tilde{f}_2 \cdot (\pi_2|_L) \\ &= \{(\tilde{f}'_1 \cdot i^1 + \tilde{f}_1) \cdot \pi_1 + \tilde{f}_2 \cdot \pi_2\} \cdot i\,. \end{aligned}$$

Put $\tilde{f} := (\tilde{f}'_1 + \tilde{f}_1) \cdot \pi_1 + \tilde{f}_2 \cdot \pi_2$. $\tilde{f}$ is a homomorphism from M to N satisfying $f = \tilde{f} \cdot i$. So the first case occurs in the diagram (1).

Corollary 1. [*Azumaya*] *Let N, M_1 and M_2 be modules. If N is M_1- and M_2-injective, then N is $M_1 \oplus M_2$-injective.*

Corollary 2. *Let N, M_1 and M_2 be modules, and let N be almost M_1- and M_2-injective. Consider a diagram:*

(*)



and put $K:=\mathrm{Ker}(f)$. *Then if* $K\underset{e}{\subset}M$, *the first case occurs in the diagram* $(*)$.

Proof. Induce the diagrams (2-k) ($k=1, 2$) from the diagram $(*)$. Since $K\underset{e}{\subset}M$ and $K\subseteq L\subseteq M$, $L\underset{e}{\subset}M$. And $\mathrm{Ker}(f|_{L_k})=K\cap L_k\underset{e}{\subset}M\cap L_k=L_k=M_k\cap L$ $\underset{e}{\subset}M_k\cap M=M_k$. Therefore Lemma B shows that the first case occurs in the diagrams (2-k). So the diagrams (3-k) for $k=1, 1$ are induced. Since $L_k\subseteq\mathrm{Ker}$ (f'_k) and $L_k\underset{e}{\subset}M_k$, $\mathrm{Ker}(f'_k)\underset{e}{\subset}M_k$. Therefore the first case also occurs in these diagrams. Thus a desired homomorphism exists in the diagram $(*)$ from Lemma C.

Proof of Theorem. (1)$\Rightarrow$(2): The first condition of (1) holds by [1], Lemma 9. We shall show the remainder condition. To show this, assume that $\mathrm{Soc}(U_0)\approx\mathrm{Soc}(U_1)\approx\mathrm{Soc}(U_2)$ and let U_0 be not U_1-injective. Let us find a direct decomposition $U_1\oplus U_2=V_1\oplus V_2$ such that $(\mathrm{Soc}(U_1))(g)\subseteq V_1$ and $V_2\neq 0$ for each isomorphism g: $\mathrm{Soc}(U_1)\to\mathrm{Soc}(U_2)$. Since V_1 is a uniform module, this means that $U_1\oplus U_2$ is extending for simple modules.

Take an isomorphism g': $\mathrm{Soc}(U_2)\to\mathrm{Soc}(U_0)$ and consider the following diagram:

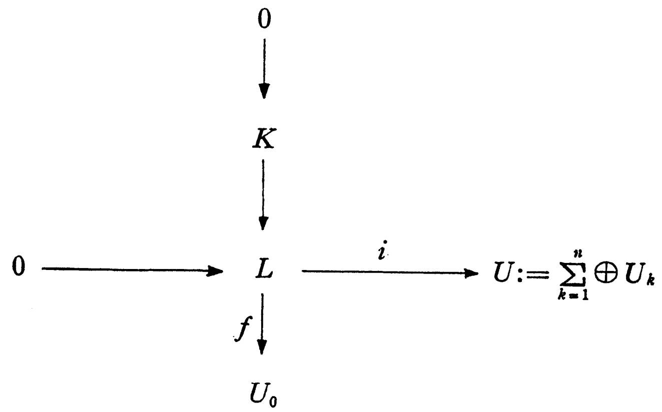
$$(※)\qquad \begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Soc}(U_1)\oplus\mathrm{Soc}(U_2) & \xrightarrow{\;i\;} & U_1\oplus U_2 \\ & & f\big\downarrow & & \\ & & U_0 & & \end{array}$$

where $f(s_1+s_2)=g'(s_2-g(s_1))$ for any s_k in $\mathrm{Soc}(U_k)$ ($k=1, 2$). Then note that $\mathrm{Ker}(f)=(\mathrm{Soc}(U_1))(g)$.

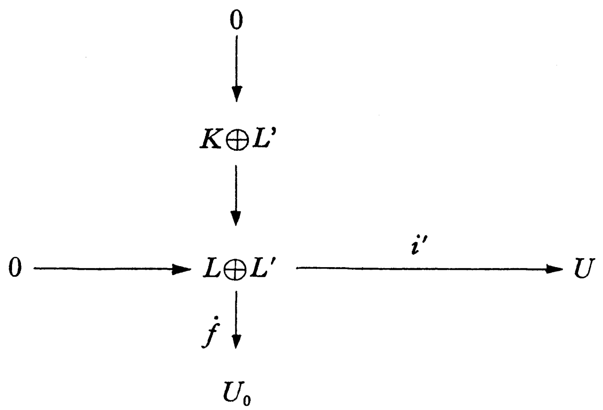
The assumption that U_0 is almost $\sum_{k=1}^{n}\oplus U_k$-injective induces that U_0 is almost $U_1\oplus U_2$-injective by [1], Lemma 9. If the first case occurs in this diagram, let $\bar{f}$: $U_1\oplus U_2\to U_0$ be a homomorphism such that $f=\bar{f}\cdot i$, then $\bar{f}|_{U_1}$: $U_1\to U_0$ is not a monomorphism since $||U_1||>||U_0||$ by Lemma A and the assumption that U_0 is not U_1-injective. Therefore $f(\mathrm{Soc}(U_1))=\bar{f}(\mathrm{Soc}(U_1))=0$. But, by the definition of f, we see $f(\mathrm{Soc}(U_1))\neq 0$. This is a contradiction. So the second case occurs in the diagram (※). Hence, by Lemma B, we have a direct decomposition $U_1\oplus U_2=V_1\oplus V_2$ such that $(\mathrm{Soc}(U_1))(g)\subseteq V_1$ and $V_2\neq 0$.

(2)$\Rightarrow$(1): We shall show this implication by induction on n. Take a diagram:

(※)



We may assume that $L\subset_e U$, since, otherwise, there is a submodule L' of U such that $L\oplus L'\subset_e U$. Then consider the following diagram:



where the homomorphism $\dot{f}$: $L\oplus L'\to U_0$ is defined as $\dot{f}(x+x')=f(x)$ for any $x\in L$ and $x'\in L'$. If the first case occurs in this diagram, let $\tilde{\dot{f}}$: $U\to U_0$ be a homomorphism such that $\dot{f}=\tilde{\dot{f}}\cdot i'$, then $\tilde{\dot{f}}\cdot i=\tilde{\dot{f}}\cdot(i'|_L)=\dot{f}|_L=f$. The first case also occurs in the original diagram (※). On the other hand, if the second case occurs in this diagram, let $0\neq U'\lessdot\oplus U$, p: $U\to U'$ be a projection and $\tilde{\dot{f}}$: $U_0\to U'$ be a homomorphism such that $p\cdot i'=\tilde{\dot{f}}\cdot\dot{f}$, then $p\cdot i=p\cdot(i'|_L)=\tilde{\dot{f}}\cdot(\dot{f}|_L)=\tilde{\dot{f}}\cdot f$. The second case also occurs in the diagram (※).

Now assume that the first case does not occurs in this diagram. And we will show that the second case occurs in it.

If $K\subset_e U$, the first case occurs in the diagram (※) by Corollary 2, a contradiction. Hence $K\not\subset_e U$. Then we may assume that $K\cap\operatorname{Soc}(U_1)=0$. Since U is a finite direct sum of uniform modules, we can take a maximal $|\{k\in\{2, 3, \cdots,$

$n\} \mid \mathrm{Soc}(U_k)\subseteq K\} \mid$ among $\mid \{k\in\{2, 3, \cdots, n\} \mid \mathrm{Soc}(U'_k)\subseteq K\} \mid$ related to the direct decomposition of U into uniform modules U'_k such that $K\cap \mathrm{Soc}(U'_1)=0$. Now we denote its direct docomposition by $\sum_{k=1}^{n}\oplus U_k$.

Since $K\cap \mathrm{Soc}(U_1)=0$, there is a homomorphism $g\colon K^*\to U_1$ with $K=K^*(g)$ where $U_*:=\sum_{k=2}^{n}\oplus U_k$, $\pi_*\colon U(=U_1\oplus U_*)\to U_*$ is the projection and $K^*:=\pi_*(K)$. Put $K_*:=K\cap U_*$. Then we have the following two cases:

$$\text{case A: } K_*\subset_e U_* .$$

$$\text{case B: } K_*\not\subset_e U_* .$$

Note that $\mathrm{Soc}(U_k)\subseteq K^*$ for any $k\in\{2, 3, \cdots, n\}$. Because, if $\mathrm{Soc}(U_k)\not\subseteq K^*$, then $f(\mathrm{Soc}(U_1)\oplus \mathrm{Soc}(U_k))$ is a direct sum of two simple submodules of U_0, a contradiction. Therefore, if $\mathrm{Soc}(U_k)\not\subseteq K_*$ for some $k\in\{2, 3, \cdots, n\}$, the socle of K^*/K_*, which is isomorphic to $\mathrm{Soc}(U_1)$ via g, is $(\mathrm{Soc}(U_k)\oplus K_*)/K_*(\approx \mathrm{Soc}(U_k))$. So $\mathrm{Soc}(U_1)\approx \mathrm{Soc}(U_k)$. On the other hand, f induces $\mathrm{Soc}(U_1)\approx \mathrm{Soc}(U_0)$ since $K\cap \mathrm{Soc}(U_1)=0$. Hence $\mathrm{Soc}(U_0)\approx \mathrm{Soc}(U_1)\approx \mathrm{Soc}(U_k)$.

In case B, if $\mathrm{Soc}(U_2)\not\subseteq K$, we have either the following two properties by assumption:

⟨2–i⟩ U_0 is U_1- and U_2-injecitve.

⟨2–ii⟩ $U_1\oplus U_2$ is extending for simple modules.

Assume that ⟨2–ii⟩ occurs. We have a direct decomposition $U_1\oplus U_2=V_1\oplus V_2$ such that $V_2\supset_e (\mathrm{Soc}(U_2))(g|_{\mathrm{Soc}(U_2)})$. Then $V_1\neq 0$ and $V_1\cap K=0$. Because, if $V_1\cap K\neq 0$, unif. $\dim((V_1\oplus V_2)\cap K)=2$ since $(\mathrm{Soc}(U_2))(g|_{\mathrm{Soc}(U_2)})\subseteq K$. But $U_1\cap K=0$ induces unif. $\dim((U_1\oplus U_2)\cap K)\leqq 1$, a contradiction. On the other hand, $\mathrm{Soc}(V_2)=(\mathrm{Soc}(U_2))\,(g|_{\mathrm{Soc}(U_2)})\subseteq K$. Therefore, we have a new direct decomposition $U=V_1\oplus V_2\oplus(\sum_{k=3}^{n}\oplus U_k)$ such that $K\cap \mathrm{Soc}(V_1)=0$. Then, since $\mathrm{Soc}(U_2)\not\subseteq K$ and $\mathrm{Soc}(V_2)\subseteq K$, the existence of this direct decomposition gives us a contradiction to the maximality of $\mid\{k\in\{2, 3, \cdots, n\} \mid \mathrm{Soc}(U_k)\subseteq K\}\mid$. Consequently, if $\mathrm{Soc}(U_2)\not\subseteq K$, ⟨2–i⟩ only occurs, i.e. U_0 is U_1- and U_2-injective.

Taking the same argument for $U_3, U_4, \cdots, U_n$ in order, we may assume that U_0 is U_1-, U_2-, $\cdots$ and U_m-injective and $\mathrm{Soc}(U_{m+1})$, $\mathrm{Soc}(U_{m+2})$, $\cdots$ and $\mathrm{Soc}(U_n)\subseteq K$ for some $m\geq 2$. (Since we are considering the case B, $m\geq 2$.) Put $M_1:=U_1\oplus U_2\oplus\cdots\oplus U_m$ and $M_2:=U_{m+1}\oplus U_{m+1}\oplus\cdots\oplus U_n$ and consider the diagrams (2–1), (2–2) and (3–1) with respect to the direct decomposition $U=M_1\oplus M_2$. Then, using Corollary 1 inductively, the first case occurs in both diagrams (2–1) and (3–1). On the other hand, in the diagram (2–2), $\mathrm{Ker}(f|_{L\cap M_2})=K\cap M_2\subset_e M_2$ since $\mathrm{Soc}(M_2)=\sum_{k=m+1}^{n}\mathrm{Soc}(U_k)\subseteq K_*$. So the second case does not occur in the diagram (2–2) by Lemma B. The first case occurs in it since U_0 is almost M_2-injective by the inductive assumption. Then the first case occurs in the diagram (※) by Lemma C, a contradiction.

In case A. Let π_1: $U(=\sum_{k=1}^{n}\oplus U_k)\to U_1$ be the projection and put $K^1:=\pi_1(K)$. For each direct decomposition of U into uniform modules U'_k in which the caes A occurs and $U'_1\cap K=0$, we obtain such K'^1. Since $||K'^1||$ is finite, we can take a minimal $||K^1||$ among $||K'^1||$ related to the direct decomposition $\sum_{k=1}^{n}\oplus U'_k$ and we denote its direct decomposition by $\sum_{k=1}^{n}\oplus U_k$.

The special case $K^1=0$ may occur. We shall first consider this case. From $K^1=0$ it follows that $K\subseteq U_*$. There are two monomorphisms: $L_1\oplus(L_*/K) \xrightarrow{\text{natural}} L/K \xrightarrow{\text{induced from} f} U_0$ and U_0 is uniform. Since $L\subset_e U$ and hence $L_1\neq 0$, and so $L_*=K$. Put $L^1:=\pi_1(L)$. since $\mathrm{Ker}(\pi_1|_L)=L_*=K=\mathrm{Ker}(f)$, there is a homomorphism f': $L^1\to U_0$ such that $f=f'\cdot(\pi_1|_L)$. So consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L^1 & \xrightarrow{\;i^1\;} & U_1 \\ & & {\scriptstyle f'}\downarrow & & \\ & & U_0 & & \end{array}$$

From the assumption that U_0 is almost U_1-injective, the first case or the second occurs in this diagram. Assume that the first case occurs and let $\bar{f}'$: $U_1\to U_0$ be a homomorphism such that $f'=\bar{f}'\cdot i^1$, put $\bar{f}:=\bar{f}'\cdot\pi_1$: $U\to U_0$, then $f=f'\cdot(\pi_1|_L)=\bar{f}'\cdot i^1\cdot(\pi_1|_L)=\bar{f}'\cdot\pi_1\cdot i=\bar{f}\cdot i$ in the diagram (※), i.e. the first case also occurs in the diagram (※), a contradiction. So the second case occurs. Let $\bar{f}'$: $U_0\to U_1$ be a homomorphism such that $i^1=\bar{f}'\cdot f'$. Then $\bar{f}'\cdot f=\bar{f}'\cdot f'\cdot(\pi_1|_L)=i^1\cdot(\pi_1|_L)=\pi_1\cdot i$ in the diagram (※), i.e. the second case also occurs in the diagram (※).

In the case $K^1\neq 0$. Consider the diagrams (2–1), (2–*) and (3–*) from the diagram (※) with respect to the direct decomposition $U=U_1\oplus U_*$.

[Claim. 1] The first case occurs in the diagram (2–*). Otherwise the second case occurs in it by inductive assumption. So there is a proepr direct summand of U_* which contains K_* by Lemma B, i.e. $K_*\not\subset_e U_*$, since $\mathrm{Ker}(f|_{L_*})=K_*$. Then the case B occurs with respect to the direct decomposition $U=U_1\oplus U_*$, a contradiction.

[Claim. 2] The first case occurs in the diagram (3–*). Otherwise the second case occurs in it by inductive assumption. So there is a proper direct summand of U which contains L_* by Lemma B, i.e. $L_* \underset{e}{\not\subset} U_*$, since $L_* \subseteq \mathrm{Ker}(f'_*)$.So $K_* \underset{e}{\not\subset} U_*$ for $K_* \subseteq L_*$. Therefore the case B also occurs, a contradiction.

Thus, we only have either the following two cases:

i) The first case occurs in the diagrams (2–1), (2–∗) and (3–∗).

ii) The second case occurs in the diagram (2–1) and the first case does in the diagram (2–∗).

In case i), the first case occurs in the diagram (※) by Lemma C, a contradiction. So we consider the case ii).

Since the second case occurs in the diagram (2–1), there exists a homomorphism p: $U_0 \to U_1$ such that $i_1 = p\cdot(f\,|_{L_1})$, i.e. $p\cdot(f\,|_{L_1}) = 1_{L_1}$. Since the first case occurs in the diagram (2–∗), there exists a homomorphism q: $U_* \to U_0$ such that $f\,|_{L_*} = q\cdot i_*$, i.e. $q\,|_{L_*} = f\,|_{L_*}$. Put $g' := p\cdot q$: $U_* \to U_1$ and $X := g^{-1}(\mathrm{Soc}(U_1))$. Then $U_*(-g') \supseteq X(g\,|_X)$. Because, since $x + g(x) \in K$ for any $x \in X$, $0 = f(x + g(x)) = f(x) + f\cdot g(x)$, i.e. $f(x) = -f\cdot g(x)$. (Note that $L \underset{e}{\subset} U$ induces $\mathrm{Soc}(U_1) \subseteq L_1$. So $g(x) \in L$, and $x \in L$ since $x + g(x) \in K \subseteq L$. Hence $f\cdot g(x)$ and $f(x)$ are defined.)

$$
\begin{aligned}
\text{Therefore } g'(x) &= p\cdot q(x) \\
&= p\cdot f(x) && (q\,|_{L_*} = f\,|_{L_*}) \\
&= p\cdot(-f\cdot g(x)) \\
&= -g(x) && (p\cdot(f\,|_{L_1}) = 1_{L_1})
\end{aligned}
$$

Hence $U_*(-g') \supseteq X(g\,|_X)$. Then $U_*(-g') \supseteq X(g\,|_X) \supsetneqq K_*(g\,|_{K_*}) = K_*$ since $X \supsetneqq K_*$. (We are considering the case $K^1 \neq 0$. So $g \neq 0$. And we have $X \supsetneqq K_*$.)

Now we consider the direct decomposition $U = U_1 \oplus (\sum_{k=2}^{n} \oplus U_k(-g'\,|_{U_k}))$. Put $K'_* := K \cap U_*(-g')$. Then $K'_* \supsetneqq K_*$ since $U_*(-g') \supsetneqq K_*$. So $K'_* \underset{e}{\subset} U_*(-g')$ for $K_* \underset{e}{\subset} U_*$. Hence the case A occurs in this direct decomposition. Let π'_1: $U(= U_1 \oplus U_*(-g')) \to U_1$ be the projection and put $K^{1\prime} := \pi'_1(K)$. Then $\|K^{1\prime}\| < \|K^1\|$, since $K'_* \supsetneqq K_*$ induces $\|K'_*\| > \|K_*\|$ and $(\|K\| =)\ \|K^1\| + \|K_*\| = \|K^{1\prime}\| + \|K'_*\|$. Therefore the direct decomposition $U = U_1 \oplus (\sum_{k=2}^{n} \oplus U_k(-g'\,|_{U_k}))$ give us a contradiction to the minimality of $\|K^1\|$.

As a consequence, taking an adequate direct decomposition of U, the special case $K^1 = 0$ occurs in the case A.

DEFINITION. Let R be a right artinian ring. We say that R is *right Co-Nakayama* if every indecomposable injective right R-module E is uniserial (i.e. E has a unique composition series.).

Corollary. *The following two conditions are equivalent*:

(1) *R is right Co-Nakayama.*

(2) *For any uniform modules* U^i *and* $U_j (i=1, \cdots, m; j=1, \cdots, n)$ *of finite composition length,* $\bigoplus_{i=1}^{m} U^i$ *is almost* $\bigoplus_{j=1}^{n} U_j$*-injective if* U^i *is almost* U_j*-injective for all* i *and* j. (*i.e. The almost injectivity among uniform modules of finite composition length is closed under finite direct sums.*)

Proof. (1)⇒(2): If $\mathrm{Soc}(U_k) \approx \mathrm{Soc}(U_l)$, $U_k \oplus U_l$ is extending for simple modules by (1). Then U^i is almost $\bigoplus_{j=1}^{n} U_j$-injective for any $i \in \{1, \cdots, m\}$ since the condition in Theorem holds. Give a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\ i\ } & \bigoplus_{j=1}^{n} U_j \\ & & f \big\downarrow & & \\ & & \bigoplus_{i=1}^{m} U^i & & \end{array}$$

Let $p_i : \bigoplus_{i=1}^{n} U^i \rightarrow U^i$ be projections $(i=1, \cdots, m)$. Consider the following diagrams for $i=1, \cdots, m$:

$$(\#\text{-}i) \qquad \begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\ i\ } & \bigoplus_{j=1}^{n} U_j \\ & & p_i \cdot f \big\downarrow & & \\ & & U^i & & \end{array}$$

If the first case occurs in all diagrams (#-i), let $\tilde{f}_i : \bigoplus_{j=1}^{n} U_j \rightarrow U^i$ with $p_i \cdot f = \tilde{f}_i \cdot i$, $f = (\bigoplus_{i=1}^{m} \tilde{f}_i) \cdot i$, i.e. the first case occurs in the given diraagm. If the second case occurs in a diagram (#-r) ($r \in \{1, \cdots, m\}$), let U' be a direct summand of $\bigoplus_{j=1}^{n} U_j$, $\pi : \bigoplus_{j=1}^{n} U_j \rightarrow U'$ be a projection and $\tilde{f}_r : U^r \rightarrow U'$ be a homomorphism such that $\pi \cdot i = \tilde{f}_r \cdot p_r \cdot f$, the second case occurs in the given diagram. Therefore $\bigoplus_{i=1}^{m} U^i$ is almost $\bigoplus_{j=1}^{n} U_j$-injective.

(2)⇒(1): Claim. For each uniform module U, $U/\mathrm{Soc}(U)$ is also uniform.

First we show this claim. Let M_1 and M_2 be submodules of U with $||M_i|| = 2$ $(i=1, 2)$. Then $\mathrm{Soc}(U)$ is almost M_1- and M_2-injective but neither M_1-nor M_2-injective. $M := M_1 \oplus M_2$ is extending for simple modules by (2) and Theorem. Let $1_s : \mathrm{Soc}(U) \rightarrow \mathrm{Soc}(U)$ be the identity map. There is an isomorphism $f : M_1 \rightarrow M_2$ such that $f|_{\mathrm{Soc}(U)} = 1_s$ by [3], Corollary 8. Let $1 : U \rightarrow U$ and $i_2 : M_2 \rightarrow U$ be the identity map and the inclusion map, respectively. Put $g := 1|_{M_1}$

$-i_2 \cdot f: M_1 \rightarrow U$. Since $f|_{\mathrm{Soc}(U)} = 1_s$, $g(\mathrm{Soc}(U)) = 0$. And so $g(M_1) \subseteq \mathrm{Soc}(U)$ for $||M_1|| = 2$. $M_1 = 1(M_1) = (i_2 \cdot f + g)(M_1) \subseteq i_2 \cdot f(M_1) + g(M_1) \subseteq M_2 + \mathrm{Soc}(U) = M_2$. Hence $M_1 = M_2$, i.e. $U/\mathrm{Soc}(U)$ is uniform.

Let E be an injective indecomposable module. Since R is right artinian, $J^n = 0$ for some n. Hence E has the finite socle series:

$$0 = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_m = E$$

for some $m \leqq n$, where S_i is the left annihilator of J^i for each i. Then apply inductively the above claim to this series to see that S_i/S_{i-1} is simple for each $i \in \{1, \cdots, m\}$, whence the assertion follows.

EXAMPLE. There is an example which shows that the Azumaya's Theorem is not able to be extended without an additional condition.

Let K be a field and

$$R = \begin{pmatrix} K & 0 & K \\ & K & K \\ 0 & & K \end{pmatrix}$$

Then, $e_{33}R$ is almost $e_{11}R$- and $e_{22}R$-injective, but not almost $e_{11}R \oplus e_{22}R$-injective, where e_{kk} are matrix units.

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