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# J-GROUPS OF SUSPENSIONS OF STUNTED LENS SPACES MOD 4

Dedicated to Professor Shôrô Araki on his 60th birthday

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## 1. Introduction

Let $L^n(q)=S^{2n+1}/\boldsymbol{Z}_q$ be the $(2n+1)$-dimensional standard lens space mod $q$. As difined in [7], we set

$$(1.1)\qquad \begin{aligned} L_q^{2n+1} &= L^n(q)\,, \\ L_q^{2n} &= \{[z_0, \cdots, z_n]\in L^n(q)\,|\, z_n \text{ is real} \geqq 0\}\,. \end{aligned}$$

In the previous paper [15], we determined the $KO$-groups $\widetilde{KO}(S^j(L_q^m/L_q^n))$ of the suspensions of the stunted lens spacs $L_q^m/L_q^n$ for $j\equiv 1 \pmod 2$. For primes $p$, the $J$-groups $\tilde{J}(S^j(L_p^m/L_p^n))$ have been determined (cf. [11] for $p=2$ and [12] for odd primes $p$). The purpose of this paper is to determine the $KO$- and $J$-groups of suspensions of stunted lens spaces mod 4.

This paper is organized as follows. In section 2 we state the main theorems: the structures of $\tilde{J}(S^j(L_{2q}^m/L_{2q}^n))$ for $j\equiv 1 \pmod 2$ are given in Theorem 1, the proof of which is similar to that for the case $q=1$ (cf. [11]) and omitted, the structures of $\widetilde{KO}(S^j(L_4^m/L_4^n))$ and $\tilde{J}(S^j(L_4^m/L_4^n))$ for $j\equiv 0 \pmod 2$ are given in Theorems 2 and 3 respectively. In section 3 we prepare some lemmas and recall known results in [8], [10] and [13]. By virtue of the results in [8], the proofs of Theorem 2 and 3 for the case $j\equiv 0 \pmod 4$ are given in section 4. Applying the method used in the corresponding parts of [8], we prove Theorems 2 and 3 for the case $j\equiv 2 \pmod 4$ in the final section.

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## 2. Satement of results

Let $\nu_p(s)$ denote the exponent of the prime $p$ in the prime power decomposition of $s$, and $\mathfrak{m}(s)$ the function defined on positive integers as follows (cf. [3]):

$$\nu_p(\mathfrak{m}(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s \not\equiv 0 \pmod{(p-1)}) \\ 1+\nu_p(s) & (p \neq 2 \text{ and } s \equiv 0 \pmod{(p-1)}) \\ 1 & (p=2 \text{ and } s \not\equiv 0 \pmod 2) \\ 2+\nu_2(s) & (p=2 \text{ and } s \equiv 0 \pmod 2) . \end{cases}$$

Let $\boldsymbol{Z}/k$ denote the cyclic group $\boldsymbol{Z}/k\boldsymbol{Z}$ of order $k$. For an integer $n$, $A(n)$ denotes the group defined by

$$(2.1) \qquad A(n) = \begin{cases} \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 & (n \equiv 0 \pmod 8) \\ \boldsymbol{Z}/2 & (n \equiv 1 \text{ or } 7 \pmod 8) \\ 0 & (\text{otherwise}) . \end{cases}$$

If $j \equiv 1 \pmod 2$, then we have

$$\widetilde{KO}(S^j(L^m_{2q}/L^n_{2q})) \cong \widetilde{KO}(S^j(RP(m)/RP(n)))$$

(cf. [15, Remark 4]), and the proof of the following theorem is similar to that for the corresponding part of the theorem in [11].

**Theorem 1.** *Let $q$, $j$, $m$ and $n$ be non-negative integers with $q \geqq 1$ and $m \geqq n+2$.*
(1) *If $j \equiv 1 \pmod 4$, then we have*

$$\tilde{J}(S^j(L^m_{2q}/L^n_{2q})) \cong \begin{cases} \boldsymbol{Z}/\mathfrak{m}((m+j)/2) \oplus A(n+j) & (m \equiv 3 \pmod 4) \\ A(n+j) & (\textit{otherwise}) . \end{cases}$$

(2) *If $j \equiv 3 \pmod 4$, then we have*

$$\tilde{J}(S^j(L^m_{2q}/L^n_{2q})) \cong \begin{cases} \boldsymbol{Z}/\mathfrak{m}((m+j)/2) & (m \equiv 1 \pmod 4) \\ \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 & (m+j \equiv 2 \pmod 8) \\ \boldsymbol{Z}/2 & (m+j \equiv 1 \textit{ or } 3 \pmod 8) \\ 0 & (\textit{otherwise}) . \end{cases}$$

REMARK. (1) In the case $m=n+1$, $S^j(L^{n+1}_q/L^n_q)$ is homeomorphic to the sphere $S^{n+j+1}$, and $J$-groups of the spheres are well-known:

$$\tilde{J}(S^k) \cong \begin{cases} \boldsymbol{Z}/\mathfrak{m}(k/2) & (k \equiv 0 \pmod 4) \\ \boldsymbol{Z}/2 & (k \equiv 1 \text{ or } 2 \pmod 8) \\ 0 & (\text{otherwise}) . \end{cases}$$

(2) If $j \equiv 1 \pmod 2$, then the above theorem and [11] imply

$$\tilde{J}(S^j(L^m_{2q}/L^n_{2q})) \cong \tilde{J}(S^j(RP(m)/RP(n)))$$

for any $q$.

In order to state the next theorem, we prepare functions $h_1$, $h_2$, $a_1$ and $b_1$ defined by

(2.2)
$$\begin{cases} h_1(n) = [n/4]+[(n+7)/8]+[(n+4)/8] \\ h_2(n) = [n/8]+[(n+6)/8]\,. \end{cases}$$

(2.3)
$$\begin{cases} a_1(m, n) = h_1(m)-[(n+1)/4]-[(n+1)/8]-[(n+6)/8] \\ b_1(m, n) = h_2(m)-[(n+7)/8]-[(n+5)/8]\,. \end{cases}$$

We denote the direct sum $\boldsymbol{Z}/n_1\oplus\cdots\oplus\boldsymbol{Z}/n_i$ by $(n_1, \cdots, n_i)$, and $\boldsymbol{Z}$ by $(\infty)$.

**Theorem 2.** *Let $j$, $m$ and $n$ be non-negative integers with $m>n$.*

(1) *Suppose* $j\equiv 0 \pmod 4$.

i) *If* $n\not\equiv 3 \pmod 4$, *then we have*

$$\widetilde{KO}(S^j(L_4^m/L_4^n))\cong\begin{cases} \boldsymbol{Z}/2^{a_1(m+j,n+j)}\oplus\boldsymbol{Z}/2^{b_1(m+j,n+j)} & (b_1(m+j, n+j)\geqq 0) \\ 0 & (b_1(m+j, n+j)<0)\,. \end{cases}$$

ii) *If* $n\equiv 3 \pmod 4$, *then we have*

$$\widetilde{KO}(S^j(L_4^m/L_4^n))\cong\begin{cases} \boldsymbol{Z}\oplus\boldsymbol{Z}/2^{a_1(m+j,n+j)}\oplus\boldsymbol{Z}/2^{b_1(m+j,n+j)} & (b_1(m+j, n+j)\geqq 0) \\ \boldsymbol{Z} & (b_1(m+j, n+j)<0)\,. \end{cases}$$

(2) *Suppose* $j\equiv 2 \pmod 4$.

i) *If* $m\geqq n+9$, *then we have*

$$\widetilde{KO}(S^j(L_4^m/L_4^n))\cong\boldsymbol{Z}/2^{[(m+j)/4]-[(n+j+1)/4]}\oplus A(m+j-1)\oplus B(n+j)\,,$$

*where $A(m)$ is the group difined by* (2.1), *and $B(n)$ is the group defined by*

$$B(n)=\begin{cases} \boldsymbol{Z} & (n\equiv 3 \pmod 4) \\ \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & (n\equiv 1 \pmod 8) \\ \boldsymbol{Z}/2 & (n\equiv 0 \text{ or } 2 \pmod 8) \\ 0 & (\textit{otherwise})\,. \end{cases}$$

ii) *If* $n+8\geqq m>n$, *then the groups* $\widetilde{KO}(S^j(L_4^m/L_4^n))$ *are isomorphic to the corresponding groups in the following table:*

| $n+j$ (mod 8) \ $m-n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 0 | (2) | (2, 2) | (2) | (2, 2) | (2, 2) | (2, 2) | (2, 2) | (4, 2, 2) |
| 1 | (2) | (2) | (4, 2) | (4, 2) | (4, 2) | (4, 2) | (4, 4, 2) | (4, 4, 2, 2) |
| 2 | 0 | (4) | (4) | (4) | (4) | (4, 4) | (4, 4, 2) | (4, 2, 2) |
| 3 | (∞) | (∞) | (∞) | (∞) | (∞, 4) | (∞, 4, 2) | (∞, 2, 2) | (∞, 2) |
| 4 | 0 | 0 | 0 | (4) | (4, 2) | (2, 2) | (2) | (4) |
| 5 | 0 | 0 | (4) | (4, 2) | (2, 2) | (2) | (4) | (4) |
| 6 | 0 | (4) | (4, 2) | (2, 2) | (2) | (4) | (4) | (4) |
| 7 | (∞) | (∞, 2) | (∞, 2) | (∞) | (∞, 2) | (∞, 2) | (∞, 2) | (∞, 2) |

REMARK. (1) Combining this theorem with [15, Theorem 2], we obtain the complete results for the groups $\widetilde{KO}(S^j(L_4^m/L_4^n))$.
(2) The partial results for the case $n=0$ of this theorem have been obtained in [8].

In order to state the next theorem, we set

$$(2.4)\quad \begin{cases} a(j, m, n) = \begin{cases} a_1(m, n) & (j=0) \\ \min\{\nu_2(j)+1,\ a_1(m+j, n+j)\} & (j>0) \end{cases} \\ b(j, m, n) = \begin{cases} b_1(m, n) & (j=0) \\ \min\{\nu_2(j)+1,\ b_1(m+j, n+j)\} & (j>0)\,. \end{cases} \end{cases}$$

Main result is the following theorem.

**Theorem 3.** *Let $j$, $m$ and $n$ be non-negative integers with $m>n$.*
(1) *Suppose* $j\equiv 0$ (mod 4).
i) *If* $n\not\equiv 3$ (mod 4), *then we have*

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong \begin{cases} \boldsymbol{Z}/2^{a(j,m,n)}\oplus \boldsymbol{Z}/2^{b(j,m,n)} & (b(j, m, n)\geqq 0) \\ 0 & (b(j, m, n)<0)\,. \end{cases}$$

ii) *In the case* $n\equiv 3$ (mod 4), *we have*

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong \begin{cases} \boldsymbol{Z}/\mathrm{m}((n+j+1)/2)\cdot 2^c\oplus \boldsymbol{Z}/2^{d+i}\oplus \boldsymbol{Z}/2^k & (b(j, m, n)\geqq 0) \\ \boldsymbol{Z}/\mathrm{m}((n+j+1)/2) & (b(j, m, n)<0)\,, \end{cases}$$

*where $i$, $k$, $c$ and $d$ are integers defined by*

$$(2.5)\quad \begin{cases} i = \begin{cases} \min\{\nu_2(n+1)-1,\ a(j, m, n)\} & (n+j\equiv 7 \ (\mathrm{mod}\ 8)) \\ \min\{\nu_2(n+1),\ a(j, m, n)\} & (n+j\equiv 3 \ (\mathrm{mod}\ 8)) \end{cases} \\ k = \min\{\nu_2(n+1)-1,\ b(j, m, n)\} \\ c = \max\{a(j, m, n)-i,\ b(j, m, n)-k\} \\ d = \min\{a(j, m, n)-i,\ b(j, m, n)-k\}\,. \end{cases}$$

(2) *Suppose* $j\equiv 2$ (mod 4).
i) *If* $m\geqq n+9$, *then we have*

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong A(m+j-1)\oplus C(n+j)\,,$$

*where $A(m)$ is the group defined by* (2.1), *and $C(n)$ is the group defined by*

$$C(n) = \begin{cases} \boldsymbol{Z}/2\mathrm{m}((n+1)/2)\oplus \boldsymbol{Z}/2 & (n\equiv 3 \ (\mathrm{mod}\ 4)) \\ \boldsymbol{Z}/4\oplus \boldsymbol{Z}/2\oplus \boldsymbol{Z}/2 & (n\equiv 1 \ (\mathrm{mod}\ 8)) \\ \boldsymbol{Z}/4\oplus \boldsymbol{Z}/2 & (n\equiv 0 \ \mathit{or}\ 2 \ (\mathrm{mod}\ 8)) \\ \boldsymbol{Z}/4 & (\mathit{otherwise})\,. \end{cases}$$

ii) *If* $n+8 \geqq m > n$, *then the groups* $\tilde{J}(S^j(L_4^m/L_4^n))$ *are isomorphic to the corresponding groups in the following table, where* $M$ *denotes the integer* $\mathfrak{m}((n+j+1)/2)$:

| $n+j$ (mod 8) \ $m-n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 0 | (2) | (2, 2) | (2) | (2, 2) | (2, 2) | (2, 2) | (2, 2) | (4, 2, 2) |
| 1 | (2) | (2) | (4, 2) | (4, 2) | (4, 2) | (4, 2) | (4, 4, 2) | (4, 4, 2, 2) |
| 2 | 0 | (4) | (4) | (4) | (4) | (4, 4) | (4, 4, 2) | (4, 2, 2) |
| 3 | $(M)$ | $(M)$ | $(M)$ | $(M)$ | $(M, 4)$ | $(M, 4, 2)$ | $(M, 2, 2)$ | $(M, 2)$ |
| 4 | 0 | 0 | 0 | (4) | (4, 2) | (2, 2) | (2) | (4) |
| 5 | 0 | 0 | (4) | (4, 2) | (2, 2) | (2) | (4) | (4) |
| 6 | 0 | (4) | (4, 2) | (2, 2) | (2) | (4) | (4) | (4) |
| 7 | $(M)$ | $(M, 2)$ | $(M, 2)$ | $(M)$ | $(M, 2)$ | $(M, 2)$ | $(M, 2)$ | $(M, 2)$ |

REMARK. (1) Combining this theorem with Theorem 1, we obtain the complete results for the groups $\tilde{J}(S^j(L_4^m/L_4^n))$.
(2) The partial results for the case $j=n=0$ of this theorem have been obtained in [9].

## 3. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 2 and 3.

**Lemma 3.1.** *Let* $j$ *be a positive integer with* $j \equiv 0 \pmod 2$ *and* $k$ *be an odd integer. Then we have*

$$k^j - 1 \equiv (k^2-1)(j/2) \qquad (\bmod\ 2^{\nu_2(j)+4})\,.$$

Proof. Since $k^2 \equiv 1 \pmod 8$, we have

$$\begin{aligned} k^j - 1 &= (k^2-1)((k^2)^{(j/2)-1} + (k^2)^{(j/2)-2} + \cdots + 1) \\ &\equiv (k^2-1)(j/2) \qquad (\bmod\ 2^6)\,. \end{aligned}$$

This proves the lemma for the case $\nu_2(j)=1$. Assume that

$$k^j - 1 \equiv (k^2-1)(j/2) \qquad (\bmod\ 2^{\nu_2(j)+4})\,.$$

Then we have

$$\begin{aligned} k^{2j} - 1 &= (k^j-1)(k^j+1) \\ &\equiv (k^2-1)(j/2)(k^j+1) && (\bmod\ 2^{\nu_2(j)+5}) \\ &\equiv (k^2-1)(j/2)(2+(k^2-1)(j/2)) && (\bmod\ 2^{2\nu_2(j)+6}) \\ &\equiv (k^2-1)(2j/2) && (\bmod\ 2^{2\nu_2(j)+4}) \end{aligned}$$

Since $\nu_2(j) \geqq 1$, this implies

$$k^{2j}-1\equiv(k^2-1)(2j/2) \qquad (\mathrm{mod}\ 2^{\nu_2(2j)+4})$$

Thus the lemma is proved by the induction with respect to $\nu_2(j)$. q.e.d.

Considering the $\boldsymbol{Z}/4$-action on $S^{2n+1}\times \boldsymbol{C}$ given by

$$\exp(2\pi\sqrt{-1}/4)(z, u) = (z\cdot\exp(2\pi\sqrt{-1}/4),\ u\cdot\exp(2\pi\sqrt{-1}/4))$$

for $(z, u)\in S^{2n+1}\times \boldsymbol{C}$, we have a complex line bundle

$$\eta\colon (S^{2n+1}\times\boldsymbol{C})/(\boldsymbol{Z}/4)\to L_4^{2n+1}\,.$$

Then we have the following elements

$$(3.2) \qquad \begin{cases} \sigma=\eta-1\in\tilde{K}(L_4^{2n+1}) \\ \sigma(1)=\eta^2-1\in\tilde{K}(L_4^{2n+1})\,. \end{cases}$$

The following proposition is well known.

**Proposition 3.3.** *If $m\geqq 2$, then we have*
(1) (Mahammed [13]) *The ring $K(L_4^m)$ is isomorphic to the truncated polynomial ring*

$$\boldsymbol{Z}[\sigma]/(\sigma^{[m/2]+1}, (\sigma+1)^4-1)\,,$$

*where $(\sigma^{[m/2]+1}, (\sigma+1)^4-1)$ means the ideal of $\boldsymbol{Z}[\sigma]$ generated by $\sigma^{[m/2]+1}$ and $(\sigma+1)^4-1$.*
(2) (Kobayashi and Sugawara [10]) *The group $\tilde{K}(L_4^m)$ is isomorphic to the direct sum of cyclic groups of order $2^{[m/2]+1}$, $2^{[m/4]}$ and $2^{[(m-2)/4]}$ generated by $\sigma$, $\sigma(1)+2^{[m/4]+1}\sigma$ and $\sigma(1)\sigma+2^{[(m+2)/4]+1}\sigma$ respectively. That is,*

$$\tilde{K}(L_4^m)\cong\langle\{\sigma, \sigma(1), \sigma(1)\sigma\}\rangle/\langle\{X_1, X_2, X_3\}\rangle\,,$$

*where $X_1=2^{[m/2]+1}\sigma$, $X_2=2^{[m/4]}\sigma(1)+2^{2[m/4]+1}\sigma$ and $X_3=2^{[(m-2)/4]}\sigma(1)\sigma+2^{2[(m+2)/4]}\sigma$.*

The following lemma is obtained by the above proposition.

**Lemma 3.4.** *Let $u$ be a positive integer. Then, in $K(L_4^m)$,*

$$\sigma^u = a_u\sigma+b_u\sigma(1)+c_u\sigma(1)\sigma\,,$$

*where $a_u$, $b_u$ and $c_u$ are integers defined by*

$$a_u=(-2)^{u-1}\,,$$

$$b_u=\begin{cases} 2(-4)^{(u/4)-1} & (u\equiv 0\ (\mathrm{mod}\ 4)) \\ 0 & (u\equiv 1\ (\mathrm{mod}\ 4)) \\ (-4)^{(u-2)/4} & (u\equiv 2\ (\mathrm{mod}\ 4)) \\ -2(-4)^{(u-3)/4} & (u\equiv 3\ (\mathrm{mod}\ 4)) \end{cases}$$

*and*

$$c_u = \begin{cases} -2^{u-2} & (u\equiv 0 \pmod 4) \\ 2^{u-2}+2(-4)^{(u-5)/4} & (u\equiv 1 \pmod 4) \\ -2^{u-2}+(-4)^{(u-2)/4} & (u\equiv 2 \pmod 4) \\ 2^{u-2}-(-4)^{(u-3)/4} & (u\equiv 3 \pmod 4)\,. \end{cases}$$

Proof. By making use of the relation $(\sigma+1)^4=1$, we obtain equalities

$$a_{u+1} = -2a_u\,,$$
$$b_{u+1} = a_u - 2c_u$$

and

$$c_{u+1} = b_u - 2c_u,$$

where $a_1=1$, $b_1=0$, and $c_1=0$. Thus the lemma is proved by the induction with respect to $u$. q.e.d.

For each integer $n$ with $0\leqq n<m$, we denote the inclusion map of $L_4^n$ into $L_4^m$ by $i_n^m$, and denote the kernel of the homomorphism

$$(i_n^m)^!\colon \tilde{K}(L_4^m) \to \tilde{K}(L_4^n)$$

by $V_n$. Then by Proposition 3.3 and Lemma 3.4, we obtain the following lemma.

**Lemma 3.5.** *Let $u$ be a positive integer with $2u<m$. Then we have*

$$\sigma^u \equiv \begin{cases} \sigma & (u=1) \\ \sigma(1)-2\sigma & (u=2) \\ (-1)^{(u-1)/2}(2^{(u-3)/2}\sigma(1)\sigma+2^{u-1}\sigma) & (u\equiv 1 \pmod 2) \text{ and } u>1) \\ (-1)^{(u-2)/2}(2^{(u-2)/2}\sigma(1)+2^{u-1}\sigma) & (u\equiv 0 \pmod 2) \text{ and } u>2) \end{cases}$$

*modulo the subgroup $V_{2u}$.*

Considering the $\boldsymbol{Z}/4$-action on $S^{2n+1}\times\boldsymbol{R}$ given by

$$\exp(2\pi\sqrt{-1}/4)(z, v) = (z\cdot\exp(2\pi\sqrt{-1}/4),\ -v)$$

for $(z, v)\in S^{2n+1}\times\boldsymbol{R}$, we have a real line bundle

$$\nu\colon (S^{2n+1}\times\boldsymbol{R})/(\boldsymbol{Z}/4) \to L_4^{2n+1}\,.$$

We set

$$\kappa = \nu - 1 \in \widetilde{KO}(L_4^{2n+1})\,.$$

It is easy to see that

$$(3.6)\qquad \begin{cases} c(\kappa)=\sigma(1) \\ r(\sigma(1))=2\kappa\,, \end{cases}$$

where $c\colon KO\to K$ is the complexification and $r\colon K\to KO$ is the real restriction. Let

$$I\colon \tilde{K}(X)\to\tilde{K}(S^2X)$$

and

$$I_R;\ \widetilde{KO}(X)\to\widetilde{KO}(S^8X)$$

be the Bott periodicity isomorphisms for $K$- and $KO$-theory respectively. Then we have the following proposition.

**Proposition 3.7.** (1) (Kobayashi and Sugawara [10]) *If $j\equiv 0 \pmod 8$ and $m\geqq 2$, then $\widetilde{KO}(S^j(L_4^m))$ is isomorphic to the dierct sum of the cyclic groups of order $2^{h_1(m)}$ and $2^{h_2(m)}$ generated by $r(I^{j/2}(\sigma))$ and $I_R^{j/8}(\kappa)+2^{[m/4]}r(I^{j/2}(\sigma))$ respectively. That is,*

$$\widetilde{KO}(S^j(L_4^m))\cong\langle\{r(I^{j/2}(\sigma)),\ I_R^{j/8}(\kappa)\}\rangle/\langle\{Y_1,\ Y_2\}\rangle\,,$$

*where $Y_1=2^{h_1(m)}r(I^{j/2}(\sigma))$ and $Y_2=2^{h_2(m)}I_R^{j/8}(\kappa)+2^{h_2(m)+[m/4]}r(I^{j/2}(\sigma))$.*

*In the case $j\equiv 0 \pmod 8$ and $m=1$, the group $\widetilde{KO}(S^jL_4^1)\cong\widetilde{KO}(S^{j+1})$ is isomorphic to $\boldsymbol{Z}/2$ generated by $I_R^{j/8}(\kappa)$.*

(2) (Kobayashi [8]) *If $j\equiv 4 \pmod 8$ and $m\geqq 4$, then the group $\widetilde{KO}(S^j(L_4^m))$ is isomorphic to the direct sum of the cyclic groups of order $2^{h_1(m+4)-2}$ and $2^{h_2(m+4)-2}$ generated by $r(I^{j/2}(\sigma))$ and $r(I^{j/2}(\sigma(1)+2^{[m/4]+1}\sigma))$ respectively. That is,*

$$\widetilde{KO}(S^j(L_4^m))\cong\langle\{r(I^{j/2}(\sigma)),\ r(I^{j/2}(\sigma(1)))\}\rangle/\langle\{Y_1,\ Y_2\}\rangle\,,$$

*where $Y_1=2^{h_1(m+4)-2}r(I^{j/2}(\sigma))$ and*

$$Y_2=2^{h_2(m+4)-2}r(I^{j/2}(\sigma(1)))+2^{h_2(m+4)+[m/4]-1}r(I^{j/2}(\sigma))\,.$$

*If $j\equiv 4 \pmod 8$ and $1\leqq m<4$, then we have $\widetilde{KO}(S^j(L_4^m))\cong 0$.*

## 4. Proof for the case $j\equiv 0 \pmod 4$

In this section we prove the parts (1) of Theorems 2 and 3. Throughout this section, $j$ denotes a non-negative integer with $j\equiv 0 \pmod 4$.

We consider the elements $y_1$ and $y_2$ of $\widetilde{KO}(S^jL_4^m)$ defined by

$$(4.1)\qquad \begin{cases} y_1=r(I^{j/2}(\sigma)) \\ y_2=\begin{cases} I_R^{j/8}(\kappa) & (j\equiv 0 \pmod 8) \\ r(I^{j/2}(\sigma(1))) & (j\equiv 4 \pmod 8)\,. \end{cases} \end{cases}$$

According to [1] and [4], we have the following lemma.

**Lemma 4.2.** *The Adams operations are given by the following formulae.*

(1) $$\psi^k(y_1)=\begin{cases} k^{j/2}y_1 & (k\equiv 1 \pmod 2)\\ 2k^{j/2}y_2 & (j\equiv 0 \pmod 8) \text{ and } k\equiv 2 \pmod 4)\\ k^{j/2}y_2 & (j\equiv 4 \pmod 8) \text{ and } k\equiv 2 \pmod 4)\\ 0 & (k\equiv 0 \pmod 4)\,. \end{cases}$$

(2) $$\psi^k(y_2)=\begin{cases} k^{j/2}y_2 & (k\equiv 1 \pmod 2)\\ 0 & (k\equiv 0 \pmod 2)\,. \end{cases}$$

For each integer $n$ with $0\leqq n<m$, we denote the kernel of the homomorphism

$$(i_n^m)^!\colon \widetilde{KO}(S^jL_4^m)\to \widetilde{KO}(S^jL_4^n)$$

by $VO_{m,n}^j$.

**Lemma 4.3.** *If* $0\leqq n<m$, *then we have*

$$VO_{m,n}^j\cong\begin{cases} \boldsymbol{Z}/2^{h_1(m+j)-h_2(n+j)-[(n+j)/4]}\oplus \boldsymbol{Z}/2^{h_2(m+j)-h_1(n+j)+[(n+j)/4]} & \\ \qquad (h_2(m+j)\geqq h_1(n+j)-[(n+j)/4]) & \\ 0 \qquad (h_2(m+j)<h_1(n+j)-[(n+j)/4])\,. \end{cases}$$

Proof. By Proposition 3.7, $VO_{m,n}^j$ is the subgroup of $\widetilde{KO}(S^jL_4^m)$ generated by $Y_1$ and $Y_2$, where

$$Y_1=\begin{cases} y_1 & (1-4[j/8]+j/2\geqq n)\\ 2^{h_1(n)}y_1 & (j\equiv 0 \pmod 8) \text{ and } n\geqq 2)\\ 2^{h_1(n+4)-2}y_1 & (j\equiv 4 \pmod 8) \text{ and } n\geqq 4) \end{cases}$$

and

$$Y_2=\begin{cases} 2y_2 & (j\equiv 0 \pmod 8) \text{ and } n=1)\\ y_2 & (j\equiv 4 \pmod 8) \text{ and } 4>n\geqq 0)\\ 2^{h_2(n)}(y_2+2^{[n/4]}y_1) & (j\equiv 0 \pmod 8) \text{ and } n\neq 1)\\ 2^{h_2(n+4)-2}(y_2+2^{[n/4]+1}y_1) & (j\equiv 4 \pmod 8) \text{ and } n\geqq 4)\,. \end{cases}$$

Consider the case $h_2(m+j)\geqq h_1(n+j)-[(n+j)/4]$. Suppose that $[(m+j)/4]+h_2(n+j)\geqq h_1(n+j)$ and $m\geqq 2$. Then we have the relations $A_i=0$ $(i=1, 2)$, where

$$A_1=\begin{cases} 2^{h_1(m)}Y_1 & (j\equiv 0 \pmod 8) \text{ and } n=1)\\ 2^{h_1(m+4)-2}Y_1 & (j\equiv 4 \pmod 8) \text{ and } 4>n\geqq 0)\\ 2^{h_1(m+j)-h_1(n+j)}Y_1 & (\text{otherwise}) \end{cases}$$

and

$$A_2=\begin{cases} 2^{h_2(m)}(y_2+2^{[m/4]}y_1) & (j\equiv 0 \pmod 8) \\ 2^{h_2(m+4)-2}(y_2+2^{[m/4]+1}y_1) & (j\equiv 4 \pmod 8)\,. \end{cases}$$

Setting

$$A_3=\begin{cases} A_1 & (1-4[j/8]+j/2\geqq n\geqq 1+2[j/8]-j/4) \\ A_1+2^{h_1(m+j)-h_2(m+j)-[(n+j)/4]}A_2 & (\text{otherwise})\,, \end{cases}$$

$$u_1=\begin{cases} Y_1 & (1-4[j/8]+j/2\geqq n\geqq 1+2[j/8]-j/4) \\ Y_2+2^{h_2(n+j)-h_1(n+j)+[(m+j)/4]}Y_1 & (\text{otherwise}) \end{cases}$$

and

$$u_2=\begin{cases} Y_2+2^{[m/4]+1}Y_1 & (1-4[j/8]+j/2\geqq n\geqq 1+2[j/8]-j/4) \\ (2^{[m/4]-[n/4]}-1)Y_1+2^{h_1(n+j)-h_2(n+j)-[(n+j)/4]}Y_2 & (\text{otherwise})\,, \end{cases}$$

we have

$$A_3=2^{h_1(m+j)-h_2(n+j)-[(n+j)/4]}u_1$$

and

$$A_2=2^{h_2(m+j)-h_1(n+j)+[(n+j)/4]}u_2\,.$$

Noting that

$$A_1=\begin{cases} A_3 & (1-4[j/8]+j/2\geqq n\geqq 1+2[j/8]-j/4) \\ A_3-2^{h_1(m+j)-h_2(m+j)-[(n+j)/4]}A_2 & (\text{otherwise})\,, \end{cases}$$

$$Y_1=\begin{cases} u_1 & (1-4[j/8]+j/2\geqq n\geqq 1+2[j/8]-j/4) \\ 2^{h_1(n+j)-h_2(n+j)-[(n+j)/4]}u_1-u_2 & (\text{otherwise}) \end{cases}$$

and

$$Y_2=\begin{cases} -2^{[m/4]+1}u_1+u_2 & (1-4[j/8]+j/2\geqq n\geqq 1+2[j/8]-j/4) \\ (1-2^{[m/4]-[n/4]})u_1+2^{h_2(n+j)-h_1(n+j)+[(m+j)/4]}u_2 & (\text{otherwise})\,, \end{cases}$$

we see that $VO^j_{m,n}$ is isomorphic to the group generated by $u_1$ and $u_2$ with relations $A_i=0$ $(i=2, 3)$. This implies the lemma for the case $[(m+j)/4]+h_2(n+j)\geqq h_1(n+j)$ and $m\geqq 2$.

Suppose that $h_2(m+j)+[(n+j)/4]\geqq h_1(n+j)>[(m+j)/4]+h_2(n+j)$ and $n\neq 1$. Then we have $n+j\equiv 1 \pmod 8$, $n+2\geqq m>n$ and $VO^j_{m,n}\cong \boldsymbol{Z}/2$ generated by $Y_2$. If $n=1$ and $2\leqq m\leqq 3$, then we have $VO^j_{m,n}\cong \boldsymbol{Z}/2$ generated by $Y_1$. If $n=0$, the lemma follows from Proposition 3.7. Thus the proof of the lemma for the case $h_2(m+j)\geqq h_1(n+j)-[(n+j)/4]$ is completed.

If $h_2(m+j)<h_1(n+j)-[(n+j)/4]$, then we have $[(m+j)/8]=[(n+j-4)/8]$. This implies $h_1(m+j)=h_1(n+j)$, $h_2(m+j)=h_2(n+j)$ and $[(m+j)/4]=[(n+j)/4]$. Hence we have $VO^j_{m,n}\cong 0$.

Thus the proof of the lemma is completed. q.e.d.

Suppose that $n\not\equiv 3 \pmod 4$. Then we have

$$a_1(m+j,\ n+j) = h_1(m+j)-h_2(n+j)-[(n+j)/4]$$

and

$$b_1(m+j,\ n+j) = h_2(m+j)-h_1(n+j)+[(n+j)/4]\,.$$

Thus the part i) of (1) of Theorem 2 is proved by making use of [15, Corollary 3] and Lemma 4.3.

**Proof of the part i) of (1) of Thoerem 3.** We set

$$UO^j_{m,n} = \sum_k (\bigcap_e k^e(\psi^k-1)VO^j_{m,n})\,. \tag{4.4}$$

Since the order of $VO^j_{m,n}$ is equal to a power of 2, we have

$$UO^j_{m,n} = \sum_{k\,:\,\text{odd}}(\psi^k-1)VO^j_{m,n} = \sum_{k\,:\,\text{odd}}(k^{j/2}-1)VO^j_{m,n} = 2^{\nu_2(j)+1}VO^j_{m,n}\,,$$

by Lemma 4.2 and Lemma 3.1. Since the order of $\widetilde{KO}(S^j(L^m_4/L^n_4))$ is finite, we have

$$\tilde{J}(S^j(L^m_4/L^n_4))\cong VO^j_{m,n}/UO^j_{m,n} = VO^j_{m,n}/2^{\nu_2(j)+1}VO^j_{m,n}\,.$$

Thus the part i) of (1) of Theorem 3 is proved by making use of Lemma 4.3. q.e.d.

Now, we turn to the case $n\equiv 3 \pmod 4$. In the rest of this section, $n$ denotes a positive integer with $n\equiv 3 \pmod 4$. It follows from [15] that we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \widetilde{KO}(S^{j+n+1}) & = & \widetilde{KO}(S^{j+n+1}) & & \\
 & & & & \downarrow{\scriptstyle\delta_1} & & \downarrow{\scriptstyle\delta_2} & & \\
(4.5)\quad 0 & \longrightarrow & VO^j_{m,n+1} & \xrightarrow{f_1} & \widetilde{KO}(S^j(L^m_4/L^n_4)) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n+1}) & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle f_3} & & \downarrow & & \\
0 & \longrightarrow & VO^j_{m,n+1} & \longrightarrow & VO^j_{m,n-1} & \xrightarrow{f_4} & \widetilde{KO}(S^j(L^{n+1}_4/L^{n-1}_4)) & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of exact sequences. Since $\widetilde{KO}(S^{j+n+1})$ is isomorphic to $\boldsymbol{Z}$, the upper row of (4.5) splits. Choose $y\in\widetilde{KO}(S^j(L^m_4/L^n_4))$ such that $\beta=f_2(y)$ generates the group $\widetilde{KO}(S^{j+n+1})$. Then we have an isomorphism

$$f\colon\ VO^j_{m,n+1}\oplus\widetilde{KO}(S^{j+n+1}) \to \widetilde{KO}(S^j(L^m_4/L^n_4))$$

defined by $f(x,\ k\beta)=f_1(x)+ky$ for every $(x,\ k)\in VO^j_{m,n+1}\oplus\boldsymbol{Z}$. This proves the part ii) of (1) of Theorem 2. Moreover, we have the following lemma.

**Lemma 4.6.** *If* $j\equiv 0 \pmod 4$ *and* $n\equiv 3 \pmod 4$, *then there is an element* $y\in\widetilde{KO}(S^j(L_4^m/L_4^n))$ *which satisfies the following conditions.*

(1) $\beta=f_2(y)$ *generates the group* $\widetilde{KO}(S^{j+n+1})$.

(2) $$f_3(y)=\begin{cases} 2^{(n-1)/2}y_1 & (n+j+1\equiv 4 \pmod 8)) \\ 2^{(n-3)/4}y_2+2^{(n-3)/2}y_1 & (n+1\equiv j\equiv 0 \pmod 8)) \\ 2^{(n-7)/4}y_2+2^{(n-3)/2}y_1 & (n+1\equiv j\equiv 4 \pmod 8) \text{ and } n>3) \\ y_1 & (j\equiv 4 \pmod 8) \text{ and } n=3)\,. \end{cases}$$

Proof. Suppose that $j\equiv 0 \pmod 8$ and $n\equiv 7 \pmod 8$. By the proof of Lemma 4.3, we have

$$VO_{m,n+1}^j=\langle\{2^{(n+1)/2}y_1,\ 2^{(n+1)/4}y_2\}\rangle$$

and

$$VO_{m,n-1}^j=\langle\{2^{(n-1)/2}y_1,\ 2^{(n-3)/4}y_2+2^{(n-3)/2}y_1\}\rangle\,.$$

Hence

$$\widetilde{KO}(S^j(L_4^{n+1}/L_4^{n-1}))\cong VO_{m,n-1}^j/VO_{m,n+1}^j\cong \boldsymbol{Z}/4$$

and the first group is generated by $f_4(2^{(n-3)/4}y_2+2^{(n-3)/2}y_1)$. It follows from the commutativity of the diagram (4.5) that the element $y$ can be chosen to satisfy $f_3(y)=2^{(n-3)/4}y_2+2^{(n-3)/2}y_1$. The proofs for the other cases are similar. q.e.d.

In the rest of this section, we fix an element $y\in\widetilde{KO}(S^j(L_4^m/L_4^n))$ which satisfies the conditions of Lemma 4.6.

**Lemma 4.7.** *If $k$ is an odd integer, then the Adams operation* $\psi^k$ *is given by*

$$\psi^k(y)=k^{(n+j+1)/2}y+((k^{j/2}-k^{(n+j+1)/2})/4)f_1(4f_3(y))\,.$$

Proof. We necessarily have

$$\psi^k(y)=uy+f_1(x)$$

for some integer $u$ and an element $x\in VO_{m,n+1}^j$. By using the $\psi$-map $f_2$, we see that $u=k^{(n+j+1)/2}$. Under $f_3$, $f_1(x)$ maps into $x$ and $y$ maps into $f_3(y)$, and we see that

$$\psi^k(f_3(y))=k^{(n+j+1)/2}f_3(y)+x\,.$$

It follows from Lemma 4.2 that

$$k^{j/2}f_3(y)=k^{(n+j+1)/2}f_3(y)+x\,.$$

This implies that

$$x=((k^{j/2}-k^{(n+j+1)/2})/4)(4f_3(y))$$

and

$$\begin{aligned}\psi^k(y)&=k^{(n+j+1)/2}y+f_1(x)\\&=k^{(n+j+1)/2}y+((k^{j/2}-k^{(n+j+1)/2})/4)f_1(4f_3(y))\,.\end{aligned}\qquad\text{q.e.d.}$$

We now recall some definition in [3]. Set $Y=\widetilde{KO}(S^j(L_4^m/L_4^n))$ and let $f$ be a function which assigns to each integer $k$ a non-negative integer $f(k)$. Given such a function $f$, we define $Y_f$ to be the subgroup of $Y$ generated by

$$\{k^{f(k)}(\psi^k-1)(y)\,|\,k\in\boldsymbol{Z},\ y\in Y\}\,;$$

that is,

$$Y_f=\langle\{k^{f(k)}(\psi^k-1)(y)\,|\,k\in\boldsymbol{Z},\ y\in Y\}\rangle\,.$$

Then the kernel of the homomorphism $J''\colon Y\to J''(Y)$ coincides with $\bigcap_f Y_f$, where the intersection runs over all functions $f$.

Suppose that $f$ satisfies

(4.8) $\quad f(k)\geqq m+\max\,\{\nu_p(\mathrm{m}((n+j+1)/2))\,|\,p$ is a prime divisor of $k\}$

for every $k\in\boldsymbol{Z}$. For each odd integer $i$, $N(i)$ denotes the integer chosen to satisfy the property

(4.9) $$iN(i)\equiv 1 \pmod{2^m}\,.$$

In the following calculation we put $(n+j+1)/2=u$ for the sake of simplicity. From Lemmas 3.1 and 4.7, we have

$$\begin{aligned}&k^{f(k)}(\psi^k-1)(y)\\&=k^{f(k)}(k^u-1)y+k^{f(k)}((k^{j/2}-k^u)/4)f_1(4f_3(y))\\&=k^{f(k)}(k^u-1)y+k^{f(k)}N(u/2^{\nu_2(u)})((u(k^{j/2}-1)-u(k^u-1))/2^{\nu_2(u)+2})f_1(4f_3(y))\\&\equiv k^{f(k)}(k^u-1)y+k^{f(k)}N(u/2^{\nu_2(u)})((((j/2)(k^u-1)-u(k^u-1))/2^{\nu_2(u)+2})f_1(4f_3(y))\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{mod}\ f_1(UO^j_{m,n+1}))\\&=(k^{f(k)}(k^u-1)/2^{\nu_2(u)+2})(2^{\nu_2(u)+2}y-N(u/2^{\nu_2(u)})((n+1)/2)f_1(4f_3(y)))\,.\end{aligned}$$

By virtue of [3; II, Theorem (2.7) and Lemma (2.12)], we have

$$\begin{aligned}&\langle f_1(UO^j_{m,n+1})\cup\{k^{f(k)}(\psi^k-1)(y)\,|\,k\in\boldsymbol{Z}\}\rangle\\&=\langle f_1(UO^j_{m,n+1})\cup\{\mathrm{m}(u)/2^{\nu_2(u)+2}(2^{\nu_2(u)+2}y-N(u/2^{\nu_2(u)})((n+1)/2)f_1(4f_3(y)))\}\rangle\end{aligned}$$

Therefore,

$$Y_f = \langle f_1(UO^j_{m,n+1}) \cup \{\mathfrak{m}((n+j+1)/2)y - Mf_1(4f_3(y))\}\rangle$$

where $M = (\mathfrak{m}((n+j+1)/2)/2^{\nu_2(n+j+1)+1})N((n+j+1)/2^{\nu_2(n+j+1)})((n+1)/2)$. Since this is true for every function $f$ which satisfies (4.8), we have

$$(4.10) \qquad J''(Y) \cong Y/\langle f_1(UO^j_{m,n+1}) \cup \{\mathfrak{m}((n+j+1)/2)y - Mf_1(4f_3(y))\}\rangle .$$

Suppose that $b(j, m, n) \geqq 0$. It follows from the proof of Lemma 4.3 that $VO^j_{m,n+1} \cong \boldsymbol{Z}/2^{a_1(m+j,n+j)} \oplus \boldsymbol{Z}/2^{b_1(m+j,n+j)}$ is generated by

$$u_1 = \begin{cases} 2^{(n+1)/4}y_2 + (2^{[(m+n+1)/4]} + 2^{(n+1)/2})y_1 & (j \equiv 0 \pmod 8) \\ 2^{(n-3)/4}y_2 + (2^{[(m+n+1)/4]} + 2^{(n+1)/2})y_1 & (j \equiv 4 \pmod 8) \end{cases}$$

and

$$u_2 = \begin{cases} 2^{(n+5)/4}y_2 + 2^{[(m+n+5)/4]}y_1 & (j \equiv n-3 \equiv 0 \pmod 8) \\ 2^{(n+1)/4}y_2 + 2^{[(m+n+5)/4]}y_1 & (j \equiv n-3 \equiv 4 \pmod 8) \\ 2^{(n+1)/4}y_2 + 2^{[(m+n+1)/4]}y_1 & (j \equiv n+1 \equiv 0 \pmod 8) \\ 2^{(n-3)/4}y_2 + 2^{[(m+n+1)/4]}y_1 & (j \equiv n+1 \equiv 4 \pmod 8)) . \end{cases}$$

By Lemma 4.6, we have

$$4f_3(y) = \begin{cases} 2u_1 - u_2 & (n+j \equiv 3 \pmod 8) \\ u_1 - u_2 & (j \equiv 4 \pmod 8 \text{ and } n=3) \\ (1-2^{[(m-n+3)/4]})u_1 + (1+2^{[(m-n+3)/4]})u_2 & (\text{otherwise}) . \end{cases}$$

Therefore

$$J''(Y) \cong \langle y, u_1, u_2\rangle / \langle \{M_0 y + M_1 u_1 + M_2 u_2,\ 2^{a(j,m,n)}u_1,\ 2^{b(j,m,n)}u_2\}\rangle ,$$

where

$$M_0 = \mathfrak{m}((n+j+1)/2) ,$$

$$M_1 = \begin{cases} -2M & (n+j \equiv 3 \pmod 8) \\ -M & (j \equiv 4 \pmod 8 \text{ and } n=3) \\ -(1-2^{[(m-n+3)/4]})M & (\text{otherwise}) \end{cases}$$

and

$$M_2 = \begin{cases} -(1+2^{[(m-n+3)/4]})M & (n+j \equiv 7 \pmod 8 \text{ and } n>3) \\ M & (\text{otherwise}) . \end{cases}$$

Set

$$i = \begin{cases} \min\{a(j, m, n), \nu_2(n+1)\} & (n+j \equiv 3 \pmod 8) \\ \min\{a(j, m, n), \nu_2(n+1)-1\} & (n+j \equiv 7 \pmod 8) \end{cases}$$

and

$$k = \min\{b(j, m, n), \nu_2(n+1)-1\}.$$

Since $\nu_2(M)=\nu_2(n+1)-1$, the greatest common divisor of $M_1$ and $2^{a(j,m,n)}$ is equal to $2^i$, and the greatest common divisor of $M_2$ and $2^{b(j,m,n)}$ is equal to $2^k$. Choose integers $e_1$, $e_2$, $e_3$ and $e_4$ with

$$e_1 2^{a(j,m,n)} + e_2 M_1 = 2^i$$

and

$$e_3 2^{b(j,m,n)} + e_4 M_2 = 2^k .$$

For the sake of simplicity, we put $a=a(j, m, n)$ and $b=b(j, m, n)$ in the following calculation. If $a-i \geqq b-k$, then we have

$$A\begin{pmatrix} M_0 y + M_1 u_1 + M_2 u_2 \\ 2^a u_1 \\ 2^b u_2 \end{pmatrix} = \begin{pmatrix} 2^{a-i} M_0 y \\ 2^{b-k+i}((e_2 M_0/2^i)y + u_1) \\ 2^k((e_4 M_0/2^k)y + (e_4 M_1/2^k)u_1 + u_2) \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2^{a-i} & -M_1/2^i & -(M_2/2^k)2^{a-b-i+k} \\ e_2 2^{b-k} & e_1 2^{b-k} & -e_2 M_2/2^k \\ e_4 & 0 & e_3 \end{pmatrix}$$

and $\det A=1$. This implies that

$$J''(Y) \cong \boldsymbol{Z}/2^{a-i} M_0 \oplus \boldsymbol{Z}/2^{b-k+i} \oplus \boldsymbol{Z}/2^k .$$

On the other hand, if $b-k > a-i$, then we have

$$B\begin{pmatrix} M_0 y + M_1 u_1 + M_2 u_2 \\ 2^a u_1 \\ 2^b u_2 \end{pmatrix} = \begin{pmatrix} 2^{b-k} M_0 y \\ 2^a u_1 \\ 2^k((e_4 M_0/2^k)y + (e_4 M_1/2^k)u_1 + u_2) \end{pmatrix},$$

where

$$B = \begin{pmatrix} 2^{b-k} & -(M_1/2^i)2^{-a+b+i-k} & -M_2/2^k \\ 0 & 1 & 0 \\ e_4 & 0 & e_3 \end{pmatrix}$$

and $\det B=1$. This implies that

$$J''(Y) \cong \boldsymbol{Z}/2^{b-k} M_0 \oplus \boldsymbol{Z}/2^a \oplus \boldsymbol{Z}/2^k .$$

Thus we have

(4.11) *If* $j\equiv 0 \pmod 4$, $n\equiv 3 \pmod 4$ *and* $b(j, m, n)\geqq 0$, *then we have*

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong \boldsymbol{Z}/\mathrm{m}((n+j+1)/2)\cdot 2^c\oplus \boldsymbol{Z}/2^{d+i}\oplus \boldsymbol{Z}/2^k ,$$

*where* $i$, $k$, $c$ *and* $d$ *are integers defined by* (2.5).

Next suppose that $b(j, m, n)<0$. It follows from Lemma 4.3 that we have $VO^j_{m,n+1}\cong 0$. This implies that the homomorphism $f_2$ in the diagram (4.5) is an isomorphism of $\psi$-groups. Thus we obtain

(4.12) *If* $j\equiv 0 \pmod 4$, $n\equiv 3 \pmod 4$ *and* $b(j, m, n)<0$ *then we have*

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong \boldsymbol{Z}/\mathrm{m}((n+j+1)/2) .$$

Now, combining (4.11) and (4.12) we obtain the part ii) of (1) of Theorem 3. Thus the proof for the case $j\equiv 0 \pmod 4$ is completed.

## 5. Proof for the case $j\equiv 2$ (mod 4)

In this section we prove the parts (2) of Theorems 2 and 3. Throughout this section $j$ denotes a positive integer with $j\equiv 2 \pmod 4$. Consider the elements $x_1$, $x_2$ and $x_3$ of $\tilde{K}(S^jL_4^m)$ defined by

$$(5.1)\qquad \begin{cases} x_1 = I^{j/2}\sigma , \\ x_2 = I^{j/2}\sigma(1) , \\ x_3 = I^{j/2}(\sigma(1)\sigma) . \end{cases}$$

According to [1], we have the following lemma.

**Lemma 5.2.** *The Adams operations are given by the following formulae.*

$$(1)\quad \psi^k(x_1) = \begin{cases} k^{j/2}(x_1+x_2+x_3) & (k\equiv 3 \pmod 4) \\ k^{j/2}x_1 & (k\equiv 1 \pmod 4) \\ k^{j/2}x_2 & (k\equiv 2 \pmod 4) \\ 0 & (k\equiv 0 \pmod 4) . \end{cases}$$

$$(2)\quad \psi^k(x_2) = \begin{cases} k^{j/2}x_2 & (k\equiv 1 \pmod 2) \\ 0 & (k\equiv 0 \pmod 2) . \end{cases}$$

$$(3)\quad \psi^k(x_3) = \begin{cases} k^{j/2}(-x_3-2x_2) & (k\equiv 3 \pmod 4) \\ k^{j/2}x_3 & (k\equiv 1 \pmod 4) \\ 0 & (k\equiv 0 \pmod 2) . \end{cases}$$

Consider the elements $X_1$, $X_2$ and $X_3$ of $\tilde{K}(S^jL_4^m)$ defined by

$$(5.3)\quad \begin{cases} X_1=\begin{cases} 2^{[(n+3)/2]}x_1 & (n\geqq 1)\\ x_3 & (n=0)\end{cases}\\ X_2=\begin{cases} 2^{[(n+1)/4]}x_2 & (n\equiv 0 \text{ or } 3 \pmod 4)\\ 2^{[(n+1)/4]}x_2+2^{[(n+1)/2]}x_1 & (n\equiv 1 \text{ or } 2 \pmod 4)\end{cases}\\ X_3=\begin{cases} 2^{[(n-1)/4]}x_3 & (n\equiv 1 \text{ or } 2 \pmod 4)\\ 2^{[(n-1)/4]}x_3+2^{[(n+1)/2]}x_1 & (n\equiv 0 \text{ or } 3 \pmod 4) \text{ and } n\geqq 3)\\ x_1 & (n=0)\,.\end{cases}\end{cases}$$

For each integer $n$ with $0\leqq n\leqq m$, we denote the kernel of the homomorphism

$$(i_n^m)^!\colon \tilde{K}(S^jL_4^m)\to \tilde{K}(S^jL_4^n)$$

by $V_{m,n}^j$. Then by Proposition 3.3, we have

$$(5.4)\qquad V_{m,2[(n+1)/2]}^j=\langle\{X_1, X_2, X_3\}\rangle\,.$$

Consider the Bott exact sequence (cf. [5] and [6, (12.2)])

$$(5.5)\qquad \to \widetilde{KO}(S^{j+2}X)\xrightarrow{c}\tilde{K}(S^{j+2}X)\xrightarrow{r\circ I^{-1}}\widetilde{KO}(S^jX)\xrightarrow{\partial}\widetilde{KO}(S^{j+1}X)\to$$

for $X=L_4^m/L_4^n$, where $\partial$ is the homomorphism defined by the exterior product with the generator of $\widetilde{KO}(S^1)$. Using the isomorphisms

$$VO_{m,2[(n+1)/2]}^{j+2}\cong\widetilde{KO}(S^{j+2}(L_4^m/L_4^{2[(n+1)/2]}))$$

and

$$V_{m,2[(n+1)/2]}^j\cong\tilde{K}(S^j(L_4^m/L_4^{2[(n+1)/2]}))\,,$$

we obtain the exact sequence

$$(5.6)\qquad \to VO_{m,2u}^{j+2}\xrightarrow{I^{-1}\circ c}V_{m,2u}^j\xrightarrow{r_1}\widetilde{KO}(S^j(L_4^m/L_4^{2u}))\xrightarrow{\partial}G\to 0\,,$$

where $u=[(n+1)/2]$ and

$$G=\begin{cases}\widetilde{KO}(S^{j+1}(L_4^m/L_4^{2u})) & (m+j\equiv 0, 1 \text{ or } 2 \pmod 8)\\ 0 & (\text{otherwise})\,.\end{cases}$$

Consider the generators $y_1$ and $y_2$ of $\widetilde{KO}(S^{j+2}L_4^m)$ defined by (4.1).

**Lemma 5.7.** (1) $I^{-1}\circ c(y_1)=2x_1+x_2+x_3$.

$$(2)\quad I^{-1}\circ c(y_2)=\begin{cases} x_2 & (j\equiv 6 \pmod 8)\\ 2x_2 & (j\equiv 2 \pmod 8)\,.\end{cases}$$

Proof. (1) By (4.1), we have

$$\begin{aligned} I^{-1}\circ c(y_1) &= I^{-1}(c\circ r(I^{(j+2)/2}(\sigma))) = I^{j/2}((1+t)(\sigma)) \\ &= I^{j/2}(2\sigma+\sigma(1)+\sigma(1)\sigma) = 2x_1+x_2+x_3 . \end{aligned}$$

(2) If $j\equiv 6 \pmod 8$, then by (3.6) we have

$$I^{-1}\circ c(y_2) = I^{-1}(I^{(j+2)/2}(c(\kappa))) = I^{j/2}(\sigma(1)) = x_2 .$$

If $j\equiv 2 \pmod 8$, then we have

$$\begin{aligned} I^{-1}\circ c(y_2) &= I^{-1}(I^{(j+2)/2}(c\circ r(\sigma(1)))) = I^{j/2}((1+t)(\sigma(1))) \\ &= I^{j/2}(2\sigma(1)) = 2x_2 . \end{aligned}$$

q.e.d.

**5.1. Proof for the case $n\equiv 0 \pmod 2$.** By Proposition 3.7 and (5.4), we have

$$VO^{j+2}_{m,n} = \begin{cases} \langle\{2^{h_1(n)}y_1,\ 2^{h_2(n)}(y_2+2^{[n/4]}y_1)\}\rangle & (j\equiv 6 \pmod 8) \\ \langle\{2^{h_1(n+4)-2}y_1,\ 2^{h_2(n+4)-2}(y_2+2^{[n/4]+1}y_1)\}\rangle & \\ & (j\equiv 2 \pmod 8 \text{ and } n\geqq 4) \\ \langle\{y_1,\ y_2\}\rangle & (j\equiv 2 \pmod 8 \text{ and } 0\leqq n\leqq 2) \end{cases}$$

and $V^j_{m,n}=\langle\{X_1, X_2, X_3\}\rangle$. Using Lemma 5.7, we obtain

(5.8) *For the homomorphism $r_1$ in the exact sequence* (5.6), *we have*

$$\operatorname{Ker} r_1 = \begin{cases} \langle\{2X_2,\ (1-2^{n/4})X_1+X_2+2^{(n+4)/4}X_3\}\rangle & (n+j\equiv 2 \pmod 8 \textit{ and } n\geqq 4) \\ \langle\{2X_2,\ X_1+X_2+2X_3\}\rangle & (j\equiv 2 \pmod 8 \textit{ and } n=0) \\ \langle\{X_2,\ (1-2^{n/4})X_1+2^{(n+4)/4}X_3\}\rangle & (n+j\equiv 6 \pmod 8 \textit{ and } n\geqq 4) \\ \langle\{X_2,\ X_1+2X_3\}\rangle & (j\equiv 6 \pmod 8 \textit{ and } n=0) \\ \langle\{2X_2-X_1,\ 2^{(n+2)/4}X_3+2X_2\}\rangle & (n+j\equiv 0 \pmod 8) \\ \langle\{2X_2-X_1,\ 2^{(n-2)/4}X_3+X_2\}\rangle & (n+j\equiv 4 \pmod 8) . \end{cases}$$

If $m\geqq n+2$, then $\operatorname{Im} r_1$ is isomorphic to the group generated by $\{X_1, X_2, X_3\}$ with relations $A_i=0$ $(1\leqq i\leqq 5)$, where

$$A_1 = \begin{cases} 2X_2 & (n+j\equiv 2 \pmod 8) \\ X_2 & (n+j\equiv 6 \pmod 8) \\ 2X_2-X_1 & (n\equiv 2 \pmod 4) , \end{cases}$$

$$A_2 = \begin{cases} (1-2^{n/4})X_1+X_2+2^{(n+4)/4}X_3 & (4\leqq n\equiv 0 \pmod 4) \\ X_1+X_2+2X_3 & (n=0) \\ 2^{(n+2)/4}X_3+2X_2 & (n+j\equiv 0 \pmod 8) \\ 2^{(n-2)/4}X_3+X_2 & (n+j\equiv 4 \pmod 8) , \end{cases}$$

$$A_3=\begin{cases} 2^{[(m-n+2)/4]}X_3+2^{[(m-n-2)/4]}(2^{[(m-n+2)/4]}-1)X_1 & (4\leqq n\equiv 0 \pmod 4)\\ 2^{[(m-2)/4]}X_1+2^{2[(m+2)/4]}X_3 & (n=0)\\ 2^{[(m-n)/4]}X_3+2^{2[(m-n)/4]}X_1 & (n\equiv 2 \pmod 4)\,, \end{cases}$$

$$A_4=\begin{cases} 2^{[(m-n)/4]}X_2+2^{2[(m-n)/4]}X_1 & (4\leqq n\equiv 0 \pmod 4)\\ 2^{[m/4]}X_2+2^{2[m/4]+1}X_3 & (n=0)\\ 2^{[(m-n-2)/4]}(2X_2+(2^{[(m-n+2)/4]}-1)X_1) & (n\equiv 2 \pmod 4) \end{cases}$$

and

$$A_5=\begin{cases} 2^{[(m+2)/2]}X_3 & (n=0)\\ 2^{[(m-n)/2]}X_1 & (\text{otherwise})\,. \end{cases}$$

Thus we obtain

(5.9) *If* $m+j-2\geqq n+j\not\equiv 2 \pmod 8$ *or* $m+j-6\geqq n+j\equiv 2 \pmod 8$, *then we have*

$$r_1(V^j_{m,n})\cong\langle\{X_1, X_2, X_3\}\rangle/\langle\{A_1, A_2, B_3\}\rangle$$
$$\cong\begin{cases} \boldsymbol{Z}/2^{[(m+j)/4]-[(n+j)/4]}\oplus\boldsymbol{Z}/2 & (n+j\equiv 0 \text{ or } 2 \pmod 8)\\ \boldsymbol{Z}/2^{[(m+j)/4]-[(n+j)/4]} & (n+j\equiv 4 \text{ or } 6 \pmod 8)\,, \end{cases}$$

*where* $B_3=2^{[(m+j)/4]-[(n+j)/4]}X_3$,

$$A_1=\begin{cases} 2X_2 & (n+j\equiv 2 \pmod 8)\\ X_2 & (n+j\equiv 6 \pmod 8)\\ 2X_2-X_1 & (n\equiv 2 \pmod 8) \end{cases}$$

*and*

$$A_2=\begin{cases} 2^{(n+4)/4}X_3+X_2+(1-2^{n/4})X_1 & (4\leqq n\equiv 0 \pmod 4)\\ 2X_3+X_2+X_1 & (n=0)\\ 2^{(n+2)/4}X_3+2X_2 & (n+j\equiv 0 \pmod 8)\\ 2^{(n-2)/4}X_3+X_2 & (n+j\equiv 4 \pmod 8)\,. \end{cases}$$

*If* $n+j\equiv 2 \pmod 8$ *and* $n+5\geqq m\geqq n+2$, *then we have*

$$r_1(V^j_{m,n})\cong\langle\{X_1, X_2, X_3\}\rangle/\langle\{B_1, X_2-2X_3, 4X_3\}\rangle\cong\boldsymbol{Z}/4\,,$$

*where*

$$B_1=\begin{cases} X_1+2X_3 & (n\geqq 4)\\ X_1 & (n=0)\,. \end{cases}$$

*In the case* $m=n+1$, *we have* $r_1(V^j_{m,n})\cong 0$.

By Lemma 5.2 and (5.8), we obtain the following.

(5.10) *The Adams operations are given by the following formulae.*

$$(1)\quad \psi^k(r_1(X_3)) = \begin{cases} k^{j/2} r_1(X_3) & (k\equiv 1 \pmod 4) \\ -k^{j/2} r_1(X_3) & (k\equiv 3 \pmod 4) \\ 0 & (k\equiv 0 \pmod 2)\,. \end{cases}$$

$$(2)\quad \psi^k(r_1(X_2)) = \begin{cases} r_1(X_2) & (n\equiv 0 \pmod 4 \text{ and } k\equiv 1 \pmod 2) \\ 0 & (n\equiv 0 \pmod 4 \text{ and } k\equiv 0 \pmod 2). \end{cases}$$

$$(3)\quad \psi^k(r_1(2^{(n-2)/4}X_3+X_2))$$
$$= \begin{cases} r_1(2^{(n-2)/4}X_3+X_2) & (n\equiv 2 \pmod 4 \text{ and } k\equiv 1 \pmod 2) \\ 0 & (n\equiv 2 \pmod 4 \text{ and } k\equiv 0 \pmod 2)\,. \end{cases}$$

By Lemma 3.1, (5.6), (5.9) and (5.10), we obtain the results for the cases $j\equiv 2 \pmod 4$, $n\equiv 0 \pmod 2$ and $m+j\equiv 3$, 4, 5, 6 or 7 (mod 8).

We now turn to the case $m+j\equiv 1 \pmod 8$. Suppose that $m\geqq n+3$, and consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & V^j_{m-2,n} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_4^{m-2}/L_4^n)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & V^j_{m,n} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_4^m/L_4^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_4^m/L_4^n)) \longrightarrow 0 \\
 & & f\uparrow & & g\uparrow & & h\uparrow \\
0 & \rightarrow & \tilde{K}(S^{m+j-1}) & \xrightarrow{r_2} & \widetilde{KO}(S^{m+j})\oplus\widetilde{KO}(S^{m+j-1}) & \xrightarrow{\partial_1\oplus\partial_2} & \widetilde{KO}(S^{m+j+1})\oplus\widetilde{KO}(S^{m+j})\rightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & 0
\end{array}$$

of exact sequences, where $\partial_1\colon \widetilde{KO}(S^{m+j})\rightarrow\widetilde{KO}(S^{m+j+1})$ is an isomorphism. We denote the generators of $\widetilde{KO}(S^{m+j})$ and $\widetilde{KO}(S^{m+j+1})$ by $\omega_1$ and $\omega_2$ respectively. Since $\widetilde{KO}(S^{m+j})\cong \boldsymbol{Z}/2$, Lemma 3.5 implies that $\tilde{K}(S^{m+j-1})\cong\boldsymbol{Z}$ has a generator $\gamma$ with

$$f(\gamma) = \begin{cases} 2^{(m-7)/4}x_3+2^{(m-3)/2}x_1 & (m\geqq 7) \\ x_1 & (m=3) \end{cases}$$

and $r_2(\gamma)=2\beta$, where $\beta$ is a generator of the group $\widetilde{KO}(S^{m+j-1})\cong\boldsymbol{Z}$. It follows from (5.9) that we have

$$\begin{aligned} 2g(\beta) &= r_1(f(\gamma)) \\ &= \begin{cases} r_1(2^{(m-7)/4}x_3+2^{(m-3)/2}x_1) & (m\geqq 7) \\ r_1(x_1) & (m=3) \end{cases} \end{aligned}$$

$$=\begin{cases} 2^{(m-n-3)/4}r_1(X_3)+2^{(m-n-7)/4}r_1(X_2) & (n+j\equiv 2 \pmod 8) \\ 2^{(m-n-3)/4}r_1(X_3) & (n+j\equiv 6 \pmod 8) \\ 2^{(m-n-5)/4}r_1(X_3) & (n\equiv 2 \pmod 4)\,. \end{cases}$$

If $m\geqq n+7$, we set $\alpha=g(\beta)-2^{((m-7)/4)-[(n+2)/4]}r_1(X_3)$. Then we have $\partial(\alpha)=h(\omega_1)$, and

$$2\alpha=\begin{cases} 0 & (m\geqq n+9) \\ r_1(X_2) & (m=n+7)\,. \end{cases}$$

By (5.10) and the fact $4g(\beta)=0$, we have

$$\begin{aligned}\psi^k(\alpha) &= k^{(m+j-1)/2}g(\beta)-\psi^k(2^{((m-7)/4)-[(n+2)/4]}r_1(X_3)) \\ &= \begin{cases} \alpha & (k\text{: odd}) \\ 0 & (k\text{: even})\,. \end{cases}\end{aligned}$$

According to [3, II], we have

$$\psi^k(\omega_i)=\begin{cases} \omega_i & (k\text{: odd}) \\ 0 & (k\text{: even}) \end{cases}$$

$(i=1, 2)$. If $m\geqq n+9$, then the short exact sequence

$$0\to r_1(V^j_{m,n})\to \widetilde{KO}(S^j(L^m_4/L^n_4))\to \widetilde{KO}(S^{j+1}(L^m_4/L^n_4))\to 0$$

of $\psi$-groups splits. Hence

$$\widetilde{KO}(S^j(L^m_4/L^n_4))\cong r_1(V^j_{m,n})\oplus \boldsymbol{Z}/2\oplus \boldsymbol{Z}/2$$

and

$$\tilde{J}(S^j(L^m_4/L^n_4))\cong J''(r_1(V^j_{m,n}))\oplus \boldsymbol{Z}/2\oplus \boldsymbol{Z}/2\,.$$

If $m=n+7$, then we have

$$\widetilde{KO}(S^j(L^m_4/L^n_4))=\langle r_1(V^j_{m,n})\cup\{\alpha, g(\omega_1)\}\rangle=\langle\{r_1(X_3), \alpha, g(\omega_1)\}\rangle\,.$$

Since ord $\widetilde{KO}(S^j(L^m_4/L^n_4))=32$ by [15], ord$\langle r_1(X_3)\rangle=$ord$\langle\alpha\rangle=4$ and ord$\langle g(\omega_1)\rangle$ $=2$, we have

$$\widetilde{KO}(S^j(L^m_4/L^n_4))\cong\tilde{J}(S^j(L^m_4/L^n_4))\cong \boldsymbol{Z}/4\oplus \boldsymbol{Z}/4\oplus \boldsymbol{Z}/2\,.$$

If $m=n+5$ or $n+3$, then we have

$$\widetilde{KO}(S^j(L^m_4/L^n_4))=\langle\{g(\beta), g(\omega_1)\}\rangle\,.$$

Since ord $\widetilde{KO}(S^j(L^m_4/L^n_4))=8$ by [15], ord$\langle g(\beta)\rangle=4$ and ord$\langle g(\omega_1)\rangle=2$, we have

$$\widetilde{KO}(S^j(L^m_4/L^n_4))\cong\tilde{J}(S^j(L^m_4/L^n_4))\cong \boldsymbol{Z}/4\oplus \boldsymbol{Z}/2\,.$$

Thus we obtain the results for the case $j\equiv 2 \pmod 4$, $n\equiv 0 \pmod 2$ and $m+j\equiv 1 \pmod 8$.

The proof for the case $j\equiv 2 \pmod 4$, $n\equiv 0 \pmod 2$ and $m+j\equiv 0 \pmod 8$ is similar to that for the above case, so we omit it.

Finally we consider the case $m+j\equiv 2 \pmod 8$. Inspect the commutative diagram

$$\begin{array}{ccccccc}
V^j_{m-2,n} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_4^{m-2}/L_4^n)) & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
V^j_{m,n} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_4^m/L_4^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_4^m/L_4^n)) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\tilde{K}(S^j(L_4^m/L_4^{m-2})) & \xrightarrow{r} & \widetilde{KO}(S^j(L_4^m/L_4^{m-2})) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_4^m/L_4^{m-2})) & \longrightarrow & 0 \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}$$

of exact sequences. Since

$$\widetilde{KO}(S^j(L_4^m/L_4^{m-2}))\cong\widetilde{KO}(S^{j+m-2}L_4^2)\cong \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2$$

by Proposition 3.7, and

$$r(\tilde{K}(S^j(L_4^m/L_4^{m-2})))\cong\widetilde{KO}(S^{j+1}(L_4^m/L_4^{m-2}))\cong\boldsymbol{Z}/2\,,$$

the short exact sequence

$$0\to r(\tilde{K}(S^j(L_4^m/L_4^{m-2})))\to\widetilde{KO}(S^j(L_4^m/L_4^{m-2}))\to\widetilde{KO}(S^{j+1}(L_4^m/L_4^{m-2}))\to 0$$

splits. The Adams operations on $\widetilde{KO}(S^j(L_4^m/L_4^{m-2}))$ or $\widetilde{KO}(S^{j+1}(L_4^m/L_4^{m-2}))$ are given by

$$\psi^k=\begin{cases} 1 & (k\text{: odd}) \\ 0 & (k\text{: even})\,. \end{cases}$$

Hence the short exact sequence

$$0\to r_1(V^j_{m,n})\to\widetilde{KO}(S^j(L_4^m/L_4^n))\to\widetilde{KO}(S^{j+1}(L_4^m/L_4^n))\to 0$$

of $\psi$-groups splits. Thus we obtain the result for the case $j\equiv 2 \pmod 4$, $n\equiv 0 \pmod 2$ and $m+j\equiv 2 \pmod 8$.

Thus the proof for the case $j\equiv 2 \pmod 4$ and $n\equiv 0 \pmod 2$ is completed.

**5.2. Proof for the case $n\equiv 3 \pmod 4$.** Consider the following commutative diagram, in which the row is exact.

$$\begin{array}{ccccccc} 0 \longrightarrow & V^j_{m,n+1} & \xrightarrow{f_1} & \tilde{K}(S^j(L^m_4/L^n_4)) & \xrightarrow{f_2} & \tilde{K}(S^{n+j+1}) & \longrightarrow 0 \\ & \| & & \downarrow f_3 & & & \\ & V^j_{m,n+1} & \hookrightarrow & \tilde{K}(S^j(L^m_4)) . & & & \end{array}$$

By Lemma 3.5, we can choose an element $x \in \tilde{K}(S^j(L^m_4/L^n_4))$ such that $f_2(x)$ generates the group $\tilde{K}(S^{n+j+1}) \cong \boldsymbol{Z}$ and

$$f_3(x) = 2^{(n-1)/2}x_1 + 2^{(n-3)/4}x_2 + 2^{(n-3)/2}x_3 .$$

Applying the method used in the proof of Lemma 4.7 to $x$, we obtain the following result by Lemma 5.2.

(5.11) *The Adams operations are given by*

$$\psi^k(x) = \begin{cases} k^u x + ((k^{j/2} - k^u)/4) f_1(4f_3(x)) & (k \equiv 1 \pmod 2) \\ k^u x - (k^u/4) f_1(4f_3(x)) & (k \equiv 0 \pmod 4) \\ k^u x + f_1(k^{j/2} 2^{(n-3)/4} X_2 - k^u f_3(x)) & (k \equiv 2 \pmod 4) , \end{cases}$$

*where* $u = (n+j+1)/2$.

This implies that $c \circ r(x) = (1 + \psi^{-1})(x) = 0$. By (5.8), we have

$$r_1(4f_3(x)) = r_1((1 - 2^{(n+1)/4})X_1 + 2X_2 + 2^{(n+5)/4}X_3) = r_1(X_2) .$$

Thus we obtain

(5.12) (1) $2r(x) = r(c \circ r(x)) = 0$.

(2) $$\psi^k(r(x)) = k^{(n+j+1)/2} r(x) = \begin{cases} r(x) & (k \equiv 1 \pmod 2) \\ 0 & (k \equiv 0 \pmod 2) . \end{cases}$$

Inspect the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & V^j_{m,n+1} & \xrightarrow{f_1} & \tilde{K}(S^j(L^m_4/L^n_4)) & \xrightarrow{f_2} & \tilde{K}(S^{n+j+1}) & \longrightarrow 0 \\ & \downarrow r_1 & & \downarrow r & & \downarrow r & \\ 0 \longrightarrow & \widetilde{KO}(S^j(L^m_4/L^{n+1}_4)) & \rightarrow & \widetilde{KO}(S^j(L^m_4/L^n_4)) & \rightarrow & \widetilde{KO}(S^{n+j+1}) & \longrightarrow 0 \\ & & & & & \downarrow & \\ & & & & & 0 & \end{array}$$

of exact sequences. Since

$$\widetilde{KO}(S^{n+j+1}) \cong \begin{cases} \boldsymbol{Z}/2 & (n+j \equiv 1 \pmod 8) \\ 0 & (n+j \equiv 5 \pmod 8) , \end{cases}$$

using (5.12) we see that the short exact sequence

$$0 \to \widetilde{KO}(S^j(L_4^m/L_4^{n+1})) \to \widetilde{KO}(S^j(L_4^m/L_4^n)) \to \widetilde{KO}(S^{n+j+1}) \to 0$$

of $\psi$-groups splits. This implies that

$$\widetilde{KO}(S^j(L_4^m/L_4^n)) \cong \widetilde{KO}(S^j(L_4^m/L_4^{n+1})) \oplus \widetilde{KO}(S^{n+j+1})$$

and

$$\tilde{J}(S^j(L_4^m/L_4^n)) \cong \tilde{J}(S^j(L_4^m/L_4^{n+1})) \oplus \tilde{J}(S^{n+j+1}).$$

Thus, results of the case $j\equiv 2 \pmod 4$ and $n\equiv 3 \pmod 4$ follow from those of the case $j\equiv 2 \pmod 4$ and $n\equiv 0 \pmod 4$.

**5.3. Proof for the case $n\equiv 1 \pmod 4$.** Consider the following commutative diagram, in which the row is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^j_{m,n+1} & \xrightarrow{f_1} & \tilde{K}(S^j(L_4^m/L_4^n)) & \xrightarrow{f_2} & \tilde{K}(S^{n+j+1}) & \longrightarrow & 0 \\ & & \| & & \downarrow f & & & & \\ & & V^j_{m,n+1} & \hookrightarrow & \tilde{K}(S^j(L_4^m)) . & & & & \end{array}$$

By Lemma 3.5, we can choose an element $x\in\tilde{K}(S^j(L_4^m/L_4^n))$ such that $f_2(x)$ generates the group $\tilde{K}(S^{n+j+1})\cong \boldsymbol{Z}$ and

$$f_3(x) = \begin{cases} 2^{(n-5)/4}x_3+2^{(n-1)/2}x_1 & (n\geqq 5) \\ x_1 & (n=1). \end{cases}$$

Applying the method used in the proof of Lemma 4.7 to $x$, we obtain the following result by Lemma 5.2.

(5.13) *The Adams operations are given by*

$$\psi^k(x) = \begin{cases} k^u x+((k^{j/2}-k^u)/4)f_1(4f_3(x)) & (k\equiv 1 \pmod 4)) \\ k^u x-((k^{j/2}+k^u)/4)f_1(4f_3(x))+k^{j/2}f_1(2^{(n-5)/4}(2X_2+2X_3-X_1)+X_1-X_2) & \\ & (k\equiv 3 \pmod 4) \textit{ and } n\geqq 5) \\ k^u x-((k^{j/2}+k^u)/4)f_1(4f_3(x))+k^{j/2}f_1(X_2+X_3) & \\ & (k\equiv 3 \pmod 4) \textit{ and } n=1) \\ k^u x-(k^u/4)f_1(4f_3(x))+(k^{j/2}/2)2^{(n-1)/4}f_1(2X_2-X_1) & \\ & (k\equiv 2 \pmod 4)) \\ k^u x-(k^u/4)f_1(4f_3(x)) & (k\equiv 0 \pmod 4)), \end{cases}$$

*where* $u=(n+j+1)/2$.

Inspect the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & & 0 & & \widetilde{KO}(S^{n+j+2}) & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
VO^{j+2}_{m,n+1} & \xrightarrow{I^{-1}\circ c} & V^{j}_{m,n+1} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L^m_4/L^{n+1}_4)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L^m_4/L^{n+1}_4)) & & \\
\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle h_1} & & \\
VO^{j+2}_{m,n} & \longrightarrow & \tilde{K}(S^j(L^m_4/L^n_4)) & \xrightarrow{r_2} & \widetilde{KO}(S^j(L^m_4/L^n_4)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L^m_4/L^n_4)) & & \\
\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2} & & \\
\widetilde{KO}(S^{n+j+3}) & \xrightarrow{I^{-1}\circ c} & \tilde{K}(S^{n+j+1}) & \xrightarrow{r} & \widetilde{KO}(S^{n+j+1}) & \xrightarrow{\partial} & \widetilde{KO}(S^{n+j+2}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & \widetilde{KO}(S^{j-1}(L^m_4/L^{n+1}_4)) & & & &
\end{array}
$$

of exact sequences. By Proposition 3.7, we have

$$
VO^{j+2}_{m,n} = \begin{cases} \langle \{2^{(n+1)/2}y_1,\ 2^{(n-5)/4}y_2+2^{(n-1)/2}y_1\}\rangle & (j\equiv 2 \pmod 8 \text{ and } n\geqq 5) \\ \langle \{2^{(n+1)/2}y_1,\ 2^{(n-1)/4}y_2+2^{(n-1)/2}y_1\}\rangle & (j\equiv 6 \pmod 8 \text{ and } n\geqq 5) \\ \langle \{y_1,\ 2y_2\}\rangle & (j\equiv 6 \pmod 8 \text{ and } n=1) \\ \langle \{y_1,\ y_2\}\rangle & (j\equiv 2 \pmod 8 \text{ and } n=1)\,. \end{cases}
$$

Using Lemma 5.7, we obtain

(5.14) $$\operatorname{Ker} r_2 = \langle \{f_1(2X_2-X_1),\ f_1(2^{(n-1)/4}X_3+X_2)\}\rangle\,.$$

If $m\geqq n+3$, then we have

$$
\operatorname{Coker} g_2 \cong \widetilde{KO}(S^{n+j+2}) \cong \begin{cases} Z/2 & (n+j\equiv 7 \pmod 8) \\ 0 & (n+j\equiv 3 \pmod 8)\,, \end{cases}
$$

and hence

$$
r(\tilde{K}(S^{n+j+1})) = g_2(\widetilde{KO}(S^j(L^m_4/L^n_4))) = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j\equiv 7 \pmod 8) \\ \widetilde{KO}(S^{n+j+1}) & (n+j\equiv 3 \pmod 8)\,. \end{cases}
$$

Since $h_1$ is a monomorphism, we have $\operatorname{Ker} g_1 \subset r_1(V^j_{m,n+1})$. Thus we obtain a split short exact sequence

$$
0\to \widetilde{KO}(S^j(L^m_4/L^{n+1}_4))/\operatorname{Ker} g_1 \xrightarrow{\bar{g}_1} \widetilde{KO}(S^j(L^m_4/L^n_4)) \xrightarrow{\bar{g}_2} Z \to 0\,,
$$

where

$$
\operatorname{Ker} g_1 = \langle r_1(2^{(n-1)/4}X_3+X_2)\rangle\,.
$$

By (5.9), we obtain

(5.15) *If* $m\geqq n+3$, *then we have*

$$r_1(V^j_{m,n+1})/\mathrm{Ker}\, g_1 \cong \langle\{X_1, X_2, X_3\}\rangle/\langle\{A_1, B_2, B_3\}\rangle,$$

*where* $A_1=2X_2-X_1$, $B_2=2^{(n-1)/4}X_3+X_2$ *and* $B_3=2^{[(m-n-1)/4]}X_3$.

Thus the group $\widetilde{KO}(S^j(L_4^m/L_4^n))$ is determined by using results of the case $j\equiv 2 \pmod 4$ and $n\equiv 2 \pmod 4$. In order to determine the group $\tilde{J}(S^j(L_4^m/L_4^n))$, we use the following fact which is obtained from (5.13) and (5.14).

(5.16) *The Adams operations are given by*

$$\psi^k(r_2(x)) = \begin{cases} k^{(n+j+1)/2}r_2(x)+((k^{j/2}-k^{(n+j+1)/2})/4)r_2(f_1(4f_3(x))) & (k\equiv 1 \pmod 4)) \\ k^{(n+j+1)/2}r_2(x)-((k^{j/2}+k^{(n+j+1)/2})/4)r_2(f_1(4f_3(x))) & (k\equiv 3 \pmod 4)) \\ k^{(n+j+1)/2}r_2(x)-(k^{(n+j+1)/2}/4)r_2(f_1(4f_3(x))) & (k\equiv 0 \pmod 2)) \end{cases}$$

Set $U=\sum_{k:\,\mathrm{odd}}(\psi^k-1)\widetilde{KO}(S^j(L_4^m/L_4^{n+1}))$. By Lemma 3.1 and (5.9), we have $U=\langle 4r_1(X_3)\rangle$. If $k\equiv\varepsilon \pmod 4$ $(\varepsilon=\pm 1)$, then we have

$$\begin{aligned} &((\varepsilon k^{j/2}-k^{(n+j+1)/2})/4)r_2(f_1(4f_3(x))) \\ &\equiv((\varepsilon k^{j/2}-k^{(n+j+1)/2})/2)g_1(r_1(X_3)) && (\mathrm{mod}\ g_1(U)) \\ &\equiv((k-\varepsilon)/2)g_1(r_1(X_3)) && (\mathrm{mod}\ g_1(U)) \\ &\equiv((k^{(n+j+1)/2}-1)/2^{\nu_2(n+j+1)})g_1(r_1(X_3)) && (\mathrm{mod}\ g_1(U)). \end{aligned}$$

Thus we have $\tilde{J}(S^j(L_4^m/L_4^n))\cong\widetilde{KO}(S^j(L_4^m/L_4^n))/U_1$, where $U_1$ is the subgroup of $\widetilde{KO}(S^j(L_4^m/L_4^n))$ generated by $4g_1(r_1(X_3))$ and $\mathfrak{m}((n+j+1)/2)r_2(x)-2g_1(r_1(X_3))$.

Suppose $m+j\equiv 3$, 4, 5, 6 or 7 (mod 8). Then we have

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong\langle\{r_2(x), g_1(r_1(X_3))\}\rangle/\langle\{A_1, A_2\}\rangle,$$

where $A_1=\mathfrak{m}((n+j+1)/2)r_2(x)-2g_1(r_1(X_3))$ and

$$A_2 = \begin{cases} 4g_1(r_1(X_3)) & (m\geqq n+9) \\ 2g_1(r_1(X_3)) & (n+8\geqq m\geqq n+5) \\ g_1(r_1(X_3)) & (n+4\geqq m\equiv n+3). \end{cases}$$

Thus we obtain the results of the cases $j\equiv 2 \pmod 4$, $n\equiv 1 \pmod 4$ and $m+j\equiv$ 3, 4, 5, 6 or 7 (mod 8).

Since Ker $g_1=r_1(\langle 2^{(n-1)/4}X_3+X_2\rangle)$, the rest of the proof is similar to that for the case $j\equiv 2 \pmod 4$ and $n\equiv 2 \pmod 4$.

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