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SEMIFIELD PLANES OF CHARACTERISTIC p ADMITTING p-PRIMITIVE BAER COLLINEATIONS

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1. Introduction

Let π denote a semifield plane of order q^2 and kernel $K\cong GF(q)$ where q is a prime power p^r. We shall say that π admits a *p-primitive Baer collineation* σ if and only if σ is a collineation which fixes a Baer subplane pointwise and $|\sigma|$ is a p-primitive divisor of $q-1$ (i.e. $|\sigma|\,|\,q-1$ but $\not| \, p^i-1$ for $1\leq i<r$).

The main result of this article is essentially that p-primitive Baer collineations are easy to come by and the class of such semifield planes characterize *all* dimension two semifield planes.

Theorem 2.1. *The class of semifield planes of dimension two and characteristic p which admit a p-primitive Baer collineation is equivalent to the general class of semifield planes of dimension two and characteristic p (either class constructs the other).*

NOTATION 1.1. Let π be any semifield plane of order q^2 and kernel $\geqq K\cong GF(q)$ (π could be Desarguesian). Represent $\pi=\{(x_1, x_2, y_1, y_2)\,|\,x_i, y_i\in K, i=1,2\}$, $x=(x_1,x_2)$, $y=(y_1,y_2)$, $\mathcal{O}=(0,0)$. If $x=\mathcal{O}$ is a shears axis then the spread for π takes the following form $x=\mathcal{O}$, $y=x\begin{bmatrix}\alpha, & \beta\\ \bar{g}(\alpha,\beta), & h(\alpha,\beta)\end{bmatrix}$ where $\bar{g}$, h are biadditive maps (a function $f\colon K\times K\to K$ is biadditive$\Leftrightarrow f(\alpha,\beta)+f(\delta,\gamma)=f(\alpha+\delta,\beta+\gamma)$).

(1.2) Extensions of π. (See Hiramine et al. [2] and Johnson [3])

With the notation of (1.1), extend K by t so that $K[t]\cong GF(q^2)$ and $t^2=t\theta+\rho$ for $\theta,\rho\in K$.

Define $g(\alpha,\beta)=\bar{g}(\alpha,\beta)+\theta h(\alpha,\beta)$. Further define $f(\alpha+\beta t)=g(\alpha,\beta)-h(\alpha,\beta)t$ for all $\alpha,\beta\in K$. Then (Johnson [3] (3.1)) $x=\mathcal{O}$, $y=x\begin{bmatrix}\alpha, & \beta\\ \bar{g}(\alpha,\beta), & h(\alpha,\beta)\end{bmatrix}$ represents the spread for π if and only if

$$x=\mathcal{O},\ y=x\begin{bmatrix}\delta+\gamma t, & \alpha+\beta t\\ g(\alpha,\beta)-h(\alpha,\beta)t, & (\delta+\gamma t)^q\end{bmatrix}=\begin{bmatrix}u, & v\\ f(v), & u^q\end{bmatrix},$$

for $u=\delta+\gamma t$, $v=\alpha+\beta t$ in $GF(q^2)$ represents a spread of a translation plane π^E of order q^4 and kernel $GF(q^2)$. The translation plane π^E is called an *extension* of π and π a *contraction of* π^E.

(1.3) p-primitive Baer collineations in extensions.

Let π be any semifield plane or oder q^2 and kernel $\geq K\cong GF(q)$. Let π^E be an extension of π as in (1.2). Then π^E is a semifield plane which admits a p-primitive Baer collineation.

Proof. Let $\begin{bmatrix} 1 & & & \\ & e & & \\ & & e & \\ & & & 1 \end{bmatrix}=\sigma\ni|e|=q+1.$ Then σ maps $y=x\begin{bmatrix} u & v \\ f(v), & u^q \end{bmatrix}$ onto $y=x\begin{bmatrix} ue, & v \\ f(v), & u^q e^{-1} \end{bmatrix}=x\begin{bmatrix} ue, & v \\ f(v), & (ue)^q \end{bmatrix}$. Thus, σ is the required p-primitive Baer collineation. Note f is additive if and only if g, and h are biadditive;

$$g(\alpha+\delta,\ \beta+\gamma)=g(\alpha,\ \beta)+g(\delta,\ \gamma) \quad \text{and}$$
$$h(\alpha+\delta,\ \beta+\gamma)=h(\alpha,\ \beta)+h(\delta,\ \gamma)$$

So, this proves part of (2.1).

The reader is referred to Hiramine, Matsumoto and Oyama [2] and Johnson [3] for further background. In particular, the ideas for this article and for [3] were generated by studying the methods of *H-M-O*.

2. Semifields admitting p-primitive Baer collineations

In this section, we complete the proof of our main result (2.1) stated in section 1. We have seen that arbitrary semifield planes of dimension two give rise to semifield planes of dimension two which admit p-primitive Baer collineations. We must show that the converse statement is valid.

Let $\sum$ denote a semifield plane of order q^2 and kernel $\geq K\cong GF(q)$, where q is a prime power p^r. Assume that $\sum$ admits a p-primitive Baer collineation σ.

In the following lemmas, assume the above conditions and notation.

Lemma 2.2. *Coordinates for $\sum$ may be chosen so that $\sum=\{(x_1, x_2, y_1, y_2)\mid x_i, y_i\in K,\ i=1, 2\}$, Fix $\sigma=\{(x_1, 0, 0, y_2)\mid x_1, y_2\in K\}$ and $\sigma=\begin{bmatrix} 1 & & & \\ & e & & \\ & & e & \\ & & & 1 \end{bmatrix}$ for some element $e\in K\ni|e|$ is a p-primitive divisor of $q-1$.*

Proof. By Foulser [1], Fix σ must be a K-space. We may assume that σ leaves invariant a shears axis of $\sum$. Choose a K-Basis $\{e_1, e_2, e_3, e_4\}$ for $\sum$ such that $\langle e_1, e_2\rangle$, $\langle e_3, e_4\rangle$, are components of $\sum$ where $\langle e_1, e_4\rangle = \mathrm{Fix}\,\sigma$ and such that $\langle e_3, e_4\rangle$ is a shears axis of $\sum$. Then relatively to this basis, $\mathrm{Fix}\,\sigma = \{(x_1, 0, 0, x_4) \mid x_1, x_4 \in K\}$, and σ fixes $x=\mathcal{O}$ and $y=\mathcal{O}$ so $\sigma = \left[\begin{array}{cc|cc} 1 & 0 & & \\ a & e & & \\ \hline & & f & b \\ & & 0 & 1 \end{array}\right]$. By Maschke's Theorem, we may assume $a=b=0$ (see the argument H-M-O [2] (4.3)).

There exist exactly q components of the form $y=x\begin{bmatrix} 0 & u \\ h(u), & s(u)\end{bmatrix}$ where $u\in K$ and h, s are additive functions on K.

$$y = x\begin{bmatrix} 0, & u \\ h(u), & s(u)\end{bmatrix} \xrightarrow{\sigma} y = x\begin{bmatrix} 0, & u \\ e^{-1}fh(u), & s(u)e^{-1}\end{bmatrix}$$

Note that $\begin{bmatrix} 0 & -u \\ h(-u), & s(-u)\end{bmatrix} = \begin{bmatrix} 0 & -u \\ -h(u), & -s(u)\end{bmatrix}$ since h, s are additive. Thus, $\begin{bmatrix} 0 & 0 \\ (e^{-1}f-1)h(u), & (e^{-1}-1)s(u)\end{bmatrix}$ is identically zero.

So, since $|e^{-1}|$ is a p-primitive divisor of $q-1$, $e^{-1}f=1$ and $s(u)=0$ (as h must be 1–1).

This proves (2.2) and also the following:

Lemma 2.3. *q components of $\sum$ have the form $y=x\begin{bmatrix} 0 & u \\ h(u), & 0\end{bmatrix}$ for all $u\in K$.*

Lemma 2.4. *There exist exactly q components of $\sum$ of the form $y=x\begin{bmatrix} v & 0 \\ 0 & m(v)\end{bmatrix}$ for all $v\in K$.*

Proof. There exist q components of the form $y=\begin{bmatrix} v & 0 \\ l(v), & m(v)\end{bmatrix}$ for all $v\in K$ and l, m additive functions on K.

$$y = x\begin{bmatrix} v & 0 \\ l(v), & m(v)\end{bmatrix} \xrightarrow{\sigma} y = x\begin{bmatrix} ve, & 0 \\ l(v), & m(v)e^{-1}\end{bmatrix}$$

Since $y=x\begin{bmatrix} -v & 0 \\ -l(v), & -m(v)\end{bmatrix}$ is also a component, we obtain $y=x\begin{bmatrix} v(e-1), & 0 \\ 0, & m(v)(e^{-1}-1)\end{bmatrix}$ is a component for all $v\in K$. Since $e\neq 1$, $K(e-1)=K$ so that $l(v)=0$ for all $v\in K$.

Lemma 2.5. *Referring to (2.4), q is square and $m(v)=cv^{\sqrt{q}}$ where c is a constant.*

Proof. By the proof to (2.4) $m(v)e^{-1}=m(ve)$ for all $v\in K$. Since m is additive, $m(v)=\sum_{i=0}^{r-1} m_i v^{p^i} (q=p^r)$ (see e.g. Vaughn [4]). Then $m_i v^{p^i} e^{p^i} = m_i v^{p^i} e^{-1}$. Assume for $1 \leq i \leq r$ that $m_i \neq 0$. Then $e^{p^i}=e^{-1}$ so that $e^{p^r+1}=1$. Since $|e|\,|p^i+1$ so that $|e|\,|(p^{2i}-1, p^r-1)=(p^{(2i,r)}-1)$. Assume first that r is odd. Then $(2i, r)=(i, r)\,|\,i$. Since $|e|\,|\,p^{(2i,r)}-1\,|\,p^i-1$ and e is p-primitive, it must be that $i=r$. But then $e^{p^r}=e=e^{-1}$ if and only if $e^2=1$. However, 2 is never a p-primitive divisor of $q-1$.

So r is even (q is square). Now $(2i, r)=2(i, r/2)$ and $|e|\,|\,p^{2(i,r/2)}-1$ implies that $2(i, r/2)\geq 2(r/2)$ so that $(i, r/2)=r/2$ which implies $i=r/2$. Thus, $m(v)=h_{r/2}u^{p^{r/2}}=h_{r/2}u^{\sqrt{q}}$. This proves (2.5).

We may now complete the proof of (2.1).

We have shown that if $\sum$ is a semifield plane of order q^2 and kernel $\geq K \cong GF(q)$, $q=p^r$ which admits a p-primitive Baer collineation then q is a square and coordinates may be chosen so the spread for $\sum$ may be represented in the form $x=\mathcal{O}$, $y=x\begin{bmatrix} v, & u \\ l(u), & cv^{\sqrt{q}} \end{bmatrix}$ where $v, u \in K \cong GF(q)$ and c is a constant in K and h is an additive function on K. Clearly, by a basis change, we may assume $c=1$.

Let $F \subseteq K$, $F \cong GF(\sqrt{q})$ and $K=F[t]$ where $t^2=t\theta+\rho$ for $\theta, \rho \in F$. Define g, h: $GF(\sqrt{q})\times GF(\sqrt{q}) \to GF(\sqrt{q})$ by $l(u=\alpha+\beta t)=g(\alpha, \beta)-h(\alpha, \beta)t$ for all $\alpha, \beta \in F$. Then by (1.2), a corresponding (contraction) translation plane $\sum^C$ of order q and kernel $\geq F \cong GF(\sqrt{q})$ is obtained whose spread is represented by $x=\mathcal{O}$, $y=x\begin{bmatrix} \alpha, & \beta \\ g(\alpha, \beta)-\theta h(\alpha, \beta), & h(\alpha, \beta) \end{bmatrix}$ for $\alpha, \beta \in F$. $\sum^C$ is a semifield plane as l is additive implies g, h are biadditive;

$$l((\alpha+\beta t)+(\delta+\gamma t)) = l(\alpha+\beta t)+l(\delta+\gamma t)$$

$\Leftrightarrow$

$$\begin{aligned} g(\alpha+\delta, \beta+\gamma) &= g(\alpha, \beta)+g(\delta, \gamma) \quad \text{and} \\ h(\alpha+\delta, \beta+\gamma) &= h(\alpha, \beta)+h(\delta, \gamma)\,. \end{aligned}$$

Let $\bar{g}(\alpha, \beta)=g(\alpha, \beta)-\theta h(\alpha, \beta)$. Then

$$\begin{aligned} \bar{g}(\alpha+\delta, \beta+\gamma) &= g(\alpha+\delta, \beta+\gamma)-\theta h(\alpha+\delta, \beta+\gamma) \\ &= g(\alpha, \beta)-\theta h(\alpha, \beta)+g(\delta, \gamma)-\theta h(\delta, \gamma) \\ &= \bar{g}(\alpha, \beta)+\bar{g}(\delta, \gamma) \Leftrightarrow l \quad \text{is additive.} \end{aligned}$$

This proves our main result (2.1).

References

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