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SUPER MANIFOLDS

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Introduction

This work is a continuation of a previous work [2] on super differential calculus. We develop herein a foundation of super manifolds according to the same principle used in [2]. That is, we describe the concepts on a super manifold in terms of the non-super differential calculus on the underlying manifold of a super manifolds. Thus, we treat a super manifold as a non-super infinite-dimensional manifold with an additional geometric structure. A model of our argument is a study of complex manifolds in which a complex manifold is treated as a real manifold with a complex structure. In section 1 we give some preliminary arguments of a non-super differential calculus on some kind of infinite-dimensional Euclidean space and some algebraic preparations on super vector spaces. Also we review the super differential calculus studied in [2] and give a new version of the Cauchy-Riemann equations, which is more practical than the previous one in [2]. Section 2 deals with the definitions of a super manifold and its underlying non-super manifold. In seciton 3 we discuss tangent vectors and show how a super manifold can be regarded as a non-super infintie-dimensional manifold with a geometric structure, called an almost suepr structure. In section 4 we study super vector fields and define a local one-parameter group of local transformations for an even super vector field. In section 5 we prove one of the main theorem in this note, the super version of Frobenius' theorem, which will serve as a basic theorem for the study of super manifolds and super Lie groups.

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1. Preliminary

1.1. Affine bundles

Let $\boldsymbol{R}^n$ denote the space of all n-column real vectors $y=(y^\nu)$ $(y^\nu\in\boldsymbol{R}$, $1\leq\nu\leq n)$. When $\boldsymbol{R}^n$ is regarded as an affine space in a natural way, it is sometimes denoted by $\boldsymbol{A}^n$. An affine mapping φ of $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ is given by $\varphi(y)=Ay+b(y\in\boldsymbol{R}^n)$ where $A=(a^\mu_\nu)$ is a real (m, n)-matrix and $b=(b^\mu)\in\boldsymbol{R}^m(1\leq\nu\leq n,$

$1 \leq \mu \leq m$). The Lie group of all affine transformations of $\boldsymbol{R}^n$ is denoted by $A(n)$, which is given by

$$A(n) = \begin{pmatrix} GL(n; \boldsymbol{R}) & \boldsymbol{R}^n \\ 0 & 1 \end{pmatrix}.$$

A vector field v on $\boldsymbol{R}^n$ is said to be *affine* if v is written as follows: $v = \sum_{\nu=1}^{n} (\sum_{\mu=1}^{n} a_\mu^\nu y^\mu + b^\nu) \frac{\partial}{\partial y^\nu}$. A smooth fibre bundle A over a base space B is called an *affine bundle* if the standard fibre is a real affine space $\boldsymbol{A}^n$ and the transition functions are $A(n)$-valued. That is, there exists a family $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ of local trivializations satisfying the following 1)~3).

1) $\{U_\alpha\}$ is an open covering of B.

2) f_α is a smooth mapping of $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$ onto $\boldsymbol{A}^n$ such that the mapping $\pi \times f_\alpha$ of $\tilde{U}_\alpha$ onto $U^n \times \boldsymbol{A}^n$ is a diffeomorphism and the following diagram is commutative.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) = \tilde{U}_\alpha & \xrightarrow{\pi \times f_\alpha} & U_\alpha \times \boldsymbol{A}^n \\ & \pi \searrow \quad \swarrow & \text{the 1st projection} \\ & U_\alpha & \end{array}$$

where π denotes the projection of A onto B.

3) The transition function $g_{\alpha\beta}$ is a smooth mapping of $U_\alpha \cap U_\beta$ into $A(n)$ such that $f_{\alpha x} = g_{\alpha\beta}(x) \circ f_{\beta x}$ on the fibre $A_x = \pi^{-1}(x)$ for $x \in U_\alpha \cap U_\beta$ where $f_{\alpha x}$ is the restriction of f_α to the fibre $A_x = \pi^{-1}(x)$.

Then each fibre $A_x = \pi^{-1}(x)$ can be regarded as an affine space. Let (ψ_α, U_α) be a local coordinate system of the manifold B. Then $\Psi_\alpha = (\psi_\alpha, f_\alpha)$ is a local coordinate on $\pi^{-1}(U_\alpha) \subset A$, which is called an *affine local coordinate* on $\pi^{-1}(U_\alpha) \subset A$. Let A and $\bar{A}$ be affine bundles over B and $\bar{B}$, respectively. A smooth bundle mapping $\tilde{\varphi}$ of A into $\bar{A}$ is said to be *affine* if the restriction $\tilde{\varphi}|_{A_x}$ of $\tilde{\varphi}$ to each fibr eA_x ($x \in B$) is an affine mapping of A_x into $\bar{A}_{\varphi(x)}$ where φ is the corresponding mapping of B into $\bar{B}$.

1.2. Non-super differential calculus

Let $\{E_N\}_{N \geq 0}$ be a family of finite dimensional real vector spaces and p_N^{N+1} a linear mapping of E_{N+1} onto E_N. Such a family will be called a *projective family* of finite dimensional real vector spaces. Then the *projective limit* $E = \varprojlim E_N$ is naturally defined as follows: $E = \{(z_N) \in \prod_{N \geq 0} E_N : p_N^{N+1}(z_{N+1}) = z_N (N \geq 0)\}$. The natural projection of E onto E_N will be denoted by p_N. For $z \in E$, $p_N(z) \in E_N$ will be denoted by z_N. Considering the natural topology on a finite dimensional vector space, the projective limit E has a Fréchet space topology so that the projection p_N of E onto E_N is continuous and open for each $N \geq 0$. For $N=0$, E_0 and p_0 and $z_0 = p_0(z)$ $(z \in E)$ will be denoted by E_B and p_B and z_B, respectively. A subset U of E will be called a *domain* in E if $U_B = p_B(U)$ is an open subset of E_B and $U = p_B^{-1}(U_B)$.

Let $\bar{E} = \lim \bar{E}_N$ be the projective limit of another projective family of finite dimensional real vector spaces and $\bar{p}_N$ the natural projection of $\bar{E}$ onto $\bar{E}_N$. Let U be a domain of E. A real-valued function f defined on U is said to be *admissible* on U if there exist some integer N and a real-valued C^∞ function g on U_N such that $f = g \circ p_N$ on U. A mapping φ of U into $\bar{E}$ is said to be *admissible* if $\bar{p}_N \circ \varphi$ is admissible on U for each $N \geq 0$. A mapping φ of U into $\bar{E}$ is said to be *projectable* if for each $N \geq 0$ there exists a C^∞-mapping φ_N on U_N into $\bar{E}_N$ such that $\varphi_N \circ p_N = \bar{p}_N \circ \varphi$ on U. In this case φ_N is called the *N-th projection* of φ. Thus a projectable mapping is admissible. A mapping φ of U into $\bar{E}$ is said to be *regular* if φ is projectable and for each $N \geq 0$ the following diagram is an affine bundle mapping:

$$\begin{array}{ccc} U_{N+1} & \xrightarrow{\varphi_{N+1}} & \bar{E}_{N+1} \\ \downarrow p_N^{N+1} & & \downarrow \bar{p}_N^{N+1} \\ U_N & \xrightarrow{\varphi_N} & \bar{E}_N \end{array}$$

where U_{N+1} and $\bar{E}_{N+1}$ are regarded as trivial affine bundles over base spaces U_N and $\bar{E}_N$, respectively. That is, for each $z_N \in U_N$, φ_{N+1} is an affine mapping of an affine subspace $(p_N^{N+1})^{-1}(z_N)$ $(\subset U_{N+1} \subset E_{N+1})$ into an affine subspace $(\bar{p}_N^{N+1})^{-1}(\varphi_N(z_N))$ $(\subset \bar{E}_{N+1})$. If a one-to-one mapping φ of a domain $U \subset E$ onto a domain $\bar{U} \subset \bar{E}$ is projectable (regular) and the inverse mapping of φ is also projectable (regular), the φ is called a *projectable* (*regular*) *diffeomorphism* of U onto $\bar{U}$.

Let φ be a projectable mapping of a domain $U \subset E$ into $\bar{E}$. For each $z \in U$, the Jacobi matrix $\mathcal{J}\varphi(z)$ of φ at z is defined as follows: $\mathcal{J}\varphi(z)h = \frac{d}{dt}\varphi(z+th)_{t=0}$ $(h \in E)$. Then the Jacobi matrix $\mathcal{J}\varphi(z)$ is a projectable linear mapping of E into $\bar{E}$. Moreover the N-th projection of $\mathcal{J}\varphi(z)$ is the ordinary Jacobi matrix $J\varphi_N$ of the N-th projection φ_N of φ: That is, as a linear mapping of E_N into $\bar{E}_N$, $(\mathcal{J}\varphi(z))_N = J\varphi_N(z_N)$ for each $z \in E$ and $N \geq 0$.

1.3. Super differential calculus

We review the super differential calculus developed in [2] and add some new results. Let $\{\zeta^N : N \geq 1\}$ be a set of countably infinite distinct letters. Λ_N denotes the Grassmann algebra of the vector space generated by $\{\zeta^1, \zeta^2, \cdots, \zeta^N\}$ over the real number field $\boldsymbol{R}$ where for $N=0$, $\Lambda_0 = \boldsymbol{R}$. The family $\{\Lambda_N : N \geq 0\}$ and the natural projection of Λ_{N+1} onto Λ_N form a projective family, which defines the projective limit Λ, called the *super number algebra*. Λ can be identified with the algebra of all formal series of the following form:

$$z = \sum_{K\in\Gamma} z_K \zeta^K$$

where $\Gamma=\{K=(k_1, \cdots, k_h)\colon 1\leq k_1<\cdots<k_h\}$, $z_K\in\boldsymbol{R}$ and $\zeta^K=\zeta^{k_1}\cdots\zeta^{k_h}(\zeta^\phi=1\in\boldsymbol{R})$. The natural projection p_N of Λ onto Λ_N maps the above $z\in\Lambda$ to the following $z_N\in\Lambda_N$:

$$z_N=\sum_{K\in\Gamma_N} z_K \zeta^K$$

where $\Gamma_N=\{K=(k_1, \cdots, k_h)\colon 1\leq k_1<\cdots<k_h\leq N\}$. For each $K=(k_1, \cdots, k_h)\in\Gamma$, the *parity* $|K|$ of K is defined by $|K|=h \bmod 2\in\boldsymbol{Z}_2=\{[0], [1]\}$. For $p\in\boldsymbol{Z}_2$, Γ_p and Λ_p are defined as follows:

$$\Gamma_p = \{K\in\Gamma\colon |K| = p\}$$
$$\Lambda_p = \{z\in\Lambda\colon z = \sum_{K\in\Gamma_p} z_K \zeta^K,\ z_K\in\boldsymbol{R}\}\ .$$

If a super number z is in Λ_p, then the *parity* $|z|$ of z is, by definition, $p\in\boldsymbol{Z}_2$. If $|z|=[0]$ ([1]), then z is said to be *even* (*odd*). The *super Euclidean space* $\boldsymbol{R}^{m|n}$ of dimension $(m|n)$ is the product space $(\Lambda_{[0]})^m\times(\Lambda_{[1]})^n$ where there are m copies of $\Lambda_{[0]}$ and n copies of $\Lambda_{[1]}$. The projection p_N of Λ onto Λ_N induces the projection of $\boldsymbol{R}^{m|n}$ onto $\boldsymbol{R}_N^{m|n}$ which is, by definition, the product space $((\Lambda_{[0]})_N)^m\times((\Lambda_{[1]})_N)^n$ where $(\Lambda_p)_N=p_N(\Lambda_p)$ $(p\in\boldsymbol{Z}_2)$. The space $\boldsymbol{R}_N^{m|n}$ is called the *N-th skeleton* of the super Euclidean space $\boldsymbol{R}^{m|n}$. The super Euclidean space $\boldsymbol{R}^{m|n}$ is identified with the projective limit of the projective family $\{\boldsymbol{R}_N^{m|n}\colon N\geq 0\}$ of finite dimensional real vector spaces. Thus $\boldsymbol{R}^{m|n}$ is a Fréchet space and the projection p_N of $\boldsymbol{R}^{m|n}$ onto $\boldsymbol{R}_N^{m|n}$ is continuous and open for $N\geq 0$. The 0-th skeleton, $\boldsymbol{R}^m$, is called the *body* of $\boldsymbol{R}^{m|n}$. The projection of $\boldsymbol{R}^{m|n}$ onto the i-th component Λ_p ($p=[0]$ ([1]) if $1\leq i\leq m$ $(m+1\leq i\leq m+n)$, respectively) will be denoted by z^i for $1\leq i\leq m+n$. For $1\leq i\leq m$ $(m+1\leq i\leq m+n)$, sometimes z^i will be denoted by x^μ (θ^p), respectively where $1\leq\mu\leq m$ and $1\leq p\leq n$. Thus as usual, each $z\in\boldsymbol{R}^{m|n}$ can be written as follows:

$$\begin{aligned} z &= (z^1, \cdots, z^{m+n}) &&= (z^i) \\ &= (x^1, \cdots, x^m, \theta^1, \cdots, \theta^n) &&= (x^\mu, \theta^p) = (x, \theta)\,. \end{aligned}$$

The *parity* $|i|$ of the coordinate index i is defined as follows: $|i|=[0]$ ([1]) if $1\leq i\leq m$ $(m+1\leq i\leq m+n)$. On the N-th skeleton $\boldsymbol{R}_N^{m|n}$ of $\boldsymbol{R}^{m|n}$ we consider the following natural coordinate system $\{z_K^i\colon 1\leq i\leq m+n, K\in\Gamma_N, |K|=|i|\}$. For each $z=(z^i)\in\boldsymbol{R}^{m|n}$, the component z^i can be written as follows:

$$z^i = \sum_{K\in\Gamma_p} z_K^i \zeta^K \quad \text{where} \quad p=|i|\,.$$

Thus $z_N=(z_N^i)\in\boldsymbol{R}_N^{m|n}$ has the coordinate $\{z_K^i\colon 1\leq i\leq m+n, K\in\Gamma_N, |K|=|i|\}$.

Formally $\{z^i_K\colon 1\leq i\leq m+n,\ K\in\Gamma,\ |K|=|i|\}$ can be regarded as a natural coordinate system of $\boldsymbol{R}^{m|n}$. Since the super Euclidean space $\boldsymbol{R}^{m|n}$ is a projective limit of $\{\boldsymbol{R}^{m|n}_N\colon N\geq 0\}$, we have the differential calculus as developed in the previous section. This differential calculus on $\boldsymbol{R}^{m|n}$ will be called the *non-super differential calculus* on $\boldsymbol{R}^{m|n}$.

Here we give a revised version of Cauchy-Riemann equations of a super smooth function. We shall follow the definitions in [2]. Let K and L be elements in Γ such that $K\cap L=\phi$. Then $K\vee L$ denotes the element in Γ such that the set of entries of $K\vee L$ is the union of K and L. Then for $K, L\in\Gamma$, we define $\varepsilon(K, L)$ as follows: If $K\cap L\neq\phi$, then $\varepsilon(K, L)=0$. If $K\cap L=\phi$, then $\varepsilon(K, L)=\pm 1$ is defined by $\zeta^K\zeta^L=\varepsilon(K, L)\zeta^{K\vee L}$. For $1\leq i\leq m+n$ and $K\in\Gamma$ with $|i|=|K|$, $\dfrac{\partial}{\partial z^i_K}$ is defined as in [2]. For $K, L\in\Gamma$, we define $\dfrac{\partial}{\partial z^i_{K+L}}$ as follows:

$$\frac{\partial}{\partial z^i_{K+L}}=\begin{cases}0 & \text{if } K\cap L\neq\phi\,,\\ \varepsilon(K, L)\dfrac{\partial}{\partial z^i_{K\vee L}} & \text{if } K\cap L=\phi\,.\end{cases}$$

Then we have the following revised Cauchy-Riemann equations.

Theorem 1.1. *Let f be a Λ-valued projectable function defined on a domain U in $\boldsymbol{R}^{m|n}$. Then the following conditions 1)∼5) are equivalent.*

1) *$f(z)$: G^1 on U.*
2) *$f(z)$ satisfies the following equations on U:*

$$\frac{\partial}{\partial x^\mu_K}f(z)=\frac{\partial}{\partial x^\mu_\phi}f(z)\cdot\zeta^K\quad(1\leq\mu\leq m,\ K\in\Gamma\colon |K|=[0])\,,$$

$$\frac{\partial}{\partial\theta^p_L}f(z)\cdot\zeta^H+\frac{\partial}{\partial\theta^p_H}f(z)\cdot\zeta^L=0\quad(1\leq p\leq n,\ L, H\in\Gamma\colon |L|=|H|=[1])\,.$$

3) *$f(z)$ satisfies the following equations on U:*

$$\frac{\partial}{\partial z^i_{K+H}}f(z)=\frac{\partial}{\partial z^i_K}f(z)\cdot\zeta^H\quad(1\leq i\leq m+n,\ K, H\in\Gamma\colon |i|=|K|,\ |H|=[0])\,.$$

4) *$f(z)$: super smooth on U.*
5) *$f(z)$ can be written as follows:*

$$f(x,\theta)=\sum_P\tilde{\phi}_P(x)\cdot\theta^P\quad(P=(p_1,\cdots,p_k)\colon 1\leq p_1<\cdots<p_k\leq n)\,,$$

where $\tilde{\phi}_P(x)$ is the Z-expansion of a Λ-valued smooth function $\phi_P(t)$ on $t\in U_B\subset\boldsymbol{R}^m$ and $\theta^P=\theta^{p_1}\cdots\theta^{p_k}$.

Proof. The conditions 1), 2), 4) and 5) are equivalent as shown in [2]. First we show that 1) implies 3). As shown in [2], if $f(z)$ is G^1 on U, then it satisfies the following on U:

$$\frac{\partial}{\partial z_K^i} f(z) = f \frac{\overleftarrow{\partial}}{\partial z^i}(z)\cdot\zeta^K \quad (1\leq i\leq m+n, K\in\Gamma\colon |i|=|K|)\,.$$

If $K\cap H\neq\phi$, then $\zeta^K\zeta^H=0$. Thus 3) holds if $K\cap H\neq\phi$. Suppose $K\cap H=\phi$. Then $\frac{\partial}{\partial z_{K+H}^i}f(z)=\varepsilon(K,H)\frac{\partial}{\partial z_{K\vee H}^i}f(z)=\varepsilon(K,H)f\frac{\overleftarrow{\partial}}{\partial z^i}(z)\cdot\zeta^{K\vee H}=f\frac{\overleftarrow{\partial}}{\partial z^i}(z)\cdot\zeta^K\cdot\zeta^H=\frac{\partial}{\partial z_K^i}f(z)\cdot\zeta^H$. Now we show that 3) implies 2). Clearly 3) implies the first equations of 2). By a straight calculation, we can show that 3) implies the following equations.

$$\left(\frac{\partial}{\partial\theta_L^p}f(z)\cdot\zeta^H+\frac{\partial}{\partial\theta_H^p}f(z)\cdot\zeta^L\right)\cdot\zeta^j = 0$$

for $1\leq j$, $1\leq p\leq n$, L, $H\in\Gamma$: $|L|=|H|=[1]$. This holds for each $j\geq 1$. Therefore the second equations of 2) hold.

We shall call the equations of 3) in the above theorem the *Cauchy-Riemann equations* of a super smooth function.

Theorem 1.2. *If $f(z)$ is a super smooth function on a domain U in $\boldsymbol{R}^{m|n}$, then $f(z)$ is a regular mapping of U into Λ in the sense of the non-super differential calculus.*

Proof. By a straight calculation, we obtain the following:

$$f_{N+1}(z_{N+1}) = f_{N+1}(z_N+(z_{N+1}-z_N)) = f_{N+1}(z_N)+\sum_{i=1}^{m+n}\left(f\frac{\overleftarrow{\partial}}{\partial z^i}\right)_N(z_N)\cdot(z_{N+1}^i-z_N^i)\,.$$

This shows that $f(z)$ is regular in the sense of the non-super differential calculus.

1.4. Super vector spaces

The notion of super vector space is given in [1], which also develops the linear algebra over super vector spaces. Here we restrict ourselves to the real case. For details, see [1]. A two-sided Λ-module S is called a $\boldsymbol{Z}_2$-graded Λ-module if S has two subspaces $S_{[0]}$ and $S_{[1]}$ such that $S=S_{[0]}+S_{[1]}$ (direct sum) and $\Lambda_p\cdot S_q\subset S_{p+q}$ and $S_p\cdot\Lambda_q\subset S_{p+q}$ for $p, q\in\boldsymbol{Z}_2$. If an element x of S is in $S_{[0]}$ or $S_{[1]}$, then x is said to be *homogeneous*. And if $x\in S_{[0]}$ ($S_{[1]}$), then x is said to be *even* (*odd*) and the *parity* $|x|$ of x is, by definition, [0] ([1]). A $\boldsymbol{Z}_2$-graded Λ-module S is called a *super vector space* if $ax=(-1)^{ax}xa$ for any homogeneous elements $a\in\Lambda$ and $x\in S$ where a and x in $(-1)^{ax}$ denote their parities $|a|$ and $|x|$. A finite set $\{u_1, \cdots, u_k\}$ of vectors in S is called a *base* of S if each element in S is written uniquely as a linear combination of $\{u_1, \cdots, u_k\}$. Then k is called the *total dimension* of the super vector space S. If each vector in a base of S is homogeneous then the base is called a *homogeneous base*. If $\{u_1, \cdots, u_m, v_1, \cdots, v_m\}$ and $\{\bar{u}_1, \cdots, \bar{u}_{\bar{m}}, \bar{v}_1, \cdots, \bar{v}_{\bar{n}}\}$ are homogeneous bases of S such that $u_i, \bar{u}_{\bar{i}}$ are even and $v_j, \bar{v}_{\bar{j}}$ are odd, then we have that $m=\bar{m}$ and $n=\bar{n}$. The pair $(m|n)$ is called the *dimension* of the super vector space S. If a super vector space S has a base, then S has a homogeneous base. Let S be a finite dimensional super vector space and $\{u_1, \cdots, u_k\}$ a base of S. We define an equivalence relation, $\widetilde{N}$, on S as follows: Let $x=\sum u_i{}^ix$ and $y=\sum u_i{}^iy$ where ${}^ix, {}^iy\in\Lambda$. Then $x\widetilde{N}y$ if and only if $({}^ix)_N=({}^iy)_N\in\Lambda_N$ for each i. This definition is independent of a choice of a base of S. Then the *N-th skeleton* S_N of S is, by definition, the quotient space $S_N=S/\widetilde{N}$ of S by the relation $\widetilde{N}$. Then S_N is a $\boldsymbol{Z}_2$-graded Λ_N-module and $\{S_N\}$ forms in a natural way a projective family of finite dimensional real vector spaces whose projective limit is S.

Lemma 1.3. *Let S be a finite dimensional super vector space and $\{u_1, \cdots, u_p\}$ a set of super vectors of S. If $\{(u_1)_B, \cdots, (u_p)_B\}$ is linearly independent over $\boldsymbol{R}$, then there exist vectors $\{v_1, \cdots, v_q\}$ in S such that $\{u_1, \cdots, u_p, v_1, \cdots, v_q\}$ forms a base of S where* dim $S=p+q$.

Proof. Let A be a $(p+q, p)$-matrix whose components are in Λ. Then if rank $A_B=p$, there exists an invertible $(p+q)$-matrix P such that $A=P\cdot\begin{pmatrix}E\\0\end{pmatrix}$ where E denotes the identity p-matrix. In fact three exists a real invertible $(p+q)$-matrix Q such that $A_B=Q\cdot\begin{pmatrix}E\\0\end{pmatrix}$. Let $P=Q+(A-A_B, 0)$ where 0 denotes the $(p+q, q)$-zero matrix. Then P has the desired property. The above lemma follows from this assertion.

A subset $\bar{S}$ of a super vector space S is called a *super subspace* of S if $\bar{S}$ is a $\boldsymbol{Z}_2$-graded Λ-submodule of S. Let S be a finite dimentional super vector space. A super subspace $\bar{S}$ is said to be *normal* if there exists a base $\{u_1, \cdots, u_k\}$ of S such that $\{u_1, \cdots, u_{\bar{k}}\}$ $(\bar{k}\leq k)$ is a base of $\bar{S}$. Then a normal super subspace $\bar{S}$ is a finite dimensional super vector space itself and if dim $S=(m|n)$ and dim $\bar{S}=(\bar{m}|\bar{n})$ and $\{u_1, \cdots, u_{\bar{m}}, v_1, \cdots, v_{\bar{n}}\}$ a homogeneous base of $\bar{S}$, then there exist vectors $u_{\bar{m}+1}, \cdots, u_m, v_{\bar{n}+1}, \cdots, v_n\in S$ such that $\{u_1, \cdots, u_m, v_1, \cdots, v_n\}$ forms a homogeneous base of S. This follows from Lemma 1.3.

Lemma 1.4. *Let S be a finite dimensional super vector space and $\bar{S}$ a normal super subspace of S. If a vector x in S satisfies that $x\varepsilon$ is in $\bar{S}$ for each $\varepsilon\in\Lambda_{[1]}$, then x is in $\bar{S}$.*

Proof. Let $\{u_1, \cdots, u_{\bar{k}}\}$ be a base of S such that $\{u_1, \cdots, u_{\bar{k}}\}$ $(\bar{k}\leq k)$ is a base of $\bar{S}$. Let $x=\sum u_i{}^ic$ where ${}^ic$ is in Λ. Then $x\varepsilon=\sum u_i({}^ic\varepsilon)\in\bar{S}$ for each $\varepsilon\in\Lambda_{[1]}$. Thus ${}^ic\varepsilon=0$ for $\varepsilon\in\Lambda_{[1]}$ and $\bar{k}<i\leq k$. Therefore ${}^ic=0$ for $\bar{k}<i\leq k$ and hence x

is in $\bar{S}$.

Let S and $\bar{S}$ be super vector spaces and Φ a mapping of S into $\bar{S}$ whose image of $x \in S$ is denoted by $\Phi(x) \in \bar{S}$. Then Φ is called a *super linear mapping* of S into $\bar{S}$ if $\Phi(x+y)=\Phi(x)+\Phi(y)$ and $\Phi(xa)=\Phi(x)a$ for x, $y \in S$ and $a \in \Lambda$. Let Φ be a super linear mapping of S into $\bar{S}$. The *parity* $|\Phi|$ of a super linear mapping Φ is defined in a natural way, which is characterized by $|\Phi(z)| = |\Phi| \cdot |z|$ $(z \in S)$. Let S and $\bar{S}$ be finite dimensional super vector spaces and Φ an even super linear mapping of S into $\bar{S}$. Then if the rank of Φ_B is equal to dim S, the image $\Phi(S)$ of S by Φ is a normal super subspace of $\bar{S}$. This follows from Lemma 1.3.

EXAMPLE 1.1. Let ${}^{m|n}\Lambda$ be a set of all $m+n$ *column* vectors $z=({}^i z)$ whose components are super numbers. For an odd super number $\varepsilon \in \Lambda_{[1]}$, the scalar multiplications εz and $z\varepsilon$ are defined as follows:

$$\varepsilon({}^i z) = ((-1)^i \varepsilon\, {}^i z)$$
$$({}^i z)\varepsilon = ({}^i z \varepsilon)$$

where i in $(-1)^i$ denotes the parity $|i|$ of the coordinate index. The addition and the scalar multiplication by an even super number are defined as usual. Let e_i be the column vector whose i-th component is 1 and others are 0. Then each $z=({}^i z) \in {}^{m|n}\Lambda$ can be written as $z=\sum e_i\, {}^i z$. Thus $\{e_i\}$ is a homogeneous base of ${}^{m|n}\Lambda$ and the dimension of ${}^{m|n}\Lambda$ is $(m|n)$.

2. Manifolds

2.1. Non-super manifolds

Let $E=\varprojlim E_N$ be a projective limit of a projective family of a finite dimensional real vector spaces. A topological space M is called a *projectable* (*regular*) *manifold* modeled after the projective limit $E=\varprojlim E_N$ if there is a local coordinate system $\{(U_\alpha, \psi_\alpha)\}$ such that 1) $\{U_\alpha\}$ is an open covering of M, 2) ψ_α is a homeomorphism of $U_\alpha \subset M$ onto a domain $\psi_\alpha(U_\alpha) \subset E$ and 3) $\psi_\alpha \circ \psi_\beta^{-1}$ is a projectable (regular) diffeomorphism of a domain $\psi_\beta(U_\alpha \cap U_\beta)$ onto a domain $\psi_\alpha(U_\alpha \cap U_\beta)$ in E. On a projectable manifold M, we define an equivalence relation, $\underset{N}{\sim}$, as follows: If x and y in M are in a coordinate neighbourhood U with a local coordinate ψ such that $\psi(x)_N=\psi(y)_N$ in E_N, then $x \underset{N}{\sim} y$. Then this relation is an equivalence relation on M. The quotient space $M/_N$ is denoted by M_N, called the *N-th skeleton* of M. The projection of M onto $M_{\tilde{N}}$ will be denoted by p_N. For $N=0$, M_0 and p_0 will be denoted by M_B and p_B, respectively. The local coordinate system $\{(U_\alpha, \psi_\alpha)\}$ of M induces a local coordinate system $\{(U_{\alpha N}, \psi_{\alpha N})\}$ of M_N, which makes M_N an ordinary smooth manifold of dimension dim E_N where $U_{\alpha N}=p_N(U_\alpha) \subset M_N$ and $\psi_{\alpha N}$ is the induced one-to-one mapping of $U_{\alpha N}$

onto $\psi_{\alpha N}(U_{\alpha N})=(\psi_\alpha(U_\alpha))_N\subset E_N$. Then M can be regarded as the projective limit $\varprojlim M_N$ of the family $\{M_N\}$ of finite dimensional smooth manifolds. A subset U of M will be called a *domain* if $U_B=p_B(U)$ is a connected open subset of M_B and $U=p_B^{-1}(U_B)$. A domain of M can be regarded as a projectable manifold modeled after $E=\varinjlim E_N$ itself. If M is a projectable manifold then M is a fibre bundle over a base space M_B and M_N is a smooth fibre bundle over M_B. Moreover if M is a regular manifold then in a natural way M_{N+1} is an affine bundle over a base space M_N for $N\geq 0$.

Let M be a projectable (regular) manifold modeled after $E=\varprojlim E_N$ and $\bar{E}=\varprojlim \bar{E}_N$ a subspace of E where $\bar{E}_N$ is a vector subspace of E_N for $N\geq 0$. Then a subset $\bar{M}$ of M is called a *projectable* (*regular*) *submanifold* of M modeled after $\bar{E}=\varprojlim \bar{E}_N$ if for each point $o\in\bar{M}$ there exists a local projectable (regular) coordinate (U,ψ) of M such that $o\in U$, $\psi(o)=0$ and $\bar{M}\cap U=\{z\in U\colon \psi(z)\in\bar{E}\}$.

Let f be a real valued function on M. Then f is said to be *admissible* if $f\circ\psi^{-1}$ is an admissible function on a domain $\psi(U)$ in E for each local coordinate (U,ψ) of M. We denote the algebra of all germs of admissible functions at z in M by $\mathcal{A}(z)$. Let M and $\bar{M}$ be projectable manifolds and φ a mapping of M into $\bar{M}$. Then φ is said to be *projectable* if for each $N\geq 0$ there exists a smooth mapping φ_N of M_N into $\bar{M}_N$ such that $\varphi_N\circ p_N=\bar{p}_N\circ\varphi$ on M_N where $\bar{p}_N$ denotes the projection of $\bar{M}$ onto $\bar{M}_N$. The mapping φ_N is called the *N-th projection* of φ. Let M and $\bar{M}$ be regular manifolds and φ a projectable mapping of M into $\bar{M}$. Then φ is said to be *regular* if the $N+1$-th projection φ_{N+1} of M_{N+1} into $\bar{M}_{N+1}$ is an affine bundle homomorphism over a base mapping φ_N of M_N into $\bar{M}_N$ for each $N\geq 0$.

2.2. Super manifolds

A topological space M is called a *super manifold* of dimension $(m|n)$ if there exists a local coordinate system $\{(U_\alpha,\psi_\alpha)\}$ such that 1) $\{U_\alpha\}$ is an open covering of M, 2) ψ_β is a homeomorphism of $U_\alpha\subset M$ onto a domain $\psi_\alpha(U_\alpha)\subset\boldsymbol{R}^{m|n}$ and 3) $\psi_\alpha\circ\psi_\beta^{-1}$ is a super diffeomorphism of a domain $\psi_\beta(U_\alpha\cap U_\beta)$ onto a domain $\psi_\alpha(U_\alpha\cap U_\beta)$ in $\boldsymbol{R}^{m|n}$. It follows from Theorem 1.2 that a super manifold of dimension $(m|n)$ can be regarded as a regular manifold modeled after $\boldsymbol{R}^{m|n}=\varprojlim \boldsymbol{R}_N^{m|n}$. This regular manifold is called the *underlying non-super manifold* of the super manifold M. Then a domain of a super manifold is a super manifold itself. A Λ-valued function f on a super manfiold M is said to be *super smooth* if $f\circ\psi^{-1}$ is a super smooth function on a domain $\psi(U)\subset\boldsymbol{R}^{m|n}$ for each local coordinate (U,ψ) of M. We denote by $\mathcal{O}(z)$ the set of all germs of super smooth functions at z in M. In a natural way $\mathcal{O}(z)$ is a super vector space. That is, $f\in\mathcal{O}(z)$ is *even* (*odd*) if the value of f is in $\Lambda_{[0]}$ ($\Lambda_{[1]}$), respectively. $\mathcal{A}(z;\Lambda)$ denotes the set of all germs of Λ-valued admissible functions at z in M, which is a super vector space containing $\mathcal{O}(z)$ as a super subspace.

Let $M(\bar{M})$ be a super manifold of dimension $(m|n)$ $((\bar{m}|\bar{n}))$, respectively and φ a mapping of M into $\bar{M}$. Then φ is said to be *super smooth* if $\bar{\psi}_\lambda\circ\varphi\circ\psi_\alpha^{-1}$ is a super smooth mapping of a domain $\psi_\alpha(U_\alpha)\subset \boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{\bar{m}|\bar{n}}$ where (U_α, ψ_α) is a local coordinate of M and $(\bar{U}_\lambda, \bar{\psi}_\lambda)$ is a local coordinate of $\bar{M}$ such that $\varphi(U_\alpha)\subset \bar{U}_\lambda$. A super smooth mapping φ is regular on the underlying non-super manifold and particularly φ induces a smooth mapping φ_N of the N-th skeleton M_N into $\bar{M}_N$, the N-th projection of φ. Let (U, ψ) be a local coordinate of a super manifold M. We denote $z^i\circ\psi$ simply by z^i. Then $\tilde{\psi}=\{z^i_K\colon 1\leq i\leq m+n, K\in\Gamma, |i|=|K|\}$ is a local coordinate of the underlying non-super manifold of M where $z^i=\sum_{K\in\Gamma} z^i_K\,\zeta^K$. Let M be a super manifold of dimension $(m|n)$. A subset $\bar{M}$ of M is called a *super submanifold* of M of dimension $(\bar{m}|\bar{n})$ if for each $o\in\bar{M}$ there exists a local coordinate (U, ψ) around o in M such that $\psi=(z^i)=(x^\mu, \theta^p)$ and $\psi(o)=0$ and $U\cap\bar{M}=\{z\in U\colon x^{\bar{m}+1}=\cdots=x^m=\theta^{\bar{n}+1}=\cdots=\theta^n=0\}$. Then $\bar{M}$ itself is a super manifold in a natural way.

For $\varepsilon>0$, $I_\varepsilon\subset\boldsymbol{R}^{1|0}$ is defined by $I_\varepsilon=\{\tau\in\boldsymbol{R}^{1|0}\colon |\tau_B|<\varepsilon\}$. A super smooth mapping of I_ε into a super manifold M is called an *even super curve* on M. By Z-expansion, a non-super curve $c(t)$ on a super manifold $M(|t|<\varepsilon)$ defines uniquely an even super curve $\tilde{c}(\tau)$ on $M(\tau\in I_\varepsilon)$ such that $\tilde{c}(t)=c(t)$ for $|t|<\varepsilon$. Conversely each even super curve on M can be obtained from a non-super curve in such a way.

3. Tangent spaces

3.1. Non-super tangent spaces

Let M be a projectable manifold modeled after $E=\varprojlim E_N$. For each $z\in M$, $T_{z_N}(M_N)$ denotes the tangent space of the manifold M_N at z_N. Then the projection p_N^{N+1} of M_{N+1} onto M_N induces the differential $(p_N^{N+1})_*$ of $T_{z_{N+1}}(M_{N+1})$ onto $T_{z_N}(M_N)$ and $\{T_{z_N}(M_N)\}$ is a projective family of finite dimensional real vector spaces. The projective limit of this projective family will be denoted by $\mathcal{T}_z(M)$, called the *tangent space* at $z\in M$ of the projectable manifold M. Then $\mathcal{T}_z(M)$ is the vector space of all derivations of the algebra $\mathcal{A}(z)$.

Let M and $\bar{M}$ be projectable manifolds and φ a projectable mapping of M into $\bar{M}$. Then a projectable linear mapping φ_* of $\mathcal{T}_z(M)$ into $\mathcal{T}_{\varphi(z)}(\bar{M})$, called the *differential* of φ at $z\in M$, is defined in a natural way so that $(\varphi_N)_*\circ(p_N)_*=(\bar{p}_N)_*\circ\varphi_*$ on $\mathcal{T}_z(M)$ for $N\geq 0$.

Let $c(t)$ be a curve on a projectable manifold M. Then as usual we can define a tangent vector $\dot{c}(t)\in\mathcal{T}_{c(t)}(M)$, called the *tangent vector of a curve* $c(t)$ at t, so that $(\dot{c}(t))_N=\dot{c}_N(t)$ in $T_{c_N(t)}(M_N)$ where $c_N(t)=(c(t))_N$ is the *N-th* projection of the curve $c(t)$.

We shall prove the following theorem of the inverse mapping.

Theorem 3.1. *Let M and $\bar{M}$ be regular manifolds and φ a regular mapping* *of M into $\bar{M}$ such that the differential φ_* of φ at each $z\in M$ is an isomorphism of $\mathcal{T}_z(M)$ onto $\mathcal{T}_{\varphi(z)}(\bar{M})$. Then for each $z\in M$ there exists a domain $U\subset M$ containing z such that φ is a regular diffeomorphism of U onto $\varphi(U)$.*

Proof. In order to prove the theorem, we condider the case only locally. Let U be a domain in E containing 0 and φ a regular mapping of U into $\bar{E}$ whose Jacobi matrix $\mathcal{J}\varphi(z)$ is a projectable linear isomorphism of E onto $\bar{E}$ for each $z\in U$. Then we have to prove that there exists a domain V of E containing 0 such that φ is a regular diffeomorphism of V onto a domain $\varphi(V)$ in $\bar{E}$. Now the Jacobi matrix of each N-th projection φ_N is invertible at each point in U since the Jacobi matrix of φ is invertible. Therefore by the ordinary inverse mapping theorem there exists an open set V_B containing 0 in E_B such that φ_B is a difeomorphism of V_B onto an poen set $\varphi_B(V_B)$ in $\bar{E}_B$. We define a domain V in E by $V=p^{-1}(V_B)$ and an open set V_N in E_N by $V_N=p_N(V)$ $(N\geq 0)$. By induction we shall prove that φ_N is a diffiomorphism of V_N onto $\varphi_N(V_N)$. Now suppose that this holds at N. Since the mapping φ_{N+1} is an affine bundle homomorphism over the base space mapping φ_N of V_N into $\bar{E}_N$, φ_{N+1} is an affine mapping on each fibre $(p_N^{N+1})^{-1}(z_N)$ which is, by assumption, invertible for each $z_N\in V_N$. Therefore φ_{N+1} is a diffeomorphism of V_{N+1} onto $\varphi_{N+1}(V_{N+1})$. And hence φ is a regular diffeomorphism of V onto $\varphi(V)$.

Let M and $\bar{M}$ be regular manifolds and φ a regular mapping of M into $\bar{M}$ such that the differential φ_* of φ at each $z\in M$ is an isomorphism of $\mathcal{T}_z(M)$ into $\mathcal{T}_{\varphi(z)}(\bar{M})$. Then for each $z\in M$ there exists a domain $U\subset M$ containing z such that φ is a regular diffeomorphism of U onto a regular submanifold $\varphi(U)$ of $\bar{M}$.

3.2. Super tangent spaces

Let M be a super manifold of dimension $(m|n)$ and z a point in M. A mapping v of $\mathcal{O}(z)$ into Λ is called a *super tangent vector* at z if v satisfies the following conditions where $f\cdot v$ denotes the image of $f\in\mathcal{O}(z)$ by v. For each f, $g\in\mathcal{O}(z)$ and $a\in\Lambda$,

1) $(f+g)\cdot v=f\cdot v+g\cdot v$
2) $(af)\cdot v=a(f\cdot v)$
3) $(fg)\cdot v=f(z)\,(g\cdot v)+(-1)^{fg}g(z)\,(f\cdot v)$

where f, g in $(-1)^{fg}$ of 3) denote the parities of f, g. We denote by $T_z(M)$ the set of super tangent vectors at $z\subset M$, called the *super tangent space* of M at $z\in M$. The *parity* $|v|$ of a super tangent vector v is defined by $|f\cdot v|=|f|\cdot|v|$ for $f\in\mathcal{O}(z)$ and $v\in T_z(M)$. Then the super tangent space $T_z(M)$ of M at $z\in M$ is a super vector space in a natural way. Let $(U,\psi=(z^i))$ be a local coordinate around $z\in M$. Then as in an ordinary way a tangent vector $\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z\in T_z(M)$ is defined: $f\cdot\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z=(f\circ\psi^{-1})\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_{\psi(z)}$ for $f\in\mathcal{O}(z)$ where the right hand side is defined in a super differential calculus [2]. Then the parity of $\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z$ is the parity $|i|$ of the coordinate index i.

Theorem 3.2. *The super tangent space $T_z(M)$ is a super vector space of dimension $(m|n)$. Moreover $\{\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z\}$ forms a homogeneous base of $T_z(M)$ and for each $v\in T_z(M)$, $v=\sum\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z {}^iv$ where ${}^iv=z^i\cdot v$ $(1\leq i\leq m+n)$.*

Proof. Applying the following lemma the theorem will be obtained as usual.

Lemma 3.3. *Let f be a super smooth function on a domain U of $\boldsymbol{R}^{m|n}$ containing 0. Then there exist super smooth functions F_{ij} on U such that for each $z\in U$*

$$f(z)=f(0)+\sum_{i=1}^{m+n}f\frac{\overleftarrow{\partial}}{\partial z^i}(0)\cdot z^i+\sum_{i\leq j}F_{ij}(z)\cdot z^i\cdot z^j\,.$$

Proof. By Theorem 1.1, $f(z)$ can be written as follows:

$$f(z)=\sum_P\tilde{\varphi}_P(x)\cdot\theta^P \qquad \text{where} \quad z=(x,\theta)\,.$$

By the ordinary differential calculus, each $\varphi_P(t)$ $(t\in U_B)$ can be written as follows:

$$\varphi_P(t)=\varphi_P(0)+\sum_{\mu=1}^{m}\frac{\partial}{\partial t^\mu}\varphi_P(0)\cdot t^\mu+\sum_{\mu\leq\nu}\varphi_{P\mu\nu}(t)\cdot t^\mu\cdot t^\nu$$

for some smooth functions $\varphi_{P\mu\nu}(t)$ $(1\leq\mu\leq\nu\leq m)$.
Therefore we have

$$f(z)=\sum_P(\tilde{\varphi}_P(0)+\sum_{\mu=1}^{m}\tilde{\varphi}_P\frac{\overleftarrow{\partial}}{\partial x^\mu}(0)\cdot x^\mu+\sum_{\mu\leq\nu}\tilde{\varphi}_{P\mu\nu}(x)\cdot x^\mu\cdot x^\nu)\cdot\theta^P\,.$$

For $P=\phi$, $\tilde{\varphi}_\phi(0)=f(0)$ and $\tilde{\varphi}_\phi\frac{\overleftarrow{\partial}}{\partial x^\mu}(0)=f\frac{\overleftarrow{\partial}}{\partial x^\mu}(0)\,.$

And for $P=(p)$, $\tilde{\varphi}_{(p)}(0)=f\frac{\overleftarrow{\partial}}{\partial\theta^p}(0)$. Thus

$$\begin{aligned}f(z)=&f(0)+\sum_{\mu=1}^{m}f\frac{\overleftarrow{\partial}}{\partial x^\mu}(0)\cdot x^\mu+\sum_{\mu\leq\nu}\tilde{\varphi}_{\phi\mu\nu}(x)\cdot x^\mu\cdot x^\nu\\&+\sum_{p=1}^{n}(f\frac{\overleftarrow{\partial}}{\partial\theta^p}(0)+\sum_{\mu=1}^{m}\tilde{\varphi}_{(p)}\frac{\overleftarrow{\partial}}{\partial x^\mu}(0)\cdot x^\mu+\sum_{\mu\leq\nu}\tilde{\varphi}_{(p)\mu\nu}(x)\cdot x^\mu\cdot x^\nu)\cdot\theta^p\\&+\sum_P{}'(\tilde{\varphi}_P(0)+\sum_{\mu=1}^{m}\tilde{\varphi}_P\frac{\overleftarrow{\partial}}{\partial x^\mu}(0)\cdot x^\mu+\sum_{\mu\leq\nu}\tilde{\varphi}_{P\mu\nu}(x)\cdot x^\mu\cdot x^\nu)\cdot\theta^P\end{aligned}$$

where in the last term the sum, $\sum'$, is taken over $\{P=(p_1, \cdots, p_h)\colon h\geq 2\}$. This completes the proof of the lemma.

For $p\in \boldsymbol{Z}_2$, the subspace $T_z(M)_p$ of $T_z(M)$ is defined by $T_z(M)_p=\{v\in T_z(M)\colon |v|=p\}$. Since the super tangent space $T_z(M)$ is a super vector space with a finite dimension, the N-th skeleton $T_z(M)_N$ is well-defined.

Let M and $\bar{M}$ be super manifolds and φ a super smooth mapping of M into $\bar{M}$. Then the *super differential* φ_* of φ is defined as usual: For each $z\in M$, φ_* is a mapping of $T_z(M)$ into $T_{\varphi(z)}(\bar{M})$ defined by $f\cdot(\varphi_* v)=(f\circ\varphi)\cdot v$ for $f\in\mathcal{O}(\varphi(z))$ and $v\in T_z(M)$. Then φ_* is an even super linear mapping of $T_z(M)$ into $T_{\varphi(z)}(\bar{M})$: That is, $\varphi_*(u+v)=\varphi_* u+\varphi_* v$ and $\varphi_*(va)=(\varphi_* v)a$ and $|\varphi_* v|=|v|$ for u, $v\in T_z(M)$ and $a\in\Lambda$. In terms of local coordinates, the super differential can be expressed as follows:

$$\varphi_*\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z=\sum_j\left(\frac{\overleftarrow{\partial}}{\partial \bar{z}^j}\right)_{\varphi(z)}\left(\varphi^j\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z$$

where (z^i) is a local coordinate around z and $(\bar{z}^j)$ is a local coordinate around $\varphi(z)$ and $\varphi^j=\bar{z}^j\circ\varphi$. Since φ_* is a super linear mapping, we have the N-th projection, $(\varphi_*)_N$, of φ_* which is a mapping of $T_z(M)_N$ into $T_{\varphi(z)}(M)_N$. In particular, the 0-th projection is called the *body* of φ_*, denoted by $(\varphi_*)_B$, which is a $\boldsymbol{R}$-linear mapping of $T_z(M)_B$ into $T_{\varphi(z)}(\bar{M})_B$ where

$$T_z(M)_B=\left\{\sum\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z {}^i v\colon {}^i v\in\boldsymbol{R}\right\}.$$

Let $\gamma(\tau)$ be an even super curve on $M(\tau\in I_\varepsilon)$. Then the super tangent vector $\dot{\gamma}(\tau)\in T_{\gamma(\tau)}(M)$ of $\gamma(\tau)$ is defined as usual: For $f\in\mathcal{O}(\gamma(\tau))$, $f\cdot\dot{\gamma}(\tau)=(f\circ\gamma)\frac{\overleftarrow{d}}{d\tau}(\tau)$. In other words, $\dot{\gamma}(\tau)=\gamma_*\left(\frac{\overleftarrow{d}}{d\tau}\right)_\tau$. Thus $\dot{\gamma}(\tau)$ is an even super tangent vector. In terms of local coordinates, $T_z(M)$ can be indentified with the super vector space ${}^{m|n}\Lambda$. Then the super differential φ_* is a super linear mapping defined by the super Jacobi matrix $J\varphi(z)$ and the body $(\varphi_*)_B$ of φ_* is a linear mapping defined by the body $(J\varphi(z))_B$ of the matrix $J\varphi(z)$.

Now we obtain the following theorem by the inverse mapping theorem in a super differential calculus [2].

Theorem 3.4. *Let φ be a super smooth mapping of a super manifold M into a super manifold $\bar{M}$ such that the super differential φ_* of φ at a point $z\in M$ is a linear isomorphism of $T_z(M)$ onto $T_{\varphi(z)}(\bar{M})$. Then there exists a domain U of M cantaining the point z such that φ is a super diffeomorphism of U onto a domain $\varphi(U)$ of $\bar{M}$.*

3.3. Almost super structures

Let M be a super manifold of dimension $(m|n)$ and $(U, \psi=(z^i))$ a local coordinate of M. Then $\{\left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z\}$ is a base of the super vector space $T_z(M)$ and the even subspace $T_z(M)_{[0]}$ of $T_z(M)$ is given by

$$T_z(M)_{[0]} = \{\sum_i \left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z {}^i v \colon {}^i v \in \Lambda_p,\ p = |i|\}\ .$$

The local coordinate $\psi=(z^i)$ of M gives a local coordinate $\tilde{\psi}=(z^i_K)$ of the underlying non-super manifold of M. That is, $\psi_N=(z^i_N)=(z^i_K)$ is a local coordinate of M_N where $K\in\Gamma_N$ and $|K|=|i|$. Therefore the tangent space $\mathcal{T}_z(M)$ of the underlying non-super manifold of M is given by

$$\mathcal{T}_z(M) = \{\sum_{i,K} a^i_K \left(\frac{\partial}{\partial z^i_K}\right)_z \colon a^i_K\in \boldsymbol{R},\ K\in\Gamma,\ |K|=|i|\}\ .$$

Then the following correspondence of $T_z(M)_{[0]}$ to $\mathcal{T}(M)$ gives an $\boldsymbol{R}$-isomorphism.

$$T_z(M)_{[0]}\ni v = \sum_i \left(\frac{\overleftarrow{\partial}}{\partial z^i}\right)_z {}^i v \to \tilde{v} = \sum_{i,K} {}^i v_K \left(\frac{\partial}{\partial z^i_K}\right)_z \in \mathcal{T}_z(M)$$

where ${}^i v = \sum_{K\in\Gamma} {}^i v_K\, \zeta^K$ $(|K|=|i|)$. By a straight computation we see that the above correspondence is independent of the choice of local coordinate. Moreover we have that $f\cdot v=\tilde{v}\cdot f$ for $v\in T_z(M)_{[0]}$ and $f\in\mathcal{O}(z)\subset\mathcal{A}(z;\Lambda)$. In fact this follows from the Cauchy-Riemann equations of a super smooth function. Let M and $\bar{M}$ be super manifolds and φ a super smooth mapping of M into $\bar{M}$. Then the following diagram is commutative.

$$\begin{array}{ccc} T_z(M)_{[0]} & \xrightarrow{\varphi_*} & T_{\varphi(z)}(\bar{M})_{[0]} \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{T}_z(M) & \xrightarrow{\varphi_*} & \mathcal{T}_{\varphi(z)}(M) \end{array}$$

For each $H\in\Gamma_{[0]}$, we define a linear endomorphism J^H of $\mathcal{T}_z(M)$ by $J^H\,\tilde{v}=\widetilde{(\zeta^H v)}$ for each $v\in T_z(M)_{[0]}$. We call the family $\{J^H\colon H\in\Gamma_{[0]}\}$ of endomorphisms of $\mathcal{T}_z(M)$ the *almost super structure* on the underlying non-super manifold of a super manifold M. In particular, we have $J^H\left(\frac{\partial}{\partial z^i_K}\right)_z = \left(\frac{\partial}{\partial z^i_{K+H}}\right)_z$ for $H\in\Gamma_{[0]}$ and $K\in\Gamma$ with $|K|=|i|$.

We can prove the following theorem from the Cauchy-Riemann equations of a super smooth function.

Theorem 3.5. *Let M and $\bar{M}$ be super manifolds and φ a projectable mapp-*

ing of M into $\bar{M}$ w.r.t. the underlying non-super manifold structures. Then φ is super smooth if and only if $\varphi_* \circ J^H = J^H \circ \varphi_*$ on the tangent space $\mathcal{T}_z(M)$ for each $z \in M$ and $H \in \Gamma_{[0]}$.

4. Vector fields

4.1. Vector fields on an affine bundle

Let A be an affine bundle over a base space B with projection π and the standard fibre $\boldsymbol{A}^n$. A vector field $\tilde{X}$ on A is said to be *projectable* if there exists a vector field X on B such that $\pi_*(\tilde{X}_y) = X_{\pi(y)}$ for each $y \in A$. A projectable vector field $\tilde{X}$ is said to be *affine* if $(f_\alpha)_*(\tilde{X}|_{A_x})$ is an affine vector field on $\boldsymbol{A}^n$ for each $x \in U_\alpha$ where $\tilde{X}|_{A_x}$ denotes the vector field defined on the fibre A_x and $(U_\alpha, f_\alpha, g_{\alpha\beta})$ is a local trivialization of the affine bundle A over B. Let $\Psi_\alpha = (\psi_\alpha = (x^i), f_\alpha = (y^\nu))$ be a local affine coordinate on $\pi^{-1}(U_\alpha) \subset A$ where $\psi_\alpha = (x^i)$ is a local coordinate on $U_\alpha \subset B$ and (y^ν) is a natural coordinate of $\boldsymbol{A}^n$. Then a vector field $\tilde{X}$ is affine if and only if $\tilde{X}$ is written as follows:

$$\tilde{X} = \sum_{i=1}^{m} c^i(x) \frac{\partial}{\partial x^i} + \sum_{\nu=1}^{n} \Big(\sum_{\mu=1}^{n} A^\nu_\mu(x) y^\mu + b^\nu(x) \Big) \frac{\partial}{\partial y^\nu}$$

where $A^\nu_\mu(x)$, $b^\nu(x)$ and $c^i(x)$ are smooth functions on U and $\dim B = m$. An affine vector field $\tilde{X}$ in the above from is said to be *parallel* if c^i and A^ν_μ vanish identically for $1 \leq i \leq m$ and $1 \leq \nu, \mu \leq n$.

Theorem 4.1. *Let $\tilde{X}$ be an affine vector field on A and $X = \pi_*(\tilde{X})$ the vector field on B. Let ϕ_t be a local one-parameter group of local transformations generating X which is defined on $|t| < \varepsilon$ and an open set $V \subset B$. Then there exists a local one-parameter group $\tilde{\phi}_t$ of local transformations generating $\tilde{X}$ which is defined on $|t| < \varepsilon$ and $\tilde{V} = \pi^{-1}(V)$ and each $\tilde{\phi}_t$ is an affine bundle mapping with the base mapping ϕ_t for each $|t| < \varepsilon$.*

Proof. Suppose that $\tilde{X}$ is written in the above from in terms of a local affine coordinate $((x^i), (y^\nu))$ on $\pi^{-1}(U_\alpha)$. Then the differential equation for $\tilde{\phi}_t(x, y)$ is given as follows:

$$\frac{d}{dt} x^i = c^i(x) \qquad (1 \leq i \leq m)$$

$$\frac{d}{dt} y^\nu = \sum_{\mu=1}^{n} A^\nu_\mu(x) y^\mu + b^\nu(x) \quad (1 \leq \nu \leq n).$$

Thus $\phi_t(x)$ is the solution of the first p equations with $\phi_0(x) = x$. Then we consider the last q equations. That is,

$$\frac{d}{dt} y = A(\phi_t(x)) y + b(\phi_t(x))$$

where $y=(y^\nu)$, $A=(A^\nu_\mu)$ and $b=(b^\nu)$ $(1\leq\nu,\ \mu\leq n)$. Let $Y(t, x)$ be a smooth mapping into $GL(n; \boldsymbol{R})$ defined on $|t|<\varepsilon$ and $x\in U_\alpha$ such that $\frac{d}{dt}Y(t, x)=A(\phi_t(x))\cdot Y(t, x)$ and $Y(0, x)=E$. Since $A(\phi_t(x))$ is smooth on $|t|<\varepsilon$ and $x\in U_\alpha$, the above $Y(t, x)$ exists uniquely. Now let

$$\psi_t(x, y) = Y(t, x)\cdot\left(y+\int_0^t Y(s, x)^{-1}\cdot b(\phi_s(x))\,ds\right).$$

Then $\tilde{\Phi}_t(x, y)$ is given by $\tilde{\Phi}_t(x, y)=(\phi_t(x), \psi_t(x, y))$. This completes the proof.

4.2. Non-super vector fields

Let M be a projectable (regular) manifold modeled after $E=\varprojlim E_N$. Then the tangent bundle $\mathcal{T}(M)=\bigcup_{z\in M}\mathcal{T}_z(M)$ of M can be regarded as a projectable (regular) manifold modeled after $E\times E=\varprojlim E_N\times E_N$ in a natural way. That is, when (U, ψ) is a local coordinate of M, the differential ψ_* induces a one-to-one mapping of $\mathcal{T}(U)$ onto $U\times E$ which gives a local coordinate of $\mathcal{T}(M)$. In other words, the tangent bundle $\mathcal{T}(M)$ is the projective limit of the family $\{T(M_N)\}$ of the tangent bundles of $\{M_N\}$. A section v of the tangent bundle $\mathcal{T}(M)$ over M is called a *projectable* (*regular*) *vector field* on M if the section is a projectable (regular) mapping of M into $\mathcal{T}(M)$. That is, in terms of local coordinate (U, ψ), the mapping $z\to\psi_*(v_z)\in\mathcal{T}_{\psi(z)}(E)=E$ is a projectable (regular) mapping of U into E. Then the N-th projection v_N of a projectable vector field v gives a vector field on M_N. As usual we denote by v_B the 0-th projection of v, a vector field on M_B. Let u and v be projectable (regular) vector fields on M. Then the vector field $[u, v]$ is defined in a natural way, so that $[u, v]_N=[u_N, v_N]$ on M_N.

Let v be a projectable vector field on a projectable manifold M. Then ϕ_t is called a *local one-parameter group of local transformation* of M generating the projectable vector field v if ϕ_t is defined on $|t|<\varepsilon$ and a domain U of M and satisfies the following conditions:

1) the mapping $(-\varepsilon, \varepsilon)\times U\ni(t, z)\to\phi_t(z)\in M$ is projectable,
2) if $|t|$, $|s|$, $|t+s|<\varepsilon$ and z, $\phi_s(z)\in U$, then

$$\phi_{t+s}(z) = \phi_t(\phi_s(z)),$$

3) for each $z\in U$, v_z is the tangent vector of the curve $\phi_t(z)$ at $t=0$.

For a regular vector field we have the following theorem.

Theorem 4.2. *Let M be a regular manifold modeled after $E=\varprojlim E_N$ and v a regular vector field on M. Let ϕ^B_t be a local one-parameter group of local transformations generating the vector field v_B on M_B such that ϕ^B_t is defined on*

$|t|<\varepsilon$ and an open set U_B of M. Then there exists a local one-parameter group ϕ_t of local transformations generating the vector field v on M such that ϕ_t is defined on $|t|<\varepsilon$ and the domain $U=p_B^{-1}(U_B)$ of M and the mapping $(t, z)\to\phi_t(z)$ is a regular mapping.

Proof. This theorem follows immediately from Theorem 4.1.

As usual the bracket of vecor fields is given as follows.

Theorem 4.3. *Let u be a regular vector field on a regular manifold M and ϕ_t a local one-parameter group of local transformations generating the vector field u. Then for each projective vector field v on M, we have*

$$[u, v] = \lim_{t\to 0}\frac{1}{t}(v-\phi_{t*}(v)) .$$

4.3. Super vector fields

Let M be a super manifold of dimension $(m|n)$. Then the super tangent bundle $T(M)=\bigcup_{z\in M} T_z(M)$ of M can be regarded as a super manifold of dimension $(2m|2n)$ in a natural way. That is, when (U, ψ) is a local coordinate of M, the differential ψ_* induces a one-to-one mapping of $T(U)$ onto $U\times \boldsymbol{R}^{m|n}\subset\boldsymbol{R}^{2m|2n}$ which gives a local coordinate of $T(M)$. A section of the super tangent bundle $T(M)$ over M is called a *super vector field* on M if the section is a super smooth mapping of M into $T(M)$. Let X be a super vector field on M. Then for $z\in M$, we have $X_z\in T_z(M)$ and for a super smooth function f on M, $f\cdot X$ is a super smooth function on M where $(f\cdot X)(z)=f\cdot X_z$ for $z\in M$. A super vector field X on M is said to be *even* (*odd*) if X_z is an even (odd) tangent vector at each $z\in M$. In terms of local coordinate $(U, \psi=(z^i))$, a super vector field X can be written as follows: $X=\sum_i \frac{\overleftarrow{\partial}}{\partial z^i}\,{}^iX$ where ${}^iX=z^i\cdot X$. A super Lie bracket of vector fields X and Y on M is defined as follows: For a super smooth function f on M, $f\cdot[X, Y]=(f\cdot X)\cdot Y-(-1)^{XY}(f\cdot Y)\cdot X$ where X and Y in $(-1)^{XY}$ denote the parities of X and Y. Then $[X, Y]$ is a super vector field on M.

Let X be an even super vector field on M. Then by the correspondence of $T_z(M)_{[0]}$ onto $\mathfrak{T}_z(M)$ at each $z\in M$, X defines a non-super regular vector field $\tilde{X}$ on the underlying non-super manifold of M. In terms of local coordinate, $\tilde{X}$ is given by $\tilde{X}=\sum_{i,K}{}^iX_K\frac{\partial}{\partial z_K^i}$ where ${}^iX=\sum_K {}^iX_K\zeta^K$ $(|i|=|K|)$. Then for even super vector fields X and Y on M we have $[\tilde{X}, \tilde{Y}]=-\widetilde{[X, Y]}$.

Theorem 4.4. *Let u be a non-super regular vector field on a super manifold M and ϕ_t a local one-parameter group of local transformation generating the regular vector field u which is defined on $|t|<\varepsilon$ and a domain $U\subset M$. Then the* *following conditions are equivalent.*

1) *There exists an even super vector field* X *on* M *such that* $u=\tilde{X}$ *on* M.
2) $[u, J^H v]=J^H[u, v]$ *for each* $H\in\Gamma_{[0]}$ *and each non-super projectable vector field* v *on* M.
3) ϕ_t *is a super smooth mapping of* U *into* M *for* $|t|<\varepsilon$.

Proof. Suppose that u is written locally as follows: $u=\sum_{i,K} u_K^i \frac{\partial}{\partial z_K^i}$. Then let $u^i=\sum_K u_K^i \zeta^K$. Then $u=\tilde{X}$ for some even super vector field X if and only if each u^i is super smooth. By a straight calculation, for $|j|=|K|$ and $|H|=[0]$, we have

$$[u, J^H\left(\frac{\partial}{\partial z_L^j}\right)]=-\sum_{i,K}\left(\frac{\partial}{\partial z_{H+L}^j}u_K^i\right)\frac{\partial}{\partial z_K^i}=-\sum_i\frac{\overleftarrow{\partial}}{\partial z^i}\left(\frac{\partial}{\partial z_{H+L}^j}u^i\right)$$

$$J^H[u, \frac{\partial}{\partial z_L^j}]=J^H\left(-\sum_{i,K}\left(\frac{\partial}{\partial z_L^j}u_K^i\right)\frac{\partial}{\partial z_K^i}\right)=-\sum_i\frac{\overleftarrow{\partial}}{\partial z^i}\left(\frac{\partial}{\partial z_L^j}u^i\right)\cdot\zeta^H$$

under the identification of $\mathcal{T}_z(M)$ with $T_z(M)_{[0]}$. Thus the equivalence of 1) and 2) follows from the Cauchy-Riemann equations of a super smooth function. It follows from Theorem 4.3 and Theorem 3.5 that 3) implies 2). Conversely, applying the usual procedure we can show that 2) implies 3).

Let X be an even super vector field on M and $\tilde{X}$ the non-super regular vector field corresponding to X and ϕ_t a local one-parameter group of local transformations generating the non-super regular vector field $\tilde{X}$ on M such that ϕ_t is defined on $|t|<\varepsilon$ and a domain $U\subset M$. Then for each $|t|<\varepsilon$, the mapping $z\to\phi_t(z)$ is super smooth by Theorem 4.4. On the other hand, for each $z\in U$, the mapping $t\to\phi_t(z)$ is a curve on M and, by Z-expansion, the curve defines an even super curve, denoted by $\Phi_\tau(z)$, defined on $\tau\in I_\varepsilon$ so that $\phi_t(z)=\Phi_t(z)$ for $|t|<\varepsilon$ and $z\in U$. Then Φ_τ satisfies the following conditions:

1) the mapping $I_\varepsilon\times U\ni(\tau, z)\to\Phi_\tau(z)\in M$ is super smooth,
2) if τ, σ and $\tau+\sigma\in I_\varepsilon$ and z, $\Phi_\sigma(z)\in U$, then

$$\Phi_{\tau+\sigma}(z)=\Phi_\tau(\Phi_\sigma(z)),$$

3) for each $z\in U$, X_z is the super tangent vector of the even super curve $\Phi_\tau(z)$ at $\tau=0$.

Φ_τ is called the *local even super one-parameter group of local super transformations* generating the even super vector field X. Therefore we have the following theorem.

Theorem 4.5. *Let* X *be an even super vector field on* M. *Let* ϕ_t^B *be a local one-parameter group of local transformations generating the vector field* $\tilde{X}_B$ *on* M_B *such that* ϕ_t^B *is defined on* $|t|<\varepsilon$ *and an open set* U_B *of* M_B. *Then there* *exists a local even super one-parameter group of local super transformations generating the even vector field X on M such that Φ_τ is defined on $\tau\in I_\varepsilon$ and the domain $U=p_B^{-1}(U_B)$ of M.*

5. Frobenius' Theorem

5.1. Frobenius' Theorem on an affine bundle

A *differential system* D of dimension r on a smooth manifold M is a subbundle of the tangent bundle $T(M)$ of M with a *local base* around each point of M. That is, for each $x\in M$ there exist vector fields $X_1, \cdots, X_r$ on a neighborhood U of x which form a base of D_y for each $y\in U$. D is said to be *involutive* if, for any vector fields X and Y belonging to D, $[X, Y]$ also belongs to D.

Let A be an affine bundle over B with standard fibre $\boldsymbol{A}^n$ and projection π. An involutive differential system $\tilde{D}$ on A is said to be *affine* if $\tilde{D}$ has a local base $\{X_i, Y_k\}$ where each X_i is an affine vector field and each Y_k is a parallel vector field such that $\{\pi_*(X_i)\}$ is linearly independent. Then an affine differential system $\tilde{D}$ on A induces an involutive differential system D on the base space B so that $\pi_*(\tilde{D})=D$ and $\{\pi_*(X_i)\}$ forms a local base for D.

Theorem 5.1. *Let $\tilde{D}$ be an affine differential system on an affine bundle A over a base space B and D the induced involutive differential system on B. Let V be an integral submanifold of D and $\tilde{o}$ a point in $\pi^{-1}(V)\subset A$. Then there exists an integral submanifold $\tilde{V}$ of $\tilde{D}$ such that $\tilde{o}\in\tilde{V}$ and $\tilde{V}$ is an affine subbundle of $A|_V$ over V where $A|_V$ is the restriction of the affine bundle A to $V\subset B$.*

Proof. This follows from the following.

Lemma 5.2. *Let $(x^1, \cdots, x^m)$ and $(x^1, \cdots, x^m, y^1, \cdots, y^n)$ be the natural coordinates on $\boldsymbol{R}^m$ and $\boldsymbol{R}^{m+n}$, respectively, and π the natural projection of $\boldsymbol{R}^{m+n}$ onto $\boldsymbol{R}^m$ and $U=\{x\in\boldsymbol{R}^m\colon |x^i|<\varepsilon\}$ and $\tilde{U}=\pi^{-1}(U)$. Let $D\,(\tilde{D})$ be an involutive differential system on $\boldsymbol{R}^m\,(\boldsymbol{R}^{m+n})$, respectively, such that $\pi_*(\tilde{D}_{(x,y)})=D_x$ for each $(x,y)\in\boldsymbol{R}^{m+n}$ and $\dim D=a$ and $\dim\tilde{D}=a+b$. Suppose that $\{x\in U: x^{a+1}=c^{a+1}, \cdots, x^m=c^m\}$ is an integral submanifold of D for each $c=(c^j)\in\boldsymbol{R}^{m-a}$ with $|c^j|<\varepsilon$ $(a+1\leq j\leq m)$ and that there exists a local base $\{X_1, \cdots, X_a, Y_1, \cdots, Y_b\}$ of $\tilde{D}$ on $\tilde{U}$ such that*

$$X_i=\frac{\partial}{\partial x^i}+\sum_{\nu=1}^{n}\alpha_i^\nu(x,y)\frac{\partial}{\partial y^\nu}\quad(1\leq i\leq a)$$

$$Y_k=\sum_{\nu=1}^{n}\beta_k^\nu(x)\frac{\partial}{\partial y^\nu}\quad(1\leq k\leq b)$$

where $\alpha_i^\nu(x,y)=\sum_{\mu=1}^{n}A_{i\mu}^\nu(x)\,y^\mu+b_i^\nu(x)\,(1\leq i\leq a,\ 1\leq\nu\leq n)$ and $A_{i\mu}^\nu(x)$, $b_i^\nu(x)$ and $\beta_k^\nu(x)$ are smooth functions on U.

Then there exist smooth mappings $\varphi(x)$ of U into $GL(n;\boldsymbol{R})$ and $\xi(x)$ of U into $\boldsymbol{R}^n$ such that $\Phi\colon \bar{x}=x,\ y=\varphi(x)y+\xi(x)$ is a diffeomorphism of $\tilde{U}$ and

$$\Phi^{-1}(\{(\bar{x},\bar{y})\in\tilde{U}\colon \bar{x}^{a+1}=c^{a+1},\cdots,\bar{x}^m=c^m,\ \bar{y}^{b+1}=d^{b+1},\cdots,\bar{y}^n=d^n\})$$

is an integral submanifold of $\tilde{D}$ for each $(c,d)\in\boldsymbol{R}^{m-a}\times\boldsymbol{R}^{n-b}$ with $|c^i|<\varepsilon$ $(a+1\le i\le m)$.

Proof. When a function is written as the above $\alpha_i^\nu(x,y)$, the function is called an affine function along each fibre. The above expression of X_i and Y_k will be written as follows.

$$(X,Y)=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\begin{pmatrix}E_a & 0\\ 0 & 0\\ \alpha(x,y) & \beta(x)\end{pmatrix}.$$

Since the rank of the (n,b)-matrix $\beta(x)$ is b, there exists a smooth mapping $C(x)$ of U into $GL(n;\boldsymbol{R})$ such that $C(x)\beta(x)=\begin{pmatrix}E_b\\0\end{pmatrix}$. Define a diffeomorphism Φ of U by $\Phi\colon \bar{x}=x,\ \bar{y}=C(x)y$. Then we have

$$\Phi_*(X,Y)=\left(\frac{\partial}{\partial\bar{x}},\frac{\partial}{\partial\bar{y}}\right)\begin{pmatrix}E_a & 0\\ 0 & 0\\ f(x,y) & E_b\\ g(x,y) & 0\end{pmatrix}$$

where each component of $f(x,y)$ and $g(x,y)$ is an affine function along each fibre. Let $(\bar{X},\bar{Y})=(X,Y)\begin{pmatrix}E_a & 0\\ -f(x,y) & E_b\end{pmatrix}$. Then $\{\bar{X}_i,\bar{Y}_k\}$ forms a local base of $\tilde{D}$ on $\tilde{U}$. Let $\bar{x}=(\bar{x}^1,\cdots,\bar{x}^a)$, $\bar{u}=(\bar{x}^{a+1},\cdots,\bar{x}^m)$, $\bar{y}=(\bar{y}^1,\cdots,\bar{y}^b)$ and $\bar{v}=(\bar{y}^{b+1},\cdots,\bar{y}^n)$. Then we have

$$\Phi_*(\bar{X},\bar{Y})=\left(\frac{\partial}{\partial\bar{x}},\frac{\partial}{\partial\bar{u}},\frac{\partial}{\partial\bar{y}},\frac{\partial}{\partial\bar{v}}\right)\begin{pmatrix}E_a & 0\\ 0 & 0\\ 0 & E_b\\ \bar{g} & 0\end{pmatrix}$$

where each component of $\bar{g}=g\circ\Phi^{-1}(\bar{x},\bar{u},\bar{y},\bar{v})$ is an affine function along each fibre. That is,

$$\Phi_*(\bar{X}_i)=\frac{\partial}{\partial\bar{x}^i}+\sum_{t=1}^{n-b}\bar{g}_i^t(\bar{x},\bar{u},\bar{y},\bar{v})\frac{\partial}{\partial\bar{v}^t}\quad(1\le i\le a)$$

$$\Phi_*(\bar{Y}_k)=\frac{\partial}{\partial\bar{y}^k}\quad(1\le k\le b).$$

Since $[\Phi_*(\bar{X}_i),\Phi_*(\bar{Y}_k)]=-\sum_{t=1}^{n-b}\left(\frac{\partial}{\partial\bar{y}^k}\bar{g}_i^t\right)\frac{\partial}{\partial\bar{v}^t}$ is a linear combination of $\{\Phi_*(\bar{X}_i),$

$\Phi_*(\bar{Y}_k)\}$, it must vanish and hence g is a function of $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v})$. Therefore g is written as follows:

$$g_i^t(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) = \sum_{s=1}^{n-b} G_{i\,s}^t(\boldsymbol{x}, \boldsymbol{u})\, \boldsymbol{v}^s + h_i^t(\boldsymbol{x}, \boldsymbol{u}) \quad (1 \leq i \leq a,\ 1 \leq t \leq n-b)$$

where $G_{i\,s}^t$ and h_i^t are smooth functions of $(\boldsymbol{x}, \boldsymbol{u})$. Since $[\Phi_*(X_i), \Phi_*(X_j)]$ is a linear combination of $\left\{\dfrac{\partial}{\partial \boldsymbol{v}^t}\right\}$ and also is a linear combination of $\{\Phi_*(X_j), \Phi_*(Y_k)\}$, it must vanish. Let G_i be the square $(n-b)$-matrix whose (t, s)-component is $G_{i\,s}^t$ and $\omega = \sum_{i=1}^{a} -G_i(\boldsymbol{x}, \boldsymbol{u})\, d\boldsymbol{x}^i$ a $\mathfrak{gl}(n-b; \boldsymbol{R})$-valued 1-form on $U_{\bar{x}} = \{\boldsymbol{x} \in \boldsymbol{R}^a: |\boldsymbol{x}^i| < \varepsilon\}$ where $\boldsymbol{u} \in \boldsymbol{R}^{m-a}$ is regarded as a parameter. Then $[\Phi_*(X_i), \Phi_*(X_j)] = 0$ implies that $d\omega + \omega \wedge \omega = 0$ on $U_{\bar{x}}$ and hence there exists a smooth mapping G of $\boldsymbol{x} \in U_{\bar{x}}$ into $GL(n-b; \boldsymbol{R})$ with parameter $\boldsymbol{u} \in \boldsymbol{R}^{m-a}$ such that $G^{-1}dG = \omega$ on $U_{\bar{x}}$. That is, $\dfrac{\partial}{\partial \boldsymbol{x}^i} G = -GG_i$ $(1 \leq i \leq m-a)$ on $U = \{(\boldsymbol{x}, \boldsymbol{u}) \in \boldsymbol{R}^m: |\boldsymbol{x}^i|, |\boldsymbol{u}^j| < \varepsilon\}$. We define a diffeomorphism Ψ of $\tilde{U}$ by Ψ: $\bar{\bar{x}} = \boldsymbol{x}$, $\bar{\bar{u}} = \boldsymbol{u}$, $\bar{\bar{y}} = \bar{y}$, $\bar{\bar{v}} = G(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{v}$. Then

$$\Psi_*\Phi_*(\bar{X}, \bar{Y}) = \Psi_* \left(\frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial \boldsymbol{u}}, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \boldsymbol{v}}\right) \begin{pmatrix} E_a & 0 \\ 0 & 0 \\ 0 & E_b \\ g & 0 \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial \bar{\bar{x}}}, \frac{\partial}{\partial \bar{\bar{u}}}, \frac{\partial}{\partial \bar{\bar{y}}}, \frac{\partial}{\partial \bar{\bar{v}}}\right) \begin{pmatrix} E_a & 0 & 0 & 0 \\ 0 & E_{p-a} & 0 & 0 \\ 0 & 0 & E_b & 0 \\ k & * & 0 & G \end{pmatrix} \begin{pmatrix} E_a & 0 \\ 0 & 0 \\ 0 & E_b \\ g & 0 \end{pmatrix}$$

where $k_i^t = -(GG_i\boldsymbol{v})^t$. Then $(k+Gg)_i^t = \sum_{s=1}^{n-b} G_s^t h_i^s$ and hence

$$\Psi_*\Phi_*(\bar{X}, \bar{Y}) = \left(\frac{\partial}{\partial \bar{\bar{x}}}, \frac{\partial}{\partial \bar{\bar{u}}}, \frac{\partial}{\partial \bar{\bar{y}}}, \frac{\partial}{\partial \bar{\bar{v}}}\right) \begin{pmatrix} E_a & 0 \\ 0 & 0 \\ 0 & E_b \\ \bar{k} & 0 \end{pmatrix}$$

where each component of $\bar{k} = \bar{k}(\bar{\bar{x}}, \bar{\bar{u}})$ is a smooth function of $(\bar{\bar{x}}, \bar{\bar{u}}) \in U$. Therefore there exists a smooth mapping $\varphi(x, u)$ of U into $GL(n; \boldsymbol{R})$ such that

$$\Phi_*(\bar{X}, \bar{Y}) = \left(\frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial \boldsymbol{u}}, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \boldsymbol{v}}\right) \begin{pmatrix} E_a & 0 \\ 0 & 0 \\ 0 & E_b \\ \bar{k} & 0 \end{pmatrix}$$

where Φ: $\boldsymbol{x} = x$, $\boldsymbol{u} = u$, $\bar{y} = y$, $\boldsymbol{v} = \varphi(x, u)v$ and $\bar{k} = \bar{k}(\boldsymbol{x}, \boldsymbol{u})$. Then the components of $\bar{k}$ satisfy the following: $\frac{\partial}{\partial \bar{x}^i}\bar{k}^t_j=\frac{\partial}{\partial \bar{x}^j}\bar{k}^t_i$ on U for $1\leq i, j\leq a$, $1\leq t\leq n-a$. Therefore there exists a smooth function $K^t(\bar{x}, \bar{u})$ on U for $1\leq t\leq q-a$ such that $\bar{k}^t_i=\frac{\partial}{\partial \bar{x}^i}K^t$ on U for $1\leq i\leq a$. Define a diffeomorphism Ψ of $\tilde{U}$ by Ψ: $\bar{\bar{x}}=\bar{x}$, $\bar{\bar{u}}=\bar{u}$, $\bar{\bar{y}}=\bar{y}$, $\bar{\bar{v}}=\bar{v}-K(\bar{x}, \bar{u})$. Then

$$\Psi_*\Phi_*(\bar{X}, \bar{Y})=\left(\frac{\partial}{\partial \bar{\bar{x}}}, \frac{\partial}{\partial \bar{\bar{u}}}, \frac{\partial}{\partial \bar{\bar{y}}}, \frac{\partial}{\partial \bar{\bar{v}}}\right)\begin{pmatrix} E_a & 0 \\ 0 & 0 \\ 0 & E_b \\ 0 & 0 \end{pmatrix}.$$

This completes the proof.

5.2. Non-super Frobenius' Theorem

Let M be a projectable manifold modeled after $E=\varprojlim E_N$. A differential system $\mathcal{D}$ on M can be defined as usual: That is, for each $z\in M$, $\mathcal{D}_z$ is a vector subspace of $\mathcal{T}_z(M)$. A differential system $\mathcal{D}$ on M is said to be *projectable* if for each $N\geq 0$ there exists a smooth differential system D_N on M_N such that $(D_N)_{z_N}=(p_N)_*(\mathcal{D}_z)\subset T_{z_N}(M_N)$ for each $z\in M$. Let $\mathcal{D}$ be a projectable differential system on M. A projectable vector field v on M is said to belong to $\mathcal{D}$ if $v_z\in\mathcal{D}_z$ for each $z\in M$. $\mathcal{D}$ is said to be *involutive* if, for any projectable vector fields u and v belonging to $\mathcal{D}$, $[u, v]$ also belongs to $\mathcal{D}$. A set $\{X_i: i\geq 1\}$ of projectable vector fields on a domain $U\subset M$ is called a *local base* of $\mathcal{D}$ over U if for each $N\geq 0$ $\{(X_i)_N: 1\leq i\leq d_N\}$ forms a local base of the differential system D_N over an open set $U_N\subset M_N$ and $(X_i)_N=0$ $(d_N+1\leq i)$ where d_N denotes the dimension of the differential system D_N. Let M be a regular manifold modeled after $E=\varprojlim E_N$ and $\mathcal{D}$ a projectable differential system on M and $\{X_i: i\geq 1\}$ a local base of $\mathcal{D}$ over a domain $U\subset M$. The local base $\{X_i: i\geq 1\}$ is said to be *regular* if each X_i is a regular vector field on U and $\{(X_i)_{N+1}: d_N+1\leq i\leq d_{N+1}\}$ are parallel vector fields on each fibre $(p_N^{N+1})^{-1}(z_N)$ where each fibre $(p_N^{N+1})^{-1}(z_N)$ is regarded as an affine space. If for each point $z\in M$ there exists a regular local base of $\mathcal{D}$ over a domain U containing z, then $\mathcal{D}$ is said to be *regular*. Let $\bar{M}$ be a projectable submanifold of M. Then $\bar{M}$ is said to be an *integral submanifold* of $\mathcal{D}$ if $\mathcal{T}_z(\bar{M})=\mathcal{D}_z$ for each $z\in\bar{M}$. The following theorem follows from Theorem 5.1.

Theorem 5.3. *Let M be a regular manifold modeled after $E=\varprojlim E_N$ and $\mathcal{D}$ an involutive regular differential system on M. Then for any point $o\in M$, there exists an integral regular submanifold of $\mathcal{D}$ through o.*

5.3. Super Frobenius' Theorem

Let M be a super manifold of dimension $(m|n)$. A *super differential system*

D of $(\bar{m}|\bar{n})$-dimension on M is a subbundle of $T(M)$ satisfying the following condition: for each $z\in M$, there exist a domain U containing z and super vector fields $\{X_1, \cdots, X_m, \Theta_1, \cdots, \Theta_n\}$ on U such that $X_\mu (1\leq \mu \leq m)$ is even and $\Theta_p (1\leq p\leq n)$ is odd and $\{X_1, \cdots, X_m, \Theta_1, \cdots, \Theta_n\}$ forms a base of $T_z(M)$ at each $z\in U$ and $\{X_1, \cdots, X_{\bar{m}}, \Theta_1, \cdots, \Theta_{\bar{n}}\}$ forms a base of D_z at each $z\in U$. Then $\{X_1, \cdots, X_{\bar{m}}, \Theta_1, \cdots, \Theta_{\bar{n}}\}$ is called a *local base* of D on U. Thus each D_z is a normal super vector subspace of the super vector space $T_z(M)$. A super differential system D on M is said to be *involutive* if, for any super vector field X and Y belonging to D, $[X, Y]$ also belongs to D. A super differential system D on M defines a differential system $\mathscr{D}$ on the non-super underlying manifold of M as follows: For $z\in M$, $\mathscr{D}_z$ is a subspace of $\mathscr{T}_z(M)$ corresponding to $D_{z[0]}$, the even space of D_z, under the identification between $T_z(M)_{[0]}$ and $\mathscr{T}_z(M)$. The differential system $\mathscr{D}$ on M is called the *associated differential system* with D. Then we can prove by Lemma 1.4 that D is involutive if and only if $\mathscr{D}$ is involutive.

Theorem 5.4. *Let D be a super differential system on a super manifold M and $\mathscr{D}$ the associated differential system. Then the differential system $\mathscr{D}$ is regular in the sense of non-super differential calculus.*

Proof. Let $\{X_1, \cdots, X_{\bar{m}}, \Theta_1, \cdots, \Theta_{\bar{n}}\}$ be a local base of D on a domain U. In terms of local coordinate $(U, \psi=(z^i))$, X_ν and Θ_q are written as follows:

$$X_\nu = \sum_i \frac{\partial}{\partial z^i}\, {}^iX_\nu, \quad \Theta_q = \sum_i \frac{\partial}{\partial z^i}\, {}^i\Theta_q \quad (1\leq \nu \leq \bar{m}, 1\leq q\leq \bar{n}).$$

For $1\leq \nu\leq \bar{m}$, $1\leq q\leq \bar{n}$, $H\in\Gamma_{[0]}$ and $L\in\Gamma_{[1]}$, let $\tilde{X}_\nu^H=\widetilde{(X_\nu \zeta^H)}$ and $\tilde{\Theta}_q^L=\widetilde{(\Theta_q\zeta^L)}$. Then $\{\tilde{X}_\nu^H, \tilde{\Theta}_q^L\}$ forms a local base for the associated differential system $\mathscr{D}$ on U: That is,

$$\{(\tilde{X}_\nu^H)_{N+1}, (\tilde{\Theta}_q^L)_{N+1}: 1\leq \nu\leq \bar{m}, 1\leq q\leq \bar{n}, H, L\in\Gamma_{N+1}, |H| = [0], |L| = [1]\}$$

is a local base of $D_{N+1}=p_{N+1*}(\mathscr{D})$ on $U_{N+1}=p_{N+1}(U)$. Among these vector fields, each of

$$\{(\tilde{X}_\nu^H)_{N+1}, (\tilde{\Theta}_q^L)_{N+1}: 1\leq \nu\leq \bar{m}, 1\leq q\leq \bar{n}, H, L\in(\Gamma_{N+1}-\Gamma_N), |H|= [0], |L|= [1]\}$$

vanishes by the projection p_N^{N+1} of U_{N+1} onto U_N. In terms of local coordinate $(U, \psi=(z^i))$, for $1\leq \nu\leq \bar{m}$, $1\leq q\leq \bar{n}$, $H\in\Gamma_{[0]}$ and $L\in\Gamma_{[1]}$.

$$(\tilde{X}_\nu^H)_{N+1} = \sum_{i,K} {}^iX_{\nu K} \frac{\partial}{\partial z^i_{K+H}} \quad \text{and} \quad (\tilde{\Theta}_q^L)_{N+1} = \sum_{i,K} {}^i\Theta_{qK} \frac{\partial}{\partial z^i_{K+L}}$$

where ${}^iX_\nu = \sum_K {}^iX_{\nu K}\,\zeta^K$ $(|K|=|i|)$ and ${}^i\Theta_q = \sum_K {}^i\Theta_{qK}\,\zeta^K$ $(|K|=|i|+1)$ and $K+H, K+L\in\Gamma_{N+1}$. If H and L are in $\Gamma_{N+1}-\Gamma_N$, then both H and L contain $N+1$ and hence all K in the above sums are in Γ_N. Therefore the coefficients of $(\tilde{X}_\nu^H)_{N+1}$ and $(\tilde{\Theta}_q^L)_{N+1}$ are functions of $z_N\in U_N$. Thus $\mathscr{D}$ is a regular differential system on M.

Theorem 5.5. *Let D be a super differential system on a super manifold M and $\bar{M}$ a regular submanifold of the underlying non-super regular manifold of M. Then if $\bar{M}$ is an integral submanifold of the associated regular differential system $\mathcal{D}$, $\bar{M}$ is a super submanifold of M.*

Proof. Let $o\in\bar{M}\subset M$ and $\{X_1, \cdots, X_{\bar{m}}, \Theta_1, \cdots, \Theta_{\bar{n}}\}$ a local base of D on a domain U containing o and $(U, \psi=(x^\mu, \theta^p))$ a local coordinate such that $(X_\nu)_o=\left(\frac{\overleftarrow{\partial}}{\partial x^\nu}\right)_o$ and $(\Theta_q)_o=\left(\frac{\overleftarrow{\partial}}{\partial \theta^q}\right)_o$ for $1\leq\nu\leq\bar{m}$, $1\leq q\leq\bar{n}$. We denote by π the projection of $\boldsymbol{R}^{m|n}$ onto $\boldsymbol{R}^{\bar{m}|\bar{n}}$ defined by π: $\bar{x}^\nu=x^\nu$, $\bar{\theta}^q=\theta^q(1\leq\nu\leq\bar{m}, 1\leq q\leq\bar{n})$. We take U so small that $\pi_*\psi_*(X_{\nu z})$ $(1\leq\nu\leq\bar{m})$ and $\pi_*\psi_*(\Theta_{qz})$ $(1\leq q\leq\bar{n})$ are linearly independent for each $z\in U$. Let $\bar{\psi}=\pi\circ\psi\circ\iota$ a regular mapping of $U\cap\bar{M}$ into $\boldsymbol{R}^{\bar{m}|\bar{n}}$ where ι denotes the inclusion of $U\cap\bar{M}$ into U. Then for each $z\in U\cap\bar{M}$, $\bar{\psi}_*$ is a $\boldsymbol{R}$-linear isomorphism of $\mathcal{T}_z(\bar{M})$ onto $\mathcal{T}_{\bar{\psi}(z)}(\boldsymbol{R}^{\bar{m}|\bar{n}})$. Thus it follows from Theorem 3.4 that if we take U sufficiently small, then $\bar{\psi}$ is a regular diffeomorphism of $U\cap\bar{M}$ onto a domain $\bar{\psi}(U\cap\bar{M})$ of $\boldsymbol{R}^{\bar{m}|\bar{n}}$. Moreover we can show that $\bar{\psi}_*\circ J^H=J^H\circ\bar{\psi}_*$ for $H\in\Gamma_{[0]}$ and hence $\bar{\psi}^{-1}$ is a super imbedding of $\bar{\psi}(U\cap\bar{M})$ into M whose image is $U\cap\bar{M}$ and hence $\bar{M}$ is a super submanifold of M.

A super submanifold $\bar{M}$ of M is called an *integral super submanifold* of a super differential system D on M if, for each $z\in\bar{M}$, $T_z(\bar{M})$ equals D_z. Then the following theorem is a straight consequence of Theorem 5.4, Theorem 5.3 and Theorem 5.5.

Theorem 5.6. *Let D be an involutive super differential system on a super manifold M and $o\in M$. Then there exists an integral super submanifold of D through o.*

References

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