Morita, T.
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ORDERS OF KNOTS IN THE ALGEBRAIC KNOT COBORDISM GROUP

TOSHIYUKI MORITA

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1. Introduction

The algebraic knot cobordism group $G_{\pm}$ was introduced by Levine [4] in order to study the cobordism groups of codimension two knots. In [5], he gave a complete set of invariants for $G_{\pm}$ and showed that $G_{\pm}$ is isomorphic to $\boldsymbol{Z}^{\infty}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{\infty}\oplus(\boldsymbol{Z}/4\boldsymbol{Z})^{\infty}$. In particular the order $a(K)$ of an odd dimensional knot K in the algebraic knot cobordism group is equal to 1, 2, 4 or infinite, and it is determined as follows.

Theorem A. ([5] Prop. 22) (1) *$a(K)$ is finite if and only if the local signature $\sigma_{\varphi}(K)$ vanishes for every symmetric irreducible real factor $\varphi(t)$ of the Alexander polynomial $\Delta(t)$ of K.*

(2) *Suppose that $a(K)$ is finite. Then $a(K)=4$ if and only if for some p-adic number field $\boldsymbol{Q}_p$, there exists a symmetric irreducible factor $\lambda(t)$ of $\Delta(t)$ over $\boldsymbol{Q}_p$, such that*

$$((-1)^d\lambda(1)\lambda(-1),\ -1)_p = -1 \quad \textit{and} \quad \varepsilon_{\lambda}(K) = 1\,.$$

Here $(\ ,\ )_p$ is the Hilbert symbol and $d=(1/2)\deg\lambda(t)$, and $\varepsilon_{\lambda}(K)$ is defined as follows. Let $\Phi(t)$ be the symmetric irreducible factor of $\Delta(t)$ over $\boldsymbol{Q}$ which has $\lambda(t)$ as an irreducible factor over $\boldsymbol{Q}_p$. Then $\varepsilon_{\lambda}(K)$ is the exponent of $\Phi(t)$ in $\Delta(t)$ modulo 2.

However, in order to determine whether $a(K)=4$ or not, we must check the Hilbert symbols for every prime number. The purpose of this paper is to prove the following theorem, which improves Theorem A and enables us to determine $a(K)$ through a finite procedure.

Theorem. *If $p\not| 2\Delta(-1)$, then $((-1)^d\lambda(1)\lambda(-1),\ -1)_p=+1$ for any symmetric irreducible factor $\lambda(t)$ of $\Delta(t)$ over $\boldsymbol{Q}_p$.*

Thus, to determine whether $a(K)=4$ or not, it suffices to check the Hilbert symbols only for prime factors of $2\Delta(-1)$. By using this theorem, we determine $a(K)$ of evrey prime classical knot K up to 10-crossings.

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2. Proof of Theorem

We need the following lemma for the proof of Theorem (see [7] p. 26, 13: 7).

Lemma. *Let $f(t)$ be the product $f_1(t)f_2(t)\cdots f_n(t)$ of irreducible polynomials $f_i(t)$ $(1\leqq i\leqq n)$ in $\boldsymbol{Q}_p[t]$ such that $f_i(0)=\pm 1$ $(1\leq i\leq n)$. If $f(t)\in\boldsymbol{Z}_p[t]$, then $f_i(t)\in\boldsymbol{Z}_p[t]$ for any i $(1\leq i\leq n)$.*

Proof of Theorem. If p-adic integers q, r are coprime with p and $p\neq 2$, then we have $(q, r)_p=+1$ (cf. [9] p. 20 Theorem 1). Hence it suffices to show that $\lambda(1)\lambda(-1)\in\boldsymbol{Z}_p$ and $\lambda(1)\lambda(-1)\not\equiv 0$ (mod $p\boldsymbol{Z}_p$) for any symmetric irreducible factor $\lambda(t)$ of $\Delta(t)$ in $\boldsymbol{Q}_p[t]$.

Since $\Delta(t)=\Delta(t^{-1})$, there is a polynomial $F(x)$ in $\boldsymbol{Z}_p[x]$ such that $\Delta(t)=F(t-2+t^{-1})$ and $F_j(0)=\Delta(1)=\pm 1$. Let $F(x)=F_1(x)F_2(x)\cdots F_n(x)$ be a prime factorization of $F(x)$ in $\boldsymbol{Q}_p[x]$. If necessary by multiplying a constant to each factor, we may assume that $F_j(0)=\pm 1$ for any $j(1\leq j\leq n)$. Then, by Lemma, $F_j(x)\in\boldsymbol{Z}_p[x]$ for any j $(1\leq j\leq n)$. Put $\lambda_j(t)=F_j(t-2+t^{-1})$. Then $\lambda_j(t)$ is symmetric and $\Delta(t)=\lambda_1(t)\lambda_2(t)\cdots\lambda_n(t)$.

Since $F_j(x)$ is irreducible in $\boldsymbol{Q}_p[x]$, $\lambda_j(t)$ can not be decomposed into symmetric irreducible polynomials in $\boldsymbol{Q}_p[t]$. Hence $\lambda_j(t)$ is irreducible or decomposed into non-symmetric irreducible polynomials in $\boldsymbol{Q}_p[t]$. Hence we may suppose that $\lambda_1(t),\cdots,\lambda_k(t)$ are irreducible and $\lambda_{k+1}(t),\cdots,\lambda_n(t)$ are decomposedinto non-symmetric irreducible polynomials in $\boldsymbol{Q}_p[t]$. Since $F_j(x)\in\boldsymbol{Z}_p[x]$, for any j $(1\leq j\leq k)$,

$$\lambda_j(1)\lambda_j(-1) = F_j(0)F_j(-4)\in\boldsymbol{Z}_p\,.$$

Since $p\nmid\Delta(-1)$,

$$\textstyle\prod_{j=1}^{n}\lambda_j(1)\lambda_j(-1) = \Delta(1)\Delta(-1)\not\equiv 0 \qquad (\mathrm{mod}\ p\boldsymbol{Z}_p)\,.$$

Hence, for any j $(1\leq j\leq k)$,

$$\lambda_j(1)\lambda_j(-1)\not\equiv 0 \qquad (\mathrm{mod}\ p\boldsymbol{Z}_p)\,.$$

This completes the proof of Theorem.

3. Application

By using our theorem, we can determine $a(K)$ of every prime knot K up to 10-crossings. To illustrate our method, we present the calculation for the knot 8_{13}. The Alexander polynomial $\Delta(t)$ of 8_{13} is $2t^4-7t^3+11t^2-7t+2$. The irreducible factorization of this polynomial in $\boldsymbol{R}[t]$ is

$$\Delta(t) = (\alpha t^2+\beta t+\gamma)(\gamma t^2+\beta t+\alpha)\,,$$

where

$$\alpha = (1+\sqrt{29}+\sqrt{2(\sqrt{29}-1)})/4\,,$$
$$\beta = (1-\sqrt{29})/2\,,$$
$$\gamma = (1+\sqrt{29}-\sqrt{2(\sqrt{29}-1)})/4\,.$$

Thus $\Delta(t)$ has no symmetric irreducible real factor and hence $a(8_{13})$ is finite by Theorem A (1). Since $\Delta(t)$ is irreducible in $\boldsymbol{Z}[t]$, $a(8_{13})\neq 1$ by [3]. So we consider whether $a(8_{13})=2$ or 4. Since $2\Delta(-1)=2\cdot 29$, it sufficies to check the Hilbert symbols only for $\boldsymbol{Q}_2$ and $\boldsymbol{Q}_{29}$ by Theorem. The irreducible factorization of $\Delta(t)$ in $\boldsymbol{Q}_2[t]$ is

$$\Delta(t) = (at+b)(ct+d)(et^2+ft+e)\,,$$

where

$$\mathrm{a} = 0+1\cdot 2+0\cdot 2^2+0\cdot 2^3+\cdots\,, \quad \mathrm{b} = 1+1\cdot 2+0\cdot 2^2+0\cdot 2^3+\cdots\,,$$
$$\mathrm{c} = 1+0\cdot 2+0\cdot 2^2+1\cdot 2^3+\cdots\,, \quad \mathrm{d} = 0+1\cdot 2+1\cdot 2^2+0\cdot 2^3+\cdots\,,$$
$$\mathrm{e} = 1+0\cdot 2+0\cdot 2^2+0\cdot 2^3+\cdots\,, \quad \mathrm{f} = 1+0\cdot 2+0\cdot 2^2+0\cdot 2^3+\cdots\,.$$

Hence, the symmetric irreducible factor of $\Delta(t)$ in $\boldsymbol{Q}_2[t]$ is only et^2+ft+e. Put $\lambda(t)=et^2+ft+e$. Then

$$\begin{aligned}((-1)^d\lambda(1)\lambda(-1),\,-1)_2 &= (-(2e+f)(2e-f),\,-1)_2\\ &= (1+0\cdot 2+1\cdot 2^2+1\cdot 2^3+\cdots,\,-1)_2\\ &= +1 \qquad \text{(cf. [9] p. 20 Theorem 1)}\,.\end{aligned}$$

Next, we check the Hilbert symbols for $\boldsymbol{Q}_{29}$. In general, if $p\equiv 1 \pmod 4$, then $(q,\,-1)_p=+1$ for any element q of $\boldsymbol{Q}_p$ (cf. [9] p. 20 Theorem 1). Since $29\equiv 1$ (mod 4),

$$((-1)^d\lambda(1)\lambda(-1),\,-1)_{29} = +1$$

for any symmetric irreducibe factor $\lambda(t)$ of $\Delta(t)$ in $\boldsymbol{Q}_{29}[t]$. Hence we obtain $a(8_{13})=2$.

The following is a table of knots up to 10-crossings in the table of [8] with finite order in the algebraic knot cobordism group. The second column ($|\Delta(-1)|$) is a list of the prime factorization of $|\Delta(-1)|$ of the Alexander polynomial $\Delta(t)$ of a knot K (cf. [1]). The third column ($\langle p,\,\lambda(t)\rangle$) is a list of a minimal prime number and a symmetric irreducible factor $\lambda(t)$ of $\Delta(t)$

over $\boldsymbol{Q}_p$ with $((-1)^d\lambda(1)\lambda(-1)_p, -1)=-1$ and $\varepsilon_\lambda(K)=1$. In the third column, the symbol "—" denotes that there is no factor and prime number with this condition. In the last column, $o(K)$ denotes the order of a knot K in the classical knot cobordism group C_1 introduced by [3]. The symbol "A" (resp. "S") denotes that the corresponding knot is amphicheiral (resp. slice). Amphicheirality is copied from [1]. Sliceness is copied from [2] (cf. [6]).

K	$\lvert\Delta(-1)\rvert$	$\langle p, \lambda(t)\rangle$	$a(K)$	$o(K)$
4_1	5	—	2	2 (A)
6_1	3^2	—	1	1 (S)
6_3	13	—	2	2 (A)
7_7	$3\cdot 7$	$\langle 3, \Delta(t)\rangle$	4	?
8_1	13	—	2	?
8_3	17	—	2	2 (A)
8_8	5^2	—	1	1 (S)
8_9	5^2	—	1	1 (S, A)
8_{12}	29	—	2	2 (A)
8_{13}	29	—	2	?
8_{17}	37	—	2	2 (A)
8_{18}	$3^2\cdot 5$	—	2	2 (A)
8_{20}	3^2	—	1	1 (S)
9_{14}	37	—	2	?
9_{19}	41	—	2	?
9_{24}	$3^2\cdot 5$	—	2	?
9_{27}	7^2	—	1	1 (S)
9_{30}	53	—	2	?
9_{33}	61	—	2	?
9_{34}	$3\cdot 23$	$\langle 3, t^2-(1+1\cdot 3+\cdots)t+1\rangle$	4	?
9_{37}	$3^2\cdot 5$	—	2	?
9_{41}	7^2	—	1	1 (S)
9_{44}	17	—	2	?
9_{46}	3^2	—	1	1 (S)
10_1	17	—	2	?
10_3	5^2	—	1	1 (S)
10_{10}	$3^2\cdot 5$	—	2	?
10_{13}	53	—	2	?
10_{17}	41	—	2	2 (A)
10_{22}	7^2	—	1	1 (S)
10_{26}	61	—	2	?
10_{28}	53	—	2	?
10_{31}	$3\cdot 19$	$\langle 3, \Delta(t)\rangle$	4	?
10_{33}	$5\cdot 13$	—	2	2 (A)

K	$\lvert\Delta(-1)\rvert$	$\langle p, \lambda(t)\rangle$	$a(K)$	$o(K)$
10_{34}	37	—	2	?
10_{35}	7^2	—	1	1 (S)
10_{37}	53	—	2	2 (A)
10_{42}	3^4	—	1	1 (S)
10_{43}	73	—	2	2 (A)
10_{45}	89	—	2	2 (A)
10_{48}	7^2	—	1	1 (S)
10_{58}	$5\cdot 13$	—	2	?
10_{60}	$5\cdot 17$	—	2	?
10_{68}	$3\cdot 19$	$\langle 3, \Delta(t)\rangle$	4	?
10_{71}	$7\cdot 11$	$\langle 7, r^2-(5+2\cdot 7+\cdots)t+1\rangle$	4	?
10_{75}	3^4	—	1	1 (S)
10_{79}	61	—	2	2 (A)
10_{81}	$5\cdot 17$	—	2	2 (A)
10_{86}	83	$\langle 83, \Delta(t)\rangle$	4	?
10_{87}	3^4	—	1	1 (S)
10_{88}	101	—	2	2 (A)
10_{90}	$7\cdot 11$	$\langle 7, t^2-(5+0\cdot 7+\cdots)t+1\rangle$	4	?
10_{91}	73	—	2	?
10_{96}	$3\cdot 31$	$\langle 3, t^2-(1+1\cdot 3+\cdots)t+1\rangle$	4	?
10_{99}	3^4	—	1	1 (S, A)
10_{102}	73	—	2	?
10_{104}	$7\cdot 11$	$\langle 7, t^2-(5+4\cdot 7+\cdots)t+1\rangle$	4	?
10_{107}	$3\cdot 31$	$\langle 3, t^4+(1+2\cdot 3+\cdots)t^3+(0+0\cdot 3+\cdots)t^2+(1+\cdots)t+1\rangle$	4	?
10_{109}	$5\cdot 17$	—	2	2 (A)
10_{115}	109	—	2	2 (A)
10_{118}	97	—	2	2 (A)
10_{119}	101	—	2	?
10_{123}	11^2	—	1	1 (S, A)
10_{129}	5^2	—	1	1 (S)
10_{135}	37	—	2	?
10_{137}	5^2	—	1	1 (S)
10_{140}	3^2	—	1	1 (S)
10_{146}	$3\cdot 11$	$\langle 3, t^2-(1+1\cdot 3+\cdots)t+1\rangle$	4	?
10_{153}	1	—	1	1 (S)
10_{155}	5^2	—	1	1 (S)
10_{158}	$3^2\cdot 5$	—	2	?
10_{165}	$3^2\cdot 5$	—	2	?

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Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka, 558, Japan
and
Sima Seiki Co., LTD.
Sakata, Wakayama,
641, Japan