Nagatomo, K.
Osaka J. Math.
25 (1988), 49-70

FORMAL POWER SERIES SOLUTIONS OF THE STATIONARY AXISYMMETRIC VACUUM EINSTEIN EQUATIONS

KIYOKAZU NAGATOMO*

(Received September 18, 1986)

Introduction

The present paper is concerned with the formal power series solution of the stationary axially symmetric vacuum (SAV) Einstein equations. By using a formal Riemann-Hilbert problem we shall present an explicit formula representing the formal solution in terms of a certain $\infty\times\infty$ matrix ξ. The space V_0 of all formal solutions corresponding to the SAV space-times which has the formal Ernst potentials will be determined exactly. The asymptotically flat SAV space-times constitute a subset of V_0.

A four-dimensional manifold M with a Lorentz metric g is said to be stationary axially symmetric if the metric g is time-independent and invariant under the rotation around an axis. Let R_{ij} be the Ricci tensor of the metric g. In addition to being stationary axially symmetric, if the Einstein equations $R_{ij}=0$ hold on M, then the space (M, g) is called a SAV space-itme. In case (M, g) is a SAV space-time, by an appropriate choice of the local coordinates (z, ρ, x^1, x^2) the Einstein equations are reduced to the differential equation for a 2×2 matrix function h of the variables, z and ρ (cf. Belinskii-Zakharov [1])

$$(1)\qquad \partial_\rho(\rho\partial_\rho h\cdot h^{-1})+\partial_z(\rho\partial_z h\cdot h^{-1})=0$$

where $\partial_z=\partial/\partial_z$ and $\partial_\rho=\partial/\partial_\rho$. Further the geometrical restrictions are imposed on h: the matrix h is symmetric and $\det(h)=-\rho^2$. We shall consider the formal power series solution $h\in\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ whose twist potentials are also elements of $\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ (cf. Section 1). The asymptotically flat SAV space-times are included in this category (cf. Introduction of Hauser-Ernst [6]).

Recently, much progress has been made on the inverse scattering approach to equation (1), see Hauser-Ernst [5], [6], Belinskii-Zakharov [1] and Maison [7]. See also Cosgrove [2] for the relationship between these works. In these

* The author was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

approaches the various Lax pairs for equation (1) are proposed to describe the symmetry group of the space of the solutions (Hauser-Ernst), the soliton solutions (Belinskii-Zakharov) and the integrability (Maison). Our present method is based on the fact that the action of the symmetry group for the solutions of the SAV Einstein equations is transitive. In particular, any solution is generated from the solution corresponding to Minkowski space (cf. Geroch [4], Hauser-Ernst [6]). In the present paper we use the Lax pair introduced in Belinskii-Zakharov [1]. However our normalization of the Ψ-potential is different from that of Belinskii-Zakharov and has the following properties;

(i) *The Ψ-potential of Minkowksi space has a simple expression; see below.*

(ii) *The solution of our formal Riemann-Hilbert problem is described by a certain $\infty\times\infty$ matrix ξ.*

By using these properties we can get an explicit formula representing the solution.

Our study of the SAV Einstein equations is as follows. By setting $U=\rho\partial_\rho h\cdot h^{-1}$ and $V=\rho\partial_z h\cdot h^{-1}$ we have the first order differential equations

$$(2)\qquad \partial_\rho U+\partial_z V=0\,,\quad \rho(\partial_z U-\partial_\rho V)+V+[U, V]=0\,.$$

If $h(z, \rho)$ is a member of V_0, then we have $U(z, \rho)$, $V(z, \rho)\in\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ (cf. Subsection 2.1). Conversely let us suppose that $U(z, \rho)$, $V(z, \rho)\in\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ is a solution of equation (2). If there exists a formal power series solution $h(z, \rho)$ of

$$(3)\qquad \rho\partial_\rho h=U\cdot h\,,\quad \rho\partial_z h=V\cdot h$$

satisfying the conditions ${}^t h=h$ and $\det(h)=-\rho^2$, then one has $h(z, \rho)\in V_0$.

The Lax equation for (2) introduced in Belinskii-Zakharov [1] is expressed as the integrability conditions of the overdetermined system of linear differential equations

$$(4)\qquad D_1\Psi(z, \rho, w)=U(z, \rho)\Psi(z, \rho, w)\,,\quad D_2\Psi(z, \rho, w)=V(z, \rho)\Psi(z, \rho, w)$$

where $D_1=w\partial_z+\rho\partial_\rho+2w\partial_w$, $D_2=\rho\partial_z-w\partial_\rho$ and w is a real parameter independent of z and ρ. In this paper we require the solution $\Psi(z, \rho, w)$, which is called the Ψ-potential, to be of the form $\Psi(z, \rho, w)=\sum_{k=0}^{\infty}\Psi_k(z, \rho)w^{-k}$, $\Psi_k(z, \rho)\in\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ with $\Psi(0, 0, w)=\mathbf{1}$, $\Psi_0(z, \rho)=\mathbf{1}$. These requirements make a dierence between our present work and Belinskii-Zakharov [1].

In our approach it suffices to find $\Psi(z, \rho, w)$ which has the property:

(P) $D_1\Psi(z, \rho, w)\cdot[\Psi(z, \rho, w)]^{-1}$ *and* $D_2\Psi(z, \rho, w)\cdot[\Psi(z, \rho, w)]^{-1}$ *are independent of the variable* w.

In the case of Minkowski space, which is the trivial example of the SAV spacetime, we have

$$\Psi^{(0)}(z, \rho, w) = \exp\left(\frac{1}{2}\cdot\frac{\rho^2}{w}\partial_z\right)\begin{bmatrix} \dfrac{w}{w-2z} & 0 \\ 0 & 1 \end{bmatrix}$$

where $\exp((\rho^2/2w)\partial_z)$ denotes the infinite order differential operator $\exp((\rho^2/2w)\partial_z) = \sum_{k=0}^{\infty}((\rho^2/2w)\partial_z)^k/k!$. We shall show that if for some invertible element $u(t)\in gl(2, \boldsymbol{R}[[t]])$, $\Psi(z, \rho, w)\cdot\exp((\rho^2/2w)\partial_z)\{u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1}\}$ contains no negative power of w (Riemann-Hilbert problem), then $\Psi(z, \rho, w)$ has the property (P) (Lemma). For any invertible element $u(t)$ of $gl(2, \boldsymbol{R}[[t]])$ the existence and uniqueness of $\Psi(z, \rho, w)$ shall be proved. Further an explicit formula representing the $\infty\times\infty$ matrix ξ associated with the Ψ-potential is given (Theorem 1), where ξ is an matrix $\Lambda\xi=(\xi_{ij})_{i\in\boldsymbol{Z}, j<0}$, $\xi_{ij}\in gl(2, \boldsymbol{R}[[z, \rho]])$ with the condition $\Lambda\xi=\xi C$, $\xi_{ij}=\delta_{ij}\mathbf{1}$ for $i, j<0$ where $\Lambda=(\delta_{i+1,j}\mathbf{1})_{i,j\in\boldsymbol{Z}}$ and

$$C=\begin{bmatrix}(\delta_{i+1,j}\mathbf{1})_{i<-1,j<0} \\ (-\Psi_{-j})_{j<0}\end{bmatrix} \qquad \text{(cf. Takasaki [9])}.$$

Therefore one may expect that the conditions for the formal solutions $h(z, \rho)$ of equation (3) satisfying ${}^t h=h$ and $\det(h)=-\rho^2$ to exists are expressed in terms of $u(t)$. In fact we shall prove that these conditions are $\det(u(t))=1$ and $u_{12}(t)=t^2u_{21}(t)$ (Theorem 2). Further any ξ-matrix associated with a member of V_0 is obtained by the formula given in Theorem 2 (Theorem 3).

Recently Nakamura [8] proposed a linearization of the SAV Einstein equations under certain Ansatz. The differential operators which describe the ρ developmen of the solution appeared in [8] and in this paper are essentially the same. But the relation between these two approaches remains unclear at the present time.

The present paper is organized as follows. The main results (Lemma and Theorem 1, 2, 3) are stated in Section 1. In Section 2 we study the initial value problem of the SAV Einstein equations in formal category, Lemma is proved in Section 3. A proof of Theorem 2 is given in Section 4. We prove Theorem 2, 3 in Section 5.

The author would like to express his hearty thanks to Professor N. Ikeda and Professor S. Tanaka for their encouragement and is heartily grateful to the referee for his valuable comments on the present work.

1. Preliminaries and statement of results

Suppose given a SAV space-time. Taking an appropriate local coordinates system (z, ρ, x^1, x^2), we can write its metric in the form $g=e^{2\Gamma}(dz^2+d\rho^2)+\sum_{a,b=1,2}h_{ab}dx^a dx^b$ where the metric coefficients Γ and h_{ab} are scalar functions of only two variables, z and ρ. Indeed, the matrix $h=(h_{ab})$ is symmetric, i.e., (1.1) $$^{t}h = h\,.$$

By using the freedom of the remaining choice of z and ρ we can impose on the matrix h the following supplementary condition:

(1.2) $$\det(h_{ab}) = -\rho^2\,.$$

In this coordinates system (Weyl's canonical coordinates system) the essential part of the Einstein equations is

(1.3) $$\partial_\rho(\rho\partial_\rho h\cdot h^{-1})+\partial_z(\rho\partial_z h\cdot h^{-1}) = 0\,.$$

The function Γ is determined by the rest of the Einstein equatioms $2\partial_\rho\Gamma=-\rho^{-1}+(4\rho)^{-1}\mathrm{Tr}(U^2-V^2)$, $2\partial_z\Gamma=(2\rho)^{-1}\mathrm{Tr}(UV)$ where $U=\rho\partial_\rho h\cdot h^{-1}$ and $V=\rho\partial_z h\cdot h^{-1}$.

By virtue of the condition (1.2) and the identity for any 2×2 amtrix g

(1.4) $$^{t}g\varepsilon g = \det(g)\varepsilon$$

with $\varepsilon=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$, equation (1.3) is rewritten in the form $\partial_\rho(\rho^{-1}h\varepsilon\partial_\rho h)+\partial_z(\rho^{-1}h\varepsilon\partial_z h)=0$. Hence there exists uniquely up to additive constants a 2×2 matrix function $J(z, \rho)$ such that

(1.5) $$\partial_\rho J = \rho^{-1}h\varepsilon\partial_z h\,, \quad \partial_z J = -\rho^{-1}\, h\varepsilon\partial_\rho h\,.$$

We call this $J(z, \rho)$ the twist potential of $h(z, \rho)$.

We let denote V_0 the set of formal power series solutions $h(z, \rho)\in \mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ whose twist potentials are also elements of $\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$. In this paper we consider the formal power series solutions $h(z, \rho)\in V_0$.

For any member of V_0, $U(z, \rho)$ and $V(z, \rho)$ are elements of $\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$ (cf. section 2) and satisfy the first order differential equations

(1.6) $$\begin{gathered}\partial_\rho U+\partial_z V = 0\,, \quad \rho(\partial_z U-\partial_\rho V)+V+[U, V] = 0\,,\\ U(z, \rho),\ V(z, \rho)\in\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])\,.\end{gathered}$$

Equation (1.6) is expressed as the compatibility conditions of the overdetermined system of differential equations

(1.7) $$D_1\Psi(z, \rho, w) = U(z, \rho)\Psi(z, \rho, w)\,, \quad D_2\Psi(z, \rho, w) = V(z, \rho)\Psi(z, \rho, w)$$

where $D_1=w\partial_z+\rho\partial_\rho+2w\partial_w$, $D_2=\rho\partial_z-w\partial_\rho$ and w is a real parameter independent of z and ρ. In these equations we require $\Psi(z, \rho, w)$ to be a formal power series of the form

(1.8)
$$\Psi(z, \rho, w) = \sum_{k=0}^{\infty} \Psi_k(z, \rho) w^{-k}, \quad \Psi(0, 0, w) = \mathbf{1},$$
$$\Psi_k(z, \rho) \in gl(2, \boldsymbol{R}[[z, \rho]]), \quad \Psi_0(z, \rho) = \mathbf{1}.$$

If $h(z, \rho)$ is a member of V_0, a formal power series solution $\Psi(z, \rho, w)$ of equation (1.7) with the condition (1.8) surely exists uniquely, which we call the Ψ-potential of h. Conversely if we can find $\{\Psi(z, \rho, w), U(z, \rho), V(z, \rho)\}$ subject to (1.7) and (1.8), then $\{U(z, \rho), V(z, \rho)\}$ is a solution of equation (1.6). Furthermore if there exists $h(z, \rho) \in gl(2, \boldsymbol{R}[[z, \rho]])$ with (1.1) and (1.2) such that

(1.9)
$$\rho\partial_\rho h = U \cdot h, \quad \rho\partial_z h = V \cdot h,$$

then $h(z, \rho)$ is a member of V_0.

In the following we shall give an explicit formula representing the large class of solutions of equation (1.6) by using the linear differential equation (1.7) and shall show that any member of V_0 is recovered by the equation (1.9) by taking appropriate U and V which are constructed through our formula.

Let us suppose a formal power sreies of the form (1.8) has the property

(P) *$D_1\Psi(z, \rho, w) \cdot [\Psi(z, \rho, w)]^{-1}$ and $D_2\Psi(z, \rho, w) \cdot [\Psi(z, \rho, w)]^{-1}$ are independent of the parameter w.*

Then we may easily see that $D_1\Psi(z, \rho, w) \cdot \Psi[(z, \rho, w)]^{-1} = \partial_z\Psi_1(z, \rho)$, $D_2\Psi(z, \rho, w) \cdot [\Psi(z, \rho, w)]^{-1} = -\partial_\rho\Psi_1(z, \rho)$. Therefore $U = \partial_z\Psi_1(z, \rho)$ and $V = -\partial_\rho\Psi_1(z, \rho)$ satisfy equation (1.6).

We first show the next Lemma, which is concerned with a solution of a formal version of a Riemann-Hilbert problem.

Lemma. *Let $u(t)$ be an invertible element of $gl(2, \boldsymbol{R}[[t]])$ and $\Psi(z, \rho, w)$ be a formal power series of the form (1.8). If $X_+(z, \rho, w)$ can be defined by*

$$X_+(z, \rho, w) = \Psi(z, \rho, w) \exp\left(\frac{1}{2} \cdot \frac{\rho^2}{w} \partial_z\right) \left\{u(w-2z) \begin{bmatrix} 1 - \frac{2z}{w} & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

and contains no negative power of w, then $\Psi(z, \rho, w)$ has the property (P).

REMARK 1.1. The operator $\exp((\rho^2/2w)\partial_z)$ denotes the infinite order differential operator defined by $\exp((\rho^2/2w)\partial_z) = \sum_{k=0}^{\infty} ((\rho^2/2w))\partial_z)^k/k!$.

REMARK 1.2. The matrix $\begin{bmatrix} 1 - \frac{2z}{w} & 0 \\ 0 & 1 \end{bmatrix}$ appeared in Lemma is a value of $[\Psi^{(0)}(z, \rho, w)]^{-1}$ at the axis $\rho = 0$ where $\Psi^{(0)}(z, \rho, w)$ is the Ψ-potential of Minkowski space-time. More precisely

$$\Psi^{(0)}(z, \rho, w) = \exp((\rho^2/2w)\partial_z)\begin{bmatrix} \dfrac{w}{w-2z} & 0 \\ 0 & 1 \end{bmatrix}.$$

By using the notion of the ξ-matrix we shall construct $\Psi(z, \rho, w)$ with property (P) for a given invertible element $u(t) \in gl(2, \boldsymbol{R}[[t]])$. Recal that the ξ-matrix is an $\infty \times \infty$ matrix $\xi = (\xi_{ij})_{i \in \boldsymbol{Z}, j<0}$, $\xi_{ij} \in gl(2, \boldsymbol{R}[[z, \rho]])$ with the condition $\Lambda\xi = \xi C$, $\xi_{ij} = \delta_{ij}\mathbf{1}$ for $i, j<0$ where $\Lambda = (\delta_{i+1,j}\mathbf{1})_{i,j \in \boldsymbol{Z}}$ and $C = \begin{bmatrix} (\delta_{i+1,j}\mathbf{1})_{i<-1,j<0} \\ (\xi_{0j})_{j<0} \end{bmatrix}$. The relation $\xi_{0,j} = -\Psi_{-j}$ $j<0$ gives us the one to one correspondence between Ψ and ξ. Let $\Psi^{(0)}(z, \rho, w)$ be the Ψ-potential of Minkowski space-time. We shall define an $\infty \times \infty$ matrix $\Xi = (\Xi_{ij})_{i \in \boldsymbol{Z}, j<0}$ by

$$\Xi_{ij}(z, \rho) = \sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(\frac{\rho^2}{2}\partial_z\right)^m W_{j-i+m}(z),$$

(1.10)

$$W_k(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \cdot u^{(m)}(-2z) \cdot \Psi^{(0)*}_{m-k}(z, 0), \qquad k \in \boldsymbol{Z}$$

where $u^{(m)} = \left(\dfrac{d}{dt}\right)^m u$, $[\Psi^{(0)}(z, \rho, w)]^{-1} = \sum_{m=0}^{\infty} \Psi^{(0)*}_m(z, \rho) w^{-k}$ and we set $\Psi^{(0)*}_m(z, \rho) = 0$ for $m<0$. Since $\Psi^{(0)*}_m(z, 0) = 0$ for $m \geq 2$, $W_k(z)$ can be defined for any $k \in \boldsymbol{Z}$. We shall show the following:

Theorem 1. *Let $u(t)$ be an invertible element of $gl(2, \boldsymbol{R}[[t]])$ and $\Xi(z, \rho)$ be the $\infty \times \infty$ matrix defined by* (1.10). *Then one has*
(1) *The $\infty \times \infty$ matrix $\Xi_{(-)}(z, \rho) = (\Xi_{ij}(z, \rho))_{i,j<0}$ is invertible and the product $\xi(z, \rho) = \Xi(z, \rho) \cdot (\Xi_{(-)}(z, \rho))^{-1}$ is well defined.*
(2) *The formal power series $\Psi(z, \rho, w) = \mathbf{1} + \sum_{k=1}^{\infty}(-\xi_{0,-k}(z, \rho))w^{-k}$ is the unique one which satisfies the assumption in Lemma with respect to $u(t) \in gl(2, \boldsymbol{R}[[z, \rho]])$. Therefore $\Psi(z, \rho, w)$ has the property (P).*
(3) *Denote $\partial_z\Psi_1(z, \rho)$ and $-\partial_\rho\Psi_1(z, \rho)$ by $U[u](z, \rho)$ and $V[u](z, \rho)$, respectively. Then $\{U[u](z, \rho), V[u](z, \rho)$ is a solution of equation* (1.6)

REMARK 1.3. The statement (3) of Theorem 1 is immediately derived from the fact (2) as before.

Let $U[u]$ and $V[u]$ be the ones given in Theorem 1. If we choose a special family of $u(t)$, then equation

$$(1.9') \qquad \rho\partial_\rho h = U[u] \cdot h, \quad \rho\partial_z h = V[u] \cdot h$$

has a solution $h[u] \in gl(2, \boldsymbol{R}[[z, \rho]])$ satisfying the conditions (1.1) and (1.2). In this case the above $h[u]$ is a member of V_0. In more detail we shall show that:

Theorem 2. *If* $\det(u(t))=1$ *and* $u_{12}(t)=t^2u_{21}(t)$, *then there exists a solution* $h[u]$ *of equation* (1.9′) *satisfying the condition* (1.1) *and* (1.2). *Hence* $h[u]$ *is a solution of the stationary axially symmetric vacuum Einstein equations.*

Equation (1.3) has the gauge transformation group which corresponds to the choice of Weyl's canonical coordinates system. By a special choice of gauge and the condition (1.2) we may assume that at the axis $\rho=0$ (cf. subsection 2.1)

$$\text{(1.11)}\quad \begin{aligned} &\text{(i)}\quad h(z,0)=\begin{bmatrix}0 & 0\\ 0 & -f(z)\end{bmatrix},\ \partial_\rho h(z,0)=0 \text{ for some } f(z)\in \boldsymbol{R}[[z]]\\ &\text{(ii)}\quad \Psi_1(z,0)=\begin{bmatrix}2z & 0\\ \chi(z) & 0\end{bmatrix},\ \partial_\rho\Psi_1(z,0)=0 \text{ and } \chi(0)=0\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } \chi(z)\in\boldsymbol{R}[[z]]. \end{aligned}$$

Under the assumption (i) and (ii) a solution h of equatoin (1.3) is uniquely determined by $f(z)$ and $\chi(z)$ (cf. subsection 2.1). We shall prove that:

Theorem 3. *Suppose* $h(z, \rho)$ *be a solution of equation* (1.1) (1.2) *and* (1.3) *satisfying* (i) *and* (ii). *Let* $u(t)$ *be the element of* $\mathfrak{gl}(2, \boldsymbol{R}[[t]])$ *such that*

$$\det(u(t))=1\,,\quad u_{12}(t)=t^2u_{21}(t)\,,$$

$$u_{11}(t)=\frac{1}{f\left(-\dfrac{t}{2}\right)}\,,\quad u_{21}(t)=\frac{\chi\left(-\dfrac{t}{2}\right)u_{11}(t)}{t}\,.$$

Then $h(z, \rho)=h[u](z, \rho)$.

REMARK 1.4. For any $h(z, \rho)\in V_0$ satisfying (i) and (ii) of (1.11), the function $\Gamma(z, \rho)$ is an element of $\mathfrak{gl}(2, \boldsymbol{R}[[z, \rho]])$.

2. Gauge conditions

2.1. The H-potential. Following Hauser-Ernst [6] we shall introduce the H-potential and give the equivalent form of equation (1.3), which is called Hauser-Ernst equations. The well-known Ernst equation is deduced as the integrability condition of the Hausre-Ernst equations. The important fact is that all components of the H-potential are uniquely determined by a solution of the Ernst equation (cf. [3]) up to the gauge transformations. In this Subsection we shall choose an appropriate gauge which is useful for our discussion.

Let $h(z, \rho)$ be a member of V_0 and $J(z, \rho)$ be the twist potential of $h(z, \rho)$.

Define a 2×2 complex matrix function $H(z, \rho)=(H_{ab}(z, \rho))$, which is called the H-potential, by $H=-h+iJ$, $i=\sqrt{-1}$. Then equation (1.3) is equivalent to

$$-i\rho\partial_z H = h\varepsilon\partial_\rho H\,, \quad i\rho\partial_\rho H = h\varepsilon\partial_z H\,. \tag{2.1}$$

This equivalence is easily verified by using the identity (1.4) and equation (1.5). Since the matrix h is symmetric, we have $H-{}^tH=i(J-{}^tJ)=-2\alpha i\varepsilon$ with some real scalar function $\alpha(z, \rho)$. With the aid of equation (1.5) and $h\varepsilon h=-\rho^2$ we get $\partial_\rho\alpha\varepsilon=0$ and $\partial_z\alpha\varepsilon=-\varepsilon$. Hence we may assume that

$$H-{}^tH = 2iz\varepsilon\,. \tag{2.2}$$

We define $\mathcal{E}:=H_{22}$, $f:=\mathrm{Re}\,\mathcal{E}=-h_{22}$ and $\chi:=J_{22}$. The equations derived from equations (1.2), (2.1) and (2.2) are as follows (see Hauser-Ernst [6]):

$$\begin{aligned}
&\partial_\rho(f^{-1}h_{12}) = \rho f^{-2}\partial_z\chi\,, \quad \partial_z(f^{-}h_{12}) = \rho f^{-2}\partial_\rho\chi\,,\\
&\partial_\rho J_{11} = f^{-1}(2\rho\partial_z h_{12}-h_{11}\partial_\rho\chi)\,,\\
&\partial_z J_{11} = f^{-1}(2h_{12}-2\rho\partial_\rho h_{12}-h_{11}\partial_z\chi)\,,\\
&\partial_\rho J_{12} = -f^{-1}(\rho\partial_z f+h_{12}\partial_\rho\chi)\,,\\
&\partial_z J_{12} = f^{-1}(\rho\partial_\rho f-h_{12}\partial_z\chi)\,,\\
&h_{11} = f^{-1}\{\rho^2-(h_{12})^2\}\,, \quad J_{21} = J_{12}-2z\,.
\end{aligned} \tag{2.3}$$

By a direct calculation we have the Ernst equation for $\mathcal{E}$

$$f(\rho\partial_\rho^2\mathcal{E}+\partial_\rho\mathcal{E}+\rho\partial_z^2\mathcal{E}) = \rho\{(\partial_\rho\mathcal{E})^2+(\partial_z\mathcal{E})^2\}\,. \tag{2.4}$$

as the integrability condition for equation (2.3). It is easily shown that all components of H are uniquely determined by $\mathcal{E}$ up to the following gauge transformations

$$H \to H+iB,\ z \to z+\frac{1}{2}(B_{12}-B_{21}) \tag{2.5}$$

and

$$H \to {}^tSHS\,, \quad z \to (\det S)z, \quad \rho \to (\det S)\rho \tag{2.6}$$

where $B=(B_{ab})$ is any real 2×2 constant matrix and S is an element of $\boldsymbol{GL}(2, \boldsymbol{R})$ (Also see Hauser-Ernst [6]). Equation (2.1) with the conditions (1.2) and (2.2) are invariant under the transformations (2.5) and (2.6). These transformations are derived from the following facts. The twist potential J is not uniquely defined by equation (1.5). It remains arbitrary up to transformation $J\to J+B$. Also if the ignorable variables x^a of Weyl's canonical coordinates are subject to transformation $(x^1, x^2)\to(x^1, x^2){}^tS^{-1}$, then we have $h\to{}^tShS$.

We can use the transformations (2.5) and (2.6) to make $H(0, 0)=\begin{bmatrix}0 & 0\\ 0 & \mathcal{E}(0, 0)\end{bmatrix}$, $\mathrm{Im}\,\mathcal{E}(0, 0)=0$. Then by using equation (2.3) and (2.4) we can prove that on the axis $\rho=0$ the H-potential H has the form (cf. [6])

$$H(z,0)=\begin{bmatrix} 0 & 0 \\ -2iz & \mathcal{E}(z,0)\end{bmatrix}, \quad \mathrm{Im}\,\mathcal{E}(0,0)=0, \tag{2.7}$$

$$(\partial_\rho H)(z,0)=0. \tag{2.8}$$

The next proposition is concerned with the initial value problem at the axis $\rho=0$.

Proposition 2.1. *Let $f(z)$, $\chi(z)\in \boldsymbol{R}[[z]]$ and $f(z)$ be invertible. Then there exists a unique solution $\mathcal{E}(z,\rho)\in \boldsymbol{C}[[z,\rho]]$ of the Ernst equation* (2.4) *such that $\mathcal{E}(z,0)=f(z)+i\chi(z)$.*

Proof. In terms of $f(z,\rho)=\mathrm{Re}\,\mathcal{E}(z,\rho)$ and $\chi(z,\rho)=\mathrm{Im}(z,\rho)$, the Ernst equation is written as

$$f(\rho\partial_\rho^2 f+\partial_\rho f+\rho\partial_z^2 f)=\rho\{(\partial_\rho f)^2+(\partial_z f^2)-(\partial_\rho\chi)^2-(\partial_z\chi)^2\},$$
$$f(\rho\partial_\rho^2\chi+\partial_\rho\chi+\rho\partial_z^2\chi)=2\rho\{\partial_\rho f\partial_\rho\chi+\partial_z f\partial_z\chi\}.$$

Substituting $f(z,\rho)=\sum_{k=0}^{\infty} f_k(z)\rho^k$ and $\chi(z,\rho)=\sum_{k=0}^{\infty}\chi_k(z)\rho^k$, we have $f_0(z)f_1(z)=0$, $f_0(z)\chi_1(z)=0$ and for any $m\geqq 0$

$$\begin{aligned}\sum_{j=0}^{m+2} k^2 f_{m+2-k}(z)f_k(z) &= \sum_{k=1}^{m+1} k(m+2-k)\{f_{m+2-k}(z)f_k(z)-\chi_{m+2-k}(z)\chi_k(z)\}\\ &+\sum_{k=0}^{m}\{\partial_z f_{m+1-k}(z)\partial_z f_k(z)-f_{m-k}(z)\partial_z^2 f_k(z)-\partial_z\chi_{m-k}(z)\partial_z\chi_k(z)\},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\sum_{k=0}^{m+2} k^2 f_{m+2-k}(z)\chi_k(z) &= \sum_{k=1}^{m+1} k(m+2-k)f_{m+2-k}(z)\chi_k(z)\\ &+\sum_{k=0}^{m+1}\{\partial_z f_{m-k}(z)\partial_z\chi_k(z)-f_{m-k}(z)\partial_z^2\chi_k(z).\end{aligned} \tag{2.10}$$

Since $f(z)$ is invertible, we can find niductively all $f_k(z)$ and $\chi_k(z)$ from equations (2.9) and (2.10). q.e.d.

Combining our gauge condition and Proposition 2.1, we have that:

Proposition 2.2. *The H-potential is uniquely determined in our gauge by the conditions on the axis $\rho=0$*

$$H(z,0)=\begin{bmatrix} 0 & 0 \\ -2iz & \mathcal{E}(z,0)\end{bmatrix}, \quad (\partial_\rho H)(z,0)=0, \quad \mathrm{Im}\,\mathcal{E}(0,0)=0.$$

2.2. The Ψ-potential. In this subsection we shall prove that for any $h(z,\rho)\in V_0$ there exists uniquely the Ψ-potential of $h(z,\rho)$. We give the conditions for the formal power series $\Psi(z,\rho,w)$ with property (P) to have the solution $h(z,\rho)\in \mathcal{gl}(2, \boldsymbol{R}[[z,\rho]])$ of equation (1.9) satisfying the conditions (1.1) and (1.2).

For any $h(z, \rho)\in V_0$, substituting the expression $\Psi(z, \rho, w)=\sum_{k=0}^{\infty}\Psi_k(z, \rho)w^{-k}$ into equation (1.7), we find infinite number of conservation laws

$$(2.11)\qquad \begin{aligned} &\partial_z\Psi_{k+1}(z, \rho)+\rho\partial_\rho\Psi_k(z, \rho)-2k\Psi_k(z, \rho) = U(z, \rho)\Psi_k(z, \rho)\,, \\ &\rho\partial_z\Psi_k(z, \rho)-\partial_\rho\Psi_{k+1}(z, \rho) = V(z, \rho)\Psi_k(z, \rho)\,, \\ &\Psi_k(0, 0) = 0 \end{aligned}$$

for $k\geq 1$. The relation for $k=0$ is

$$(2.12)\qquad U(z, \rho) = \partial_z\Psi_1(z, \rho)\,, \quad V(z, \rho) = -\partial_\rho\Psi_1(z, \rho)\,.$$

The potential $\Psi_1(z, \rho)$ and the twist potential $J(z, \rho)$ are essentially same. Recall that $J(z, \rho)$ is defined as a solution of differential equations $\partial_z J=-\rho^{-1}h\varepsilon\partial_\rho h$ and $\partial_\rho J=\rho^{-1}h\varepsilon\partial_z h$. By using the identity $h\varepsilon h=-\rho^2\varepsilon$, we have $\partial_z{}^tJ\varepsilon=\rho\partial_\rho h\cdot h^{-1}$ and $\partial_\rho{}^tJ\varepsilon=-\rho\partial_z h\cdot h^{-1}$. Hence in our gauge conditions we get

$$(2.13)\qquad \Psi_1(z, \rho) = {}^tJ(z, \rho)\varepsilon\,.$$

The coefficients $\Psi_k(z, \rho)$ of w^{-k} of $\Psi(z, \rho, w)$ for $k\geq 2$ is uniquely determined by equation (2.11).

Substituting (2.12) into the second of equation (1.6), we have

$$(2.14)\qquad \rho(\partial_z^2\Psi_1+\partial_\rho^2\Psi_1)-\partial_\rho\Psi_1-[\partial_z\Psi_1, \partial^a\Psi_1] = 0\,.$$

The gauge conditions for the potential $\Psi_1(z, \rho)$ are

$$(2.15)\qquad \Psi_1(z, 0) = \begin{bmatrix} 2z & 0 \\ -\chi(z, 0) & 0 \end{bmatrix}, \quad \partial_\rho\Psi_1(z, 0) = 0\,, \quad \Psi_1(0, 0) = 0\,.$$

Hence the condition (1.2) is equivalent to

$$(2.16)\qquad \mathrm{Tr}(\Psi_1(z, \rho)) = 2z\,.$$

The condition for $\Psi_1(z, \rho)$ corresponding to the condition (1.1) is rather complicated. To find this we need the next proposition.

Proposition 2.3. *There exists a unique solution of equation* (2.16) *such that on the axis* $\rho=0$

$$\Psi_1(z, 0) = \begin{bmatrix} 2z & 0 \\ \Psi_{21}(z) & 0 \end{bmatrix}, \quad \partial_\rho\Psi_1(z, 0) = 0\,,$$

$$(\partial_\rho^2\Psi_{11})(z, 0) = -(\partial_\rho^2\Psi_{22})(z, 0) = \Psi_{11}^0(z)\,, \quad (\partial_\rho^4\Psi_{12})(z, 0) = \Psi_{12}^2(z)$$

for any $\Psi_{21}(z)$, $\Psi_{22}^0(z)$ *and* $\Psi_{12}^2(z)\in\boldsymbol{R}[[z]]$ *where we denote*

$$\Psi_1(z, \rho) = \begin{bmatrix} \Psi_{11}(z, \rho) & \Psi_{12}(z, \rho) \\ \Psi_{21}(z, \rho) & \Psi_{22}(z, \rho) \end{bmatrix}.$$

Proof. A proof is given in Appendix A and B. q.e.d.

By using equation (2.3) and (2.4) we can calculate $(\partial_\rho^2 J_{12})(z, 0)$ and $(\partial_\rho^4 J_{11})(z, 0)$:

$$(\partial_\rho^2 J_{12})(z, 0) = -[f(z)]^{-1}\partial_z f(z), \quad (\partial_\rho^4 J_{11})(z, 0) = -3[f(z)]^{-2}(\partial_z^2 \chi(z))$$

where we denote $f(z)=f(z, 0)$ and $\chi(z)=\chi(z, 0)$. In terms of $\Psi_1(z, \rho)$, it follows that

$$(\partial_\rho^2 \Psi_{22})(z, 0) = -[f(z)]^{-1}\partial_z f(z), \tag{2.17}$$

$$(\partial_\rho^4 \Psi_{12})(z, 0) = -3[f(z)]^{-2}(\partial_z^2 \chi(z)). \tag{2.18}$$

Conversely, given $\chi(z)$, $f(z) \in \boldsymbol{R}[[z]]$ such that $\chi(0)=0$, $f(0) \neq 0$ for any $\Psi_1(z, \rho)$ subject to (2.15), (2.17) and (2.18), we can find a symmetric matrix function $h(z, \rho)$ which satisfies equation (1.9) with the condition (1.2) as follows. Find the solution of the Enrst equation such that $\mathcal{E}(z, 0)=f(z)+i\chi(z)$ and define the mettic coefficients h_{ab} by equation (2.3) and the initial conditions (2.7): indeed, $h(z, \rho)$ is symmetric. Then the uniqueness of $\Psi_1(z, \rho)$ implies that $\partial_z \Psi_1 = \rho \partial_\rho h \cdot h^{-1}$ and $\partial_\rho \Psi_1 = -\rho \partial_z h \cdot h^{-1}$.

In our fixed gauge conditions the above discussion proves that:

Proposition 2.4. *Let $h(z, \rho) \in V_0$ and $\Psi_1(z, \rho)$ be the coefficient of w^{-1} of the Ψ-potential of h. If $h(z, \rho)$ and $\Psi_1(z, \rho)$ satisfy* (i) *and* (ii) *of* (1.11), *then $\Psi_1(z, \rho)$ satisfies* (2.17) *and* (2.18). *Conversely, for given $\chi(z)$, $f(z) \in \boldsymbol{R}[[z]]$ such that $\chi(0)=0$ and $f(0) \neq 0$, let $\Psi_1(z, \rho)$ be the solution of equation* (2.14) *with* (2.15), (2.17) *and* (2.18). *Then there exists uniquely an $h(z, \rho) \in gl(2, \boldsymbol{R}[[z]])$ satisfying $\partial_z \Psi_1 = \rho \partial_\rho h \cdot h^{-1}$, $\partial_\rho \Psi_1 = -\rho \partial_z h \cdot h^{-1}$ and ${}^t h = h$,* $\det(h) = -\rho^2$.

3. Proof of Lemma

Before we prove Lemma, we give some properties of the Ψ-potential for Minkowski space-time. The metric of Minkowski space-time takes the form in cylindrical coordinates

$$ds^2 = -(dt)^2 + \rho^2(d\theta)^2 + (dz)^2 + (d\rho)^2.$$

In our gauge we have $h=\begin{bmatrix} \rho^2 & 0 \\ 0 & -1 \end{bmatrix}$, $U=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $V=0$. Its Ψ-potential $\Psi^{(0)}(z, \rho, w)$ is

$$\Psi^{(0)}(z, \rho, w) = \begin{bmatrix} (1-2z/w-\rho^2/w^2)^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

By using the infinite order differential operator, we can have an expression of $1-2z/w-\rho^2/w^2$

$$1-\frac{2z}{w}-\frac{\rho^2}{w^2}=\exp\left(\frac{1}{2}\cdot\frac{\rho^2}{w}\partial_z\right)\left(1-\frac{2z}{w}\right).$$

Hence we obtain

$$[\Psi^{(0)}(z, \rho, w)]^{-1}=\exp\left(\frac{1}{2}\cdot\frac{\rho^2}{w}\partial_z\right)\begin{bmatrix}1-2z/w & 0\\ 0 & 1\end{bmatrix}.$$

Proof of Lemma. By virtue of the commutator relation

$$[\partial_\rho, \exp((\rho^2/2w)\partial_z)]=(\rho/w)\exp((\rho^2/2w)\partial_z)\partial_z$$

and

$$[\partial_w, \exp((\rho^2/2w)\partial_z)]=-(\rho^2/2w^2)\exp((\rho^2/2w)\partial_z)\partial_z,$$

we find that

$$\begin{aligned}D_1X_+(z, \rho, w)&=D_1\Psi(z, \rho, w)\cdot[\Psi(z, \rho, w)]^{-1}\cdot X_+(z, \rho, w)\\ &+\Psi(z, \rho, w)\cdot\exp\left(\frac{1}{2}\cdot\frac{\rho^2}{w}\partial_z\right)(w\partial_z+2w\partial_w)\{u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1}\}\\ &=D_1\Psi(z, \rho, w)\cdot[\Psi(z, \rho, w)]^{-1}\cdot X(z, \rho, w)\\ &-\Psi(z, \rho, w)\exp((\rho^2/2w)\partial_z)\{u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1}\cdot U^{(0)}(z, 0)\}.\end{aligned}$$

Since $U^{(0)}(z, 0)=\partial_z\Psi_1(z, 0)$ is the constant matrix, we have

$$\begin{aligned}D_1X_+(z, \rho, w)[X_+(z, \rho, w)]^{-1}+X_+(z, \rho, w)\cdot U^{(0)}(z, 0)\cdot[X_+(z, \rho, w)]^{-1}\\ =D_1\Psi(z, \rho, w)\cdot[\Psi(z, \rho, w)]^{-1}.\end{aligned}$$

Similarly we obtain

$$D_2X_+(z, \rho, w)\cdot[X_+(x, \rho, w)]^{-1}=D_2\Psi(z, \rho, w)\cdot[\Psi(z, \rho, w)]^{-1}.$$

Since $X_+(0, 0, w)=u(w)$ and $u(t)$ is invertible in $gl(2, \boldsymbol{R}[[t]])$, the inverse matrix $(X_+(z, \rho, w))^{-1}$ contains no negative power of w. Hence the left hand sides of the above relations contain no negative power of w. Therefore the right hand sides are independent of the variable w. q.e.d.

4. Proof of Theorem 1

The second statement of Theorem 2 is easily proved by using the first statement as follows. The property (P) is equivalent to $(\Phi_{-j})_{j\in Z}\cdot\Xi=0$ where we set $\Psi_{-j}=0$ for $j<0$. By the assumption $\Psi_0=\mathbf{1}$ we have $(\Psi_{-j})_{j<0}\cdot\Xi_{(-)}=-(\Xi_{0j})_{j<0}$. Hence $(\Psi_{-j})_{j<0}$ is unique and is expressed as

$$(\Psi_{-j})_{j<0}=-(\Xi_{0j})_{j<0}\cdot(\Xi_{(-)})^{-1}. \tag{4.1}$$

Proof of the first statement is composed of the next two steps. In each step we shall prove the invertibility of $\Xi_{(-)}(z, 0)$ and $\Xi_{(-)}(z, \rho)$, respectively.

The 1st step. Recall that the Ψ-potential of Minkowski space-time is $\Psi^{(0)}(z, \rho, w) = \begin{bmatrix} (1-2z/w-\rho^2/w^2)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$. Since $\Psi_k^{(0)*}(z, 0)=0$ for $k\geq 2$, it follows that $W_k(z)$ can be defined and have the form

$$(4.2) \qquad W_k(z) = \begin{cases} 0 & k\leq -2 \\ u^{(0)}(-2z)\Psi_1^{(0)*}(z, 0) & k=-1 \\ u^{(0)}(-2z)/k!+u^{(k+1)}(-2z)\Psi_1^{(0)*}(z, 0)/(k+1)! & k\geq 0 \end{cases}$$

where as given in the previous section

$$(4.3) \qquad \Psi_1^{(0)*}(z, 0) = \begin{bmatrix} -2z & 0 \\ 0 & 0 \end{bmatrix}.$$

From the definition, $\Xi_{(-)}(z, 0)$ has the expression

$$\begin{aligned} \Xi_{(-)}(z, 0) &= (\{u^{(0)}(-2z)+u^{(1)}(-2z)\Psi_1^{(0)*}(z, 0)\}\delta_{ij})+K \\ &= I+K \end{aligned}$$

where K is an $\infty\times\infty$ matrix $(K_{ij})_{i<0}$. Easily we can see that $K_{ij}=0$ for $j-i\leq -2$, $K_{ii}=0$ and $\mathrm{ord}(K_{ij})\geq 1$ for $j-i=-1$. Here for any $f\in gl(2, \boldsymbol{R}[[z]])$ we say $\mathrm{ord}(f)\geq n$ when f has the form $f=z^n g$ for some $g\in gl(2, \boldsymbol{R}[[z]])$.

Lemma 4.1. *Let $(K^n)_{ij}$ be the (i, j) component of K^n. Then*

$$(1) \qquad (K^n)_{ij} = 0\,, \qquad j-i\leq -n-1$$

$$(2) \qquad \mathrm{ord}(K^n)_{ij}\geq n-m\,, \qquad j-i = -n+2m\,, \quad -n+2m+1$$
$$m = 0, 1, \cdots, n-1\,.$$

Proof. We prove these by induction on n. For $n=1$, the statement is trivial. Assume (1) and (2) are true for n. Then $(K^{n+1})_{ij}=\sum_{p\in Z} K_{ip}(K^n)_{pj}$. By the assumption we see that

$$(K^{n+1})_{ij} = \textstyle\sum_{p=i-1}^{n+j} K_{ip}(K^n)_{pj}\,.$$

Hence K^{n+1} can be defined and the first statement immediately follows. Further, since $K_{ii}=0$, we have

$$(K^{n+1})_{ij} = K_{i,i-1}(K^n)_{i-1,j}+\textstyle\sum_{p=i+1}^{n+j} K_{ip}(K^n)_{pj}\,.$$

In the case of $j-i=-(n+1)+2m$ for $m=0, 1, \cdots, n$, $j-p\leq j-(i+1)=-n+2(m-1)$ and so $\mathrm{ord}(K^n)_{pj}\geq n+1-m$ for $m\neq 0$. Therefore by using $\mathrm{ord}(K_{i,i-1})\geq 1$ and $\mathrm{ord}(K^{n+1})\geq n-m$ for $m\neq n$, we obtain

$$\operatorname{ord}(K^{n+1})_{ij} \geq n+1-m \qquad \text{for } j-i=-(n+1)+2m\,, \quad m=1, 2, \cdots, n\,.$$

If $m=0$, then $n+j=i-1$. Hence $(K^{n+1})_{ij}=K_{i,i-1}(K^n)_{i-1,j}$ and $\operatorname{ord}(K^{n+1}) \geq n+1$. Similarly we can prove the second statement in the case of $m=n$. q.e.d.

Now we can define the inverse of $\Xi_{(-)}(z, 0)$ by Neumann power series, namely,

$$(\Xi_{(-)}(z, 0))^{-1} = \sum_{n=0}^{\infty} (-IK)^n \cdot I^{-1}\,.$$

From Lemma 4.1, $\operatorname{ord}(K^n)_{ij} \geq [(n+i-j)/2]$, hence the above Neumann power series is well defined in $\boldsymbol{R}[[z]]$.

The 2nd step. By using the next Lemma $\Xi_{(-)}(z, \rho)$ can be expressed as

$$\Xi_{(-)}(z, \rho) = (\Xi_{(-)}(z, 0)) \cdot (\mathbf{1}+H)$$

where $H=\sum_{k=1}^{\infty} (\Xi_{(-)}(z, 0))^{-1} \cdot [(\partial_z/2)^k W_{j-i+k}(z)]\rho^{2k}/k!$.

Lemma 4.2. *For any $m \in \boldsymbol{N}$ the product*

$$(\Xi_{(-)}(z, 0))^{-1} \cdot \partial_z^m W_{j-i+m}(z)$$

is well defined.

Proof. Since we have an expression

$$(\Xi_{(-)}(z, 0))^{-1} = \sum_{n=0}^{\infty} K^n\,, \quad \operatorname{ord}(K^n) \geq [(n+i-j)/2]$$

for some matrix K, we obtain

$$(\Xi_{(-)}(z, 0))^{-1} \cdot \partial_z^m W_{j-i+m}(z) = \sum_{n=0}^{\infty} K^n \cdot \partial_z^m W_{j-i+m}(z)\,.$$

Here

$$K^n \cdot \partial_z^m W_{j-i+m}(z) = \sum_{p=i-n}^{j+m+1} (K^n)_{ip} \cdot \partial_z^m W_{j-p+m}(z)\,.$$

Hence for large n, we have

$$\operatorname{ord}(K^n \partial_z^m (W_{j-1+m}))_{ij} \geq [(n+i-j-m-1)/2]\,.$$

q.e.d.

From the proof of Lemma 4.2, we can easily show that H has the form

$$\text{(4.4)} \qquad \begin{aligned} &H(z, \rho) = \sum_{m=1, n=0} H_{m,n}(z)\rho^{2m}, \ \operatorname{ord}(H_{m,n})_{ij} \geq \left[\frac{n+i-j-m-1}{2}\right] \\ &(H_{m,n}(z))_{ij} = 0 \qquad \text{for } \ j-i \leq -n-m-2\,. \end{aligned}$$

To see that H^2 is well defined, it is sufficient to prove that we can define

$\sum_{n_i=0}^{\infty} H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z)$. By virtue of the second of (4.4) we first have

$$(H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z))_{ij}=\sum_{\substack{k<0\\ a\le k\le b}}(H_{m_1,n_1}(z))_{ik}\cdot(H_{m_2,n_2}(z))_{kj}$$

where $a=i-n_1-m_1-1$ and $b=j+n_2+m_2+1$. The first of (4.4) asserts that

$$\operatorname{ord}(H_{m_1,n_1}(z))_{ik}\ge\left[\frac{n_1+i-k-m_1-1}{2}\right]\ge\left[\frac{n_1+i-m_1}{2}\right]$$
$$\operatorname{ord}(H_{m_2,n_2}(z))_{kj}\ge\left[\frac{n_2+k-j-m_2-1}{2}\right]\ge\left[\frac{n_2-n_1+i-j-m_1-m_2-2}{2}\right]$$

for $a\le k\le b$, $k<0$. Therefore it follows that

$$\operatorname{ord}(H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z))_{ij}\ge\left[\frac{n_1+i-m_1}{2}\right],\ \left[\frac{n_2-n_1+i-j-m_1-m_2-2}{2}\right]. \tag{4.5}$$

Suppose that $\operatorname{ord}(H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z))_{ij}\le r$. Then by using (4.5) we can easily show that

$$n_1\le 2r-i+m_1+2,\ n_2\le 4r-2i+2m_1+m_2+5\,.$$

Hence the infinite sum $\sum_{n_i=0}^{\infty} H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z)$ can be defined.

In the following we prove that the product $H^2\cdot(\Xi_{(-)}(z,0))^{-1}$ can be defined. It is sufficient to prove that we can define the product $\{\sum_{n_i=0}^{\infty}H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z)\}\cdot(\Xi_{(-)}(z,0))^{-1}$. From (4.5) we have

$$\operatorname{ord}(\{H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z)\}_{ik}\cdot(\Xi_{(-)}(z,0))^{-1}_{kj})$$
$$\ge\left[\frac{n_1+i-m_1}{2}\right],\ \left[\frac{n_2-n_1+i-k-m_1-m_2-2}{2}\right]$$

for any $k<0$. Also suppose that

$$\operatorname{ord}(H_{m_1,n_1}(z)\cdot H_{m_2,n_2}(z)\}_{ik}\cdot(\Xi_{(-)}(z,0))^{-1}_{kj})\le r\,.$$

Then we can see that $n_1\le 2r-i+m_1+2$, $n_2\le 4r-2i+2m_1+m_2+5$ and $k\ge -4r+2i-2m_1-m_2-6$. Thus the product $\sum_{n_i\infty 0}^{\infty}\{H_{m_1\ n_1}(z)\cdot H_{m_2,n_2}(z)\}\cdot(\Xi_{(-)}(z,0))^{-1})$ is well defined.

Similarly we can prove that H^n and $H^n\cdot(\Xi_{(-)}(z,0))^{-1}$ is well defined. Therefore we a can define $(\Xi_{(-)}(z,\rho))^{-1}$ by

$$\Xi_{(-)}((z,\rho))^{-1}=\sum_{n=0}^{\infty}(-H)^n\cdot(\Xi_{(-)}(z,0))^{-1}$$

where we use the fact $\operatorname{ord}_\rho(H^n)\ge n$.

Finally we prove the well-definedness of the product $\Xi(z,\rho)\cdot\Xi((_{(-)}(z,\rho))^{-1}$. The matrix $\Xi(z,\rho)$ has an expression

$$\Xi(z, \rho) = \sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(\frac{\rho^2}{2}\right)^m \cdot \partial_z^m \Xi_{(m)}(z, 0)$$

where $(\Xi_{(m)}(z, 0) = (\Xi_{i-m,j}(z, 0))_{i \in Z, j<0}$. By using the fact (4.2) we have

$$(\Xi_{(m)}(z, 0))_{i,j} = 0 \qquad \text{for } j-i+m \leq -2 .$$

Hence the product $\Xi(z, \rho) \cdot (\Xi_{(-)}(z, \rho))^{-1}$ can be defined.

5. Proofs of Theorems 2 and 3

In Section 3 and 4 we proved that for any invertible $u(t) \in gl(2, \boldsymbol{R}[[t]])$, there exists uniquely $\Psi(z, \rho, w)$ such that

$$(5.1) \quad X_+(z, \rho, w) = \Psi(z, \rho, w) \exp\left(\frac{1}{2} \cdot \frac{\rho^2}{w} \partial_z\right) \{u(w-2z) \cdot [\Psi^{(0)}(z, 0, w)]^{-1}\}$$

contains no negative power of w. Suppose that $u(t)$ is an element of $gl(2, \boldsymbol{R}[[t]])$ satisfying the assumption of Theorem 2. Let $\hat{\Psi}_1(z, \rho)$ be the solution of equation (2.14) determined by the initial conditions (cf. Proposition 2.3)

$$(5.2) \quad \begin{aligned} &\hat{\Psi}_1(z, 0) = \begin{bmatrix} 2z & 0 \\ -\hat{\chi}(z) & 0 \end{bmatrix}, \quad (\partial_\rho \hat{\Psi}_1)(z, 0) = 0 , \\ &(\partial_\rho^2 \hat{\Psi}_{11})(z, 0) = -(\partial_\rho^2 \hat{\Psi}_{22})(z, 0) = [\hat{f}(z)]^{-1} \partial_z \hat{f}(z) , \\ &(\partial_\rho^4 \hat{\Psi}_{12})(z, 0) = -3[\hat{f}(z)]^{-2} \partial_z \hat{\chi}(z) . \end{aligned}$$

where $\hat{f}(z) = 1/u_{11}(-2z)$ and $\hat{\chi}(z) = -2z u_{21}(-2z)/u_{11}(-2z)$. From Proposition 2.4, $\hat{\Psi}_1(z, \rho)$ is the coefficient of w^{-1} of the Ψ-potential $\hat{\Psi}(z, \rho, w)$ associated with some $h(z, \rho) \in V_0$. In the following we shall show tht that $\Psi_1(z, \rho)$ satisfies the same initial conditions (5.2). Then by using Proposition 2.3 we have $\hat{\Psi}(z, \rho, w) = \Psi(z, \rho, w)$. Thus Theorem 2 and Theorem 3 are proved at the same time. We first show that:

Lemma 5.1. *One has* $\hat{\Psi}(z, 0, w) = \Psi(z, 0, w)$.

Proof. Set $\rho = 0$ in (5.1). Then $\Psi(z, 0, w)$ is the unique one such that $X_+(z, 0, w)$ has no negative power of w. The uniqueness is easily verified by using the fact that $\Xi_{(-)}(z, 0)$ defined in the previous section is invertible. Therefore it is sufficient to show tht that $\hat{X}_+(z, 0, w) = \hat{\Psi}(z, 0, w) \cdot u(w-2z) \cdot [\Psi^{(0)}(z, 0, w)]^{-1}$ has no negative power of w. Recall that the Ψ-potential $\hat{\Psi}(z, \rho, w)$ is a solution of the pair of differential equations

$$(5.3) \quad \begin{aligned} &(w\partial_z + \rho\partial_\rho + 2w\partial_w) \hat{\Psi}(z, \rho, w) = (\partial_z \hat{\Psi}_1(z, \rho)) \hat{\Psi}(z, \rho, w) \\ &(\rho\partial_z - w\partial_\rho) \hat{\Psi}(z, \rho, w) = -(\partial_\rho \hat{\Psi}_1(z, \rho)) \hat{\Psi}(z, \rho, w) . \end{aligned}$$

Hence we easily obtain

$$\hat{\Psi}(z, 0, w)=\begin{pmatrix} \dfrac{w}{w-2z} & 0 \\ -\dfrac{\hat{\chi}(z)}{w-2z} & 1 \end{pmatrix}.$$

Thus we have

$$\hat{X}_+(z, 0, w)=\begin{bmatrix} u_{11}(w-2z) & \dfrac{w}{w-2z}\cdot u_{12}(w-2z) \\ (\hat{X}_+)_{21}(z, 0, w) & (\hat{X}_+)_{22}(z, 0, w) \end{bmatrix}$$

where

$$(\hat{X}_+)_{21}(z, 0, w)=-\frac{\hat{\chi}(z)}{w}u_{11}(w-2z)+\left(1-\frac{2z}{w}\right)u_{21}(w-2z),$$

$$(\hat{X}_+)_{22}(z, 0, w)=-\frac{\hat{\chi}(z)}{w-2z}\cdot u_{12}(w-2z)+u_{22}(w-2z).$$

Here we note tht that for any $v\in \mathfrak{gl}(2, \boldsymbol{R}[[z]])$ the product $\dfrac{1}{w-2z}\cdot v$ can be defined because $\dfrac{1}{w-2z}=\dfrac{1}{w}\sum_{k=0}^{\infty}\left(\dfrac{2z}{w}\right)^k$. By the assumption $u_{12}(0)=0$, all components except $(X_+)_{21}(z, 0, w)$ have no negative power of w. The (2,1) component is

$$(\hat{X}_+)_{21}(z, 0, w)$$
$$=-\hat{\chi}(z)\sum_{k=0}^{\infty}\frac{1}{k!}\cdot u_{11}^{(k)}(-2z)w^{k-1}-2z\sum_{k=0}^{\infty}\frac{1}{k!}\cdot u_{21}^{(k)}(-2z)w^{k-1}+u_{21}(w-2z).$$

Hence $-\hat{\chi}(z)u_{11}(-2z)-2zu_{21}(-2z)$, which is the coefficient of w^{-1}, equals zero. q.e.d.

In the next step we shall prove that $(\partial_\rho\Psi)(z, 0, w)=0$. Because

$$\partial_\rho X_+(z, \rho, w)=\partial_\rho\Psi(z, \rho, w)\cdot\boldsymbol{E}\cdot\{u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1}\}$$
$$+\frac{\rho}{w}\cdot\Psi(z, \rho, w)\cdot\partial_z\boldsymbol{E}\cdot\{u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1}\}$$

where

$$\boldsymbol{E}=\exp\left(\frac{1}{2}\cdot\frac{\rho^2}{w}\partial_z\right),$$

we obtain

$$\partial_\rho X_+(z, 0, w)=\partial_\rho\Psi(z, 0, w)\cdot u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1}.$$

Therefore also by using the invertibility of $\Xi_{(-)}(z, 0)$ we have $\partial_\rho\Psi(z, 0, w)=0$.

Here we will prove

$$(\partial_\rho^2\Psi_{11})(z, 0) = -(\partial_\rho^2\Psi_{22})(z, 0) = [\hat{f}(z)]^{-1}\partial_z\hat{f}(z)\,.$$

Proof. Operate ∂_ρ to the second of equation (5.3) and set $\rho=0$, then

(5.4) $$w\partial_\rho^2\Psi(z, 0, w) = \partial_z\Psi(z, 0, w)+[\partial_\rho^2\Psi_1(z, 0)]\cdot\Psi(z, 0, w)\,.$$

Similarly in the second step we have

$$\partial_\rho^2 X_+(z, 0, w) = \partial_\rho^2\Psi(z, 0, w)\cdot u(w-2z)\cdot[\Psi^{(0)}(z, 0, w)]^{-1} + \frac{\Psi(z, 0, w)}{w}\cdot\partial_z\{u(w-2z)[\Psi^{(0)}(z, 0, w)]^{-1}\}\,.$$

In Lemma 5.1 we proved that $\hat{\Psi}(z, 0, w)=\Psi(z, w, 0)$. Therefore if $\partial_\rho^2\hat{X}_+(z, 0, w)$ has no negative power of w, then we have $\partial_\rho^2\hat{\Psi}(z, 0, w)=\partial_\rho^2\Psi(z, 0, w)$. By equations (2.3) and (2.4) one can show that $\partial_\rho^2\hat{\Psi}_1(z, 0)$ has the form

(5.5) $$\partial_\rho^2\hat{\Psi}_1(z, 0) = \begin{bmatrix} -F(z) & 0 \\ G(z) & F(z) \end{bmatrix}$$

where

$$F(z) = -[\hat{f}(z)]^{-1}\partial_z\hat{f}(z),\ G(z) = \partial_z^2\hat{\chi}(z)/2-[\hat{f}(z)]^{-1}\partial_z\hat{f}(z)\partial_z\hat{\chi}(z)\,.$$

Substituting (5.5) into (5.4) we have

(5.6) $$w\partial_\rho^2\hat{\Psi}(z, 0, w) = \begin{bmatrix} \dfrac{2w}{(w-2z)^2}-\dfrac{wF(z)}{w-2z} & 0 \\ w\partial_\rho^2\hat{\Psi}_{21}(z, 0, w) & F(z) \end{bmatrix}$$

where $w\partial_\rho^2\hat{\Psi}_{21}(z, 0, w)=\dfrac{1}{w-2z}\left\{\dfrac{-2\hat{\chi}(z)}{w-2z}-\partial_z\hat{\chi}(z)+wG(z)-\hat{\chi}(z)F(z)\right\}$. By virtue of (5.6) we have

$$\partial_\rho^2(\hat{X}_+)_{11}(z, 0, w) = -\frac{2}{w}\cdot u_{11}^{(1)}(w-2z)-\frac{F(z)}{w}\cdot u_{11}(w-2z)\,,$$

$$\partial_\rho^2(\hat{X}_+)_{12}(z, 0, w) = \partial_z\left\{\frac{1}{w-2z}\cdot u_{12}(w-2z)\right\}-\frac{F(z)}{w-2z}\cdot u_{12}(w-2z)\,,$$

$$\begin{aligned}\partial_\rho^2(\hat{X}_+)_{21}(z, 0, w) = &-\frac{1}{w^2}\cdot\partial_z\hat{\chi}(z)\cdot u_{11}(w-2z)+\frac{G(z)}{w}\cdot u_{11}(w-2z)\\ &+\frac{2\hat{\chi}(z)}{w^2}\cdot u_{11}^{(1)}(w-2z)+\frac{1}{w}\cdot\left(1-\frac{2z}{w}\right)F(z)u_{21}(w-2z)\\ &-\frac{1}{w^2}\cdot\hat{\chi}(z)F(z)u_{11}(w-2z)-\frac{2}{w^2}\cdot u_{21}(w-2z)\\ &-\frac{2}{w}\cdot\left(1-\frac{2z}{w}\right)u_{21}^{(1)}(w-2z)\end{aligned}$$

$$\partial_\rho^2(\hat{X}_+)_{22}(z,\,0,\,w) = -\frac{1}{w}\partial_z\left\{\frac{\hat{\chi}(z)}{w-2z}\cdot u_{12}(w-2z)\right\}-\frac{G(z)}{w-2z}\cdot u_{12}(w-2z)$$
$$-\frac{\hat{\chi}(z)F(z)}{w(w-2z)}\cdot u_{12}(w-2z)+\frac{F(z)}{w}\cdot u_{22}(w-2z)-\frac{2}{w}\cdot u_{22}^{(1)}(w-2z)\,.$$

By using the conditions $u_{11}(-2z)\hat{f}(z)=1$, $\hat{\chi}(z)=-2zu_{21}(-2z)/u_{11}(-2z)$ and $u_{12}(0)=0$, one can show that $\partial_\rho^2(\hat{X}_+)_{11}(z,\,0,\,w)$ and $\partial_\rho^2(\hat{X}_+)_{12}(z,\,0,\,w)$ have no negative power of w and the coefficients of w^{-2} and w^{-1} in $\partial_\rho^2(\hat{X}_+)_{21}(z,\,0,\,w)$ vanishes. Finally the coefficient of w^{-1} in $\partial_\rho^2(\hat{X}_+)_{22}(z,\,0,\,w)$ is

$$\frac{2}{u_{11}(-2z)}\{u_{21}u_{12}-u_{11}u_{22}\}^{(1)}(-2z)$$

which vanishes by virtue of $\det(u(t))=1$. q.e.d.

Further assume $u_{12}(t)=t^2u_{21}(t)$. Finally we will show that

$$(\partial_\rho^4\Psi_{12})(z,\,0) = -3[\hat{f}(z)]^{-2}\partial_z^2\hat{\chi}(z)\,.$$

By Appendix A and the previous results, $(\partial_\rho^4\Psi_1)$ $(z,\,0)$ takes the form

$$(\partial_\rho^4\Psi_1)(z,\,0)=\begin{bmatrix}H(z) & I(z)\\ K(z) & -H(z)\end{bmatrix}$$

for some $H,\ I,\ K\in\boldsymbol{R}[[z]]$. Similarly as before we have

$$w\partial_\rho^4\Psi(z,\,0,\,w) = 3\partial_z\partial_\rho^2\Psi(z,\,0,\,w)+(\partial_\rho^4\Psi_1(z,\,0))\Psi(z,\,0,\,w)$$
$$+3(\partial_\rho^2\Psi_1(z,\,0))\partial_\rho^2\Psi(z,\,0,\,w)\,,$$
$$\partial_\rho^4(X_+)(z,\,0,\,w) = (\partial_\rho^4\Psi(z,\,0,\,w))\cdot u(w-2z)\cdot[\Psi^{(0)}(z,\,0,\,w)]^{-1}$$
$$+\frac{6}{w}\partial_\rho^2\Psi(z,\,0,\,w)\partial_z\{u(w-2z)\cdot[\Psi^{(0)}(z,\,0,\,w)]^{-1}\}$$
$$+\frac{3}{w^2}\cdot\Psi(z,\,0,\,w)\partial_z^2\{u(w-2z)\cdot[\Psi^{(0)}(z,\,0,\,w)]^{-1}\}\,.$$

Since the coefficient of w^{-1} in the (1,2) component of $\partial_\rho^4X_+(z,\,0,\,w)$ vanishes, we get

$$I(z) = -3u_{11}(-2z)\partial_z^2\left\{-\frac{u_{11}^{(2)}(-2z)}{2z}\right\}+6F(z)u_{11}(-2z)\partial_z\left\{-\frac{u_{12}(-2z)}{2z}\right\}$$
$$-3u_{11}(-2z)\left\{2[F(z)]^2-\frac{4u_{11}^{(2)}(2-2z)}{u_{11}(-2z)}\right\}\cdot\left\{-\frac{u_{12}(-2z)}{2z}\right\}.$$

On the other hand

$$-3[\hat{f}(z)]^{-2}\partial_z^2\hat{\chi}_1(z) = -3u_{11}(-2z)\partial_z^2\{-2zu_{21}(-2z)\}$$

$$+6F(z)u_{11}(-2z)\partial_z\{-2zu_{21}(-2z)\}$$
$$-3u_{11}(-2z)\left\{2[F(z)]^2-\frac{4u_{11}^{(2)}(-2z)}{u_{11}(-2z)}\right\}\cdot\{-2zu_{21}(-2z)\}\,.$$

Hence when $u_{12}(t)=t^2u_{21}(t)$ we have

$$I(z)=(\partial_\rho^4\Psi_{12})(z,0)=-3[\hat{f}(z)]^{-2}\partial_z^2\hat{\chi}(z)\,.$$

Therefore $\hat{\Psi}_1(z,\rho)$ and $\Psi_1(z,\rho)$ satisfy the same initial conditions and hence $\hat{\Psi}(z,\rho,w)=\Psi(z,\rho,w)$.

Appendix A

Let $\Psi_1(z,\rho)$ be a solution of equation (2.14) such that

$$\Psi_1(z,0)=\begin{bmatrix}2z & 0\\ -\chi(z) & 0\end{bmatrix},\quad (\partial_\rho\Psi_1)(z,0)=0$$

and $(\partial_\rho^2\Psi_{11})(z,0)+(\partial_\rho^2\Psi_{22})(z,0)=0$. Then

$$\Psi_{11}(z,\rho)+\Psi_{22}(z,\rho)=2z\,.$$

Proof. Taking the trace of equation (2.14), we have

$$(2.14)' \qquad \rho\partial_z^2\gamma(z,\rho)+\rho\partial_\rho^2\gamma(z,\rho)-\partial_\rho\gamma(z,\rho)=0$$

where we denote $\gamma(z,\rho)=\Psi_{11}(z,\rho)+\Psi_{22}(z,\rho)$. Since $\gamma(z,0)=2z$, $\partial_\rho\gamma(z,0)=0$ and $\partial_\rho^2\gamma(z,\rho)=0$, we can express $\gamma(z,\rho)$ as $\gamma(z,\rho)=2z+\sum_{k=0}^\infty\gamma_k(z)\rho^{k+3}$. Substituting this expression into (2.14)$'$ we have $\gamma_0(z)=\gamma_1(z)=0$ and $(k+1)(k+3)\gamma_k(z)+\partial_z^2\gamma_{k-2}(z)=0$ for $k\geq 2$. Hence it follows that $\gamma_k(z)=0$ for all $k\in\boldsymbol{N}$. q.e.d.

Appendix B

The proof of Proposition 2.3. By virtue of Appendix A, equation (2.14) is equivalent to

$$(\text{B.1})\qquad \rho\partial_z^2\Psi_{22}+\rho\partial_\rho^2\Psi_{22}-\partial_\rho\Psi_{22}+\partial_\rho\Psi_{21}\partial_z\Psi_{12}-\partial_z\Psi_{21}\partial_\rho\Psi_{12}=0\,,$$
$$(\text{B.2})\qquad \rho\partial_z^2\Psi_{12}+\rho\partial_\rho^2\Psi_{12}-3\partial_\rho\Psi_{12}+2(\partial_z\Psi_{12}\partial_z\Psi_{22}-\partial_z\Psi_{12}\partial_\rho\Psi_{22})=0\,,$$
$$(\text{B.3})\qquad \rho\partial_z^2\Psi_{21}+\rho\partial_\rho^2\Psi_{21}+\partial_\rho\Psi_{21}+2(\partial_\rho\Psi_{22}\partial_z\Psi_{21}-\partial_z\Psi_{22}\partial_\rho\Psi_{21})=0\,.$$

Under the assumptions in Proposition 2.3 the components Ψ_{22}, Ψ_{21} and Ψ_{12} take the form

$$\Psi_{22}(z,\rho)=\sum_{k=0}^\infty\frac{1}{(k+1)!}\cdot\Psi_{22}^k(z)\rho^{k+2}\,,$$
$$\Psi_{12}(z,\rho)=\sum_{k=0}^\infty\frac{1}{(k+2)!}\cdot\Psi_{12}^k(z)\rho^{k+2}\,,$$

$$\Psi_{21}(z, \rho) = \Psi_{21}(z) + \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \cdot \Psi_{21}^{k}(z)\rho^{k+2} .$$

Substituting these into (B.1), (B.2) and (B.3) we have for any k, m, $n \geqq 0$

$$\text{(B.4)} \quad \frac{k}{(k+1)!} \cdot \Psi_{22}^{k}(z) + \frac{1}{k!} \cdot \partial_z^2 \Psi_{22}^{k-2}(z) - \frac{2}{(k+1)!} \cdot \partial_z \Psi_{21}(z) \cdot \Psi_{12}^{k}(z)$$
$$+ 2 \sum_{m+n=k-2} \left\{ \frac{1}{(m+1)!(n+2)!} \cdot \Psi_{21}^{m}(z) \cdot \partial_z \Psi_{22}^{n}(z) \right.$$
$$\left. - \frac{1}{(m+2)!(n+1)!} \cdot \partial_z \Psi_{21}^{m}(z) \cdot \Psi_{12}^{n}(z) \right\} = 0 ,$$

$$\text{(B.5)} \quad \frac{k}{(k+1)!} \Psi_{12}^{k}(z) + \frac{1}{k!} \cdot \partial_z^2 \Psi_{12}^{k-2}(z)$$
$$+ 2 \sum_{m+n=k-2} \left\{ \frac{1}{(m+1)!(n+2)!} \cdot \Psi_{12}^{m}(z) \cdot \partial_z \Psi_{22}^{n}(z) \right.$$
$$\left. - \frac{1}{(m+2)!(n+1)!} \cdot \partial_z \Psi_{12}^{m}(z) \cdot \Psi_{22}^{n}(z) \right\} = 0 ,$$

$$\text{(B.6)} \quad \frac{k+2}{(k+1)!} \cdot \Psi_{21}^{k}(z) + \frac{2}{(k+1)!} \cdot \Psi_{22}^{k}(z) \cdot \partial_z \Psi_{21} + \frac{1}{k!} \cdot \partial_z^2 \Psi_{21}^{k-2}(z)$$
$$+ 2 \sum_{m+n=k-2} \left\{ \frac{1}{(m+1)!(n+2)!} \cdot \Psi_{22}^{m}(z) \cdot \partial_z \Psi_{21}^{n}(z) \right.$$
$$\left. - \frac{1}{(m+2)!(n+1)!} \cdot \partial_z \Psi_{22}^{m}(z) \cdot \Psi_{21}^{n}(z) \right\} = 0 ,$$

where we set $\Psi_{ij}^{k}(z) = 0$ for $k < 0$.

Since $\Psi_{21}(z)$, Ψ_{22}^{0} and Ψ_{12}^{2} are given as the initial data, equations (B.4), (B.5) and (B.6) uniquely determine the solutions of equation (2.14). q.e.d.

References

[1] V.A. Belinskii and V.E. Zakharov: *Integration of the Einstein equations by means of the inverse scattering problem technique*, Sov. Phys. J.E.T.P. **48** (1978), 985–994.
———: *Stationary gravitational solitons with axial symmetry*, Sov. Phys. J.E.T.P. **50** (1979), 1-9.

[2] C.M. Cosgrove: *Relationship between the inverse scattering techniques of Belinskii-Zakharov and Hauser-Ernst in general relativity*, J. Mathematical Phys. **23** (1982), 615–633.

[3] F.J. Ernst: *Complex potential formulation fo the axially symmetric gravitational fields problem*, J. Mathematical Phys. **15** (1974), 1409–1412.

[4] R. Geroch: *A new method of generating new solutions of Einstein's equation. II.*, J. Mathematical Phys. **13** (1972), 394–404.

[5] I. Hauser and F.J. Ernst: *Integral equation method for effecting Kinnersley-Chitre transformations*, Phys. Rev. D **20** (1979), 362–369.

———: *A Homogeneous Hilbert problem for the Kinnersley-Chitre transformations,* J. Mathematical Phys. **21** (1979), 1126–1139.

[6] ———: *Proof of a Geroch conjecture,* J. Mathematical Phys. **22** (1981), 1151–1163.

[7] D. Maison: *On the complete integrability of the stationary axially symmetric Einstein equations,* J. Mathematical Phys. **20** (1973), 871–877.

[8] Y. Nakamura: *On a linearization of the stationary axially symmetric Einstein equations,* Class. Quantum Grav. **4** (1987), 437–440.

[9] K. Takasaki: *A new approach to the self-dual Ynag-Mills equations,* Commun. math. Phys. **94** (1984), 35–59.

———: *ditto II.*, Saitama Math. J. **3** (1985), 11–40.

———: *Aspect of the integrability in self-dual Einstein metric and related equations,* Publ. Res. Inst. Math. Sci. **22** (1986), 949–990.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan