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REGULARITY IN TIME OF THE SOLUTION OF PARABOLIC INITIAL-BOUNDARY VALUE PROBLEM IN L^1 SPACE

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1. Introduction

This paper is concerned with the regularity in t of the solution of the initial-boundary value problem of the linear parabolic partial differential equation

(1.1) $\partial u(x, t)/\partial t+A(x, t, D)u(x, t) = f(x, t)\,, \qquad \Omega\times(0, T]\,,$

(1.2) $B_j(x, t, D)u(x, t) = 0\,, \quad j = 1, \cdots, m/2\,, \qquad \partial\Omega\times(0, T]\,,$

(1.3) $u(x, 0) = u_0(x)\,, \qquad \Omega\,.$

Here Ω is a not necessarily bounded domain in R^N with boundary $\partial\Omega$ satisfying a certain smoothness hypothesis. For each $t\in[0, T]$ $A(x, t, D)$ is a strongly elliptic linear differential operator of order m, and $\{B_j(x, t, D)\}_{j=1}^{m/2}$ is a normal set of linear differential operators of respective orders $m_j<m$. It is assumed that the realization $-A_p(t)$ of $-A(x, t, D)$ in $L^p(\Omega)$ under the boundary conditions $B_j(x, t, D)u|_{\partial\Omega}=0$, $j=1, \cdots, m/2$, generates an analytic semigroup in $L^p(\Omega)$ for any $p\in(1, \infty)$. A sufficient condition for that, which is also necessary when $p=2$, is given in S. Agmon [1]. Assuming moreover that the coefficients of $A(x, t, D)$, $\{B_j(x, t, D)\}_{j=1}^{m/2}$ and some of their derivatives in x belong to Gevrey's class $\{M_k\}$ ([4], [6], [7]) as functions of t and f also belongs to the same class as a function with values in $L^1(\Omega)$, we show that the same is true of the solution of (1.1)–(1.3) considered as an evolution equation in $L^1(\Omega)$ for any initial value $u_0\in L^1(\Omega)$. It should be noted here that if $m_j=m-1$, the boundary condition $B_j(x, t, D)u|_{\partial\Omega}=0$ is satisfied only in a variational sense.

In order to prove the result stated above we show that there exist positive constants K_0, K such that

$$\|(\partial/\partial t)^n(A(t)-\lambda)^{-1}\|\leqq K_0K^nM_n/|\lambda| \tag{1.4}$$

for any $n=0, 1, 2, \cdots$, $t\in[0, T]$ and λ in the sector Σ: $|\arg\lambda|\geqq\theta_0$, $0<\theta_0<\pi/2$, where $A(t)$ is the realization of the operator $A(x, t, D)$ in $L^1(\Omega)$ under the boundary conditions $B_j(x, t, D)u|_{\partial\Omega}=0$, $j=1, \cdots, m/2$. Once (1.4) is established, one can apply the result of [9] to show that the estimates

$$(1.5)\qquad \|(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m(\partial/\partial s)^k U(t, s)\| \leqq L_0 L^{n+m+k} M_{n+m+k}(t-s)^{-n-k}$$

hold for n, m, $k=0, 1, 2, \cdots$ for the evolution operator $U(t, s)$ to the equation

$$(1.6)\qquad du(t)/dt+A(t)u(t)=f(t)\,, \qquad 0<t\leqq T\,,$$

where L_0, L are some positive constants independent of n, m, k, t, s. As for the solution $u(t)$ of the inhomogeneous equation (1.6) satisfying the initial condition $u(0)=u_0$, if u_0 is an arbitrary element of $L^1(\Omega)$ and $f(t)$ is an infinitely differentiable function with values in $L^1(\Omega)$ such that

$$(1.7)\qquad \|d^n f(t)/dt\| \leqq F_0 F^n M_n\,, \qquad 0\leqq t\leqq T, \quad n=0, 1, 2, \cdots,$$

for some constants F_0, F, then we have

$$(1.8)\qquad \|d^n u(t)/dt^n\| \leqq L_0 L^n M_n \|u_0\| t^{-n} + \bar{F}_0 \bar{F}^n M_n t^{1-n}\,, \qquad 0<t\leqq T$$

for $n=0, 1, 2, \cdots$, where $\bar{F}_0$, $\bar{F}$ are constants depending only on d_1, F_0, F, L_0, L, T.

Analogous results on the same equation in $L^p(\Omega)$, $1<p<\infty$, were proved in [9]. It was shown in [8] that the evolution operator $U(t, s)$ of (1.6) exists if the coefficients of $A(x, t, D)$, $\{B_j(x, t, D)\}_{j=1}^{m/2}$ and some of their derivatives in x are once continuously differentiable in t.

In [10] with the aid of the idea of R. Beals [2] and L. Hörmander [5] the estimates of the kernels $G(x, y, \tau)$, $K_\lambda(x, y)$ of operators $\exp(-\tau A_p)$, $(A_p-\lambda)^{-1}$ were established for $1<p<\infty$, where $A_p=A_p(t)$ for some fixed $t\in[0, T]$. The operator $\exp(-\tau A)$ in $L^1(\Omega)$ was then defined as an integral operator with kernel $G(x, y, \tau)$, and was shown to be an analytic semigroup with the infinitesimal generator $-A=-A(t)$.

We use the same method to estimate the derivatives in t of the kernel of $(A(t)-\lambda)^{-1}$. In order to make the paper self-contained we reproduce part of the argument of [10] which is relevant to the proof of our main result.

2. Assumptions and main theorem

Let Ω be a not necessarily bounded domain of R^N locally regular of class C^{2m} and uniformly regular of class C^m in the sense of F.E. Browder [3]. The boundary of Ω is denoted by $\partial\Omega$. We put $D=(\partial/\partial x_1, \cdots, \partial/\partial x_N)$.

Let

$$A(x, t, D)=\sum_{|\alpha|\leqq m} a_\alpha(x, t)D^\alpha$$

be a linear differential operator of even order m with coefficients defined in $\bar{\Omega}$ for each fixed $t\in[0, T]$, and let

$$B_j(x, t, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x, t) D^\beta, \qquad j = 1, \cdots, m/2$$

be a set of linear differential operators of respective orders $m_j < m$ with coefficients defined on $\partial\Omega$ for each fixed $t \in [0, T]$.

The principal parts of $A(x, t, D)$ and $B_j(x, t, D)$ are denoted by $A^\#(x, t, D)$ and $B_j^\#(x, t, D)$ respectively.

Let $\{M_k, k=0, 1, 2, \cdots\}$ be a sequence of positive numbers which satisfy the following conditions ([4], [6], [7]): for some positive constants d_0, d_1, d_2

(2.1) $M_{k+1} \leqq d_0^k M_k$ for all $k \geqq 0$,

(2.2) $\binom{k}{j} M_{k-j} M_j \leqq d_1 M_k$ for all k, j such that $0 \leqq j \leqq k$,

(2.3) $M_k \leqq M_{k+1}$ for all $k \geqq 0$,

(2.4) $M_{j+k} \leqq d_2^{j+k} M_j M_k$ for all $j, k \geqq 0$.

We assume the following:

(A.1) For each $t \in [0, T]$ $A(x, t, D)$ is strongly elliptic, i.e. for all real vectors $\xi \neq 0$, all $(x, t) \in \bar{\Omega} \times [0, T]$

$$(-1)^{m/2} \operatorname{Re} A^\#(x, t, \xi) > 0 .$$

(A.2) $\{B_j(x, t, D)\}_{j=1}^{m/2}$ is a normal set of boundary operators, i.e. $\partial\Omega$ is noncharacteristic for each $B_j(x, t, D)$ and $m_j \neq m_k$ for $j \neq k$.

(A.3) For any $(x, t) \in \partial\Omega \times [0, T]$ let ν be the normal to $\partial\Omega$ at x and $\xi \neq 0$ be parallel to $\partial\Omega$ at x. The polynomials in τ

$$B_j^\#(x, t, \xi + \tau\nu), \qquad j = 1, \cdots, m/2,$$

are linearly independent modulo the polynomial in τ, $\prod_{k=1}^{m/2} (\tau - \tau_k^+(\xi, \lambda; x, t))$ for any complex number λ with non-positive real part where $\tau_k^+(\xi, \lambda; x, t)$ are the roots with positive imaginary part of the polynomial in τ, $(-1)^{m/2} A^\#(x, t, \xi + \tau\nu) - \lambda$.

(A.4) For each $t \in [0, T]$ the formal adjoint

$$A'(x, t, D) = \sum_{|\alpha| \leq m} a'_\alpha(x, t) D^\alpha$$

and the adjoint system of boundary operators

$$B'_j(x, t, D) = \sum_{|\beta| \leq m'_j} b'_{j\beta}(x, t) D^\beta, \qquad j = 1, \cdots, m/2$$

can be constructed.

(A.5) For $|\alpha| = m$ $a_\alpha(x, t)$ are uniformly continuous in $\bar{\Omega} \times [0, T]$. For $|\alpha| \leqq m a_\alpha(x, t)$, $a'_\alpha(x, t)$ have continuous derivatives in t of all orders in $\bar{\Omega} \times [0, T]$, and there exist positive constants B_0, B such that

$$(2.5)\qquad |(\partial/\partial t)^k a_\alpha(x, t)| \leqq B_0 B^k M_k$$
$$\qquad\qquad (x, t)\in\bar{\Omega}\times[0, T]$$
$$(2.6)\qquad |(\partial/\partial t)^k a'_\alpha(x, t)| \leqq B_0 B^k M_k$$

for $k=0, 1, 2, \cdots$. For $j=1, \cdots, m/2$ $D^\gamma b_{j\beta}(x, t)$, $|\gamma|\leqq m-m_j$, $|\beta|\leqq m_j$, and $D^\gamma b'_{j\beta}(x, t)$, $|\gamma|\leqq m-m'_j$, $|\beta|\leqq m'_j$, have continuous derivatives in t of all orders on $\partial\Omega\times[0, T]$, and

$$(2.7)\qquad |(\partial/\partial t)^k D^\gamma b_{j\beta}(x, t)| \leqq B_0 B^k M_k$$
$$\qquad\qquad (x, t)\in\partial\Omega\times[0, T]$$
$$(2.8)\qquad |(\partial/\partial t)^k D^\gamma b'_{j\beta}(x, t)| \leqq B_0 B^k M_k$$

for $k=0, 1, 2, \cdots$.

Let $W^{m,p}(\Omega)$ be the Banach space consisting of measurable functions defined in Ω whose distribution derivatives of order up to m belong to $L^p(\Omega)$. The norm of $W^{m,p}(\Omega)$ is defined and denoted by

$$||u||_{m,p} = \Big(\sum_{|\alpha|\leqq m}\int_\Omega |D^\alpha u|^p dx\Big)^{1/p}.$$

We simply write $||\ ||_p$ instead of $||\ ||_{0,p}$ to denote L^p-norm. We use the notation $||\ ||$ to denote both the norm of $L^1(\Omega)$ and that of bounded linear operators from $L^1(\Omega)$ to itself.

For each $t\in[0, T]$ $A(t)$ is the operator defined as follows.

The domain $D(A(t))$ is the totality of functions u satisfying the following three conditions:

(i) $u\in W^{m-1,q}(\Omega)$ for any q with $1\leqq q<N/(N-1)$,

(ii) $A(x, t, D)u\in L^1(\Omega)$ in the sense of distributions,

(iii) for any p with $0<(N/m)(1-1/p)<1$ and any $v\in W^{m,p'}(\Omega)$, $p'=p/(p-1)$ satisfying $B'_j(x, t, D)v|_{\partial\Omega}=0, j=1, \cdots, m/2$,

$$(A(x, t, D)u, v) = (u, A'(x, t, D)v).$$

For $u\in D(A(t))$

$$(A(t)u)(x) = A(x, t, D)u(x).$$

We note that the boundary value of $B_j(x, t, D)u$ is defined and vanishes if $m_j<m-1$ for $u\in D(A(t))$.

It is known that $-A(t)$ generates an analytic semigroup in $L^1(\Omega)$. Hence there exist an angle $\theta_0\in(0, \pi/2)$ and positive constants C_1, C_2 such that

$$(2.9)\qquad \rho(A(t))\supset\Sigma\cap\{\lambda: |\lambda|\geqq C_1\},$$
$$(2.10)\qquad ||(\lambda-A(t))^{-1}||\leqq C_2/|\lambda| \quad \text{for } \lambda\in\Sigma, |\lambda|\geqq C_1,$$

where $\rho(A(t))$ stands for the resolvent set of $A(t)$ and Σ is the closed sector

$\{\lambda: \theta_0 \leqq \arg\lambda \leqq 2\pi - \theta_0\} \cup \{0\}$.

We write (1.1)–(1.3) as an evolution equation in $L^1(\Omega)$:

$$du(t)/dt + A(t)u(t) = f(t), \qquad 0 < t \leqq T, \tag{2.11}$$

$$u(0) = u_0. \tag{2.12}$$

Let $U(t, s)$ be the evolution operator of (2.11) which is a bounded operator valued function defined in $\bar{\Delta}$ satisfying

$$\begin{aligned} &\partial U(t, s)/\partial t + A(t)U(t, s) = 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (s, t) \in \Delta, \\ &\partial U(t, s)/\partial s - U(t, s)A(s) = 0, \\ &\qquad U(s, s) = I \qquad\qquad 0 \leqq s \leqq T, \end{aligned}$$

where $\Delta = \{(s, t): 0 \leqq s < t \leqq T\}$ and $\bar{\Delta} = \{(s, t): 0 \leqq s \leqq t \leqq T\}$. The existence of such an operator is known by [8].

Our main result is the following:

Theorem. *Under the assumptions stated above the evolution operator* $U(t, s)$ *of* (2.11) *is infinitely differentiable in* $(s, t) \in \Delta$. *There exist constants* L_0, L *such that*

$$\begin{aligned} &\|(\partial/\partial t)^n(\partial/\partial t + \partial/\partial s)^m(\partial/\partial s)^k U(t, s)\| \\ &\qquad \leqq L_0 L^{n+m+k} M_{n+m+k}(t-s)^{-n-k}, \qquad (s, t) \in \Delta \end{aligned}$$

for $n, m, k = 0, 1, 2, \cdots$.

Let $u(t)$ *be the solution of the initial value problem* (2.11), (2.12). *If* u_0 *is an arbitrary element of* $L^1(\Omega)$ *and* $f(t)$ *is an infinitely differentiable function with values in* $L^1(\Omega)$ *such that*

$$\|d^n f(t)/dt^n\| \leqq F_0 F^n M_n, \qquad 0 \leqq t \leqq T, \quad n = 0, 1, 2, \cdots,$$

for some constants F_0, F, *then we have*

$$\|d^n u(t)/dt^n\| \leqq L_0 L^n M_n \|u_0\| t^{-n} + \bar{F}_0 \bar{F}^n M_n t^{1-n}, \qquad 0 \leqq t \leqq T,$$

for $n = 0, 1, 2, \cdots$, *where* $\bar{F}_0$, $\bar{F}$ *are constants depending only on* d_1, F_0, F, L_0, L, T.

According to [9] it suffices to prove the following proposition in order to establish the above theorem.

Proposition. *For any complex number* λ *such that* $\lambda \in \Sigma$ *and* $|\lambda| \geqq C_1$, $(A(t)-\lambda)^{-1}$ *is infinitely differentiable in* $t \in [0, T]$, *and there exist positive constants* K_0, K *such that for* $n = 0, 1, 2, \cdots$

$$\|(\partial/\partial t)^n (A(t)-\lambda)^{-1}\| \leqq K_0 K^n M_n / |\lambda|. \tag{2.13}$$

3. Preliminaries

For $1<p<\infty$ the operator $A_p(t)$ is defined as follows:

$$D(A_p(t)) = \{u \in W^{m,p}(\Omega):\ B_j(x,t,D)u = 0 \quad \text{on } \partial\Omega \text{ for } \ j=1,\cdots,m/2\},$$
$$(A_p(t)u)(x) = A(x,t,D)u(x) \qquad \text{for} \quad u \in D(A_p(t)).$$

Replacing $A(x,t,D)$ and $\{B_j(x,t,D)\}_{j=1}^{m/2}$ by $A'(x,t,D)$ and $\{B'_j(x,t,D)\}_{j=1}^{m/2}$ the operator $A'_p(t)$ is defined. According to S. Agmon [1] $-A_p(t)$ generates an analytic semigroup in $L^p(\Omega)$, and with the aid of the argument of F.E. Browder [3] it is shown that the relation $A_p^*(t)=A'_{p'}(t)$ holds where the left member stands for the adjoint operator of $A_p(t)$.

In what follows we assume that the coefficients of $B_j(x,t,D)$, $B'_j(x,t,D)$, $j=1,\cdots,m/2$, are extended to the whole of $\Omega\times[0,T]$ so that (2.7), (2.8) hold there.

Slightly extending the argument of S. Agmon [1] it can be shown that there exist an angle $\theta_0\in(0,\pi/2)$ and a constant $C_p>0$ for each $p\in(1,\infty)$ such that for any $u\in W^{m,p}(\Omega)$, a complex number λ satisfying $\theta_0\leqq\arg\lambda\leqq 2\pi-\theta_0$, $|\lambda|>C_p$, and $t\in[0,T]$

$$\text{(3.1)} \qquad \sum_{j=0}^{m} |\lambda|^{(m-j)/m}\|u\|_{j,p} \leqq C_p\{\|(A(x,t,D)-\lambda)u\|_p + \sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\|g_j\|_p + \sum_{j=1}^{m/2}\|g_j\|_{m-m_j,p}\},$$

where g_j is an arbitrary function in $W^{m-m_j,p}(\Omega)$ such that $B_j(x,t,D)u=g_j$ on $\partial\Omega$ for each $j=1,\cdots,m/2$.

For a complex vector $\eta\in \boldsymbol{C}^N$ put

$$A(x,t,D+\eta) = \sum_{|\alpha|\leqq m} a_\alpha(x,t)(D+\eta)^\alpha,$$
$$B_j(x,t,D+\eta) = \sum_{|\beta|\leqq m_j} b_{j\beta}(x,t)(D+\eta)^\beta$$

(cf. L. Hörmander [5]). As is easily seen the adjoint system of

$$(A(x,t,D+\eta),\ \{B_j(x,t,D+\eta)\}) \quad \text{is} \quad (A'(x,t,D-\bar\eta),\ \{B'_j(x,t,D-\bar\eta)\}).$$

For $1<p<\infty$ $A_p^\eta(t)$, $A_p'^\eta(t)$ are the operators defined by

$$D(A_p^\eta(t)) = \{u\in W^{m,p}(\Omega):\ B_j(x,t,D+\eta)u=0 \quad \text{on } \partial\Omega \quad \text{for} \quad j=1,\cdots,m/2\},$$
$$(A_p^\eta(t)u)(x) = A(x,t,D+\eta)u(x) \qquad \text{for} \quad u\in D(A_p^\eta(t)),$$
$$D(A_p'^\eta(t)) = \{u\in W^{m,p}(\Omega):\ B'_j(x,t,D+\eta)u=0 \quad \text{on } \partial\Omega \text{ for } \ j=1,\cdots,m/2\},$$
$$(A_p'^\eta(t)u)(x) = A'(x,t,D+\eta)u(x) \qquad \text{for} \quad u\in D(A_p'^\eta(t)).$$

Lemma 3.1. *For any $p\in(1,\infty)$ there exist positive constants C'_p, δ_p such that for $\theta_0\leqq\arg\lambda\leqq 2\pi-\theta_0$, $|\lambda|>C'_p$, $t\in[0,T]$, $|\eta|\leqq\delta_p|\lambda|^{1/m}$ the following ine-*

qualities hold:

(i) *for* $u\in W^{m,p}(\Omega)$, $g_j\in W^{m-m_j,p}(\Omega)$ *such that* $B_j(x, t, D+\eta)u=g_j$ *on* $\partial\Omega$, $j=1, \cdots, m/2$,

$$\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u\|_{i,p}\leqq C'_p\{\|(A(x, t, D+\eta)-\lambda)u\|_p+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\|g_j\|_p+\sum_{j=1}^{m/2}\|g_j\|_{m-m_j,p}\};$$

(ii) *for* $v\in W^{m,p}(\Omega)$, $h_j\in W^{m-m'_j,p}(\Omega)$ *such that* $B'_j(x, t, D+\eta)v=h_j$ *on* $\partial\Omega$, $j=1, \cdots, m/2$,

$$\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|v\|_{i,p}\leqq C'_p\{\|(A'(x, t, D+\eta)-\lambda)v\|_p+\sum_{j=1}^{m/2}|\lambda|^{(m-m'_j)/m}\|h_j\|_p+\sum_{j=1}^{m/2}\|h_j\|_{m-m'_j,p}\}.$$

Proof. In the proof of (i) we denote by C constants depending only on N, m, B_0, the upperbounds of the coefficients of $A(x, t, D)$ and the derivatives in x of the coefficients of $B_j(x, t, D)$ of order up to $m-m_j$, $j=1, \cdots, m/2$. As is easily seen

$$\begin{aligned}&\|(A(x, t, D)-\lambda)u\|_p\\ &\quad\leqq\|(A(x, t, D+\eta)-\lambda)u\|_p+\|(A(x, t, D+\eta)-A(x, t, D))u\|_p\\ &\quad\leqq\|(A(x, t, D+\eta)-\lambda)u\|_p+C\sum_{i=0}^{m-1}|\eta|^{m-i}\|u\|_{i,p}.\end{aligned}$$

If we put

$$g'_j=(B_j(x, t, D)-B_j(x, t, D+\eta))u+g_j,$$

then $g'_j\in W^{m-m_j,p}(\Omega)$ and $g'_j=B_j(x, t, D)u$ on $\partial\Omega$, and

$$\|g'_j\|_p\leqq C\sum_{i=0}^{m_j-1}|\eta|^{m_j-i}\|u\|_{i,p}+\|g_j\|_p,$$

$$\|g'_j\|_{m-m_j,p}\leqq C\sum_{i=m-m_j}^{m-1}|\eta|^{m-i}\|u\|_{i,p}+\|g_j\|_{m-m_j,p}.$$

In view of (3.1) and the above inequalities

$$\begin{aligned}\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u\|_{i,p}\leqq C_p\{&\|(A(x, t, D+\eta)-\lambda)u\|_p\\ &+C\sum_{i=0}^{m-1}|\eta|^{m-i}\|u\|_{i,p}+C\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\sum_{i=0}^{m_j-1}|\eta|^{m_j-i}\|u\|_{i,p}\\ &+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\|g_j\|_p+C\sum_{j=1}^{m/2}\sum_{i=m-m_j}^{m-1}|\eta|^{m-i}\|u\|_{i,p}\\ &+\sum_{j=1}^{m/2}\|g_j\|_{m-m_j,p}\}.\end{aligned}$$

If $0<\delta_p\leqq 1$ and $|\eta|\leqq\delta_p|\lambda|^{1/m}$ the right member of the above inequality does not exceed

$$C_p\{||(A(x,\ t,\ D+\eta)-\lambda)u||_p+C\delta_p\sum_{i=0}^{m-1}|\lambda|^{(m-i)/m}||u||_{i,p}$$
$$+C\delta_p\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\sum_{i=0}^{m_j-1}|\lambda|^{(m_j-i)/m}||u||_{i,p}$$
$$+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}||g_j||_p+C\delta_p\sum_{j=1}^{m/2}\sum_{i=m-m_j}^{m-1}|\lambda|^{(m-i)/m}||u||_{i,p}$$
$$+\sum_{j=1}^{m/2}||g_j||_{m-m_j,p}\}$$
$$\leqq C_p\{||(A(x,\ t,\ D+\eta)-\lambda)u||_p+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}||g_j||_p$$
$$+\sum_{j=1}^{m/2}||g_j||_{m-m_j,p}+C\delta_p\sum_{i=0}^{m}|\lambda|^{(m-i)/m}||u||_{i,p}\}\ .$$

Choosing δ_p sufficiently small we easily complete the proof of (i). The proof of (ii) is similar.

Especially if $u\in D(A_p^\eta(t))$, $v\in D(A_p'^\eta(t))$ then we can choose $g_j=0$, $h_j=0$ in Lemma 3.1. Hence we obtain:

Corollary. *If* $\theta_0\leqq\arg\lambda\leqq 2\pi-\theta_0$, $|\lambda|>C_p'$, $t\in[0,\ T]$, $|\eta|\leqq\delta_p|\lambda|^{1/m}$, *then* $\lambda\in\rho(A_p^\eta(t))$, $\lambda\in\rho(A_p'^\eta(t))$ *and the following inequalities hold*:

(3.2) $$||(A_p^\eta(t)-\lambda)^{-1}||_{B(L^p,L^p)}\leqq C_p'/|\lambda|\,,$$

(3.3) $$||(A_p^\eta(t)-\lambda)^{-1}||_{B(L^p,W^{m,p})}\leqq C_p'\,,$$

(3.4) $$||(A_p'^\eta(t)-\lambda)^{-1}||_{B(L^p,W^{m,p})}\leqq C_p'\,.$$

(3.5) $$(A_p^\eta(t))^*=A_{p'}'^{-\bar\eta}(t)\,.$$

Here and in what follows $B(L^p,\ L^p)$, $B(L^p,\ W^{m,p})$ stand for the sets of all bounded linear operators from $L^p(\Omega)$ to $L^p(\Omega)$, $W^{m,p}(\Omega)$ respectively.

Lemma 3.2. *For any* $p\in(1,\ \infty)$ *there exist constants* $C_{3,p}$, $C_{4,p}$ *such that the following inequalities hold for any* $n=0,\ 1,\ 2,\ \cdots$, $\arg\lambda\in[\theta_0,\ 2\pi-\theta_0]$, $|\lambda|>C_p'$, $|\eta|\leqq\delta_p|\lambda|^{1/m}$, $t\in[0\ ,T]$:

(3.6) $$||(\partial/\partial t)^n(A_p^\eta(t)-\lambda)^{-1}||_{B(L^p,L^p)}\leqq C_{3,p}C_{4,p}{}^nM_n/|\lambda|\,,$$

(3.7) $$||(\partial/\partial t)^n(A_p^\eta(t)-\lambda)^{-1}||_{B(L^p,W^{m,p})}\leqq C_{3,p}C_{4,p}{}^nM_n\,,$$

(3.8) $$||(\partial/\partial t)^n(A_p'^\eta(t)-\lambda)^{-1}||_{B(L^p,W^{m,p})}\leqq C_{3,p}C_{4,p}{}^nM_n\,.$$

Proof. In the proof of this lemma we use the notation C to denote constandts depending only on m and N. Letting f be an arbitrary element of $L^p(\Omega)$, we put $u(t)=(A_p^\eta(t)-\lambda)^{-1}f$. Then

$$(3.9) \qquad (A(x, t, D+\eta)-\lambda)u(x, t) = f(x)\,, \qquad x\in\Omega$$

$$(3.10) \qquad B_j(x, t, D+\eta)u(x, t) = 0\,, \qquad x\in\partial\Omega, \quad j=1, \cdots, m/2\,.$$

Differentiating both sides of (3.9), (3.10) n times with repect to t we get

$$(A(x, t, D+\eta)-\lambda)u^{(n)}(x, t) = -\sum_{k=0}^{n-1}\binom{n}{k}A^{(n-k)}(x, t, D+\eta)u^{(k)}(x, t)\,,$$

$$B_j(x, t, D+\eta)u^{(n)}(x, t) = -\sum_{k=0}^{n-1}\binom{n}{k}B_j^{(n-k)}(x, t, D+\eta)u^{(k)}(x, t)\,,$$

where $A^{(n-k)}$ and $B_j^{(n-k)}$ are differential operators obtained by differentiating $n-k$ times the coefficients of A, B_j with respect to t and $u^{(n)}=(\partial/\partial t)^n u$. Applying Lemma 3.1 we get

$$(3.11) \qquad \sum_{i=0}^{m}|\lambda|^{(m-i)/m}||u^{(n)}(t)||_{i,p}$$
$$\leqq C_p'\{||\sum_{k=0}^{n-1}\binom{n}{k}A^{(n-k)}(x, t, D+\eta)u^{(k)}(t)||_p$$
$$+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}||\sum_{k=0}^{n-1}\binom{n}{k}B_j^{(n-k)}(x, t, D+\eta)u^{(k)}(t)||_p$$
$$+\sum_{j=1}^{m/2}||\sum_{k=0}^{n-1}\binom{n}{k}B_j^{(n-k)}(x, t, D+\eta)u^{(k)}(t)||_{m-m_j,p}\}\,.$$

The first term in the bracket of the right side of (3.11) does not exceed

$$(3.12) \qquad C\sum_{k=0}^{n-1}\binom{n}{k}B_0B^{n-k}M_{n-k}\sum_{i=0}^{m}|\eta|^{m-i}||u^{(k)}(t)||_{i,p}$$
$$\leqq C\sum_{k=0}^{n-1}\binom{n}{k}B_0B^{n-k}M_{n-k}\sum_{i=0}^{m}|\lambda|^{(m-i)/m}||u^{(k)}(t)||_{i,p}\,.$$

It is easy to show that remaining terms in the bracket of the right side of (3.11) are not larger than the right side of (3.12). Hence

$$\sum_{i=0}^{m}|\lambda|^{(m-i)/m}||u^{(n)}(t)||_{i,p}$$
$$\leqq CC_p'\sum_{k=0}^{n-1}\binom{n}{k}B_0B^{n-k}M_{n-k}\sum_{i=0}^{m}|\lambda|^{(m-i)/m}||u^{(k)}(t)||_{i,p}\,.$$

Arguing as in [9: p. 542] we show the existence of constants $C_{3,p}$, $C_{4,p}$ such that

$$||u^{(n)}(t)||_{m,p}+|\lambda|\,||u^{(n)}(t)||_p\leqq C_{3,p}C_{4,p}{}^nM_n||f||_p$$

for $n=0, 1, 2, \cdots$. Hence we have established (3.6), (3.7). The proof of (3.8) is similar.

We choose natural numbers l, s and exponents $2=q_1<q_2<\cdots<q_s<q_{s+1}=\infty$,

$2=r_1<r_2<\cdots<r_{l-s}<r_{l-s+1}=\infty$ as follows (R. Beals [2]):

(i) in case $m>N/2$. $l=2$ and $s=1$, hence $2=q_1<q_2=\infty$ and $2=r_1<r_2=\infty$;

(ii) in case $m<N/2$. $s>N/2m$, $l-s>N/2m$, $q_j^{-1}-q_{j+1}^{-1}<m/N$ for $j=1, \cdots, s-1$, $q_{s-1}^{-1}>m/N>q_s^{-1}$, $m-N/q_s$ is not a non-negative integer, $r_j^{-1}-r_{j+1}^{-1}<m/N$ for $j=1, \cdots, l-s-1$, $r_{l-s-1}^{-1}>m/N>r_{l-s}^{-1}$, $m-N/r_{l-s}$ is not a non-negative integer;

(iii) in case $m=N/2$. $l=4$, $s=2$, $2=q_1<q_2<q_3=\infty$, $2=r_1<r_2<r_3=\infty$.

Adding some positive constant to $A(x, t, D)$ if necessary, we may suppose in view of Lemma 3.2 that for any non-negative integer n, complex number $\lambda\in\Sigma=\{\lambda: \arg\lambda\in[\theta_0, 2\pi-\theta_0]\}\cup\{0\}$, complex vector $\eta\in\boldsymbol{C}^N$ such that $|\eta|\leqq\delta|\lambda|^{1/m}$ and $t\in[0, T]$

$$\|(\partial/\partial t)^n(A_p^\eta(t)-\lambda)^{-1}\|_{B(L^p,L^p)}\leqq C_3C_4^nM_n/|\lambda|, \tag{3.13}$$

$$\|(\partial/\partial t)^n(A_p^\eta(t)-\lambda)^{-1}\|_{B(L^p,W^{m,p})}\leqq C_3C_4^nM_n \tag{3.14}$$

for $p=q_1, q_2, \cdots, q_s$, and

$$\|(\partial/\partial t)^n(A_p'^\eta(t)-\lambda)^{-1}\|_{B(L^p,L^p)}\leqq C_3C_4^nM_n/|\lambda|, \tag{3.15}$$

$$\|(\partial/\partial t)^n(A_p'^\eta(t)-\lambda)^{-1}\|_{B(L^p,W^{m,p})}\leqq C_3C_4^nM_n \tag{3.16}$$

for $p=r_1, r_2, \cdots, r_{l-s}$, where C_3, C_4 and δ are some positive constants.

According to Sobolev's imbedding theorem there exists a positive constant γ such that for $j=1, \cdots, s$

$$W^{m,q_j}(\Omega)\subset L^{q_{j+1}}(\Omega) \quad\text{and}\quad \|u\|_{q_{j+1}}\leqq\gamma\|u\|_{m,q_j}^{a_j}\|u\|_{q_j}^{1-a_j} \tag{3.17}$$

where $0<a_j=(N/m)(q_j^{-1}-q_{j+1}^{-1})<1$, and for $j=1, \cdots, l-s$

$$W^{m,r_j}(\Omega)\subset L^{r_{j+1}}(\Omega) \quad\text{and}\quad \|u\|_{r_{j+1}}\leqq\gamma\|u\|_{m,r_j}^{a_{s+j}}\|u\|_{r_j}^{1-a_{s+j}} \tag{3.18}$$

where $0<a_{s+j}=(N/m)(r_j^{-1}-r_{j+1}^{-1})<1$.

4. Estimates of the kernel of the derivatives of $\exp(-\tau A(t))$ (1)

In what follows we only consider the case (ii) of the previous section.

For complex numbers $\lambda_1, \cdots, \lambda_l\in\Sigma$, a complex vector $\eta\in\boldsymbol{C}^N$ such that

$$|\eta|\leqq\delta\min\{|\lambda_1|^{1/m}, \cdots, |\lambda_l|^{1/m}\}, \tag{4.1}$$

and $t\in[0, T]$ we put

$$S(t)=(A_2^\eta(t)-\lambda_s)^{-1}\cdots(A_2^\eta(t)-\lambda_1)^{-1}, \tag{4.2}$$

$$T(t)=(A_2^\eta(t)-\lambda_{s+1})^{-1}\cdots(A_2^\eta(t)-\lambda_l)^{-1}. \tag{4.3}$$

In view of (3.17)

$$R((A_2^\eta(t)-\lambda_1)^{-1})\subset W^{m,2}(\Omega)=W^{m,q_1}(\Omega)\subset L^{q_2}(\Omega).$$

Hence, we may replace $(A_2^\eta(t)-\lambda_2)^{-1}$ in (4.2) by $(A_{q_2}^\eta(t)-\lambda_2)^{-1}$. Continuing this process we get

$$S(t)=(A_{q_s}^\eta(t)-\lambda_s)^{-1}\cdots(A_{q_2}^\eta(t)-\lambda_2)^{-1}(A_2^\eta(t)-\lambda_1)^{-1}. \tag{4.4}$$

By virtue of Sobolev's imbedding theorem we get

$$R(S(t))\subset R((A_{q_s}^\eta(t)-\lambda_s)^{-1})\subset B^{m-N/q_s}(\bar{\Omega}) \tag{4.5}$$

where $B^{m-N/q_s}(\bar{\Omega})$ is the set of all functions which have bounded, continuous derivatives of order up to $[m-N/q_s]$ in $\bar{\Omega}$ and have derivatives of order $[m-N/q_s]$ uniformly Hölder continuous with exponent $m-N/q_s-[m-N/q_s]$.

With the aid of (3.13), (3.14), (3.17) we get

$$\begin{aligned}(4.6)\qquad &\|(\partial/\partial t)^n(A_{q_j}^\eta(t)-\lambda_j)^{-1}f\|_{q_{j+1}}\\ &\leqq\gamma\|(\partial/\partial t)^n(A_{q_j}^\eta(t)-\lambda_j)^{-1}f\|_{m,q_j}^{a_j}\|(\partial/\partial t)^n(A_{q_j}^\eta(t)-\lambda_j)^{-1}f\|_{q_j}^{1-a_j}\\ &\leqq\gamma C_3C_4^nM_n|\lambda_j|^{a_j-1}\|f\|_{q_j}.\end{aligned}$$

Using (4.4) and (4.6) for $n=0$ we obtain

$$\|S(t)\|_{B(L^2,L^\infty)}\leqq(\gamma C_3M_0)^s\prod_{j=1}^s|\lambda_j|^{a_j-1}. \tag{4.7}$$

Similarly we see that

$$R(T^*(t))\subset B^{m-N/r_{l-s}}(\bar{\Omega}), \tag{4.8}$$

$$\begin{aligned}(4.9)\qquad \|T^*(t)\|_{B(L^2,L^\infty)}&=\|(A'_{r_{l-s}}{}^{-\bar{\eta}}(t)-\bar{\lambda}_l)^{-1}\cdots(A'_{r_1}{}^{-\bar{\eta}}(t)-\bar{\lambda}_{s+1})^{-1}\|_{B(L^2,L^\infty)}\\ &\leqq(\gamma C_3M_0)^{l-s}\prod_{j=s+1}^{l}|\lambda_j|^{a_j-1}.\end{aligned}$$

Lemma 4.1 ([2]). *Let S and T be bounded linear operators in $L^2(\Omega)$ such that $R(S)\subset L^\infty(\Omega)$ and $R(T^*)\subset L^\infty(\Omega)$. Then ST has a kernel $k\in L^\infty(\Omega\times\Omega)$ satisfying*

$$\|k\|_\infty\leqq\|S\|_{B(L^2,L^\infty)}\|T^*\|_{B(L^2,L^\infty)}.$$

In view of (4.5), (4.7), (4.8), (4.9) and Lemma 4.1

$$S(t)T(t)=(A_2^\eta(t)-\lambda_1)^{-1}\cdots(A_2^\eta(t)-\lambda_l)^{-1}$$

has a continuous kernel $K_{\lambda_1,\dots,\lambda_l}{}^\eta(x,y;t)$ satisfying

$$|K_{\lambda_1,\dots,\lambda_l}{}^\eta(x,y;t)|\leqq(\gamma C_3M_0)^l\prod_{j=1}^{l}|\lambda_j|^{a_j-1}. \tag{4.10}$$

If η is pure imaginary, $e^{\cdot\eta}f\in L^p(\Omega)$ if and only if $f\in L^p(\Omega)$, and hence

$$(A_p^\eta(t)-\lambda)^{-1}f=e^{-\cdot\eta}(A_q(t)-\lambda)^{-1}(e^{\cdot\eta}f),$$

which implies

$$S(t)T(t)f = e^{-\cdot\eta}(A_2(t)-\lambda_1)^{-1}\cdots(A_2(t)-\lambda_l)^{-1}(e^{\cdot\eta}f)\,.$$

Hence, if we denote the kernel of

$$(A_2(t)-\lambda_1)^{-1}\cdots(A_2(t)-\lambda_l)^{-1}$$

by $K_{\lambda_1,\cdots,\lambda_l}(x, y; t)$, we have

$$K_{\lambda_1,\cdots,\lambda_l}(x, y;\, t) = e^{(x-y)\eta}K_{\lambda_1,\cdots,\lambda_l}{}^{\eta}(x, y;\, t) \tag{4.11}$$

if η is pure imaginary. As is easily seen $S(t)T(t)$ is a holomorphic function of η in $|\eta|\leqq\delta\min\{|\lambda_1|^{1/m}, \cdots, |\lambda_l|^{1/m}\}$, and hence so is $K_{\lambda_1,\cdots,\lambda_l}{}^{\eta}(x, y; t)$. Thus (4.11) also holds for complex vector η. With the aid of (4.10), (4.11) we get when η is real

$$|K_{\lambda_1,\cdots,\lambda_l}(x, y;\, t)|\leqq(\gamma C_3M_0)^l e^{(x-y)\eta}\prod_{j=1}^{l}|\lambda_j|^{a_j-1}\,.$$

Minimizing the right side of this inequality with respect to η we obtain

$$\begin{aligned}(4.12)\quad &|K_{\lambda_1,\cdots,\lambda_l}(x, y;\, t)|\\ &\leqq(\gamma C_3M_0)^l \exp\,[-\delta\min\{|\lambda_1|^{1/m}, \cdots, |\lambda_l|^{1/m}\}\,|x-y|]\prod_{j=1}^{l}|\lambda_j|^{a_j-1}\,.\end{aligned}$$

Next, we estimate the derivatives of $K_{\lambda_1,\cdots,\lambda_l}(x,y; t)$ in t. For that purpose we first estimate the kernel of

$$(\partial/\partial t)^n(S(t)T(t)) = \sum_{k=0}^{n}\binom{n}{k}(\partial/\partial t)^{n-k}S(t)(\partial/\partial t)^kT(t)\,. \tag{4.13}$$

Using (4.4), (4.6),

$$\begin{aligned}&||(\partial/\partial t)^{n-k}S(t)||_{B(L^2,L^\infty)}\\ &=||\sum_{k_1+\cdots+k_s=n-k}\frac{(n-k)!}{k_1!\cdots k_s!}(\partial/\partial t)^{k_s}(A^{\eta}_{q_s}(t)-\lambda_s)^{-1}\cdots\\ &\qquad\cdots(\partial/\partial t)^{k_1}(A^{\eta}_{q_1}(t)-\lambda_1)^{-1}||_{B(L^p,L^\infty)}\\ &\leqq\sum_{k_1+\cdots+k_s=n-k}\frac{(n-k)!}{k_1!\cdots k_s!}\prod_{j=1}^{s}||(\partial/\partial t)^{k_j}(A^{\eta}_{q_j}(t)-\lambda_j)^{-1}||_{B(L^{q_j},L^{q_{j+1}})}\\ &\leqq\sum_{k_1+\cdots+k_s=n-k}\frac{(n-k)!}{k_1!\cdots k_s!}\prod_{j=1}^{s}\gamma C_3C_4{}^{k_j}M_{k_j}|\lambda_j|^{a_j-1}\\ &=(\gamma C_3)^sC_4^{n-k}\sum_{k_1+\cdots+k_s=n-k}\frac{(n-k)!}{k_1!\cdots k_s!}M_{k_1}\cdots M_{k_s}\prod_{j=1}^{s}|\lambda_j|^{a_j-1}\,.\end{aligned}$$

Noting

$$\frac{(k_1+\cdots+k_s)!}{k_1!\cdots k_s!}M_{k_1}\cdots M_{k_s}\leqq d_1^{s-1}M_{k_1+\cdots+k_s}$$

which can be easily shown by induction, we get

$$(4.14)\qquad \|(\partial/\partial t)^{n-k}S(t)\|_{B(L^2,L^\infty)}$$
$$\leqq(\gamma C_3)^s d_1^{s-1}C_4^{n-k}M_{n-k}\prod_{j=1}^{s}|\lambda_j|^{a_j-1}\sum_{k_1+\cdots+k_s=n-k}1$$
$$\leqq(\gamma C_3)^s d_1^{s-1}(C_4 s)^{n-k}M_{n-k}\prod_{j=1}^{s}|\lambda_j|^{a_j-1}.$$

Similarly

$$(4.15)\qquad \|(\partial/\partial t)^k T^*(t)\|_{B(L^2,L^\infty)}$$
$$\leqq(\gamma C_3)^{l-s}d_1^{l-s-1}(C_4(l-s))^k M_k\sum_{j=s+1}^{l}|\lambda_j|^{a_j-1}.$$

With the aid of (4.10), (4.13), (4.14), (4.15) we get

$$|(\partial/\partial t)^n K_{\lambda_1,\cdots,\lambda_l}{}^{\eta}(x,y;t)|$$
$$\leqq\sum_{k=0}^{n}\binom{n}{k}(\gamma C_3)^l d_1^{l-2}C_4^n s^{n-k}(l-s)^k M_{n-k}M_k\prod_{j=1}^{l}|\lambda_j|^{a_j-1}$$
$$\leqq(\gamma C_3)^l d_1^{l-1}C_4^n\sum_{k=0}^{n}s^{n-k}(l-s)^k M_n\prod_{j=1}^{l}|\lambda_j|^{a_j-1}$$
$$\leqq(\gamma C_3)^l d_1^{l-1}C_4^n(n+1)(\max\{s,\ l-s\})^n M_n\prod_{i=1}^{l}|\lambda_j|^{a_j-1}$$
$$\leqq C_5C_6^n M_n\prod_{j=1}^{l}|\lambda_j|^{a_j-1},$$

where $C_5=(\gamma C_3)^l d_1^{l-1}$, $C_6=eC_4\max\{s,\ l-s\}$, where we used $n+1<e^n$. By the argument through which we derived (4.12) we obtain

$$(4.16)\qquad |(\partial/\partial t)^n K_{\lambda_1,\cdots,\lambda_l}(x,y;t)|$$
$$\leqq C_5C_6^n M_n\exp\left[-\delta\min\{|\lambda_1|^{1/m},\ \cdots,\ |\lambda_l|^{1/m}\}|x-y|\right]\prod_{j=1}^{l}|\lambda_j|^{a_j-1}$$
$$\leqq C_5C_6^n M_n\sum_{k=1}^{l}\exp(-\delta|\lambda_k|^{1/m}|x-y|)\prod_{j=1}^{l}|\lambda_j|^{a_j-1}.$$

5. Estimates of the kernel of the derivatives of $\exp(-\tau A(t))$ (2)

We denote the kernel of $\exp(-\tau A(t))$ by $G(x, y, \tau; t)$ which is also the kernel of $\exp(-\tau A_p(t))$, $1<p<\infty$.

Let Γ be a smooth contour running in $\Sigma-\{0\}$ from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$. Then for $|\arg\tau|<\pi/2-\theta_0$, $t\in[0, T]$

$$\exp(-l\tau A_2(t))=\left\{\frac{1}{2\pi i}\int_\Gamma e^{-\lambda\tau}(A_2(t)-\lambda)^{-1}d\lambda\right\}^l$$
$$=\left(\frac{1}{2\pi i}\right)^l\int_\Gamma\cdots\int_\Gamma e^{-\lambda_1\tau-\cdots-\lambda_l\tau}(A_2(t)-\lambda_1)^{-1}\cdots(A_2(t)-\lambda_l)^{-1}d\lambda_1\cdots d\lambda_l.$$

Hence

$$G(x, y, l\tau; t) = \left(\frac{1}{2\pi i}\right)^l \int_\Gamma \cdots \int_\Gamma e^{-\lambda_1\tau-\cdots-\lambda_l\tau} K_{\lambda_1,\cdots,\lambda_l}(x, y; t) d\lambda_1 \cdots d\lambda_l . \tag{5.1}$$

For any fixed x, y, τ, let $\Gamma_{x,y,\tau}$ be the contour defined by

$$\Gamma_{x,y,\tau} = \{\lambda : |\arg \lambda| = \theta_0 ,\ |\lambda| \geqq a\} \cup \{\lambda : \lambda = ae^{i\theta} ,\ \theta_0 \leqq \theta \leqq 2\pi - \theta_0\}$$

where

$$a = \varepsilon(|x-y|/|\tau|)^{m/(m-1)} = \varepsilon\rho/|\tau| ,\quad \rho = |x-y|^{m/(m-1)}/|\tau|^{1/(m-1)} \tag{5.2}$$

and ε is a positive constant which will be fixed later. If $|\arg \lambda| = \theta_0$ and hence $\lambda = re^{\pm i\theta_0}$, $r > 0$, then

$$\begin{aligned} \operatorname{Re} \lambda\tau &= \operatorname{Re} \lambda \operatorname{Re} \tau - \operatorname{Im} \lambda \operatorname{Im} \tau \\ &= r \operatorname{Re} \tau(\cos \theta_0 \mp \sin \theta_0(\operatorname{Im} \tau/\operatorname{Re} \tau)) \\ &\geqq r \operatorname{Re} \tau(\cos \theta_0 - \sin \theta_0(|\operatorname{Im} \tau|/\operatorname{Re} \tau)) . \end{aligned}$$

Thus if τ is in the region

$$\frac{|\operatorname{Im} \tau|}{\operatorname{Re} \tau} \leqq (1-\varepsilon_0)\frac{\cos \theta_0}{\sin \theta_0} \tag{5.3}$$

for some ε_0, $0 < \varepsilon_0 < 1$, then

$$\operatorname{Re} \lambda\tau \geqq r \operatorname{Re} \tau \cdot \varepsilon_0 \cos \theta_0 \geqq c_1 r |\tau| \tag{5.4}$$

where c_1 is some positive constant depending only on ε_0, θ_0. Differentiating both sides of (5.1) n times with respect to t, deforming the contour Γ to $\Gamma_{x,y,\tau}$ and using (4.16) we get

$$\begin{aligned} &|(\partial/\partial t)^n G(x, y, l\tau; t)| \\ &\leqq (1/2\pi)^l \int_{\Gamma_{x,y,\tau}} \cdots \int_{\Gamma_{x,y,\tau}} |e^{-\lambda_1\tau-\cdots-\lambda_l\tau} (\partial/\partial t)^n K_{\lambda_1,\cdots,\lambda_l}(x, y; t) d\lambda_1 \cdots d\lambda_l| \\ &\leqq (1/2\pi)^l \int_{\Gamma_{x,y,\tau}} \cdots \int_{\Gamma_{x,y,\tau}} e^{-\operatorname{Re}\lambda_1\tau-\cdots-\operatorname{Re}\lambda_l\tau} \\ &\quad \times C_5 C_6^n M_n \sum_{k=1}^{l} \exp(-\delta|\lambda_k|^{1/m}|x-y|) \prod_{j=1}^{l} |\lambda_j|^{a_j-1} |d\lambda_1| \cdots |d\lambda_l| \\ &= (1/2\pi)^l C_5 C_6^n M_n \sum_{k=1}^{l} \int_{\Gamma_{x,y,\tau}} \cdots \int_{\Gamma_{x,y,\tau}} e^{-\operatorname{Re}\lambda_1\tau-\cdots-\operatorname{Re}\lambda_1\tau} \\ &\quad \times \exp(-\delta|\lambda_k|^{1/m}|x-y|) \prod_{j=1}^{l} |\lambda_j|^{a_j-1} |d\lambda_1| \cdots |d\lambda_l| . \end{aligned} \tag{5.5}$$

The summand with $k=1$ in the last member of (5.5) is

$$\int_{\Gamma_{x,y,\tau}} e^{-\mathrm{Re}\,\lambda_1\tau} \exp(-\delta|\lambda_1|^{1/m}|x-y|)|\lambda_1|^{a_1-1}|d\lambda_1| \times \prod_{j=2}^{l}\int_{\Gamma_{x,y,\tau}} e^{-\mathrm{Re}\,\lambda_j\tau}|\lambda_j|^{a_j-1}|d\lambda_j|. \tag{5.6}$$

If we write

$$\int_{\Gamma_{x,y,\tau}} e^{-\mathrm{Re}\,\lambda_1\tau} \exp(-\delta|\lambda_1|^{1/m}|x-y|)|\lambda_1|^{a_1-1}|d\lambda_1| = \int_{|\lambda_1|=a} + \int_{|\lambda_1|>a} = I+II, \tag{5.7}$$

then

$$\begin{aligned} I &\leqq 2\pi a^{a_1} \exp(a|\tau|-\delta a^{1/m}|x-y|) \\ &= 2\pi(\varepsilon\rho/|\tau|)^{a_1} \exp(\varepsilon\rho-\delta\varepsilon^{1/m}\rho) \\ &\leqq C_7|\tau|^{-a_1} \exp(2\varepsilon\rho-\delta\varepsilon^{1/m}\rho) \end{aligned} \tag{5.8}$$

where C_7 is a positive constant such that

$$2\pi\sigma^{a_j} \leqq C_7 e^{\sigma} \qquad \text{for} \quad \sigma>0, \quad j=1, \cdots, l,$$

and in view of (5.4)

$$II \leqq 2\int_a^\infty \exp(-c_1 r|\tau|-\delta r^{1/m}|x-y|) r^{a_1-1} dr.$$

Suppose that $x \neq y$. By the change of the variable $r=a\sigma$ we get

$$\begin{aligned} II &= 2a^{a_1}\int_1^\infty \sigma^{a_1-1} \exp(-c_1\varepsilon\rho\sigma-\delta\varepsilon^{1/m}\rho\sigma^{1/m})d\sigma \\ &\leqq 2a^{a_1} \exp(-\delta\rho\varepsilon^{1/m}) \int_1^\infty \sigma^{a_1-1} \exp(-c_1\varepsilon\rho\sigma)d\sigma \\ &= 2(\varepsilon\rho/|\tau|)^{a_1} \exp(-\delta\rho\varepsilon^{1/m}) \frac{\Gamma(a_1)}{(c_1\varepsilon\rho)^{a_1}} = \frac{2\Gamma(a_1)}{(c_1|\tau|)^{a_1}} \exp(-\delta\rho\varepsilon^{1/m}). \end{aligned} \tag{5.9}$$

It is easy to show that (5.9) holds also in case $x=y$. Combining (5.7), (5.8), (5.9) we get

$$\int_{\Gamma_{x,y,\tau}} e^{-\mathrm{Re}\,\lambda_1\tau} \exp(-\delta|\lambda_1|^{1/m}|x-y|)|\lambda_1|^{a_1-1}|d\lambda_1| \leqq C_8|\tau|^{-a_1} \exp(2\varepsilon\rho-\delta\varepsilon^{1/m}\rho), \tag{5.10}$$

where $C_8=C_7+2\max\{\Gamma(a_j)c_1^{-a_j}; j=1, \cdots, l\}$.

For $j=2, \cdots, l$

$$\int_{\Gamma_{x,y,\tau}} e^{-\mathrm{Re}\,\lambda_j\tau}|\lambda_j|^{a_j-1}|d\lambda_j| = \int_{|\lambda_j|=a} + \int_{|\lambda_j|\geqq a} \tag{5.11}$$

$$\leqq e^{a|\tau|}a^{a_j-1}2\pi a+2\int_a^\infty e^{-c_1 r|\tau|}r^{a_j-1}dr$$
$$=2\pi(\varepsilon\rho/|\tau|)^{a_j}e^{\varepsilon\rho}+2\Gamma(a_j)(c_1|\tau|)^{-a_j}$$
$$\leqq C_7|\tau|^{-a_j}e^{2\varepsilon\rho}+2\Gamma(a_j)c_1^{-a_j}|\tau|^{-a_j}\leqq C_9|\tau|^{-a_j}e^{2\varepsilon\rho},$$

where $C_9=C_7+\max\{2\Gamma(a_j)c_1^{-a_j};j=1,\cdots,l\}$.

Combining (5.10) and (5.11), and noting $\sum_{j=1}^{l}a_j=N/m$, we see that (5.6) is dominated by

$$C_{10}|\tau|^{-N/m}\exp\{(2l\varepsilon-\delta\varepsilon^{1/m})\rho\}.$$

Other summands in the last member of (5.5) is analogously estimated, and so we get

$$|(\partial/\partial t)^n G(x,y,l\tau;t)|\leqq(2\pi)^{-l}lC_5C_{10}C_6^nM_n|\tau|^{-N/m}\exp\{(2l\varepsilon-\delta\varepsilon^{1/m})\rho\}.$$

Choosing ε so small that $c_2=\delta\varepsilon^{1/m}-2l\varepsilon>0$, and replacing τ by τ/l we obtain

$$(5.12)\qquad |(\partial/\partial t)^n G(x,y,\tau;t)|\leqq C_{11}C_6^nM_n|\tau|^{-N/m}\exp\left(-c_2\frac{|x-y|^{m/(m-1)}}{|\tau|^{1/(m-1)}}\right)$$

for τ in the region (5.3), where $C_{11}=(2\pi)^{-l}l^{1+N/m}C_5C_{10}$.

6. Estimates of the derivatives of the kernel of $(A(t)-\lambda)^{-1}$

If we denote the kernel of $(A(t)-\lambda)^{-1}$ by $K_\lambda(x,y;t)$, then

$$(6.1)\qquad K_\lambda(x,y;t)=\int_0^\infty e^{\lambda\tau}G(x,y,\tau;t)d\tau.$$

First let λ be in the region

$$(6.2)\quad \{\lambda:\mathrm{Re}\lambda>0,\ \mathrm{Im}\lambda>0,\ \mathrm{Re}\lambda/\mathrm{Im}\lambda\leqq(1-\varepsilon_1)\tan\theta_1\}\cup\{\lambda:\mathrm{Re}\lambda\leqq0,\ \mathrm{Im}\lambda>0\}$$

where ε_1 and θ_1 are arbitrary fixed constants such that $0<\theta_1<\pi/2-\theta_0$, $0<\varepsilon_1<1$. Then the integral path in the right side of (6.1) may be altered to the ray $\tau=re^{i\theta_1}$, $0<r<\infty$, since $\mathrm{Re}\lambda\tau\leqq-c_3r|\lambda|$ for some positive constant c_3 on the ray. In view of (5.12)

$$(6.3)\qquad |(\partial/\partial t)^n K_\lambda(x,y;t)|=\left|\int e^{\lambda\tau}(\partial/\partial t)^n G(x,y,\tau;t)d\tau\right|$$
$$\leqq C_{11}C_6^nM_n\int_0^\infty\exp(-c_3r|\lambda|)r^{-N/m}\exp(-c_2|x-y|^{m/(m-1)}r^{-1/(m-1)})dr.$$

First we consider the case $N>m$. If $x\neq y$, by the change of variable $r=|x-y|^m s$

$$\int_0^\infty\exp(-c_3r|\lambda|)r^{-N/m}\exp(-c_2|x-y|^{m/(m-1)}r^{-1/(m-1)})dr$$
$$=|x-y|^{m-N}\int_0^\infty s^{-N/m}\exp(-c_2s^{-1/(m-1)}-c_3|\lambda||x-y|^m s)ds.$$

Putting $h=(|\lambda|^{1/m}|x-y|)^{1-m}$ we see that the right side of the above equality does not exceed

$$|x-y|^{m-N}\int_0^h s^{-N/m}\exp(-c_2h^{-1/(m-1)})ds$$
$$+|x-y|^{m-N}\int_h^\infty s^{-N/m}\exp(-c_3h^{-m/(m-1)}s)ds = I+II.$$

If $|\lambda|^{1/m}|x-y|<1$, then

$$\int_0^h s^{-N/m}\exp(-c_2s^{-1/(m-1)})ds\leqq\int_0^\infty s^{-N/m}\exp(-c_2s^{-1/(m-1)})ds$$
$$=C_{12}\leqq C_{12}e\cdot\exp(-|\lambda|^{1/m}|x-y|).$$

Note that $C_{12}<\infty$ since $N/m>1$. If $|\lambda|^{1/m}|x-y|\geqq 1$, then letting C_{13} be a constant such that $\sigma^{(m-1)N/m}\leqq C_{13}e^{c_3\sigma/2}$ for any $\sigma>0$ and noting that $h\leqq 1$

$$\int_0^h s^{-N/m}\exp(-c_2s^{-1/(m-1)})ds\leqq C_{13}\int_0^h\exp(-2^{-1}c_2s^{-1/(m-1)})ds$$
$$\leqq C_{13}h\exp(-2^{-1}c_2h^{-1/(m-1)})\leqq C_{13}\exp(-2^{-1}c_2h^{-1/(m-1)})$$
$$=C_{13}\exp(-2^{-1}c_2|\lambda|^{1/m}|x-y|).$$

Thus

$$I\leqq C_{14}|x-y|^{m-N}\exp(-2^{-1}c_2|\lambda|^{1/m}|x-y|) \tag{6.4}$$

where $C_{14}=\max(C_{12}e, C_{13})$. Next,

$$\int_h^\infty s^{-N/m}\exp(-c_3h^{-m/(m-1)}s)ds\leqq\exp(-c_3h^{-1/(m-1)})\int_h^\infty s^{-N/m}ds$$
$$=(m/(N-m))h^{-(N-m)/m}\exp(-c_3h^{-1/(m-1)})$$
$$\leqq C_{15}\exp(-2^{-1}c_3h^{-1/(m-1)})$$

where C_{15} is a constant such that

$$(m/(N-m))\sigma^{(m-1)(N-m)/m}\leqq C_{15}\exp(2^{-1}c_3\sigma)\qquad\text{for any}\quad\sigma>0.$$

Hence

$$II\leqq C_{15}|x-y|^{m-N}\exp(-2^{-1}c_3|\lambda|^{1/m}|x-y|). \tag{6.5}$$

Combining (6.3), (6.4), (6.5) we obtain

$$|(\partial/\partial t)^n K_\lambda(x, y; t)|\leqq C_{16}C_6^nM_n|x-y|^{m-N}\exp(-c_4|\lambda|^{1/m}|x-y|), \tag{6.6}$$

in case $N>m$, where $C_{16}=C_{14}+C_{15}$, $c_4=\min(c_2, c_3)/2$.

Next, we consider the case $N=m$. If $x\neq y$, by the change of variable $r=|x-y|^m s$

$$\int_0^\infty r^{-1}\exp(-c_3r|\lambda|)\exp(-c_2|x-y|^{m/(m-1)}r^{-1/(m-1)})dr \tag{6.7}$$

$$= \int_0^\infty s^{-1} \exp(-c_3 h^{-m/(m-1)} s - c_2 s^{-1/(m-1)}) ds$$

where $h = (|\lambda|^{1/m} |x-y|)^{1-m}$ as before.

If $|\lambda|^{1/m}|x-y|<1$, then putting $b=(|\lambda||x-y|^m)^{-1}=h^{m/(m-1)}>1$ we see that the right side of (6.7) does not exceed

$$\begin{aligned}(6.8)\quad & \int_0^b s^{-1}\exp(-c_2 s^{-1/(m-1)})ds + \int_b^\infty s^{-1}\exp(-c_3 b^{-1}s)ds \\ & \leqq \int_0^1 s^{-1}\exp(-c_2 s^{-1/(m-1)})ds + \int_1^b s^{-1}ds + \int_1^\infty s^{-1}\exp(-c_3 s)ds \\ & = C_{17} + \log b = C_{17} + m\log(|\lambda|^{1/m}|x-y|)^{-1}\end{aligned}$$

where C_{17} is a constant defined by the above relation. If $h^{1/(1-m)}=|\lambda|^{1/m}|x-y|\geqq 1$, then the right side of (6.7) does not exceed

$$\begin{aligned}(6.9)\quad & \int_0^h s^{-1}\exp(-c_2 s^{-1/(m-1)})ds + \int_h^\infty s^{-1}\exp(-c_3 h^{-m/(m-1)}s - c_2 s^{-1/(m-1)})ds \\ & \leqq \exp(-2^{-1}c_2 h^{-1/(m-1)}) \int_0^h s^{-1}\exp(-2^{-1}c_2 s^{-1/(m-1)})ds \\ & \quad + \exp(-2^{-1}c_3 h^{-1/(m-1)}) \{\int_h^1 s^{-1}\exp(-c_2 s^{-1/(m-1)})ds \\ & \quad + \int_1^\infty s^{-1}\exp(-2^{-1}c_3 s)ds\} \leqq C_{18}\exp(-c_4|\lambda|^{1/m}|x-y|),\end{aligned}$$

where $C_{18} = \int_0^1 s^{-1}\exp(-2^{-1}c_2 s^{-1/(m-1)})ds + \int_0^1 s^{-1}\exp(-c_2 s^{-1/(m-1)})ds + \int_1^\infty s^{-1}\exp(-2^{-1}c_3 s)ds$. In view of (6.8), (6.9) the right side of (6.7) is not greater than

$$C_{19}\exp(-c_4|\lambda|^{1/m}|x-y|)\{1+\log^+(|\lambda|^{1/m}|x-y|)^{-1}\},$$

where $C_{19}=\max(C_{17}e^{c_4}, me^{c_4}, C_{18})$ and $\log^+\sigma=\log\sigma$ if $\sigma\geqq 1$, $\log^+\sigma=0$ if $\sigma<1$. Thus in case $N=m$

$$\begin{aligned}(6.10)\quad & |(\partial/\partial t)^n K_\lambda(x, y;\ t)| \\ & \leqq C_{11}C_{19}C_6^n M_n \exp(-c_4|\lambda|^{1/m}|x-y|)\{1+\log^+(|\lambda|^{1/m}|x-y|)^{-1}\}.\end{aligned}$$

Finally we consider the case $N<m$. Changing the variable by $r=s/|\lambda|$ and putting $\tilde{h}=|\lambda|^{1/m}|x-y|$

$$\begin{aligned}& \int_0^\infty \exp(-c_3 r|\lambda|)r^{-N/m}\exp(-c_2|x-y|^{m/(m-1)}r^{-1/(m-1)})dr \\ & = |\lambda|^{N/m-1}\int_0^\infty s^{-N/m}\exp(-c_3 s)\exp(-c_2\tilde{h}^{m/(m-1)}s^{-1/(m-1)})ds \\ & \leqq |\lambda|^{N/m-1}\exp(-c_2\tilde{h})\int_0^{\tilde{h}} s^{-N/m}\exp(-c_3 s)ds\end{aligned}$$

$$+|\lambda|^{N/m-1}\exp(-2^{-1}c_3\tilde{h})\int_{\tilde{h}}^{\infty}s^{-N/m}\exp(-c_3s/2)ds$$
$$\leqq c_3^{N/m-1}\Gamma(1-N/m)|\lambda|^{N/m-1}\exp(-c_2\tilde{h})$$
$$+(c_3/2)^{N/m-1}\Gamma(1-N/m)|\lambda|^{N/m-1}\exp(-2^{-1}c_3\tilde{h})$$
$$\leqq C_{20}|\lambda|^{N/m-1}\exp(-c_4|\lambda|^{1/m}|x-y|),$$

where $C_{20}=\{c_3^{N/m-1}+(c_3/2)^{N/m-1}\}\Gamma(1-N/m)$. Thus in case $N<m$

(6.11) $$|(\partial/\partial t)^n K_\lambda(x, y; t)|\leqq C_{11}C_{20}C_6^n M_n|\lambda|^{N/m-1}\exp(-c_4|\lambda|^{1/m}|x-y|).$$

Summing up we see that the following estimate holds

(6.12) $$|(\partial/\partial t)^n K_\lambda(x, y; t)|$$
$$\leqq C_{21}C_6^n M_n \exp(-c_4|\lambda|^{1/m}|x-y|)\times\begin{cases}|x-y|^{m-N} & \text{if } N>m\\ 1+\log^+(|\lambda|^{1/m}|x-y|)^{-1} & \text{if } N=m\\ |\lambda|^{N/m-1} & \text{if } N<m\end{cases}$$

for any $n=0, 1, 2, \cdots, (x, y)\in\bar{\Omega}\times\bar{\Omega}$, $t\in[0, T]$, λ in the region (6.2) where $C_{21}=\max(C_{16}, C_{11}C_{19}, C_{11}C_{20})$. It is clear that the same estimate holds for λ in the region

$$\{\lambda: \operatorname{Re}\lambda>0, \operatorname{Im}\lambda<0, \operatorname{Re}\lambda/|\operatorname{Im}\lambda|\leqq(1-\varepsilon_1)\tan\theta_1\}\cup\{\lambda: \operatorname{Re}\lambda\leqq 0, \operatorname{Im}\lambda<0\}.$$

It follows readily from (6.12) that

(6.13) $$\|(\partial/\partial t)^n(A(t)-\lambda)^{-1}\|_{B(L^1,L^1)}\leqq C_{22}C_6^n M_n/|\lambda|$$

for any $n=0, 1, 2, \cdots$, $t\in[0, T]$, and λ in the region

(6.14) $$\{\lambda: \operatorname{Re}\lambda>0, \operatorname{Re}\lambda/|\operatorname{Im}\lambda|\leqq(1-\varepsilon_1)\tan\theta_1\}\cup\{\lambda: \operatorname{Re}\lambda\leqq 0\}.$$

Due to the closedness of Σ and the arbitrariness of $\varepsilon_1\in(0, 1)$, $\theta_1\in(0, \pi/2-\theta_0)$ we see that there exist constants K_0, K such that (2.13) holds for any $n=0, 1, 2, \cdots$, $\lambda\in\Sigma$, $t\in[0, T]$, and the proof of the proposition and hence that of the main theorem is complete.

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