Mabuchi T.
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EINSTEIN-KÄHLER FORMS, FUTAKI INVARIANTS AND CONVEX GEOMETRY ON TORIC FANO VARIETIES

TOSHIKI MABUCHI

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0. Introduction

Throughout this paper, we assume that X is a nonsingular n-dimensional toric Fano variety (defined over $\boldsymbol{C}$), i.e., X is an n-dimensional connected projective algebraic manifold satisfying the following conditions:

(a) X admits an effective almost homogeneous algebraic group action of $(\boldsymbol{G}_m)^n$ $(\cong(\boldsymbol{C}^*)^n$ as a complex Lie group).
(b) The set $\mathcal{K}$ of all Kähler forms on X in the de Rham cohomology class $2\pi\, c_1(X)_{\boldsymbol{R}}$ is non-empty.

For each $\omega\in\mathcal{K}$, by writing it as $\omega=\sqrt{-1}\sum g(\omega)_{\alpha\bar{\beta}}\, dz^{\alpha}\wedge d\bar{z}^{\beta}$ in terms of holomorphic local coordinates $(z^1, z^2, \cdots, z^n)$ of X, we have the corresponding Ricci form $\mathrm{Ric}(\omega)$ cohomologous to ω:

$$\mathrm{Ric}(\omega):=\sqrt{-1}\,\bar{\partial}\partial \log \det(g(\omega)_{\alpha\bar{\beta}})\,.$$

Then an element ω of $\mathcal{K}$ is called an Einstein-Kähler form if $\mathrm{Ric}(\omega)=\omega$. We now pose the following:

Problem 0.1*). *Classify all X which admit, at least, one Einstein-Kähler form.*

Obviously, the Fubini-Study form on $\boldsymbol{P}^n(\boldsymbol{C})$ is a typical Einstein-Kähler form. This settles Problem 0.1 for $n=1$, because the only possible X with $n=1$ is $\boldsymbol{P}^1(\boldsymbol{C})$. However, the real difficulty comes up even at $n=2$: Let S_i be the projective algebraic surface obtained from $\boldsymbol{P}^2(\boldsymbol{C})$ by blowing up i points in general position (where $1\leqq i\leqq 3$). Then, in spite of lots of efforts by differential geometers, it is still unknown whether or not the nonsingular toric Fano variety S_3 admits an Einstein-Kähler form.

The purpose of this paper is to give a brief survey of recent progress on Problem 0.1 together with our related new results. Especially, in Sections 1∼6

*) This is also posed by T. Oda and Y.T. Siu.

(though they are somewhat of expository nature), several key ideas are introduced often without proofs, while technical details are given in the subsequent four appendices. In particular, in Appendix C (see (9.2.3) for the most general statement), we shall show that the Futaki invariants of an anti-canonically (relatively) polarized toric bundle Y over W can be regarded as the barycentre of $\boldsymbol{m}(Y)$ in terms of "Duistermaat-Heckman's measure", where $\boldsymbol{m}\colon Y \to \boldsymbol{R}^n$ ($n = \dim_{\boldsymbol{C}} Y - \dim_{\boldsymbol{C}} W$) denotes the associated "relative" moment map defined, in Appendix B, without any ambiguity of translations (cf. (8.2)). Finally, in Appendix D, a very explicit description of Einstein-Kähler metrics for Sakane-Koiso's examples will be given (cf. (10.3.2), Step 4 of (10.3)).

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1. Notation, conventions and preliminaries

Let $\boldsymbol{Z}_+$ (resp. $\boldsymbol{Z}_0$) be the set of positive (resp. non-negative) integers and $\boldsymbol{R}_+$ (resp. $\boldsymbol{R}_0$) be the set of positive (resp. non-negative) real numbers. We now put:

$$G := (\boldsymbol{G}_m)^n = \{(t_1, t_2, \cdots, t_n) \mid t_i \in \boldsymbol{C}^*\}\,,$$
$$M := \{\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \mid a_i \in \boldsymbol{Z}\}\ (\cong \boldsymbol{Z}^n)\,,$$
$$N := \left\{\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \middle|\ b_j \in \boldsymbol{Z}\right\}\ (\cong \boldsymbol{Z}^n)\,.$$

For $\boldsymbol{a} \in M$ and $\boldsymbol{b} \in N$ as above, we define $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{Z}$, $\chi^{\boldsymbol{a}} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(G, \boldsymbol{G}_m)$ and $\lambda_{\boldsymbol{b}} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(\boldsymbol{G}_m, G)$ by

$$(\boldsymbol{a}, \boldsymbol{b}) := \textstyle\sum_{i=1}^n a_i b_i\,,$$
$$\chi^{\boldsymbol{a}}((t_1, t_2, \cdots, t_n)) := t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}\,,$$
$$\lambda_{\boldsymbol{b}}(t) := (t^{b_1}, t^{b_2}, \cdots, t^{b_n})\,,$$

where $t, t_1, \cdots, t_n \in \boldsymbol{G}_m (= \boldsymbol{C}^*)$. Then the correspondence $\boldsymbol{a} \mapsto \chi^{\boldsymbol{a}}$ (resp. $\boldsymbol{b} \mapsto \lambda_{\boldsymbol{b}}$) canonically induces an isomorphism between the additive group M (resp. N) and the multiplicative group $\mathrm{Hom}_{\mathrm{alg\ gp}}(G, \boldsymbol{G}_m)$ (resp. $\mathrm{Hom}_{\mathrm{alg\ gp}}(\boldsymbol{G}_m, G)$). Note that

$$\chi^{\boldsymbol{a}}(\lambda_{\boldsymbol{b}}(t)) = t^{(\boldsymbol{a}, \boldsymbol{b})} \qquad \text{for all} \quad t \in \boldsymbol{G}_m (= \boldsymbol{C}^*)\,.$$

DEFINITION 1.1. A non-empty subset σ of N is called a *cone**) if the following conditions are satisfied:
(a) If $\boldsymbol{b}\in N$ satisfies $\beta\,\boldsymbol{b}\in\sigma$ for some $\beta\in\boldsymbol{Z}_+$, then $\boldsymbol{b}\in\sigma$.
(b) If $0\neq\boldsymbol{b}\in\sigma$, then $-\boldsymbol{b}\notin\sigma$.
(c) $0\in\sigma$.
(d) In terms of the natural additive structure of N, σ is a semigroup generated by a finite subset.

For a cone σ, there exists a unique irredundant finite subset $\{\boldsymbol{b}^1, \boldsymbol{b}^2, \cdots, \boldsymbol{b}^m\}$ of σ such that $\sigma=\sum_{k=1}^m \boldsymbol{Z}_0\,\boldsymbol{b}^k$. These $\boldsymbol{b}^1, \boldsymbol{b}^2, \cdots, \boldsymbol{b}^m$ are called the *fundamental generators* of the cone σ.

DEFINITION 1.2. A non-empty subset τ of a cone σ is called a *face* of σ, denoted by $\tau\leqq\sigma$, if there exists an element $\boldsymbol{a}$ of M such that $(\boldsymbol{a}, \boldsymbol{b})\geqq 0$ for all $\boldsymbol{b}$ in σ and that $\tau=\{\boldsymbol{b}\in\sigma\,|\,(\boldsymbol{a}, \boldsymbol{b})=0\}$. A *finite polyhedral decomposition* of N is a finite set Δ of cones in N such that
(a) if $\tau\leqq\sigma\in\Delta$, then $\tau\in\Delta$;
(b) if σ, $\tau\in\Delta$, then $\sigma\cap\tau\leqq\sigma$ and $\sigma\cap\tau\leqq\tau$;
(c) $N=\bigcup_{\sigma\in\Delta}\sigma$.

For every finite polyhedral decomposition Δ of N, we put

$$\Delta(i):=\{\sigma\in\Delta\,|\,\dim\sigma=i\}\ ,\quad 0\leqq i\leqq n\ ,$$

where $\dim\sigma$ denotes the dimension of the real vector space spanned by σ in $N_{\boldsymbol{R}}:=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$.

DEFINITION 1.3. A finite polyhedral decomposition Δ of N is said to be *nonsingular* if for each $\sigma\in\Delta(n)$, the set of fundamental generators of σ consists of n elements and forms a $\boldsymbol{Z}$-basis for N. For every nonsingular Δ, the set of fundamental generators of each element of $\Delta(i)$ consists of exactly i elements and can be completed to a $\boldsymbol{Z}$-basis for N.

We shall now quote the following fundamental results due to Demazure [6], Miyake and Oda [18], and Mumford et al. [19]:

Theorem 1.4. *To every nonsingular finite polyhedral decomposition Δ of N, one can uniquely associate an n-dimensional irreducible nonsingular G-equivariant compactification G_Δ of G possessing the following two properties:*

(a) *To each $\sigma\in\Delta(i)$, $0\leqq i\leqq n$, there corresponds a unique $(n-i)$-dimensional G-orbit, denoted by $\boldsymbol{O}^\sigma$, such that G_Δ is expressible as*

$$G_\Delta=\bigcup_{\sigma\in\Delta}\boldsymbol{O}^\sigma\quad (\textit{disjoint union}).$$

*) This notion of cones is slightly different from the ordinary one.

Furthermore, the closure $D(\sigma)$ of $\boldsymbol{O}^\sigma$ in G_Δ is an irreducible nonsingular $(n-i)$-dimensional G-stable subvariety of G_Δ written in the form

$$D(\sigma)=\bigcup_{\tau\geqq\sigma}\boldsymbol{O}^\tau \quad (\textit{disjoint union}).$$

(b) *For each $\sigma\in\Delta(n)$, $U_\sigma:=\bigcup_{\tau\leqq\sigma}\boldsymbol{O}^\tau$ forms an affine open G-stable neighbourhood of $\boldsymbol{O}^\sigma$ in G_Δ satisfying the conditions*

$$G\subseteqq U_\sigma\cong\boldsymbol{A}^n(\boldsymbol{C})$$

and

$$G_\Delta=\bigcup_{\sigma\in\Delta(n)}U_\sigma\,.$$

Let $\{\boldsymbol{b}(\sigma)^1, \boldsymbol{b}(\sigma)^2, \cdots, \boldsymbol{b}(\sigma)^n\}$ be the set of fundamental generators of σ (which forms a $\boldsymbol{Z}$-basis for N), and let $\{\boldsymbol{a}(\sigma)^1, \boldsymbol{a}(\sigma)^2, \cdots, \boldsymbol{a}(\sigma)^n\}$ be the dual basis for M defined by the relation $(\boldsymbol{a}(\sigma)^i, \boldsymbol{b}(\sigma)^j)=\delta_{ij}$. Then the corresponding characters

$$\chi_{\sigma\,;\,i}:=\chi^{\boldsymbol{a}(\sigma)^i}\in\mathrm{Hom}_{\mathrm{alg\,gp}}(G,\boldsymbol{G}_m)\,,\quad 1\leqq i\leqq n\,,$$

extend to rational functions on G_Δ, which are all regular on U_σ, forming a system of coordinate functions on U_σ by the isomorphism

$$\begin{aligned}&U_\sigma\cong\boldsymbol{A}^n(\boldsymbol{C})\\&u\mapsto(\chi_{\sigma\,;\,1}(u),\chi_{\sigma\,;\,2}(u),\cdots,\chi_{\sigma\,;\,n}(u))\,.\end{aligned}$$

In terms of these coordinates, the G-action on U_σ is described by

$$\begin{aligned}&(\chi_{\sigma\,;\,1}(g\cdot u),\chi_{\sigma\,;\,2}(g\cdot u),\cdots,\chi_{\sigma\,;\,n}(g\cdot u))\\&\quad=(\chi_{\sigma\,;\,1}(g)\cdot\chi_{\sigma\,;\,1}(u),\chi_{\sigma\,;\,2}(g)\cdot\chi_{\sigma\,;\,2}(u),\cdots,\chi_{\sigma\,;\,n}(g)\cdot\chi_{\sigma\,;\,n}(u))\,,\end{aligned}$$

where both $g\in G$ and $u\in U_\sigma$ are arbitrary.

Theorem 1.5. *Every n-dimensional irreducible nonsingular complete variety endowed with an effective regular G-action is G-equivariantly isomorphic to G_Δ for some nonsingular finite polyhedral decomposition Δ of N.*

Finally, we remark the following:

(1.6) In terms of the holomorphic coordinates $(t_1, t_2, \cdots, t_n)$ for $G=\{(t_1,\cdots,t_n)\,|\,t_i\in\boldsymbol{C}^*\}$, the G-invariant vector fields

$$t_i\,\partial/\partial t_i\,,\quad i=1,2,\cdots,n\,.$$

on G form a $\boldsymbol{C}$-basis for Lie(G). Furthermore, these naturally extend to holomorphic vector fields on G_Δ.

2. Demazure's results on toric varieties

Throughout this section, we fix a nonsingular finite polyhedral decomposition Δ of N. Put $M_{\boldsymbol{R}}:=M\otimes_{\boldsymbol{Z}}\boldsymbol{R}$. Furthermore, for each $\rho\in\Delta(1)$, let $\boldsymbol{b}_\rho$ denote the unique fundamental generator of ρ. We now consider the divisor

$$K:=-\sum_{\rho\in\Delta(1)} D(\rho)$$

on G_Δ. Recall the following fact due to Demazure [6]:

Theorem 2.1. *K is a canonical divisor of G_Δ. Moreover, the following are equivalent*:
(a) *G_Δ is a toric Fano variety.*
(b) *$-K$ is ample.*
(c) *$-K$ is very ample.*
(d) *$\sum_{-K}:=\{\boldsymbol{a}\in M_{\boldsymbol{R}}\,|\,(\boldsymbol{a},\boldsymbol{b}_\rho)\leqq 1$ for all $\rho\in\Delta(1)\}$ is an n-dimensional compact convex polyhedron whose vertices are exactly $\{\boldsymbol{a}_\tau\,|\,\tau\in\Delta(n)\}$, where each $\boldsymbol{a}_\tau$ denotes the unique element of M such that $(\boldsymbol{a}_\tau,\boldsymbol{b})=1$ for all fundamental generators $\boldsymbol{b}$ of τ.*

REMARK 2.2. It is easily seen that $\boldsymbol{P}^2(\boldsymbol{C})$, $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$, $S_i(1\leqq i\leqq 3)$ are the only possible 2-dimensional nonsingular toric Fano varieties. Recently, for dimension three also, all nonsingular toric Fano varieties were completely classified (cf. Batyrev [4], K. Watanabe and M. Watanabe [24]).

DEFINITION 2.3 (Demazure [6; p. 571]). An element $\boldsymbol{a}$ of M is called a *root* if there exists $\rho\in\Delta(1)$ such that $(\boldsymbol{a},\boldsymbol{b}_\rho)=1$ and that $(\boldsymbol{a},\boldsymbol{b}_\sigma)\leqq 0$ for all $\sigma\in\Delta(1)$ with $\sigma\neq\rho$. Let $R(\Delta)$ be the set of all roots in M.

Now, as an immediate consequence of a result of Demazure [6; p. 581], one obtains:

Theorem 2.4. *Let* Aut (G_Δ) *be the group of all holomorphic automorphisms of G_Δ. Then* Aut(G_Δ) *is a reductive algebraic group if and only if $-R(\Delta):=\{-\boldsymbol{a}\,|\,\boldsymbol{a}\in R(\Delta)\}$ coincides with $R(\Delta)$.*

REMARK 2.5. In view of this theorem and (2.2), it is now possible to determine all 3-dimensional nonsingular toric Fano varieties G_Δ with reductive Aut(G_Δ). Such a G_Δ is, actually, isomorphic to one of the following (we owe the computation to T. Ashikaga):

$$\boldsymbol{P}^3(\boldsymbol{C}),\ \boldsymbol{P}^2(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}),\ \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}),$$
$$\boldsymbol{P}^1(\boldsymbol{C})\times S_3,\ \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1)),\ F_1^5,$$

where we used the notation of K. Watanabe and M. Watanabe [24]. Obviously, the first three varieties admit an Einstein-Kähler form. Note that, for the last three varieties, $\mathrm{Aut}(G_\Delta)$ cannot act transitively on G_Δ. However, $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1))$ still admits an Einstein-Kähler form by virtue of a result of Sakane [22], partly because in this case, every maximal compact subgroup of $\mathrm{Aut}(G_\Delta)$ acts on G_Δ with principal orbits of real codimension one (cf. Appendix D).

The importance of (2.4) comes from the following theorem in differential geometry due to Matsushima [17]:

Theorem 2.6. *Let Y be a compact complex connected manifold with* $\dim_{\boldsymbol{C}} \mathrm{Aut}^\circ(Y)>0$ (*where* $\mathrm{Aut}^\circ(Y)$ *denotes the identity component of the group* $\mathrm{Aut}(Y)$ *of holomorphic automorphisms of Y*). *If Y admits an Einstein-Kähler form, then* $\mathrm{Aut}(Y)$ *is a reductive algebraic group and furthermore, the group of holomorphic isometries with respect to the corresponding Einstein-Kähler metric in* $\mathrm{Aut}^\circ(Y)$ *is a maximal compact subgroup of* $\mathrm{Aut}^\circ(Y)$.

3. The Einstein equation

For X as in Introduction, there exists a nonsingular finite polyhedral decomposition Δ of N such that $X=G_\Delta$ and that Δ satisfies the condition (d) of (2.1) (see (1.5) and (2.1)). In view of the inclusion

$$\{(t_1, \cdots, t_n)\,|\, t_i\in\boldsymbol{C}^*\} = G \subset G_\Delta\,,$$

we may regard each t_i as a rational function on G_Δ. Consider the real-valued C^∞ functions $x_1, x_2, \cdots, x_n$ on G defined by

$$(*)\qquad t_i\,\bar{t}_i = |t_i|^2 = \exp(-x_i)\,, \quad 1\leqq i\leqq n\,.$$

Since $\partial t_i=dt_i$, we have $\partial x_i=-dt_i/t_i$ and $\bar\partial x_i=-d\bar{t}_i/\bar{t}_i$. Therefore, for each C^∞ function $u=u(x_1, \cdots, x_n)$ defined on $\boldsymbol{R}^n=\{(x_1, \cdots, x_n)\,|\,x_i\in\boldsymbol{R}\}$, the following identity holds:

$$(3.1)\qquad \partial\bar\partial u = \textstyle\sum_{i,j}(\partial^2 u/\partial x_i\,\partial x_j)\,(dt_i/t_i)\wedge(d\bar{t}_j/\bar{t}_j)\,.$$

Let G_c be the maximal compact subgroup

$$\{(t_1, \cdots, t_n)\in(\boldsymbol{C}^*)^n\,|\,|t_i|=1\}\ (\cong(S^1)^n)$$

of G. Since the anti-canonical bundle K_X^{-1} of X is ample, there exists a G_c-invariant fibre metric Ω for K_X^{-1} such that the corresponding first Chern form is a positive definite (1, 1)-form. Namely, there exists a real-valued C^∞ function $u=u(x_1, \cdots, x_n)$ on $\boldsymbol{R}^n$ such that:

(3.2) $\quad\exp(-u)\prod_{i=1}^n(\sqrt{-1}\;dt_i\wedge d\bar{t}_i/|t_i|^2)$ extends to a volume form on the whole $X = G_\Delta$;

(3.3) $\sqrt{-1}\,\partial\bar{\partial}\,u$ extends to a Kähler form on G_Δ .

Note that the volume form in (3.2) is naturally identified with Ω above (and is denoted by the same Ω). In view of (3.1), the statement (3.3) in particular implies:

(3.4) At each point of $\boldsymbol{R}^n$, the matrix $(\partial^2 u/\partial x_i\,\partial x_j)$ is positive definite.

Suppose now that X admits an Einstein-Kähler form $\omega\in\mathcal{K}$. Then by Theorem (2.6), we may assume that ω is G_c-invariant. Applying the above argument to $\Omega=\omega^n$, we obtain a real-valued C^∞ function $u=u(x_1,\cdots,x_n)$ on $\boldsymbol{R}^n$ satisfying the conditions (3.2), (3.4) and furthermore, by $\mathrm{Ric}(\omega)=\omega$,

(3.5) $$\det(\partial^2 u/\partial x_i\,\partial x_j)=\exp(-u)\quad\text{on } \boldsymbol{R}^n .$$

Conversely, suppose that a real-valued C^∞ function u on $\boldsymbol{R}^n$ satisfies (3.2), (3.4) and (3.5), where we return to our original situation that $X(=G_\Delta)$ is just a nonsingular n-dimensional toric Fano variety without any assumption as to the existence of Einstein-Kähler forms. Then $\omega:=\sqrt{-1}\,\partial\bar{\partial}u$ turns out to be an Einstein-Kähler form on X. We now define:

DEFINITION 3.6. The equation (3.5) above (together with the "boundary" condition (3.2) and the convexity (3.4) for u) is called the *Einstein equation* for the toric Fano variety $X=G_\Delta$.

4. Moment maps on toric varieties

Fix a nonsingular finite polyhedral decomposition Δ of N. In this section, we study the moment map (cf. Atiyah [1], Guillemin and Sternberg [11]) of the toric variety G_Δ in terms of a suitable Kähler metric, if any, on G_Δ.

(4.1) We first assume that G_Δ is a (toric) Fano variety. Then in view of Section 3, there exists a real-valued C^∞ function u on $\boldsymbol{R}^n$ satisfying (3.2) and (3.3). Now, by the relation (*) in that section, we write each x_i as $x_i(\boldsymbol{t})$ with $\boldsymbol{t}=(t_1,\cdots,t_n)\in G$. Hence, every C^∞ function $f=f(x_1,\cdots,x_n)$ on $\boldsymbol{R}^n$ is regarded as a C^∞ function on G by $f(\boldsymbol{t}):=f(x_1(\boldsymbol{t}),\cdots,x_n(\boldsymbol{t}))$ for $\boldsymbol{t}\in G$. Recall that $M_{\boldsymbol{R}}$ is naturally identified with $\boldsymbol{R}^n$ (cf. Section 1). We now define the mapping $\boldsymbol{m}_u\colon G\to M_{\boldsymbol{R}}$ $(=\boldsymbol{R}^n)$ by

$$\boldsymbol{m}_u(\boldsymbol{t}):=((\partial u/\partial x_1)(\boldsymbol{t}),\cdots,(\partial u/\partial x_n)(\boldsymbol{t}))\,,\quad \boldsymbol{t}\in G\,.$$

Then the work of Atiyah [1] is reformulated in the following slightly stronger form:

Theorem 4.2*). *Assume that G_Δ is a nonsingular toric Fano variety.*

*) A more general statement will be proven in (8.2).

Let Q be the closure of the image $\boldsymbol{m}_u(G)$ in $M_{\boldsymbol{R}}$. Then $Q=\sum_{-K}$ (cf. (2.1)). Furthermore, $\boldsymbol{m}_u$: $G\to M_{\boldsymbol{R}}$ extends to a C^∞ map $\overline{\boldsymbol{m}}_u$: $G_\Delta\to M_{\boldsymbol{R}}$. This $\overline{\boldsymbol{m}}_u$ satisfies

(a) *the inverse image $\overline{\boldsymbol{m}}_u^{-1}(\gamma)$ of each open face γ of $\sum_{-K}$ is a single G-orbit;*
(b) *$\overline{\boldsymbol{m}}_u$ induces a diffeomorphism (including the boundaries) between the manifolds G_Δ/G_c and $\sum_{-K}$ with corners.*

REMARK 4.3. (i) It is easily checked that $\overline{\boldsymbol{m}}_u$ above coincides with the moment map: $G_\Delta\to\mathrm{Lie}(G_c)^*\cong M_{\boldsymbol{R}}$ (cf. Atiyah [1], Guillemin and Sternberg [11]) associated with the Kähler form $\sqrt{-1}\,\partial\bar\partial u\in\mathcal{K}$. (See Appendix B for the proof.)
(ii) Consider the subgroup $G_{\boldsymbol{R}}:=\{(t_1,\cdots,t_n)\in G\,|\,t_i\in\boldsymbol{R}_+\}(\cong(\boldsymbol{R}_+)^n)$ of G. Then by the natural inclusions $G_{\boldsymbol{R}}\subset G\subset G_\Delta$, we may regard $G_{\boldsymbol{R}}$ as a subset of G_Δ. Then the closure $\bar G_{\boldsymbol{R}}$ of $G_{\boldsymbol{R}}$ in G_Δ is a *manifold with corners* in the sense of Borel-Serre (cf. Oda [20]) and has a natural differentiable structure as described in Step 3 of (8.2). Note that G_Δ/G_c above is endowed with such a structure via the natural identification of G_Δ/G_c with $\bar G_{\boldsymbol{R}}$.
(iii) The difference of (4.2) from Atiyah's result [1; Theorem 2] is that the mapping between G_Δ/G_c and Q is, in our case, a diffeomorphism (instead of a homeomorphism) even along their boundaries. This diffeomorphism is essentially obtained from the ampleness of $K_{G_\Delta}^{-1}$ by the fact that a combination of (3.2) and (3.3) keeps the Jacobian of $\overline{\boldsymbol{m}}_u|_{\bar G_{\boldsymbol{R}}}$: $\bar G_{\boldsymbol{R}}\to M_{\boldsymbol{R}}$ nonvanishing also along the boundary $\bar G_{\boldsymbol{R}}-G_{\boldsymbol{R}}$.

(4.4) We now assume that G_Δ is a projective variety (where G_Δ is not necessarily a Fano variety). Note that the corresponding hyperplane bundle $L:=\mathcal{O}_{G_\Delta}(1)$ is written as $\mathcal{O}_{G_\Delta}(\sum_{\sigma\in\Delta(1)}\nu_\sigma D(\sigma))$ for some $\nu_\sigma\in\boldsymbol{Z}_0$. Then

$$\textstyle\sum_L:=\{\boldsymbol{a}\in M_{\boldsymbol{R}}\,|\,(\boldsymbol{a},\boldsymbol{b}_\sigma)\leqq\nu_\sigma\quad\text{for all }\sigma\in\Delta(1)\}$$

is an n-dimensional compact convex polyhedron (cf. Oda [21]). Since L is ample, there exists a G_c-invariant fibre metric h for L such that the corresponding first Chern form is positive definite. Therefore, we obtain a real-valued C^∞ function u on $\boldsymbol{R}^n$ satisfying the condition (3.3) and also

$$h|_G=\exp(-u)\,\xi^*\otimes\bar\xi^*\,,$$

where ξ denotes the unique holomorphic section to L over Y identified, over G, with the trivial section of constant value 1 in $\mathcal{O}_G$ via the natural isomorphism $\mathcal{O}_{G_\Delta}(\sum_{\sigma\in\Delta(1)}\nu_\sigma D(\sigma))|_G\cong\mathcal{O}_G$. Then by exactly the same formula as in (4.1), we have a mapping $\boldsymbol{m}_{u,L}$: $G\to M_{\boldsymbol{R}}$ (we put L as a subscript to emphasize the line bundle L). Now, in Theorem 4.2, replace the assumption of ampleness of $K_{G_\Delta}^{-1}$ by that of L. Then (4.2) is still valid when we further replace $\boldsymbol{m}_u$, $\overline{\boldsymbol{m}}_u$, $\sum_{-K}$, respectively by $\boldsymbol{m}_{u,L}$, $\overline{\boldsymbol{m}}_{u,L}$, $\sum_L$ (cf. (8.2)).

5. Futaki invariants for toric varieties

In [10], Futaki introduced an obstruction to the existence of Einstein-Kähler forms as follows: Let Y be a compact connected complex manifold and ω a Kähler form on Y, if any, in the cohomology class $2\pi c_1(Y)_{\boldsymbol{R}}$. Note that the space $\mathcal{X}(Y)$ of all holomorphic vector fields on Y forms a Lie algebra. Then a fundamental theorem of Futaki [10] states the following:

Theorem 5.1. *Let f_ω be the real-valued C^∞ function on Y defined uniquely, up to constant, by $Ric(\omega)-\omega=\sqrt{-1}\,\partial\bar{\partial}f_\omega$. Put $c:=((2\pi\, c_1(Y))^n[Y])^{-1}$, where $n=\dim_{\boldsymbol{C}} Y$. We further define a linear map $F=F_Y\colon \mathcal{X}(Y)\to\boldsymbol{R}$ by*

$$F(V):=c\,\mathrm{Re}\,(\int_Y (Vf_\omega)\,\omega^n)\,,\quad V\in\mathcal{X}(Y)\,.$$

Then this map F does not depend on the choice of ω. Moreover,
(a) *F is trivial on the commutator subalgebra of $\mathcal{X}(Y)$.*
(b) *If Y admits an Einstein-Kähler form, then F is trivial.*

In order to compute this F for toric varieties, we introduce the following quantities:

DEFINITION 5.2. Let Δ be a nonsingular finite polyhedral decomposition of N. If G_Δ is a Fano variety (resp. a projective variety with its hyperplane bundle L), then we define an element $\boldsymbol{a}_\Delta$ (resp. $\boldsymbol{a}_{\Delta,L}$) of $M_{\boldsymbol{R}}$ to be the barycentre of the polyhedron $\sum_{-K}$ (resp. $\sum_L$). Nemaly, the i-th component of the vector $\boldsymbol{a}_\Delta$ (resp. $\boldsymbol{a}_{\Delta,L}$) in the vector space $M_{\boldsymbol{R}}(=\boldsymbol{R}^n)$ is

$$\int_{\Sigma_{-K}} x_i\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n \Big/ \int_{\Sigma_{-K}} dx_1\wedge dx_2\wedge\cdots\wedge dx_n\,,$$
$$(\text{resp.}\int_{\Sigma_L} x_i\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n \Big/ \int_{\Sigma_L} dx_1\wedge dx_2\wedge\cdots\wedge dx_n)\,,$$

where $(x_1, x_2, \cdots, x_n)$ is the system of standard coordinates of $M_{\boldsymbol{R}}(=\boldsymbol{R}^n)$. Obviously, $\boldsymbol{a}_\Delta$ (resp. $\boldsymbol{a}_{\Delta,L}$) is in $M_{\boldsymbol{Q}}:=M\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$.

For toric Fano varieties, we can deduce from (4.2) the following simple formula:

Theorem 5.3. *Let G_Δ be a nonsingular toric Fano viariety. In the notation of* (1.6) *and* (5.1), *we put $\tilde{a}_i:=F(t_i\partial/\partial t_i)$ for each $i=1, 2, \cdots, n$. Then*

$$\boldsymbol{a}_\Delta=(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)\,.$$

REMARK 5.4. (i) In Appendix C, we shall prove a more general version of (5.3) above (cf. (9.2.3)).
(ii) We identify each element $\boldsymbol{a}=(a_1, a_2, \cdots, a_n)$ of $M_{\boldsymbol{R}}$ with $\sum_{i=1}^n a_i\,dt_i/t_i\in$

Lie$(G)^*$. Then Theorem (5.3) shows that, for any nonsingular toric Fano variety G_Δ, the restriction $F|_{\mathrm{Lie}(G)}$ of $F\colon \mathcal{X}(G_\Delta)\to \boldsymbol{R}$ to Lie(G) coincides with $\boldsymbol{a}_\Delta$.

In view of (5.3) and (5.4), we call the element $\boldsymbol{a}_\Delta$ of $M_{\boldsymbol{R}}$ the *Futaki invariant* of the toric Fano variety G_Δ. Recall that, for a reductive algebraic group H,

$$\mathrm{Lie}(\mathrm{Center}(H)) + [\mathrm{Lie}(H), \mathrm{Lie}(H)] = \mathrm{Lie}(H)\,,$$

and $\mathrm{Lie}(\mathrm{Center}(H))\subseteq \mathrm{Lie}(T)$ for every maximal torus T of H. Since G is a maximal torus of Aut(G_Δ), (a) of (5.1) together with (5.3) implies

Corollary 5.5. *Let G be a nonsingular toric Fano variety such that* Aut(G_Δ) *is reductive. Then $F\colon \mathcal{X}(G_\Delta)\to \boldsymbol{R}$ is trivial if and only if $\boldsymbol{a}_\Delta=0$.*

Finally, note the following:

REMARK 5.6. Suppose that G_Δ is a nonsingular projective variety with the corresponding very ample line bundle L (where G_Δ is not necessarily a Fano variety). Even in this case, we have a theorem similar to (5.3). Actually, $\boldsymbol{a}_{\Delta,L}$ coincides with

$$((2\pi\, c_1(L))^n[G_\Delta])^{-1}(r_L)_*|_{\mathrm{Lie}(G)}$$

in the notation in Appendix A (see also (9.2.4)).

6. Concluding remarks

A finite polyhedral decomposition Δ of N is called *canonically symmetric* if the following conditions are satisfied:
(i) Δ is nonsingular;
(ii) Δ has the property (d) of (2.1);
(iii) $-R(\Delta)=R(\Delta)$;
(iv) $\boldsymbol{a}_\Delta=0$.
Now, combining (1.5), (2.1), (2.4), (2.6), (b) of (5.1), (5.5), we obtain:

Theorem 6.1. *Let X be as in Introduction. If X admits an Einstein-Kähler form, then there exists a canonically symmetric finite polyhedral decomposition Δ of N such that X is G-equivariantly isomorphic to G_Δ.*

In view of this theorem, (0.1) in Introduction is divided into the following two problems:

Problem 6.2. *Classify all canonically symmetric finite polyhedral decompositions of N (up to isomorphism).*

Problme 6.3. *Let Δ be a canonically symmetric finite polyhedral decomposition of N. Then does G_Δ admit an Einstein-Kähler metric?*

As for (6.2), if $n \geqq 4$, no definitive results are known so far except Voskresenskiĭ and Klyachko [23] classified all centrally symmetric finite polyhedral decompositions Δ of N satisfying (i) and (ii) above (where such a Δ is always canonically symmetric). In the case $n \leqq 3$, we can classify all canonically symmetric finite polyhedral decompositions Δ of N. Namely, the corresponding G_Δ is one of the following:

(a) For $n=1$: $\boldsymbol{P}^1(\boldsymbol{C})$.
(b) For $n=2$: $\boldsymbol{P}^2(\boldsymbol{C})$, $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, S_3.
(c) For $n=3$: $\boldsymbol{P}^3(\boldsymbol{C})$, $\boldsymbol{P}^2(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, $\boldsymbol{P}^1(\boldsymbol{C}) \times S_3$, $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1, -1))$.

If $n=3$, for instance, this classification easily follows from (2.5), since we can eliminate the possibility of F_1^5 as follows: Let $\boldsymbol{b}'$, $\boldsymbol{b}''$, $\boldsymbol{b}^{(k)}$ $(0 \leqq k \leqq 6)$ be vectors in $N(=\boldsymbol{R}^3)$ defined as

$$\boldsymbol{b}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}'' = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{b}^{(0)} = \boldsymbol{b}^{(6)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\boldsymbol{b}^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{b}^{(3)} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}^{(4)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{b}^{(5)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

In terms of these vectors, Δ for F_1^5 is characterized by

$$\Delta(3) = \{\boldsymbol{Z}_0\boldsymbol{b}' + \boldsymbol{Z}_0\boldsymbol{b}^{(k-1)} + \boldsymbol{Z}_0\boldsymbol{b}^{(k)},\ \boldsymbol{Z}_0\boldsymbol{b}'' + \boldsymbol{Z}_0\boldsymbol{b}^{(k-1)} + \boldsymbol{Z}_0\boldsymbol{b}^{(k)} \mid 1 \leqq k \leqq 6\},$$

and hence the associated compact convex polyhedron $\sum_{-K}$ has exactly 12 vertices:

$$(1, 1, 1), (1, 0, 1), (1, -1, 0), (1, -1, -1), (1, 0, -1), (1, 1, 0),$$
$$(-2, 1, 1), (-2, 0, 1), (-1, -1, 0), (0, -1, -1), (0, 0, -1), (-1, 1, 0).$$

It then follows that $\boldsymbol{a}_\Delta \neq 0$.

As for (6.3), we have some results on S_3 and $\boldsymbol{P}^1(\boldsymbol{C}) \times S_3$ (cf. [16]) by the method of Section 3, though we do not go into details.

7. Appendix A

We here fix, once for all, a holomorphic line bundle L over a d-dimensional compact complex connected manifold Y. Assume that a complex Lie subgroup S of $\mathrm{Aut}(Y)$ acts holomorphically on L as bundle isomorphisms covering the S-action on Y. (If $L = K_Y^{-1}$, then our S-action on L is always assumed to be the standard one on K_Y^{-1}.) Let H be the set of all C^∞ Hermitian fibre metrics of the line bundle L over Y. For each $H \in h$, we denote by $c_1(L; h)$ the first

Chern form $(\sqrt{-1}/2\pi)\,\bar{\partial}\partial\log(h)$ of the metric h. Furthermore, note that S acts on H (from the right) by

$$H\times S\ni(h,s)\mapsto s^*h\in H\,,$$

where s^*h is defined by $(s^*h)\,(l_1,l_2):=h(s(l_1),\,s(l_2))$ for all $l_1,\,l_2\in L$ in the same fibre of L over Y. Now, to each pair $(h',h'')\in H\times H$, we associate a real number $R_L(h',h'')\in\boldsymbol{R}$ by

$$R_L(h',h''):=\int_a^b\Bigl(\frac{1}{2}\int_Y h_t^{-1}\frac{\partial h_t}{\partial t}\,(2\pi\,c_1(L;h_t))^d\Bigr)\,dt\,,$$

$\{h_t\,|\,a\leqq t\leqq b\}$ being an arbitrary piecewise smooth path in H such that $h_a=h'$ and $h_b=h''$. Then by a result of Donaldson[*)] applied to the line bundle L, the number $R_L(h',h'')$ above is independent of the choice of the path $\{h_t\,|\,a\leqq t\leqq b\}$ and therefore well-defined. Moreover, R_L is S-invariant, i.e.,

$$R_L(s^*h',s^*h'')=R_L(h',h'')\qquad\text{for all}\quad s\in S\quad\text{and all}\quad h',h''\in H\,,$$

and satisfies the 1-cocycle condition, i.e.,

$$\begin{aligned}&\text{(i)}\quad R_L(h',h'')+R_L(h'',h')=0\quad\text{and}\\&\text{(ii)}\quad R_L(h,h')+R_L(h',h'')+R_L(h'',h)=0\,,\end{aligned}$$

for all $h,\,h',\,h''\in H$. In particular, the number $R_L(h,s^*h)$ depends only on s and is independent of the choice of $h\in H$. Now, by setting

$$r_L(s):=\exp(R_L(h,s^*h))\,,\quad s\in S\,,$$

one easily obtains (see, for instance, [14; §5]):

Proposition 7.1. *$r_L\colon S\to\boldsymbol{R}_+$ is a Lie group homomorphism from S to the multiplicative group $\boldsymbol{R}_+$ of positive real numbers.*

Let $(r_L)_*\colon\mathrm{Lie}(S)\to\boldsymbol{R}$ be the Lie algebra homomorphism associated with r_L, where we always regard $\mathrm{Lie}(S)$ as a Lie subalgebra of $\mathcal{X}(Y)$ (cf. §5). For each holomorphic vector field $V\in\mathcal{X}(Y)$, we denote by $V_{\boldsymbol{R}}$ the corresponding real vector field $V+\bar{V}$ on Y. Then,

Proposition 7.2. (i) *Let $D(\subsetneqq Y)$ be an S-stable closed analytic subset of Y. Suppose there exists an S-invariant holomorphic section b over $Y-D$ to the dual bundle L^* of L. For each $h\in H$, let u_h be the real-valued C^∞ function on $Y-D$ such that $h=\exp(-u_h)\,b\otimes\bar{b}$ on $Y-D$. Then*

[*)] See Proposition 6 of S.K. Donaldson's paper "Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles", Proc. London Math. Soc. 50 (1985), 1–26.

$$(7.2.1) \qquad (r_L)_*(V) = -\frac{1}{2}\int_{Y-D} V_{\boldsymbol{R}}(u_h)\,(\sqrt{-1}\,\partial\bar{\partial}u_h)^d$$

for all $h \in H$ and all $V \in Lie(S)$.
(ii) *Under the same assumption as in* (i) *above, we consider the case where $L = K_Y^{-1}$. Suppose further that L is ample. Then the restriction $F_Y|_{\mathrm{Lie}(S)}$ of F_Y* (cf. (5.1)) *to $Lie(S)$ satisfies*

$$(7.2.2) \qquad F_Y|_{\mathrm{Lie}(S)} = ((2\pi\, c_1(L))^d[Y])^{-1}(r_L)_*\,.$$

Proof. Since (7.2.1) is straightforward from the definition of R_L, it suffices to show (7.2.2). From the assumption of ampleness of L, there exists a metric $h \in H$ for $L = K_Y^{-1}$ such that $\omega := \sqrt{-1}\,\partial\bar{\partial}u_h$ extends to a Kähler form on Y in the cohomology class $2\pi\, c_1(Y)_{\boldsymbol{R}}$. Put $\Omega := (\sqrt{-1})^d(-1)^{d(d-1)/2}\exp(-u_h)\, b\wedge\bar{b}$. Then Ω is a volume form on Y satisfying

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\,\partial\bar{\partial}f\,,$$

where $f := \log(\Omega/\omega^d)$. In view of $\omega^d = \exp(-f)\,\Omega$, we obtain

$$\begin{aligned} 0 &= -\int_Y (\text{Lie deriv. of } \exp(-f)\,\Omega \text{ w.r.t. } V_{\boldsymbol{R}}) \\ &= \int_Y V_{\boldsymbol{R}}(f)\,\omega^d - \int_Y \exp(-f)\,(\text{Lie deriv. of } \Omega \text{ w.r.t. } V_{\boldsymbol{R}}) \\ &= \int_Y V_{\boldsymbol{R}}(f)\,\omega^d + \int_Y V_{\boldsymbol{R}}(u_h)\,\omega^d = 2\,\mathrm{Re}\,(\int_Y V(f)\,\omega^d) + \int_Y V_{\boldsymbol{R}}(u_h)\,\omega^d\,. \end{aligned}$$

This together with (7.2.1) implies (7.2.2).

REMARK 7.3. In a forthcoming paper (cf. Bando and Mabuchi [3]), we shall give a little more systematic treatment of (7.2) above.

REMARK 7.4. In view of the definition of R_L, it is easy to extend the formula (7.2.1) to the following slightly general case:

Fact. *Let D, b, h, u_h be the same as in* (i) *of* (7.2). *We further assume that there exists an S-invariant morphism $\zeta\colon Y \to W$ of Y into a complex manifold W. Fix an arbitrary line bundle L' on W and let h' be a C^∞ Hermitian metric for L'. Put $L'' := \zeta^*L' \otimes L$. Then for all $h \in H$ and all $V \in Lie(S)$, we have*:

$$(7.4.1) \qquad (r_{L''})_*(V) = -\frac{1}{2}\int_{Y-D} V_{\boldsymbol{R}}(u_h)\,(\sqrt{-1}\,\partial\bar{\partial}u_h + 2\pi\zeta^*c_1(L';\,h'))^d\,.$$

REMARK 7.5. We here denote $(r_L)_*$ by $(r_{L,Y})_*$ to emphasize the base space Y. Furthermore, assume that there exists a surjective S-equivariant morphism $\lambda\colon \tilde{Y} \to Y$ from a compact complex connected manifold $\tilde{Y}$ endowed with a holomorphic S-action. Put $\tilde{L}:=\lambda^*L$. Note that the S-action on L naturally induces one on $\tilde{L}$. Then obviously,

$$(7.5.1) \qquad (r_{\tilde{L},\tilde{Y}})_* = (\deg \lambda)\,(r_{L,Y})_* .$$

8. Appendix B

The purpose of this appendix is to prove a relative version of (4.2) and (4.4). Let G (resp. G_c) be as in Section 1 (resp. 3), and P be a holomorphic princiapl bundle over a complex connected manifold W with structure group G. (Recall that, by standard definition, G acts on P from the right.) In our case, however, G acts on P from the left by

$$G\times P\ni(g,p)\mapsto g\cdot p:=p\cdot g\in P .$$

(Since G is abelian, there is no essential difference between left and right G-actions.) Note that P is locally trivial, i.e., W is written as a union of its open neighbourhoods W_α, $\alpha\in A$, such that for each α, we have a G-equivariant isomorphism

$$\iota_\alpha\colon P|_{W_\alpha}\cong W_\alpha\times G .$$

Let $pr_2\colon W_\alpha\times G\to G$ be the natural projection to the second factor and write G as $\{(t_1,\cdots,t_n)\,|\,t_i\in \boldsymbol{C}^*\}$ (cf. Section 1).

(8.1) Let Y be a complex manifold with an effective holomorphic G-action containing P as a G-stable Zariski-open dense subset. We further assume that there exists a G-invariant morphism $\zeta\colon Y\to W$ satisfying the following conditions:

(8.1.1) The restriction $\zeta|_P\colon P\to W$ coincides with the original principal bunlde P over W;

(8.1.2) $P_w:=(\zeta|_P)^{-1}(w)$ is Zariski-open and dense in $Y_w:=\zeta^{-1}(w)$ for each $w\in W$;

(8.1.3) ζ is a projective morphism with the corresponding ζ-very ample line bundle $L:=\mathcal{O}_Y(1)\in\mathrm{Pic}(Y)$;

(8.1.4) L is expressible as $\mathcal{O}_Y(D)$ for some effective divisor D on Y satisfying $\mathrm{Supp}\,(D)\subset Y-P$.

We first observe that the G-action on Y naturally lifts to a linear G-action on the line bundle L such that the following holds:

(8.1.5) Let ξ be the holomorphic section*) to L over Y which is identified, over P, with the trivial section of constant value 1 in $\mathcal{O}_P$ via the natural isomorphism $\mathcal{O}_Y(D)|_P\cong\mathcal{O}_P$. Then G acts identically on ξ.

*) This section ξ vanishes along Supp(D) so that zero$(\xi)=D$.

Note also that the cohomology class $2\pi c_1(L)_{\boldsymbol{R}}$ is represented by a G_c-invariant C^∞ (1, 1)-form ω on Y such that the pullback of ω to Y_w, denoted by ω_w, is a Kähler form on Y_w for each $w \in W$. Then there exists a G_c-invariant Hermitian C^∞ metric h for L satisfying

$$h|_P = \exp(-u)\, \xi^* \otimes \bar{\xi}^* , \quad \text{and} \tag{8.1.6}$$

$$\omega|_P = \sqrt{-1}\, \partial\bar{\partial} u \tag{8.1.7}$$

for some G_c-invariant C^∞ function u on P. We shall now define $\boldsymbol{m}\colon P \to M_{\boldsymbol{R}}$, $\Delta = \Delta_w$, $\sum = \sum_w$ ($w \in W$) as follows: For each $\alpha \in A$, put

$$t_i^{(\alpha)} := (pr_2 \circ \iota_\alpha)^*(t_i) , \quad 1 \leqq i \leqq n ,$$

and consider the real-valued C^∞ functions $x_1^{(\alpha)}, x_2^{(\alpha)}, \cdots, x_n^{(\alpha)}$ on $P|_{W_\alpha}$ defined by

$$t_i^{(\alpha)}\, \bar{t}_i^{(\alpha)} = |t_i^{(\alpha)}|^2 = \exp(-x_i^{(\alpha)}) , \quad 1 \leqq i \leqq n .$$

Now, on $P|_{W_\alpha}$, u above is regarded as a function $u(w, x_1^{(\alpha)}, \cdots, x_n^{(\alpha)})$ in $w, x_1^{(\alpha)}, \cdots, x_n^{(\alpha)}$. By the same argument as in Section 3,

$$\partial\bar{\partial} u_w = \textstyle\sum_{i,j} (\partial^2 u / \partial x_i^{(\alpha)} \partial x_j^{(\alpha)})\, (dt_i^{(\alpha)}/t_i^{(\alpha)}) \wedge (d\bar{t}_j^{(\alpha)}/\bar{t}_j^{(\alpha)}) \quad \text{on} \quad P_w \ (w \in W_\alpha) , \tag{8.1.8}$$

where $u_w := u|_{P_w}$. Let $\boldsymbol{m}^{(\alpha)}\colon P|_{W_\alpha} \to M_{\boldsymbol{R}} (= \boldsymbol{R}^n)$ be the mapping defined by

$$\boldsymbol{m}^{(\alpha)}(p) := ((\partial u/\partial x_1^{(\alpha)})(p), \cdots, (\partial u/\partial x_n^{(\alpha)})(p)) , \quad p \in P .$$

Then it is easily seen that $\boldsymbol{m}^{(\alpha)}$, $\alpha \in A$, are glued together defining a global mapping $\boldsymbol{m}\colon P \to M_{\boldsymbol{R}} (= \boldsymbol{R}^n)$ such that the restriction of $\boldsymbol{m}$ to each $P|_{W_\alpha}$ coincides with $\boldsymbol{m}^{(\alpha)}$. Now, let w be an arbitrary point of W and choose an $\alpha \in A$ such that $w \in W_\alpha$. We can then regard Y_w as a nonsingular toric variety by

$$G \ni (t_1^{(\alpha)}(p), \cdots, t_n^{(\alpha)}(p)) \overset{\cong}{\longleftrightarrow} p \in P_w \subset Y_w .$$

Hence, there exists a unique nonsingular finite polyhedral decomposition $\Delta = \Delta_w$ of N such that

(1) Δ can depend only on w and is independent of the choice of α.

(2) $Y_w \cong G_\Delta$ as a toric variety.

Furthermore, $L_w := L|_{Y_w}$ is written in the form

$$L_w = \mathcal{O}_{G_\Delta}(\textstyle\sum_{\rho \in \Delta(1)} \nu_\rho\, D(\rho)) \qquad \text{for some } \nu_\rho\text{'s in } \boldsymbol{Z}_0 ,$$

via the identification of Y_w with G_Δ. Letting $\boldsymbol{b}_\rho$ be as in Section 2, we now define an n-dimensional compact convex polyhedron $\sum = \sum_w$ in $M_{\boldsymbol{R}}$ by

$$\textstyle\sum := \{\boldsymbol{a} \in M_{\boldsymbol{R}} \mid (\boldsymbol{a}, \boldsymbol{b}_\rho) \leqq \nu_\rho \text{ for all } \rho \in \Delta(1)\} . \tag{8.1.9}$$

Since L_w is ample, the vertices of $\sum$ are exactly $\{\boldsymbol{a}_\sigma \mid \sigma \in \Delta(n)\}$, where each $\boldsymbol{a}_\sigma$ denotes the unique element of M such that $(\boldsymbol{a}_\sigma, \boldsymbol{b}_\rho) = \nu_\rho$ for all $\rho \in \Delta(1)$ with

$\rho \leqq \sigma$ (cf. Oda [21]). Then we have:

Theorem 8.2. *Let Q be the closure of the image $\boldsymbol{m}(P)$ in $M_{\boldsymbol{R}}$. Then $Q=\sum_w$ for all $w \in W$. (In particular, $\sum=\sum_w$ and $\Delta=\Delta_w$ are both independent of w.) Furthermore, $\boldsymbol{m}: P \to M_{\boldsymbol{R}}$ naturally extends to a C^∞ map $\bar{\boldsymbol{m}}: Y \to M_{\boldsymbol{R}}$. Let w be an arbitrary point of W. Then $\bar{\boldsymbol{m}}$ satisfies*

(a) $\bar{\boldsymbol{m}}^{-1}(\gamma) \cap Y_w$ *is a single G-orbit for each open face γ of $\sum$;*
(b) $\bar{\boldsymbol{m}}$ *induces a diffeomorphism (including boundaries) between manifolds Y_w/G_c and $\sum (=\sum_w)$ with corners;*
(c) $\bar{\boldsymbol{m}}|_{Y_w}$: $Y_w \to M_{\boldsymbol{R}}$ *coincides with the mapping $\bar{\boldsymbol{m}}_{u_w, L_w}$ in (4.4) via the identification of Y_w with G_Δ and is just the moment map: $Y_w \to Lie(G_c)^*(\cong M_{\boldsymbol{R}})$ associated with the Kähler form $\omega_w(=\sqrt{-1}\,\partial\bar{\partial} u_w)$ on Y_w.*

REMARK 8.2.1. Consider the case where W consists of a single point. Then (8.2) above implies (4.4). If we further assume $L=K_Y^{-1}$, then (8.2) shows nothing but (4.2) and (4.3).

Proof of (8.2). *Step* 1. Fix an $\alpha \in A$ such that $w \in W_\alpha$. For simplicity, put $z_i := t_i^{(\alpha)}$ and $x_i := x_i^{(\alpha)}$, $i=1, 2, \cdots, n$. Let $0 \leqq \theta_i < 2\pi$ be such that $z_i = \exp((-x_i/2)+\sqrt{-1}\,\theta_i)$. Then $(z_1, \cdots, z_n)$ (resp. $(x_1, \cdots, x_n, \theta_1, \cdots, \theta_n)$) forms a system of holomorphic local coordinates (resp. real local coordinates) of Y_w. Note that

$$z_i \partial/\partial z_i + \bar{z}_i \partial/\partial \bar{z}_i = -2\,\partial/\partial x_i\,, \quad 1 \leqq i \leqq n\,. \tag{8.2.2}$$

We now write the Kähler form ω_w as $\sqrt{-1}\sum_{i,j} u_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ on P_w, where $u_{i\bar{j}} := \partial_i\,\partial_{\bar{j}}(u_w)$. Put

$$V_i := t_i \partial/\partial t_i \in \mathrm{Lie}(G) \subseteq \mathfrak{X}(Y)\,, \quad 1 \leqq i \leqq n\,,$$

in terms of the coordinates $t_1, \cdots, t_n$ for $G=\{(t_1, \cdots, t_n) \mid t_i \in \boldsymbol{C}^*\}$. Then there exist real-valued C^∞ functions $\varphi_{w,i}$, $i=1, 2, \cdots, n$, on Y_w such that

$$V_i|_{Y_w} = \sum_{j,k} u^{\bar{j}k}(\partial_{\bar{j}}\,\varphi_{w,i})\,\partial/\partial z_k\,, \quad 1 \leqq i \leqq n\,, \tag{8.2.3}$$

$(u^{\bar{j}k})$ being the inverse matrix of $(u_{i\bar{j}})$ (see, for instance, Kobayashi [12; p.94]). On the other hand, by (8.2.2), the real vector field $(V_i)_{\boldsymbol{R}}$ (cf. Appendix A) is written as

$$(V_i)_{\boldsymbol{R}} = -2\,\partial/\partial x_i\,, \quad 1 \leqq i \leqq n\,, \tag{8.2.4}$$

on Y_w. Now, on P_w, (8.2.3) above implies

$$(\text{Lie deriv. of } \omega_w \text{ w.r.t. } (V_i)_{\boldsymbol{R}}) = 2\sqrt{-1}\,\partial\bar{\partial}\varphi_{w,i}\,.$$

Moreover, by (8.2.4),

$$\text{(Lie deriv. of } \omega_w \text{ w.r.t. } (V_i)_{\boldsymbol{R}}) = -2\sqrt{-1}\,\partial\bar{\partial}(\partial u_w/\partial x_i)\,.$$

Therefore, $\partial u_w/\partial x_i = -\varphi_{w,i} + C_{w,i}$ on P_w for some real constant $C_{w,i} \in \boldsymbol{R}$. Hence $\boldsymbol{m}|_{P_w}$ and $-(\varphi_{w,1}, \cdots, \varphi_{w,n})$ coincide up to translation, which implies the latter half of (c). Since the former half of (c) is obvious, this proves (c).

Step 2. Put $\tilde{\varphi}_{w,i} := -\varphi_{w,i} + C_{w,i}$. Note that, for each i, $\tilde{\varphi}_{w,i}$ depends smoothly on w, because both $\partial\bar{\partial}\tilde{\varphi}_{w,i}$($=$Lie deriv. of $-2^{-1}\omega_w$ w.r.t. $(V_i)_{\boldsymbol{R}}$) and $\tilde{\varphi}_{w,i}|_{P_w}$ ($=\partial u_w/\partial x_i$) depend smoothly on w. We then have a natural extension of $\boldsymbol{m}$ to a C^∞ mapping $\bar{\boldsymbol{m}}$: $Y \to M_{\boldsymbol{R}}$ by setting, for each fibre Y_w ($w \in W$),

$$\bar{\boldsymbol{m}}(y) := (\tilde{\varphi}_{w,1}(y), \cdots, \tilde{\varphi}_{w,n}(y))\,, \quad y \in Y_w\,.$$

Let Q_w be the image $\bar{\boldsymbol{m}}(Y_w)$ of Y_w under this mapping $\bar{\boldsymbol{m}}$. Then by a result of Atiyah [1; Theorem 2] applied to the compact Kähler manifold (Y_w, ω_w), our Q_w forms a compact convex polyhedron in $M_{\boldsymbol{R}}$ such that

(a)′ $\bar{\boldsymbol{m}}^{-1}(\gamma) \cap Y_w$ is a single G-orbit for each open face γ of Q_w;
(b)′ $\bar{\boldsymbol{m}}$ induces a homeomorphism of Y_w/G_c onto Q_w.

(Without using Atiyah's result, we can prove this by modifying the arguments in Steps 3 and 4.) We now observe that $\sum_w$ is an n-dimensional compact convex polyhedron in $M_{\boldsymbol{R}}$ only with integral vertices$\in M$. Therefore, if $Q_w = \sum_w$ ($w \in W$), then the C^∞ dependence of $\boldsymbol{m}|_{Y_w}$ on w implies that $\sum_w$ does not depend on w at all. Thus, the proof of (8.2) is reduced to showing the following:
(a)″ $Q_w = \sum_w$;
(b)″ $\bar{\boldsymbol{m}}$ induces a diffeomorphism (including boundaries) between manifolds Y_w/G_c and Q_w with corners.

Step 3. We may now assume without loss of generality that W consists of a single point. Therefore, we may further assume $P = G$ and $Y = G_\Delta$. Let $G_{\boldsymbol{R}}$ and $\bar{G}_{\boldsymbol{R}}$ be the same as in (ii) of (4.3). Then $\bar{G}_{\boldsymbol{R}}$ is naturally identified with Y/G_c. Note that

$$\bar{G}_{\boldsymbol{R}} = \bigcup_{\sigma \in \Delta(n)} U_\sigma^{\boldsymbol{R}}$$

in terms of the notation in (1.4), where $U_\sigma^{\boldsymbol{R}} := U_\sigma \cap \bar{G}_{\boldsymbol{R}}$ is a coordinate open subset of $\bar{G}_{\boldsymbol{R}}$ identified (diffeomorphically) with the product $(\boldsymbol{R}_0)^n$ of n-copies of $\boldsymbol{R}_0$ by

$$U_\sigma^{\boldsymbol{R}} \cong (\boldsymbol{R}_0)^n, \quad y \mapsto (|\chi_{\sigma;1}(y)|^2, |\chi_{\sigma;2}(y)|^2, \cdots, |\chi_{\sigma;n}(y)|^2)\,.$$

Now, fix an arbitrary element σ of $\Delta(n)$. Recall that the real-valued C^∞ functions $x_i = x_i(\boldsymbol{t})$, $i = 1, 2, \cdots, n$, on G are defined by $|t_i|^2 = \exp(-x_i)$ for $\boldsymbol{t} = (t_1, \cdots, t_n) \in G$. Similarly, to the function $\chi_{\sigma;i} = \chi_{\sigma;i}(\boldsymbol{t})$, we associate a new function $\tilde{x}_i = \tilde{x}_i(\boldsymbol{t})$ on G by

$$|\chi_{\sigma;i}(t)|^2 = \exp(-\tilde{x}_i)\,, \quad t\in G\,.$$

Then, in terms of the notation in (1.4), we have

$$\tilde{x}_i = (\boldsymbol{a}(\sigma)^i, \boldsymbol{x})\,, \quad 1\leqq i\leqq n\,, \tag{8.2.5}$$

where

$$\boldsymbol{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Furthermore, put

$$\rho^i := \boldsymbol{Z}_0\, \boldsymbol{b}(\sigma)^i \in \Delta(1)\,, \quad 1\leqq i\leqq n\,.$$

Since $\exp(-u)\,\xi^*\otimes\bar{\xi}^*$ (cf. (8.1.6)) extends to a C^∞ Hermitian metric for $L=\mathcal{O}_{G_\Delta}(\sum_{\rho\in\Delta(1)} \nu_\rho D(\rho))$, there exists a real-valued C^∞ function H: $(\boldsymbol{R}_0)^n \to \boldsymbol{R}$ such that

$$u = \textstyle\sum_{i=1}^n \nu_i\, \tilde{x}_i + H(r_1, \cdots, r_n) \quad \text{on} \quad U_\sigma^{\boldsymbol{R}}\,,$$

where $r_i := |\chi_{\sigma;i}|^2$ $(=\exp(-\tilde{x}_i))$ and $\nu_i := \nu_{\rho_i}$. We can now give a closer look at the function $u=u(x_1, \cdots, x_n)=u(\tilde{x}_1, \cdots, \tilde{x}_n)$. For example, their first and second derivatives with respect to $\tilde{x}_1, \cdots, \tilde{x}_n$ are computed immediately:

(i) $$\textstyle\sum_{i=1}^n(\partial u/\partial x_i)\,(\partial x_i/\partial \tilde{x}_j) = \partial u/\partial \tilde{x}_j = \nu_j - (\partial H/\partial r_j)\, r_j\,,$$

(ii) $$\partial^2 u/\partial \tilde{x}_i \partial \tilde{x}_j = (\partial^2 H/\partial r_i \partial r_j)\, r_i\, r_j + \delta_{ij}(\partial H/\partial r_j)\, r_j\,.$$

Recall that $(\boldsymbol{a}(\sigma)^i, \boldsymbol{b}(\sigma)^j)=\delta_{ij}$. Hence, combining (i) with (8.2.5), we obtain

(i)′ $$(\bar{\boldsymbol{m}}, \boldsymbol{b}(\sigma)^j) = \nu_j - (\partial H/\partial r_j)\, r_j\,, \quad 1\leqq j\leqq n\,.$$

Let p_σ be the point $\in U_\sigma^{\boldsymbol{R}}$ corresponding to the origin of $(\boldsymbol{R}_0)^n$ (i.e., $r_1(p_\sigma)=r_2(p_\sigma)=\cdots=r_n(p_\sigma)=0$). Then by (i)′, $(\bar{\boldsymbol{m}}(p_\sigma), \boldsymbol{b}(\sigma)^j)=\nu_j$ for all j. Thus,

$$\bar{\boldsymbol{m}}(p_\sigma) = \boldsymbol{a}_\sigma\,. \tag{8.2.6}$$

Now, fix an arbitrary point y of $U_\sigma^{\boldsymbol{R}}$ and put $I := \{i\in\{1, 2, \cdots, n\}\,|\, r_i(y)=0\}$. Then we may assume without loss of generality that $I=\{1, 2, \cdots, q\}$ for some q with $0\leqq q\leqq n$ (where if $q=0$, we always assume $I=\phi$). In view of (8.1.7) and (8.1.8),

$$\omega = \sqrt{-1}\sum_{i,j=1}^n (\partial^2 u/\partial \tilde{x}_i \partial \tilde{x}_j)\,(d\,\chi_{\sigma;i}/\chi_{\sigma;i}) \wedge (d\,\bar{\chi}_{\sigma;j}/\bar{\chi}_{\sigma;j})$$

on U_σ in terms of holomorphic local coordinates $(\chi_{\sigma;1}, \cdots, \chi_{\sigma;n})$. Rewrite this identity, using (ii) above. Then, when evaluated at y,

$$\omega(y)=\sqrt{-1}\,\textstyle\sum_{i\in I}(\partial H/\partial r_i)(y)\,d\chi_{\sigma;i}\wedge d\bar{\chi}_{\sigma;i} \\ +\sqrt{-1}\sum_{i,j>q}(\partial^2 u/\partial\tilde{x}_i\partial\tilde{x}_j)(y)\,(d\chi_{\sigma;i}/\chi_{\sigma;i})\wedge(d\bar{\chi}_{\sigma;j}/\bar{\chi}_{\sigma;j}),$$

where the last summation is taken over all $i, j\in\{1, 2, \cdots, n\}$ such that $i>q$ and $j>q$. Since ω is a Kähler form, it follows that:

(8.2.7) $(\partial H/\partial r_i)(y)>0$ for all $i\in I$, and

(8.2.8) $((\partial^2 u/\partial\tilde{x}_i\partial\tilde{x}_j)(y))_{q<i,j\leqq n}$ is a positive definite matrix.

On the other hand, the Jacobian $J(\bar{m})_y$ of the mapping $\bar{m}$: $U_\sigma^{\mathbf{R}}\to M_{\mathbf{R}}$ at the point y in terms of the coordinates $(r_1, \cdots, r_n)$ for $U_\sigma^{\mathbf{R}}$ is computed as follows:

$$J(\bar{m})_y=\det\left(\frac{\partial(\partial u/\partial x_i)}{\partial r_j}(y)\right)_{1\leqq i,j\leqq n}=\pm\det\left(\frac{\partial(\partial u/\partial\tilde{x}_i)}{\partial r_j}(y)\right)_{1\leqq i,j\leqq n}$$

$$=\pm\det\left(\begin{array}{ccc|c} -(\partial H/\partial r_1)(y) & & & \\ & -(\partial H/\partial r_2)(y) & 0 & * \\ & \ddots & & \\ 0 & & -(\partial H/\partial r_q)(y) & \\ \hline & 0 & & \left(\dfrac{-1}{r_j}\dfrac{\partial^2 u}{\partial\tilde{x}_i\,\partial\tilde{x}_j}(y)\right)_{q<i,j\leqq n} \end{array}\right),$$

where the last identity follows from

$$\frac{\partial(\partial u/\partial x_i)}{\partial r_j}(y)=-(\partial^2 H/\partial r_i\partial r_j)\,r_i-\delta_{ij}(\partial H/\partial r_j)\,,\quad\text{(cf. (ii))}\,.$$

Now, in view of (8.2.7) and (8.2.8), we obtain $J(\bar{m})_y\neq 0$. This together with (b)′ (cf. Step 2) yields (b)″. Hence, it suffices to show (a)″, i.e., $Q=\sum$. For each j, let y_j be the point in $U_\sigma^{\mathbf{R}}$ such that $r_i(y_j)=(1-\delta_{ij})\,r_i(y)$, $1\leqq i\leqq n$. Then by (i)′, $(\bar{m}(y_j), \boldsymbol{b}(\sigma)^j)=\nu_j$. On the other hand, by (i), (i)′ and (8.2.8),

$$-r_j\frac{\partial(\bar{m}, \boldsymbol{b}(\sigma)^j)}{\partial r_j}\left(=\frac{\partial(\bar{m}, \boldsymbol{b}(\sigma)^j)}{\partial\tilde{x}_j}=\partial^2 u/\partial\tilde{x}_j^2\right)\geqq 0\quad\text{on}\quad U_\sigma^{\mathbf{R}}\,.$$

Therefore, we have

(8.2.9) $(\bar{m}(y), \boldsymbol{b}(\sigma)^j)\leqq(\bar{m}(y_j), \boldsymbol{b}(\sigma)^j)=\nu_j\,,\quad 1\leqq j\leqq n\,.$

Step 4. In this final step, we complete the proof of $Q=\sum$, assuming that W is a single point. Let y be an arbitrary point of $G_{\mathbf{R}}$. Then $y\in U_\sigma^{\mathbf{R}}$ for all $\sigma\in\Delta(n)$. Hence, by (8.2.9), $(\bar{m}(y), \boldsymbol{b}(\sigma)^j)\leqq\nu_j$ for all σ and j, i.e., $\bar{m}(y)\in\sum$. Since Q is the closure of $\bar{m}(G_{\mathbf{R}})$ $(=\boldsymbol{m}(G))$ in $M_{\mathbf{R}}$, we now obtain $Q\subseteq\sum$. Recall that Q is a compact convex polyhedron in $M_{\mathbf{R}}$ (cf. Step 2). Therefore, (8.2.6) immediately implies $Q=\sum$.

9. Appendix C

In this appendix, by using a measure $d\mu$ of Duistermaat-Heckman's type (cf. [7]), we shall generalize the integral formula of Koiso and Sakane [13] on Futaki invariants. Our present result includes, at the same time, (5.3) and (5.6) in the earlier section as special cases.

DEFINITION 9.1.1. Let Y be a complex connected manifold endowed with an effective holomorphic G-action, and Δ a nonsingular finite polyhedral decomposition of N. Furthermore, let $\zeta\colon Y\to W$ be a proper G-invariant morphism of Y onto a connected complex manifold W. Then a pair ($\zeta\colon Y\to W$, G_Δ) is called a *toric bundle* if the following conditions are satisfied:

(a) ρ is locally trivial, i.e., W is a union $\bigcup_{\alpha\in A} W_\alpha$ of its open subsets W_α, $\alpha\in A$, such that for each α, there exists a G-equivariant isomorphism $\iota_\alpha\colon \zeta^{-1}(W_\alpha)\cong W_\alpha\times G_\Delta$.

(b) If α, $\beta\in A$ are such that $W_\alpha\cap W_\beta\neq\phi$, then there exists a holomorphic G-valued function $t_{\alpha\beta}=t_{\alpha\beta}(w)$ on $W_\alpha\cap W_\beta$ such that

$$\iota_\alpha\circ\iota_\beta^{-1}(w, x) = (w, t_{\alpha\beta}(w)\cdot x)$$

for all $w\in W_\alpha\cap W_\beta$ and all $x\in G_\Delta$.

REMARK 9.1.2. In the above, let $pr_{1,\alpha}\colon W_\alpha\times G_\Delta\to G_\Delta$ be the natural projection to the second factor. Put $P\colon=\bigcup_{\alpha\in A}(pr_{1,\alpha}\circ\iota_\alpha)^{-1}(G)$. Then $\zeta|_P\colon P\to W$ is naturally regarded as a principal bundle with structure group G.

DEFINITION 9.1.3. Let ($\zeta\colon Y\to W$, G_Δ) be a toric bundle and L a line bundle over Y. Then a triple ($\zeta\colon Y\to W$, G_Δ, L) is called a *polarized toric bundle* if there exists an effective divisor D on Y such that

(a) $L=\mathcal{O}_Y(D)$;

(b) $\mathrm{Supp}(D)\subset Y-P$, where P is as in (9.1.2);

(c) $D|_{Y_w}$ is an ample (or equivalently, very ample) divisor on Y_w for each $w\in W$.

REMARK 9.1.4. For a polarized toric bundle ($\zeta\colon Y\to W$, G_Δ, L), one can easily check that Y, W, P, L, D above always satisfy the conditions (8.1.1)$\sim$(8.1.4) in Appendix B. Conversely, let Y, W, P, L, D be as in Appendix B (satisfying the conditions (8.1.1)$\sim$(8.1.4)). Then by Theorem (8.2), the corresponding $\Delta=\Delta_w$ is independent of w, and it easily follows that the associated triple ($\zeta\colon Y\to W$, G_Δ, L) forms a polarized toric bundle.

(9.2) We now fix a polarized toric bundle ($\zeta\colon Y\to W$, G_Δ, L). Then for each $\rho\in\Delta(1)$, the subsets $(pr_{1,\alpha}\circ\iota_\alpha)^{-1}(D(\rho))$, $\alpha\in A$, of Y are glued together defining a global prime divisor, denoted by $\tilde{D}(\rho)$, on Y. Hence, the divisor D (cf. (a) of (9.1.3)) is written as $\sum_{\rho\in\Delta(1)}\nu_\rho\tilde{D}(\rho)$ for some ν_ρ's in $\boldsymbol{Z}_0$. We thus have the corresponding n-dimensional compact convex polyhedron $\sum$ in $M_{\boldsymbol{R}}$ defined by (8.1.9).

REMARK 9.2.1. Let $\boldsymbol{a}_k$, $k=0, 1, \cdots, s$, be the integral points in $\sum$, i.e., $\sum\cap M=\{\boldsymbol{a}_k\,|\,0\leqq k\leqq s\}$. Furthermore, put

$$\chi_k := \chi^{-\boldsymbol{a}_k}, \quad 0\leqq k\leqq s,$$

where on the right-hand side, we used the notation in Section 1. Then the mapping

$$G\ni \boldsymbol{t}\mapsto (\chi_0(\boldsymbol{t}): \chi_1(\boldsymbol{t}): \cdots : \chi_s(\boldsymbol{t}))\in \boldsymbol{P}^s(\boldsymbol{C})$$

extends to an embedding: $G_\Delta \subsetneq \boldsymbol{P}^s(\boldsymbol{C})$ such that the corresponding hyperplane bundle on G_Δ is $\mathcal{O}_{G_\Delta}(\sum_{\rho\in\Delta(1)} \nu_\rho D(\rho))$ (cf. Oda [21]). In particular, the pullback $(=\sqrt{-1}\,\partial\bar\partial \log(\sum_{k=0}^{s}|\chi_k|^2))$ of the Fubini-Study form on $\boldsymbol{P}^s(\boldsymbol{C})$ to G_Δ is positive definite everywhere on G_Δ.

DEFINITION 9.2.2. Since $G=(\boldsymbol{C}^*)^n$, we can componentwise express $t_{\alpha\beta}=t_{\alpha\beta}(w)$ in (b) of (9.1.1) in the form

$$t_{\alpha\beta}(w) = (t^{(1)}_{\alpha\beta}(w), t^{(2)}_{\alpha\beta}(w), \cdots, t^{(n)}_{\alpha\beta}(w)), \quad w\in W_\alpha\cap W_\beta .$$

Hence for each i, the system of transition functions $\{t^{(i)}_{\alpha\beta}\}_{\alpha,\beta\in A}$ defines a holomorphic line bundle $L^{(i)}$ over W. Let $P^{(i)}(:=L^{(i)}-(\text{zero section}))$ be the $\boldsymbol{C}^*$-bundle over W corresponding to $L^{(i)}$. Then, in terms of the natural identification

$$P = P^{(1)}\times_W P^{(2)}\times_W\cdots\times_W P^{(n)},$$

we can write each point p of P as

$$p = (p^{(1)}, p^{(2)}, \cdots, p^{(n)})$$

with $p^{(i)}\in P^{(i)}$, $i=1, 2, \cdots, n$. For each i, fix an arbitrary C^∞ Hermitian metric h_i on $L^{(i)}$ and define a C^∞ function $\tilde{x}_i=\tilde{x}_i(p)$ on P by

$$\exp(-\tilde{x}_i(p)) = h_i(p^{(i)}, p^{(i)}), \quad p\in P .$$

We shall now show the following formula:

Theorem 9.2.3. *Put* $e:=\dim_{\boldsymbol{C}} W$ *and* $\gamma_{n,e}:=(n+e)!/e!$. *Let* L' *be an arbitrary line bundle over* W *and put* $L'':=\zeta^*L'\otimes L$. *We now assume that* W *is compact. Furthermore, let* $x=(x_1, x_2, \cdots, x_n)$ *be the system of standard coordinates on* $M_{\boldsymbol{R}}(=\boldsymbol{R}^n)$, *and let* $T=T(x)$ *be the polynomial in* $x_1, \cdots, x_n$ *defined by* $T(x):=\gamma_{n,e}(c_1(L')+\sum_{j=1}^{n} x_j\, c_1(L^{(j)}))^e[W]$. *Then in terms of the notation in* (1.6) *and Appendix* A, *we have*:

(a) $$(r_{L''})_*(t_i\partial/\partial t_i) = (2\pi)^{n+e}\int_{\Sigma} x_i\, d\mu\,, \quad 1\leqq i\leqq n,$$

(b) $$c_1(L'')^{n+e}[Y] = \int_{\Sigma} d\mu\,,$$

where $d\mu := T(x)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

REMARK 9.2.4. In (9.2.3) above, assume that W is a single point. Then by $e=0$, $T(x)$ is nothing but the constant function 1 on $M_{\boldsymbol{R}}$. Hence, (5.6) is straightforward from (9.2.3) above. We further obtain (5.3) by setting $L=K_Y^{-1}$ (see also (7.2.2)).

REMARK 9.2.5. Note that $d\mu$ is a polynomial measure on $M_{\boldsymbol{R}}$. If L is ample on the whole space Y, then this fact is already observed by Duistermaat and Heckman [7] (see especially their formula (1.11)).

Proof of (9.2.3). *Step* 1. Let $u=u(\tilde{x}_1(p), \cdots, \tilde{x}_n(p))$ be the C^∞ function in $\tilde{x}_1=\tilde{x}_1(p), \cdots, \tilde{x}_n=\tilde{x}_n(p)$ defined by

$$u := \log(\textstyle\sum_{k=0}^{s} \exp(\boldsymbol{a}_k, \tilde{\boldsymbol{x}}(p)))\,,$$

where

$$\tilde{\boldsymbol{x}}(p) := \begin{pmatrix} \tilde{x}_1(p) \\ \tilde{x}_2(p) \\ \vdots \\ \tilde{x}_n(p) \end{pmatrix} \qquad (p \in P)\,.$$

Let ξ be the holomorphic section to L over Y as in (8.1.5). Then, in view of (9.2.1), the metric $\exp(-u)\, \xi^* \otimes \bar{\xi}^*$ for $L|_P$ extends to a G_c-invariant C^∞ Hermitian metric, denoted by h, for the whole line bundle L such that the pullback of $c_1(L, h)$ to each fibre Y_w is positive definite. We now have the corresponding $\boldsymbol{m}: P \to M_{\boldsymbol{R}}$ as in (8.1). Note that, for each $w \in W$, the image $\boldsymbol{m}(P_w)$ is just the interior of Σ. Furthermore, one can easily check that the mapping $\boldsymbol{m}$ is given by

$$\boldsymbol{m}(p) = ((\partial u/\partial \tilde{x}_1)(p), \cdots, (\partial u/\partial \tilde{x}_n)(p))\,, \quad p \in P\,.$$

Step 2. Fix an arbitrary point w' of W, and let U be a sufficiently small neighbourhood of w' in W. Over this U, choose a holomorphic local base s_i for each line bundle $L^{(i)}$ and write $h^{(i)}$ as $f_i(w)\, s_i^* \otimes \bar{s}_i^*$ for some positive C^∞ function $f_i = f_i(w)$ on U. Note that, by a suitable choice of $\{s_i\}$, we may assume

$$f_i(w') = 1 \quad \text{and} \quad (df_i)(w') = 0 \qquad \text{for all } i.$$

We now choose a system $(w_1, \cdots, w_e)$ of holomorphic local coordinates on U and write each point w of U as $w=(w_1, \cdots, w_e)$ in terms of these coordinates. Then by the isomorphism

$$P|_U(=P^{(1)}\times_W\cdots\times_W P^{(n)}|_U)\cong U\times G$$
$$(t_1 s_1(w), \cdots, t_n s_n(w)) \leftrightarrow (w, t=(t_1, \cdots, t_n)),$$

we may regard $(w_1, \cdots, w_e, t_1, \cdots, t_n)$ as a system of holomorphic local coordinates on $P|_U$. Since

$$\partial\tilde{x}_j = -(dt_j/t_j)-\zeta^*(\partial f_j/f_j) \quad \text{and} \quad \bar\partial\tilde{x}_j = -(d\bar{t}_j/\bar{t}_j)-\zeta^*(\bar\partial f_j/f_j),$$

the following holds at each point of the fibre $P_{w'}$:

$$\begin{aligned}\partial\bar\partial u &= \partial\{\textstyle\sum_{j=1}^n(\partial u/\partial\tilde{x}_j)\,(-(d\bar{t}_j/\bar{t}_j)-\zeta^*(\bar\partial f_j/f_j))\}\\ &= \textstyle\sum_{i,j}(\partial^2u/\partial\tilde{x}_i\partial\tilde{x}_j)\,(dt_i/t_i)\wedge(d\bar{t}_j/\bar{t}_j)+\sum_{j=1}^n(\partial u/\partial\tilde{x}_j)\,\zeta^*\bar\partial\partial\log(f_j).\end{aligned}$$

Now, define real-valued functions $0\leqq\theta_j<2\pi$ on $P_{w'}$ by

$$t_j = \exp((-\tilde{x}_j/2)+\sqrt{-1}\,\theta_j), \quad j=1, 2, \cdots, n,$$

and set $V^i:=t_i\partial/\partial t_i$. Furthermore, let h' be a C^∞ Hermitian metric for L' and put:

$$\begin{aligned}\tau' &:= \gamma_{n,e}\{c_1(L'; h')+\textstyle\sum_{j=1}^n(\partial u/\partial\tilde{x}_j)\,c_1(L^{(j)}; h^{(j)})\}^e,\\ \tau'' &:= \gamma_{n,e}\{c_1(L'; h')+\textstyle\sum_{j=1}^n x_j\,c_1(L^{(j)}; h^{(j)})\}^e.\end{aligned}$$

Then in view of (cf. (8.2.2))

$$dt_j\wedge d\bar{t}_j/|t_j|^2 = \sqrt{-1}\,d\tilde{x}_j\wedge d\theta_j \quad \text{and} \quad (V^i)_{\boldsymbol{R}}(u) = -2\,\partial u/\partial\tilde{x}_i,$$

we have:

$$\begin{aligned}\text{(c)}\quad &(-1/2)\int_{P_{w'}}(V^i)_{\boldsymbol{R}}(u)\,(\sqrt{-1}\,\partial\bar\partial\,u+2\pi\zeta^*c_1(L'; h'))^{n+e}\\ &= (2\pi)^e\int_{P_{w'}}(\partial u/\partial\tilde{x}_i)\det(\partial^2u/\partial\tilde{x}_k\partial\tilde{x}_l)\,(\textstyle\prod_{j=1}^n(\sqrt{-1}\,dt_j\wedge d\bar{t}_j/|t_j|^2))\wedge\zeta^*(\tau')\\ &= (2\pi)^{n+e}\int_{\tilde{x}\in\boldsymbol{R}^n}\{(\partial u/\partial\tilde{x}_i)\det(\partial^2u/\partial\tilde{x}_k\partial\tilde{x}_l)\,\tau'(w')\}\,d\tilde{x}_1\wedge d\tilde{x}_2\wedge\cdots\wedge d\tilde{x}_n\\ &= (2\pi)^{n+e}\int_\Sigma\{x_i\,\tau''(w')\}\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n,\end{aligned}$$

where we obtain the last identity by setting $x_j=\partial u/\partial\tilde{x}_j$, $j=1, 2, \cdots, n$. Similar computations also show that:

$$\text{(d)}\quad \int_{P_{w'}}((\sqrt{-1}/2\pi)\,\partial\bar\partial u+\zeta^*c_1(L'; h'))^{n+e} = \int_\Sigma\tau''(w')\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n.$$

Step 3. In view of (7.4.1), the integration of (c) over W yields (a). Since $(\sqrt{-1}/2\pi)\,\partial\bar\partial u+c_1(L'; h')$ represents $c_1(L'')$, we obtain (b) by integrating (d) over W.

(9.3) We here assume that $n=1$, i.e., $G=\boldsymbol{C}^*$. Fix a holomorphic line bundle L_1 over a compact complex connected manifold W and consider the vector bunlde $E:=\mathcal{O}_W\oplus L_1$ of rank 2 over W (where vector bundles and locally free sheaves are used interchangeably if there is no fear of confusion). We now put $Y:=\boldsymbol{P}(E^*)$ and let $\zeta\colon Y\to W$ be the natural projection. Then $Y=(E-$ (zero section))$/\boldsymbol{C}^*$, and L_1 is regarded as a Zariski-open subset of Y by

$$L_1 \hookrightarrow \boldsymbol{P}(E^*)\,(=Y)\,,\quad l\mapsto (1\oplus l) \text{ modulo } \boldsymbol{C}^*\,.$$

Via this inclusion, the zero section of L_1 defines an effective prime divisor, denoted by D_0, on Y. Note that we have another divisor $D_\infty:=Y-L_1\in\mathrm{Div}(Y)$ on Y. Put $P:=L_1-D_0$. Then the natural $\boldsymbol{C}^*$-action on the line bundle L_1 extends to a holomorphic action of $G=\boldsymbol{C}^*$ on Y with the fixed point set $D_0\cup D_\infty$. Furthermore, P is regarded as a principal bundle over W with structure group G. Let $(n', n'')\ (\neq(0,0))$ be a pair of nonnegative integers which will be specified later. Put $D:=n'D_0+n''D_\infty\in\mathrm{Div}(Y)$. Then $L:=\mathcal{O}_Y(D)$ is a ζ-very ample line bundle on Y. We thus have a polarized toric bundle ($\zeta\colon Y\to W$, $\boldsymbol{P}^1(\boldsymbol{C})$, L).

REMARK 9.3.1. Fix an arbitrary C^∞ Hermitian metric h_1 for the line bundle L_1. Now, recall the arguments in Step 1 of the proof of (9.2.3). Then, in view of (9.2.2), we can define real-valued C^∞ functions $\tilde{x}=\tilde{x}(p)$ and $u=u(p)$ on P by

$$\begin{aligned}\exp(-\tilde{x}(p))&:=h_1(p,p) &&(p\in P)\,,\\ u(p)&:=\log\Big(\sum_{k=-n''}^{n'}\exp(k\tilde{x}(p))\Big) &&(p\in P)\,.\end{aligned}$$

We also have the corresponding mapping $\boldsymbol{m}\colon P\to M_{\boldsymbol{R}}(=\boldsymbol{R})$ as in (8.1) and moreover, it is given by

$$\boldsymbol{m}(p)=(\partial u/\partial\tilde{x})(p)\,,\quad p\in P\,.$$

Note that, for each $w\in W$, the image $\boldsymbol{m}(P_w)$ is the interior of the closed interval $\Sigma=[-n'', n']$.

DEFINITION 9.3.2. Let $Y^{(1)}$ (resp. $Y^{(2)}$) be a compact complex connected manifold on which G acts holomorphically and effectively with the corresponding fixed point set $D^{(1)}$ (resp. $D^{(2)}$). Furthermore, let $\{D_i^{(2)}\,|\,i\in I\}$ be the set of all connected components of $D^{(2)}$. Then a surjective G-equivariant morphism λ: $Y^{(1)}\to Y^{(2)}$ is called a *G-collapsing* if the following conditions are satisfied:

(1) λ maps $Y^{(1)}-D^{(1)}$ isomorphically onto $Y^{(2)}-D^{(2)}$.

(2) There exists a (possibly empty) subset J of I such that $\lambda\colon Y^{(1)}\to Y^{(2)}$ is the monoidal transformation of $Y^{(2)}$ with centre $\bigcup_{j\in J}D_j^{(2)}$. (If J is empty, then λ is nothing but an isomorphism of $Y^{(1)}$ onto $Y^{(2)}$.)

We now fix an arbitrary G-collapsing $\lambda\colon Y\to\tilde{Y}$ for Y above, and let n', n''

be respectively the (complex) codimension of $\lambda(D_0)$, $\lambda(D_\infty)$ in Y. Write G as $\{t \mid t \in \boldsymbol{C}^*\}$. Then, Theorem (9.2.3) allows us to obtain the following refinement of the integral formula of Koiso and Sakane [13] on Futaki invariants:

Theorem 9.3.3. *Put* $e := \dim_{\boldsymbol{C}} W$. *Writing for brevity* $K_{\tilde{Y}}^{-1}$ *as* $\tilde{L}$, *we have*:

(a) $$(r_{\tilde{L},\tilde{Y}})_*(t\partial/\partial t) = (2\pi)^{e+1}(e+1)\int_{-n''}^{n'} x(c_1(W)+x\,c_1(L_1))^e[W]\,dx\,.$$

Suppose now that $\tilde{Y}$ *is a Fano manifold, i.e.,* $\tilde{L}$ *is ample. Let* $F_{\tilde{Y}}|_{\mathrm{Lie}(G)}$ *be the restriction of* $F_{\tilde{Y}}\colon \mathfrak{X}(\tilde{Y}) \to \boldsymbol{R}$ *to* $\mathrm{Lie}(G)$ (*cf.* (5.1)). *Then*

(b) $$F_{\tilde{Y}}|_{\mathrm{Lie}(G)} = 0 \textit{ if and only if } \int_{-n''}^{n'} x(c_1(W)+x\,c_1(L_1))^e[W]\,dx = 0\,.$$

Proof. Note that $\mathcal{O}_Y(\lambda^*\tilde{L}) = \mathcal{O}_Y(K_{\tilde{Y}}^{-1}) \otimes \mathcal{O}_Y((n'-1)\,D_0 + (n''-1)\,D_\infty) = \mathcal{O}_Y((\zeta^* K_W^{-1}) \otimes L)$. Hence by (9.2.3) applied to $L' = K_W^{-1}$, the right-hand side of (a) is $(r_{\lambda^*\tilde{L},Y})_*(t\partial/\partial t)$. This together with (7.5.1) yields (a). Now, (b) is straightforward from (a) in view of (7.2.2) applied to $S=G$.

(9.4) Now, let Y be a q-dimensional compact complex connected manifold endowed with a holomorphic effective action of $G = (\boldsymbol{C}^*)^n$. Assume that there exists an ample line bundle L on Y with a fibrewise-linear holomorphic G-action which covers the action on Y. Then we have a Kähler form ω on Y representing $2\pi\, c_1(L)_{\boldsymbol{R}}$. Express ω as $\sqrt{-1}\sum g_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$ in terms of holomorphic local coordinates $(z^1, z^2, \cdots, z^q)$ on Y. Let $V_i \in \mathfrak{X}(Y)$ be the image of $t_i\partial/\partial t_i \in \mathrm{Lie}(G)$ under the natural inclusion $\mathrm{Lie}(G) \subset \mathfrak{X}(Y)$. Now, for each i, there exists a real-valued C^∞ function φ_i (which is unique up to an additive constant) such that

$$V_i = \textstyle\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}\, \partial_{\bar{\beta}}\, \varphi_i\, \partial/\partial z_\alpha \quad \text{(cf. Step 1 of the proof of (8.2))}\,.$$

For each $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in \boldsymbol{R}^n (= M_{\boldsymbol{R}})$, we define a mapping $\boldsymbol{m}^a\colon Y \to M_{\boldsymbol{R}}$ by

$$\boldsymbol{m}^a(y) = (-\varphi_1(y)+a_1, -\varphi_2(y)+a_2, \cdots, -\varphi_n(y)+a_n)\,, \quad y \in Y\,.$$

Then the image $\Sigma^a := \boldsymbol{m}^a(Y)$ is an n-dimensional compact convex polyhedron in $M_{\boldsymbol{R}}$ (cf. Atiyah [1]). Recall that the push-forward by $\boldsymbol{m}^a$ of the symplectic measure $(\omega/2\pi)^q$ is a piecewise polynomial measure, denoted by $d\mu$, on $M_{\boldsymbol{R}}$ of finite total volume $c_1(L)^q[Y]$ (cf. Duistermaat and Heckman [7], Atiyah and Bott [2]).

DEFINITION 9.4.1. Let $\boldsymbol{a}$ be the unique element of $M_{\boldsymbol{R}}$ such that

$$(2\pi)^q \int_{\Sigma^a} x_i\, d\mu = (r_L)_*(t_i\partial/\partial t_i)\,, \quad 1 \leqq i \leqq n\,,$$

where $(x_1, \cdots, x_n)$ is the system of standard coordinates on $M_{\boldsymbol{R}} (= \boldsymbol{R}^n)$. We then denote $\boldsymbol{m}^a$ by $\boldsymbol{m}$. Now, the mapping $\boldsymbol{m}$: $Y \to M_{\boldsymbol{R}}$ is called the *strict moment map* associated with the Hodge metric ω on Y. Note that, in view of Theorem (9.2.3), this $\mathbf{m}$ is compatible with the one defined in Appendix B.

REMARK 9.4.2. Suppose that the Kähler form ω represents $2\pi\, c_1(Y)_{\boldsymbol{R}}$. In this special case, one has the following fact (which I owe to a suggestion by A. Futaki): Let $\tilde{\omega}$ be the Kähler form on Y such that $\mathrm{Ric}(\tilde{\omega})=\omega$ and that $\tilde{\omega}$ is cohomologous to ω. Then the strict moment map $\boldsymbol{m}$: $Y \to M_{\boldsymbol{R}}(=\boldsymbol{R}^n)$ associated with ω is characterized by

$$\boldsymbol{m}(y) = (-\tilde{\varphi}_1(y), -\tilde{\varphi}_2(y), \cdots, -\tilde{\varphi}_n(y)), \quad y \in Y,$$

where each $\tilde{\varphi}_i$ is a real-valued C^∞ function on Y such that the following conditions are satisfied:

(a) $\tilde{\varphi}_i$ coincides with φ_i up to an additive constant;

(b) $\int_Y \tilde{\varphi}_i\, \tilde{\omega}^n = 0$.

10. Appendix D

In [22], Sakane constructed examples of Einstein-Kähler metrics on non-homogeneous Fano manifolds. Afterwards, these were reformulated and generalized by Koiso and Sakane [13; Theorem 4.2], where almost at the same time, the author found a very simple proof for their results. (A little later, Bando also obtained a similar proof independently.) Since this new proof has the advantage of describing Einstein-Kähler metrics very explicitly, we here explain the detail.

Assume now that $n=1$, i.e., $G=\boldsymbol{C}^*$. Let $\tilde{Y}$ be a compact complex connected manifold endowed with a holomorphic effective G-action such that the corresponding fixed point set consists of just two connected components $\tilde{D}_0$ and $\tilde{D}_\infty$. Furthermore, assume that $\tilde{Y}$ is of class $\mathcal{C}$, i.e., $\tilde{Y}$ is bimeromorphic to a compact Kähler manifold. Note that, via isotropy representation, our G-action on $\tilde{Y}$ naturally induces a G-action on the normal bundle $N(\tilde{D}_0 : \tilde{Y})$ (resp. $N(\tilde{D}_\infty : \tilde{Y})$) of $\tilde{D}_0$ (resp. $\tilde{D}_\infty$) in $\tilde{Y}$. We finally assume that each element of G acts on both $N(\tilde{D}_0 : \tilde{Y})$ and $N(\tilde{D}_\infty : \tilde{Y})$ as scalar multiplication of the vector bundles.

REMARK 10.1. Blow up $\tilde{Y}$ along $\tilde{D}_0$ and $\tilde{D}_\infty$. We then have a G-collapsing λ: $Y \to \tilde{Y}$ (cf. (9.3.2)) such that $D_0 := \lambda^{-1}(\tilde{D}_0)$ and $D_\infty := \lambda^{-1}(\tilde{D}_\infty)$ are nonsingular irreducible divisors on Y fixed by the G-action. Put $P := Y - (D_0 \cup D_\infty)$. Then by the generalized Bialynicki-Birula decomposition of Fujiki [8] (see also Fujiki [9; (6.10)], Carrell and Sommese [5]), we have a natural G-equivariant identification of $P \cup D_0$ (resp. $P \cup D_\infty$) with $N(D_0 : Y)$ (resp. $N(D_\infty : Y)$) (cf. [15]). Hence, by reversing the G-action, one obtains from $N(D_0 : Y)-$(zero section) the

$\boldsymbol{C}^*$-bundle $N(D_\infty\colon Y)$—(zero section) over $W\colon=P/\boldsymbol{C}^*\cong D_0\cong D_\infty$. There now exists a line bundle L_1 over W such that $L_1=N(D_0\colon Y)$ and that $L_1^{-1}=N(D_\infty\colon Y)$. Put $E\colon=\mathcal{O}_W\oplus L_1$. We can thus regard Y as $\boldsymbol{P}(E^*)$ and furthermore, exactly the same situation as in (9.3) happens. (Therefore, until the end of this appendix, we freely use the notation of (9.3).) Let $e\colon=\dim_{\boldsymbol{C}} Y-1$. Then by (b) of (9.3.3),

$$(10.1.1)\quad F_{\tilde{Y}}|_{\mathrm{Lie}(G)}=0 \textit{ if and only if } \int_{-n''}^{n'} x(c_1(W)+x\,c_1(L_1))^e[W]\,dx=0\,,$$

where n' and n'' are respectively the (complex) codimension of $\tilde{D}_0$ and $\tilde{D}_\infty$ in $\tilde{Y}$.

DEFINITION 10.2. For simplicity, put $\tilde{P}\colon=\lambda(P)$. Recall that every element of G acts on both $N(\tilde{D}_0\colon\tilde{Y})$ and $N(\tilde{D}_\infty\colon\tilde{Y})$ as scalar multiplication. Hence, applying again the generalized Bialynicki-Birula decomposition of Fujiki [8] (see also Fujiki [9; (6.10)]), we have a natural G-equivariant identification of $\tilde{P}\cup\tilde{D}_0$ (resp. $\tilde{P}\cup\tilde{D}_\infty$) with $N(\tilde{D}_0\colon\tilde{Y})$ (resp. $N(\tilde{D}_\infty\colon\tilde{Y})$). Now, let h be an arbitrary C^∞ Hermitian metric on L_1. Note that this h naturally induces a Hermitian metric, denoted by h^{-1}, on the dual bundle L_1^{-1} of L_1. In view of the identifications

$$(L_1\text{—(zero section)})=P\cong\tilde{P}=(N(\tilde{D}_0\colon\tilde{Y})\text{—(zero section)})$$

and

$$(L_1^{-1}\text{—(zero section)})=P\cong\tilde{P}=(N(\tilde{D}_\infty\colon\tilde{Y})\text{—(zero section)})\,,$$

the Hermitian norm $\|\ \ \|_h$ (resp. $\|\ \ \|_{h^{-1}}$) on L_1 (resp. L_1^{-1}) induces a norm on $N(\tilde{D}_0\colon\tilde{Y})$ (resp. $N(\tilde{D}_\infty\colon\tilde{Y})$). Then for a Kähler form ω on W, (h,ω) is said to be a *tight pair* if the following conditions are satisfied:

(1) The norms on $N(\tilde{D}_0\colon\tilde{Y})$ and $N(\tilde{D}_\infty\colon\tilde{Y})$ induced from h are those associated with some C^∞ Hermitian metrics of respective vector bundles.
(2) ω is an Einstein-Kähler form satisfying $\mathrm{Ric}(\omega)=\omega$.
(3) The eigenvalues of $c_1(L_1;h)$ with respect to ω are constant on W.
(4) $\lambda^{-1*}\{\rho^{2(n'-1)}(\zeta^*\omega)^e\wedge\partial\rho\wedge\bar\partial\rho\}$ (resp. $\lambda^{-1*}\{\tau^{2(n''-1)}(\zeta^*\omega)^e\wedge\partial\tau\wedge\bar\partial\tau\}$) on $\tilde{P}$ extends to a C^∞ (nonvanishing) $(e+1,e+1)$-form on $N(\tilde{D}_0\colon\tilde{Y})\,(=\tilde{P}\cup\tilde{D}_0)$ (resp. $N(\tilde{D}_\infty\colon\tilde{Y})\,(=\tilde{P}\cup\tilde{D}_\infty)$),

where $\zeta\colon Y(=\boldsymbol{P}(E^*))\to W$ is the natural projection and $\rho\colon L_1\to\boldsymbol{R}$ (resp. $\tau\colon L_1^{-1}\to\boldsymbol{R}$) denotes the norm function defined by $\rho(x)\colon=\|x\|_h$ (resp. $\tau(x)\colon=\|x\|_{h^{-1}}$) for x in L_1 (resp. L_1^{-1}). In particular, if $n'=n''=1$, then (h,ω) is a tight pair if and only if (2) and (3) are satisfied.

We shall now give a slight modification of the result of Koiso and Sakane [13; Theorem 4.2]:

Theorem 10.3. *Assume that $\tilde{Y}$ is a Fano manifold, i.e., $K_{\tilde{Y}}^{-1}$ is ample. If there exists a tight pair (h, ω), then the following are equivalent:*
(a) $F_{\tilde{Y}}|_{\mathrm{Lie}(G)}=0$;
(b) *$\tilde{Y}$ admits an Einstein-Kähler form.*

Proof. In view of (5.1), it suffices to show that (a) implies (b) under the assumption that (h, ω) as above exists. The proof consists of four steps.

Step 1. Let $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_e$ be the constant eigenvalues of $2\pi\, c_1(L_1; h)$ with respect to ω. Put $D:=n'D_0+n''D_\infty$ and $L:=\mathcal{O}_Y(D)$. Then $\lambda^* K_{\tilde{Y}}^{-1}=L\otimes\zeta^* K_{\tilde{W}}^{-1}$ (see the proof of (9.3.3)). Hence, via the identification of D_0 (resp. D_∞) with W, we have:

$$\lambda^* K_{\tilde{Y}}^{-1}|_{D_0} = L\otimes\zeta^* K_{\tilde{W}}^{-1}|_{D_0} = L_1^{\otimes n'}\otimes K_{\tilde{W}}^{-1}\,,$$
$$(\text{resp. } \lambda^* K_{\tilde{Y}}^{-1}|_{D_\infty} = (L_1^{-1})^{\otimes n''}\otimes K_{\tilde{W}}^{-1})\,.$$

Therefore, via the identification of W with D_0 (resp. D_∞), the cohomology class $n'c_1(L_1)_{\boldsymbol{R}}+c_1(W)_{\boldsymbol{R}}$ (resp. $-n''c_1(L_1)_{\boldsymbol{R}}+c_1(W)_{\boldsymbol{R}}$) in $H^2(D_0\colon \boldsymbol{R})$ (resp. $H^2(D_\infty\colon \boldsymbol{R})$) is represented by $\lambda^*\theta_0$ (resp. $\lambda^*\theta_\infty$) for some positive definite (1, 1)-form θ_0 (resp. θ_∞) on $\tilde{D}_0$ (resp. $\tilde{D}_\infty$). On the other hand, $2\pi c_1(W)_{\boldsymbol{R}}$ is represented by the Kähler form ω. We now have the following:

1) If $-n''<x<n'$, then $(\omega^e[W])\prod_{k=1}^{e}(1+\mu_k x)=\{2\pi\,(c_1(W)+xc_1(L_1))\}^e[W]>0$ and in particular $1+\mu_k x>0$ for all k.
2) The smallest nonnegative integer m such that $(c_1(W)+n'c_1(L_1))^{m+1}$ (resp. $(c_1(W)-n''c_1(L_1))^{m+1}$) is numerically trivial is $\dim_{\boldsymbol{C}}\tilde{D}_0$ (resp. $\dim_{\boldsymbol{C}}\tilde{D}_\infty$). Hence the order of zeroes of $\prod_{k=1}^{e}(1+\mu_k x)$ at $x=n'$ (resp. $x=-n''$) is $n'-1$ (resp. $n''-1$).

Step 2. Define a polynomial $A=A(x)$ in x by

$$A(x):=-\int_{-n''}^{x} s\prod_{k=1}^{e}(1+\mu_k s)\,ds\,.$$

Note that, by our condition (a), we have $A(n')=A(-n'')=0$ (cf. (10.1.1)). In view of 2) of Step 1, the order of zeroes of $A(x)$ at $x=n'$ (resp. $x=-n''$) is n' (resp. n''). Furthermore, by 1) of Step 1, both $0<A(x)\leqq A(0)$ and $A'(x)/x<0$ hold for all nonzero x with $-n''<x<n'$. In particular, the rational function $A'(x)/(xA(x))$ is free from poles and zeroes over the open interval $(-n'', n')$, and has a pole of order 1 at both $x=n'$ and $x=-n''$. Now,

$$B(x):=-\int_0^x A'(s)/(sA(s))\,ds$$

is monotone increasing over the interval $(-n'', n')$ and moreover, B maps $(-n'', n')$ diffeomorphically onto $\boldsymbol{R}$, because in a neighbourhood of $x=n'$ (resp. $x=$

$-n''$), $B(x)$ is written as $-\log(n'-x)+$ real analytic function (resp. $\log(x+n'')$ $+$ real analytic function). Let $B^{-1}\colon \boldsymbol{R}\to(-n'', n')$ be the inverse function of $B\colon(-n'', n')\to\boldsymbol{R}$, and define a real-valued C^∞ function $r=r(\tilde{p})$ on $\tilde{P}$ by

$$\exp(-r(\tilde{p})) = \{(\lambda^{-1*}\rho)(\tilde{p})\}^2 \,(= \{(\lambda^{-1*}\tau)(\tilde{p})\}^{-2}), \quad \tilde{p}\in\tilde{P}.$$

Note here that, since (h, ω) is a tight pair, (1) of (10.2) shows that $(\lambda^{-1*}\rho)^2$ (resp. $(\lambda^{-1*}\tau)^2$) extends to a C^∞ function on $\tilde{P}\cup\tilde{D}_0$ (resp. $\tilde{P}\cup\tilde{D}_\infty$). We now define a C^∞ function $x=x(r)$ in r by

$$x(r) := B^{-1}(r) \quad (\text{i.e., } r = B(x(r))).$$

Then $u(r) := -\log(A(x(r)))$ satisfies (cf. (10.3.1))

$$(*) \qquad u''(r)\textstyle\prod_{k=1}^{e}(1+\mu_k u'(r)) = \exp(-u(r)),$$

since we have the identities $x'(r)=-x(r)A(x(r))/A'(x(r))$, $u'(r)=x(r)$ and $A'(x(r))$ $=-x(r)\prod_{k=1}^{e}(1+\mu_k x(r))$.

Step 3. Now, let η be the $(e+1, e+1)$-form on $\tilde{P}$ defined by

$$\begin{aligned}\eta :&= \sqrt{-1}\,4(e+1)\exp(-u(r))\,\lambda^{-1*}((\zeta^*\omega)^e\wedge\partial\rho\wedge\bar\partial\rho/\rho^2)\\ &(= \sqrt{-1}\,4(e+1)\exp(-u(r))\,\lambda^{-1*}((\zeta^*\omega)^e\wedge\partial\tau\wedge\bar\partial\tau/\tau^2)).\end{aligned}$$

In this step, we shall show that η extends to a volume form on $\tilde{Y}$. First, in view of Step 2,

$$\begin{aligned}&r = -\log(n'-x(r)) + \text{real analytic function in } x(r),\\ &(\text{resp. } r = \log(n''+x(r)) + \text{real analytic function in } x(r)).\end{aligned}$$

Hence, $(\lambda^{-1*}\rho)^2$ (resp. $(\lambda^{-1*}\tau)^2$) is written as a real analytic function in $x(r)$ with a simple zero at $x(r)=n'$ (resp. $-n''$). On the other hand, Step 2 shows also that $\exp(-u(r))$ is a real analytic function in $x(r)$ with zeroes of order exactly n' (resp. n'') at $x(r)=n'$ (resp. $-n''$). Thus, in a neighbourhood of D_0 (resp. D_∞), $(\lambda^{-1*}\rho)^{-2n'}\exp(-u(r))$ (resp. $(\lambda^{-1*}\tau)^{-2n''}\exp(-u(r))$) is written as a nonvanishing real analytic function in $(\lambda^{-1*}\rho)^2$ (resp. $(\lambda^{-1*}\tau)^2$). Since (h, ω) is a tight pair, (4) of (10.2) now implies that η extends to a volume form on $\tilde{Y}$.

Step 4. Regarding η as a volume form on $\tilde{Y}$ (cf. Step 3), we shall finally show that $\tilde{\omega} := \sqrt{-1}\,\bar\partial\partial\log\eta$ is an Einstein-Kähler form on $\tilde{Y}$. Fix an arbitrary point w_0 of W. Then over a sufficiently small open neighbourhood U of w_0 in W, there exist a holomorphic local base σ for L_1 and a system $(z_1, z_2, \cdots, z_e)$ of holomorphic local coordinates on U such that

1) $h|_U=H(w)\sigma^*\otimes\bar\sigma^*$ for some positive real-valued C^∞ function $H=H(w)$ on U satisfying both $H(w_0)=1$ and $(dH)(w_0)=0$;

2) $\omega(w_0)=\sqrt{-1}\sum_{k=1}^{e} dz_k\wedge d\bar{z}_k$;
3) $(\bar{\partial}\partial H)(w_0)=\sqrt{-1}\sum_{k=1}^{e}\mu_k\, dz_k\wedge d\bar{z}_k$.

Via the identification

$$U\times \boldsymbol{C}^*\cong P|_U$$
$$(w,t)\leftrightarrow t\cdot\sigma(w)\,,$$

we regard $(z_1, z_2, \cdots, z_e, t)$ as a system of holomorphic local coordinates on the open subset $P|_U$ of Y. Then over $P|_U$,

$$\lambda^*\eta=\sqrt{-1}\,(e+1)\,\lambda^*(\exp(-u(r)))\,(\zeta^*\omega)^e\wedge dt\wedge d\bar{t}/|t|^2\,.$$

Note that $\mathrm{Ric}(\omega)=\sqrt{-1}\,\bar{\partial}\partial\log\omega^e=\omega$. Hence along the fibre P_{w_0},

$$\begin{aligned}\lambda^*\tilde{\omega}&=\sqrt{-1}\,\partial\bar{\partial}\,\lambda^*(u(r))+\zeta^*\omega\\&=\sqrt{-1}\,\lambda^*(u''(r))\,dt\wedge d\bar{t}/|t|^2+\sqrt{-1}\,\lambda^*(u'(r))\,\bar{\partial}\partial\log H+\zeta^*\omega\,,\end{aligned}$$

(see, for similar computations, Step 2 of the proof of (9.2.3)). Therefore, when restricted to $\lambda(P_{w_0})$, the (1, 1)-form $\tilde{\omega}$ is written in the form

$$\sqrt{-1}\,u''(r)\,\lambda^{-1*}(dt\wedge d\bar{t}/|t|^2)+\sqrt{-1}\sum_{k=1}^{e}(1+\mu_k u'(r))\,\lambda^{-1*}(dz_k\wedge d\bar{z}_k)\,,$$

which is positive definite in view of (*) of Step 2. Consequently, along $\lambda(P_{w_0})$, we can express $\tilde{\omega}^{e+1}$ as

$$\sqrt{-1}\,(e+1)\,u''(r)\,\{\prod_{k=1}^{e}(1+\mu_k u'(r))\}\,\lambda^{-1*}\{(\sum_{k=1}^{e}\sqrt{-1}\,dz_k\wedge d\bar{z}_k)^e\wedge dt\wedge d\bar{t}/|t|^2\}\,,$$

and hence $\tilde{\omega}^{e+1}=\eta$ (cf. (*) of Step 2). Since w_0 is an arbitrary point of W, we now have $\mathrm{Ric}(\tilde{\omega})=\tilde{\omega}$ everywhere on $\tilde{Y}$. Thus, $\tilde{\omega}$ is an Einstein-Kähler form on $\tilde{Y}$.

REMARK 10.3.1. Let $K\in\boldsymbol{R}_+$ and $\mu_k\in\boldsymbol{R}$ $(k=1, 2, \cdots, e)$. Furthermore, let a, b, c be real numbers such that $1+\mu_k c\neq 0$ for any k. Then, for a sufficiently small $\varepsilon>0$, we can here give a complete solution of the ordinary differential equation

$$y''(x)\prod_{k=1}^{e}(1+\mu_k y'(x))=K\exp(-y(x))\,,\quad a-\varepsilon<x<a+\varepsilon\,,\tag{1}$$

with the initial conditions

$$y(a)=b\quad\text{and}\quad y'(a)=c\,.$$

In order to solve this, we put $s:=y'(x)$ and $A:=\exp(-y(x))$. Since $y''(x)$ does not change its sign over the interval $(a-\varepsilon, a+\varepsilon)$, the inverse function theorem allows us to regard x as a C^∞ function $x(s)$ in s. Consequently, A is also regarded as a C^∞ function $A(s)$ in s. Then

$$A'(s)\,y''(x) = (dA/ds)\,(ds/dx) = dA/dx = -sA(s)\,.$$

In particular, multiplying both sides of (1) by $A'(s)/A(s)$, we have

$$-s\,\textstyle\prod_{k=1}^{e}(1+\mu_k\,s) = K\cdot A'(s)\,.$$

Thus, x and $y(x)$ are written in terms of the parameter s as follows:

$$y(x) = -\log A(s)\,, \tag{2}$$

where $A(s)$ is the polynomial $\exp(-b) - K^{-1}\int_c^s t\,\prod_{k=1}^{e}(1+\mu_k\,t)\,dt$ in s. As for x, we have

$$ds/dx = y''(x) = (\textstyle\prod_{k=1}^{e}(1+\mu_k\,s))^{-1}\,K\cdot A(s)\,, \quad \text{(cf. (1))}\,,$$

and therefore,

$$x = a + \int_c^s (\textstyle\prod_{k=1}^{e}(1+\mu_k\,t))\,K^{-1}A(t)^{-1}\,dt\,. \tag{3}$$

Now, $(x, y(x))$ moves along the curve parametrized by (2) and (3) above.

REMARK 10.3.2. We apply the above construction of Einstein-Kähler metrics to the case where $Y=\tilde{Y}=\boldsymbol{P}(E^*)$ with $E:=\mathcal{O}_W\oplus\mathcal{O}_W(k,-k)$ and $W:=\boldsymbol{P}^m(\boldsymbol{C})\times\boldsymbol{P}^m(\boldsymbol{C})$ ($m\in\boldsymbol{Z}_+$, $1\leqq k\leqq m$). Note that $L_1:=\mathcal{O}_W(k,-k)$ denotes the line bundle $pr_1^*\mathcal{O}_{\boldsymbol{P}^m}(k)\otimes pr_2^*\mathcal{O}_{\boldsymbol{P}^m}(-k)$ over W, where pr_i: $\boldsymbol{P}^m(\boldsymbol{C})\times\boldsymbol{P}^m(\boldsymbol{C})\to\boldsymbol{P}^m(\boldsymbol{C})$ is the natural projection to the i-th factor ($i=1, 2$). Now, let σ: $Q_0(\boldsymbol{C}^{m+1})\to\boldsymbol{C}^{m+1}$ be the blowing-up of $\boldsymbol{C}^{m+1}$ at the origin $\boldsymbol{0}=(0, \cdots, 0)$ of $\boldsymbol{C}^{m+1}$, and let

$$\begin{aligned} p\colon \boldsymbol{C}^{m+1}-\{\boldsymbol{0}\} &\to \boldsymbol{P}^m(\boldsymbol{C}) \\ (z_0, z_1, \cdots, z_m) &\mapsto (z_0 : z_1 : \cdots : z_m) \end{aligned}$$

be the natural projection. Then the rational map $p\circ\sigma$: $Q_0(\boldsymbol{C}^{m+1})\to\boldsymbol{P}^m(\boldsymbol{C})$ easily turns out to be a morphism, and via this morphism, we can regard $Q_0(\boldsymbol{C}^{m+1})$ as the line bundle $F:=\mathcal{O}_{\boldsymbol{P}^m}(-1)$ over $\boldsymbol{P}^m(\boldsymbol{C})$. Hence, via the identification of $\boldsymbol{C}^{m+1}-\{\boldsymbol{0}\}$ with $F-$(zero section), the function

$$\boldsymbol{C}^{m+1}-\{\boldsymbol{0}\}\ni(z_0, z_1, \cdots, z_m)\mapsto\sqrt{|z_0|^2+|z_1|^2+\cdots+|z_m|^2}\in\boldsymbol{R}$$

is viewed as a Hermitian norm of the line bundle F. Since $L_1=pr_1^*(F^{\otimes -k})\otimes pr_2^*(F^{\otimes k})$, this Hermitian norm on F induces a natural norm $\|\ \|_h$ on L_1 associated with a Hermitian metric h for L_1. We can now define ρ: $L_1\to\boldsymbol{R}$ by $\rho(l):=\|l\|_h$ ($l\in L_1$). Note moreover that the Fubini-Study form ω_0 on $\boldsymbol{P}^m(\boldsymbol{C})$ is defined by

$$p^*\omega_0 = \sqrt{-1}\,\partial\bar{\partial}\log(\textstyle\sum_{i=0}^{m}|z_i|^2)\,.$$

Then, $\omega:=(m+1)(pr_1^*\omega_0+pr_2^*\omega_0)$ is an Einstein-Kähler form on W such that (h, ω) is a tight pair (cf. (10.2)), because the eigenvalues $\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_{2m}$ of $2\pi\, c_1(L_1; h)$ with respect to ω are all constant. In fact, we have

$$-\mu_1=-\mu_2=\cdots=-\mu_m=\mu_{m+1}=\mu_{m+2}=\cdots=\mu_{2m}=k/(m+1)\,.$$

Recall that $G(:=\boldsymbol{C}^*)$ acts on the line bundle L_1 by scalar multiplication and that $Y(=\tilde{Y})$ is naturally a G-equivariant compactification of L_1 (cf. (9.3)). Now by

$$\int_{-1}^{1} v(c_1(W)+v\,c_1(L_1))^{2m}[W]\,dv=(c_1(W))^{2m}[W]\int_{-1}^{1} v(1-k^2\,v^2(m+1)^{-2})^m\,dv=0\,,$$

we have $F_Y|_{\mathrm{Lie}(G)}=0$. Hence we can find an Einstein-Kähler metric on Y as constructed in the proof of (10.3) (see also Sakane [22]). Let $A(s)$ be the polynomial in s defined by

$$A(s):=-\int_{-1}^{s} v(1-k^2\,v^2(m+1)^{-2})^m\,dv\,.$$

Furthermore, define a C^∞ function $x=x(\rho)$ in ρ by

$$\rho^2=\exp\left\{-\int_0^x (1-k^2\,s^2(m+1)^{-2})^m/A(s)\,ds\right\}.$$

Then $\eta:=\sqrt{-1}\,(8m+4)\,A(x(\rho))\,(\zeta^*\omega)^{2m}\wedge\partial\rho\wedge\bar{\partial}\rho/\rho^2$ extends to a volume form on Y, where $\zeta\colon L_1\to W$ denotes the natural projection (cf. Step 3 of the proof of (10.3)). Then in view of Step 4 of the proof of (10.3), we can now conclude that $\tilde{\omega}:=\sqrt{-1}\,\bar{\partial}\partial\log\eta$ is the Einstein-Kähler form on Y such that $\tilde{\omega}^{2m+1}=\eta$.

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Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan