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ON THE THEOREMS OF GASHÜTZ AND WILLEMS

Dedicated to Professor H. Tominaga on his 60th birthday

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1. Introduction

Let p be a prime and $F=\boldsymbol{Z}/(p)$. Let G be a finite group. By *a p-chief factor*, we mean a chief factor group $V=H/K$ which is a p-group, where $H\supset K$ are normal subgroups of G. Since V is elementary, it is regarded as an irreducible right FG-module. If V has a complement in G/K, then it is called complemented. Now let us fix a chief series of G;

$$E\colon 1=G_0\subset G_1\subset\cdots\subset G_n=G\,.$$

For an irreducible FG-module U we put

$$m(E,\,U)=\,|\{i;\,G_i/G_{i-1}\simeq U \text{ and } G_i/G_{i-1} \text{ is complemented.}\}\,|$$

Let J be the radical of FG and let e be a primitive idempotent of FG such that eFG/eJ is isomorphic to the trivial FG-module F. Recently it is shown that

Theorem 1 (Willems [3]). *If V is a complemented p-chief factor of G, then it appears as a component of eJ/eJ^2 with multiplicity at least $m(E,V)$.*

On the other hand the following result is known as a theorem of Gashütz (see [1] Theorem 15.5 and Remark 15.6).

Theorem 2. *Suppose that G is p-solvable. Then if U is a simple component of eJ/eJ^2, U must be isomorphic to a complemented p-chief factor of G and it appears exactly $m(E,U)$ times in eJ/eJ^2.*

All known proofs of Gashütz's theorem involve essentially cohomological arguments and take some efforts to understand. In this short note we shall give an elementary approach to both theorems, which will be very lucid as well.

2. Preliminary Lemmas

In this section we prove several Lemmas. Some of them are possibly known, but we give proofs for the completeness. The notation used in the introduction will be fixed throughout. In addition we use $I(G)$ to denote the augmentation ideal of FG. Note that $eI(G)=eJ$.

Lemma 1. *Let $G \triangleright H$. Then there exists an FG-epimorphism*

$$\alpha\colon I(H)/I(H)^2 \to eI(H)FG/eJI(H)\,,$$

where $I(H)/I(H)^2$ is considered as an FG-module by conjugation.

Proof. Define $\beta\colon I(H) \to eI(H)\,FG/eJI(H)$ by $\beta(h-1)=e(h-1)+eJI(H)$ for $h\in H$. We observe that for $h\in H$ and $g\in G$

(1) $e(h-1)^g=e(h-1)g+e(1-g)(h^g-1)$ and $e(1-g)(h^g-1)\in eJI(H)$,

(2) $eg(1-h)=e(1-h)+e(g-1)(1-h)$.

The first equality implies that β is an FG-homomorphism, while the second one implies that β is an epimorphism since $I(H)FG=FGI(H)$. Since Ker $\beta\supset I(H)^2$, β yields α as desired.

Lemma 2. *Let V be a minimal normal p-subgroup of G which is complemented. Then we have the following:*

(1) *For any primitive idempotent f of FG, $fI(V)FG\not\subset fJ^2$,*

(2) $eI(V)FG+eJ^2/eJ^2\simeq V$.

Proof. Let T be a complement of V in G; $G=TV$, $T\cap V=1$. We have that $FG=FT\oplus I(V)FT$ and $J=J_0\oplus I(V)FT$, where J_0 denotes the radical of FT. Nothing changes in the above statements if we replace f and e with their conjugates by units of FG. In particular we may assume that f, $e\in FT$ by the lifting idempotent theorem since $I(V)FT\subset J$ (see Landrock [2] Theorem 1.5).

We have $J^2=I(V)^2FT+I(V)J_0+J_0I(V)+J_0^2$. Note that $I(V)FT=FTI(V)\supset J_0I(V)$ and $I(V)FT\cap J_0=0$. So that if $fI(V)FG=fI(V)FT\subset fJ^2$, we get the following equality from the above:

$$fI(V)\,FT=fI(V)^2\,FT+fI(V)\,J_0+fJ_0\,I(V)\,.$$

Let t be any non-zero element in the socle of FTf. By multiplying both sides by t, we get $tI(V)\,FT=tI(V)^2\,FT+tI(V)\,J_0$, whence it follows that $tI(V)FT=tI(V)^2\,FT$ by Nakayama's Lemma. Furthermore since both of them are FV-modules, we get $tFTI(V)=0$ by Nakayama's Lemma again. This is impossible and hence $fI(V)\,FG\not\subset fJ^2$.

To show the second, recall that $V\simeq I(V)/I(V)^2$ (see Willems [3]). This, together with Lemma 1, yields an FG-epimorphism

$$\varphi\colon V\to eI(V)\,FG/eJI(V)\to eI(V)\,FG+eJ^2/eJ^2\,.$$

Since V is irreducible and $eI(V)FG+eJ^2/eJ^2$ is not zero from the above, φ must be an isomorphism. This completes the proof of Lemma 2.

Lemma 3. *Let V be a minimal normal p-subgroup of G. If $eI(V)\,FG \not\subset eJ^2$, then V is complemented.*

Proof. There exists an FG-submodule L of eJ such that

$$eJ/eJ^2 = L/eJ^2 \oplus eI(V)\,FG + eJ^2/eJ^2\,.$$

Let $T=\{x\in G;\ e(x-1)\in L\}$. We want to show that T is a complement of V. It is clear that T is a subgroup of G, since $e(xy-1)=e(x-1)\,y + e(y-1)$ for any x, $y\in G$. It is also clear that $T\not\supset V$ from the assumption. So it remains only to show that $G=TV$. Note that $FG=FV + \sum_x (x-1)\,FV$, where x runs through a set of coset representatives of V in G. It follows from this that $eI(V)\,FG=eI(V) + eJI(V)$. Therefore we have $eJ=eI(V)\,FG + L= eI(V) + L$. Let $x\in G$. Then $e(x-1)=ea + b$, where $a\in I(V)$ and $b\in L$. Since the isomorphism $V\simeq I(V)/I(V)^2$ is given by $v \mapsto (v-1) + I(V)^2$ for $v\in V$, we may write $a=(v-1) + u$ with $v\in V$ and $u\in I(V)^2$. Thus $e(x-v)\in L$ and we get $e(xv^{-1}-1)\in L$. This implies that $xv^{-1}\in T$ and so $G=TV$, as asserted.

Lemma 4. *Suppose that G is p-solvable and let $H\supset K$ be normal subgroups of G. If H/K is a p'-group, then $eI(H)\,FG=eI(K)\,FG$.*

Proof. If we put $\bar{G}=G/K$, then $I(H)\,FG/I(K)\,FG\simeq I(\bar{H})\,F\bar{G}$. So it is sufficient to show that $eI(H)\,FG=0$ by assuming $K=1$. Let M be a p-complement in G. Then we can take e as $e=\frac{1}{|M|}\sum_{x\in M} x$. Since $M\supset H$, we get $eI(H)=0$, as asserted.

3. Proof of the theorems

Using the notation in the introduction, we have an ascending series of right ideals;

$$0\subset\cdots\subset eI(G_{i-1})\,FG + eJ^2\subset eI(G_i)\,FG + eJ^2\subset\cdots\subset eI(G_n)\,FG = eFG\,.$$

If G_i/G_{i-1} is a complemented p-chief factor of $\bar{G}=G/G_{i-1}$, then we have by Lemma 2

$$eI(G_i)\,FG + eJ^2/eI(G_{i-1})\,FG + eJ^2\simeq\bar{e}I(\bar{G}_i)\,F\bar{G} + \bar{e}\bar{J}^2/\bar{e}\bar{J}^2\simeq G_i/G_{i-1}\,.$$

So the Theorem 1 is obvious, while the Theorem 2 is immediate from this and Lemmas 3 and 4.

References

[1] Huppert B. and Blackburn N.: Finite groups II, Springer-Verlag, Berlin/Heidelberg/New York, 1982.

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