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ON SMOOTH *SL*(2, *C*) ACTIONS ON 3-MANIFOLDS

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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1. Introduction

Real analytic actions of the special linear group $SL(n, F)$ on analytic manifolds M are classified by C.R. Schneider [6] in case that $n=2$, $F=R$ and M is any closed surface, and by F. Uchida [7–8] in case that M is the m-sphere S^m with $5\leqq n\leqq m\leqq 2n-2$ if $F=R$ and $14\leqq 2n\leqq m\leqq 4n-2$ if $F=C$.

In this paper we are concerned with smooth $SL(2, C)$-actions on closed 3-manifolds M^3. Note that $SL(2, C)$ is simple and contains $SU(2)$ as a maximal compact subgroup. Then we have the following (cf. [1; Th.1.3])

(1.1) *If $SL(2, C)$ acts non-trivially on M^3, then so does $SU(2)$ and M^3 is a quotient space of S^3 or $S^2\times S^1$; S^3/Z_n or $(S^2\times S^1)/Z_2$ (Z_n is a cyclic group of order n).*

By this reason we are concerned mainly for the case $M^3=S^3$, and we study the equivariant homeomorphism classes of such actions.

In case of transitive actions, we see the following

Theorem 1.2. *There are real analytic $SL(2, C)$-actions ϕ_r on S^3 for $r\in R$, which are not equivariantly homeomorphic to each other, and any transitive $SL(2, C)$-action on S^3 is equivariantly diffeomorphic to some ϕ_r (see (4.1) for the definition of ϕ_r).*

In case of non-transitive actions, the classification of $SL(2, C)$-action on S^3 can be reduced to that of pairs of a one-parameter transformation group on $S^1(\subset C)$ and a real valued smooth function on $S^1-\{\pm 1\}$; and further to that of triads of subsets A and B_i ($i=1, 2$) of S^1 satisfying

(1.3) (A1) *$A(\neq\phi)$ is a finite union of closed intervals, $A\cap J(A)=\phi$ and the components of A alternate with those of $J(A)$, where J is the reflections on S^1 in the real line.*
(A2) *$B_i(i=1, 2)$ are open in S^1 and $B_1\cup B_2\subset A-\partial A$.*

Such triads (A, B_i) and (A', B'_i) are called A-*equivalent if there is an orientation preserving homeomorphis Φ of S^1 onto itself such that $\Phi J=J\Phi$ and*

(1) $\Phi(A) = A'$, $\Phi(B_i) = B_i'$ *or* (2) $\Phi(A) = J(A')$, $\Phi(B_i) = J(B_{3-i}')$ $(i = 1, 2)$.

Theorem 1.4. *There is a one-to-one correspondence between the equivariant homeomorphism classes of non-transitive smooth $SL(2, C)$-actions on S^3 and the* A-*equivalence classes of triads with* (A1–2).

As the corollary to these theorems, we see the following

Corollary 1.5. (i) *Any smooth $SL(2, C)$-actions on S^3 has no fixed points, and it is transitive if and only if so is its restricted $SU(2)$-action.*

(ii) *There are infinitely many* (*non-equivalent*) *smooth $SL(2, C)$-actions on S^3 which are not equivariantly homeomorphic to any real analytic one.*

(iii) *Any real analytic $SL(2, C)$-action on S^3 has a finite* (*odd*) *number of orbits, and non-transitive real analytic ones are determined by the number of their orbits; among them the unique linear action has five orbits.*

After preparing some results on subalgebras of $\mathfrak{sl}(2, C)$ in §2 and subgroups of $SL(2, C)$ in §3, we prove Theorem 1.2 in §4. In §5 we recall some basic facts on smooth actions. The proof of Theorem 1.4 consists of two parts: We first study in §§6–8 the relation between the equivariant homeomorphism classes of our non-transitive actions and B-equivalence classes defined in (6.8) (see Theorem 8.1), and secondly that between A- and B-equivalence ones in §§9–11 (see Theorem 11.1).

2. Subalgebras of $\mathfrak{sl}(2, C)$

Let $\mathfrak{g}$ be a semi-simple Lie algebra over R, and let ad: $\mathfrak{g}\to GL(\mathfrak{g})$ and $B\colon \mathfrak{g}\times\mathfrak{g}\to R$ denote respectively the adjoint representation and the Killing form of $\mathfrak{g}$, i.e. $\mathrm{ad}(X)Y=[X,Y]$ and $B(X,Y)=\mathrm{Trace}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$ for $X,Y\in\mathfrak{g}$. Then we have a direct sum decomposition (a Cartan decomposition)

$$\mathfrak{g} = \mathfrak{k}+\mathfrak{p} \quad (\mathfrak{k} \text{ is a subalgebra, and } \mathfrak{p} \text{ is a subspace})$$

satisfying the following conditions (cf. [3; Ch. III, Prop. 7.4]).

(2.1) *Let $s\colon \mathfrak{g}\to\mathfrak{g}$ and $B_s\colon \mathfrak{g}\times\mathfrak{g}\to R$ be defined by*

$$s(X+Y) = X-Y(X\in\mathfrak{k},\ Y\in\mathfrak{p}),\ B_s(X, Y) = -B(X, s(Y))\ (X, Y\in\mathfrak{g}).$$

Then s is an involutive automorphism of $\mathfrak{g}$, and B_s is a positive definite symmetric bilinear form of $\mathfrak{g}$. Therefore

(i) $[\mathfrak{k}, \mathfrak{p}]=\mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}]=\mathfrak{k}$ *and*

(ii) $B_s(X, Y)=B_s(s(X), s(Y))$, $B_s(X, \mathrm{ad}(Y)Z)=-B_s(\mathrm{ad}(sY)X, Z)$ $(X, Y, Z\in\mathfrak{g})$.

In the followings, we consider the orthogonality with respect to B_s in (2.1).

Lemma 2.2. (i) *$\mathfrak{k}$ is orthogonal to $\mathfrak{p}$.*
(ii) *Any $Y\in\mathfrak{g}$ is orthogonal to* $\mathrm{ad}(X)Y$ *if $X\in\mathfrak{k}$.*

Proof. The lemma follows immediately from (2.1) (ii). *q.e.d.*

Let $\mathfrak{u}$ be a subalgebra of $\mathfrak{g}$, and set

(2.3) $\mathfrak{k}'=\mathfrak{k}\cap\mathfrak{u}$, $\mathfrak{p}'=\mathfrak{p}\cap\mathfrak{u}$ *and* $\mathfrak{m}$=*the orthogonal complement of* $\mathfrak{k}'+\mathfrak{p}'$ *in* $\mathfrak{u}$.

Hence $\mathfrak{u}=\mathfrak{k}'+\mathfrak{p}'+\mathfrak{m}$, where $\mathfrak{k}'$ and $\mathfrak{k}'+\mathfrak{p}'$ are s-invariant subalgebras of $\mathfrak{u}$.

Lemma 2.4. (i) *The projections p_1: $\mathfrak{g}\to\mathfrak{k}$ and p_2: $\mathfrak{g}\to\mathfrak{p}$ are injective on $\mathfrak{m}$.*
(ii) $\mathfrak{m}\subset\mathfrak{m}_1+\mathfrak{m}_2$ *for* $\mathfrak{m}_i=p_i(\mathfrak{m})$ $(i=1, 2)$, *and* $\mathfrak{k}'$ (*resp.* $\mathfrak{p}'$) *is orthogonal to* $\mathfrak{m}_1$ (*resp.* $\mathfrak{m}_2$).
(iii) $\mathfrak{m}$ *is* $\mathrm{ad}(\mathfrak{k}'+\mathfrak{p}')$-*invariant, and* $\mathfrak{m}_i$ *are* $\mathrm{ad}(\mathfrak{k}')$-*invariant.*

Proof. (i) follows from $\mathrm{Ker}\, p_i\cap\mathfrak{m}=\{0\}$ $(i=1, 2)$. (ii) $\mathfrak{m}\subset\mathfrak{m}_1+\mathfrak{m}_2$ is clear, and the latter half follows from Lemma 2.2 (i).

(iii) $\mathrm{ad}(sX)Y\in\mathfrak{k}'+\mathfrak{p}'$ holds for X, $Y\in\mathfrak{k}'+\mathfrak{p}'$, since $\mathfrak{k}'+\mathfrak{p}'$ is an s-invariant subalgebra. Then by (2.1) (ii)

$$B_s(Y, \mathrm{ad}(X)\, Z) = -B_s(\mathrm{ad}(sX)\; Y, Z) = 0 \quad \text{for any} \quad Z\in\mathfrak{m}\,.$$

This shows $\mathrm{ad}(X)\, Z\in\mathfrak{m}$, and hence $\mathrm{ad}(X)\,\mathfrak{m}\subset\mathfrak{m}$. Since $\mathrm{ad}(X)\, Z=\mathrm{ad}(X)\, Z_1+\mathrm{ad}(X)\, Z_2(Z_i=p_i(Z))$, we see that $\mathrm{ad}(X)\, Z_i\in\mathfrak{m}_i$ if $X\in\mathfrak{k}'$ by (2.1) (i), and thus $\mathfrak{m}_i$ are $\mathrm{ad}(\mathfrak{k}')$-invariant. *q.e.d.*

In the rest of this section we consider the case that

$$\mathfrak{g} = \mathfrak{sl}(2, C) = \{X\in M(2, C);\ \mathrm{Trace}\ X = 0\}$$

with the bracket operation $[X, Y]=XY-YX$, which is the Lie algebra of $G=SL(2, C)$. This has a R-basis

$$K_1=\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, K_2=\frac{i}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, K_3=\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, H_j=iK_j(j=1, 2, 3)\,.$$

These satisfy the following relations:

$$\text{(2.5)}\qquad \begin{aligned} &-[K_a, K_b] = K_c = [H_a, H_b],\ -[K_a, H_b] = H_c = [K_b, H_a] \quad \textit{and} \\ &[K_a, H_a] = 0 \quad \textit{for} \quad (a, b, c) = (1, 2, 3), (2, 3, 1)\ \textit{and}\ (3, 1, 2)\,. \end{aligned}$$

Now we have a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ by setting

$$\text{(2.6)}\qquad \begin{aligned} &\mathfrak{k} = \mathfrak{su}(2) = \{X\in\mathfrak{g};\, X+X^* = 0\} = \langle K_1, K_2, K_3\rangle \quad \textit{and} \\ &\mathfrak{p} = \{X\in\mathfrak{g};\, X = X^*\} = \langle H_1, H_2, H_3\rangle\,, \end{aligned}$$

where $\langle\ \ \rangle$ denotes a R-vector space spanned by the elements in the angle bracket. By (2.5–6) we see immediately the followings

(2.7) (i) $[\sum_{i=1}^3 a_i H_i, \sum_{i=1}^3 b_i H_i] = -[\sum_{i=1}^3 a_i K_i, \sum_{i=1}^3 b_i K_i]$
$= (a_2 b_3 - a_3 b_2) K_1 + (a_3 b_1 - a_1 b_3) K_2 + (a_1 b_2 - a_2 b_1) K_3$.
(ii) $\{K_i, H_i; 1 \leqq i \leqq 3\}$ *is an orthogonal basis of* $\mathfrak{g}$.

Let Ad be the adjoint representation of G,

$$\mathrm{Ad}\colon G \to GL(\mathfrak{g}) \quad \text{given by} \quad \mathrm{Ad}(g)\, X = gXg^{-1} \quad (g \in G, X \in \mathfrak{g}).$$

We say that subalgebras $\mathfrak{u}$ and $\mathfrak{u}'$ are conjugate in G if $\mathrm{Ad}(g)\,\mathfrak{u} = \mathfrak{u}'$ for some $g \in G$. We notice the following

(2.8) *For any* $X \in \mathfrak{k}$ (*resp.* $X \in \mathfrak{p}$), *there exists* $g \in K$ *such that* $\mathrm{Ad}(g)\, X \in \langle K_1 \rangle$ (*resp.* $\mathrm{Ad}(g)\, X \in \langle H_1 \rangle$).

We prove the following Lemmas 2.9–11 under the condition:

(C1) $\mathfrak{u}$ *is a proper subalgebra of* $\mathfrak{g}$ *with* $\dim \mathfrak{u} \geqq 3$, *and* $\mathfrak{k}'$, $\mathfrak{p}'$ *and* $\mathfrak{m}$ *are given by* (2.3) *for* $\mathfrak{k}$ *and* $\mathfrak{p}$ *in* (2.6).

Lemma 2.9. (i) $\dim \mathfrak{k}' \neq 2$. *If* $\dim \mathfrak{k}' = 3$, *then* $\mathfrak{u} = \mathfrak{k}$.
(ii) $\dim \mathfrak{p}' \leqq 2$. *If* $\dim \mathfrak{p}' = 2$, *then* $\dim \mathfrak{k}' = 1$.
(iii) $1 \neq \dim \mathfrak{m} \leqq 3$. *If* $\dim \mathfrak{m} = 3$, *then* $\mathfrak{k}' + \mathfrak{p}' = \{0\}$.

Proof. (i) The first half follows from (2.7) (i). Assume that $\dim \mathfrak{k}' = 3$, i.e. $\mathfrak{k}' = \mathfrak{k}$. By (2.5) we see that $[\mathfrak{k}, X] = \mathfrak{p}$ for $0 \neq X \in \mathfrak{p}$, whence $\mathfrak{p}' = \{0\}$ by (C1). Furthermore $\mathfrak{m} = \{0\}$ follows from $\mathfrak{m}_1 = \{0\}$ and Lemma 2.4 (i). Therefore $\mathfrak{u} = \mathfrak{k}$.
(ii) Assume that $\dim \mathfrak{p}' = 3$, i.e. $\mathfrak{p}' = \mathfrak{p}$. Then (2.7) (i) implies $\mathfrak{u} = \mathfrak{g}$, and this is contrary to (C1). Thus $\dim \mathfrak{p}' \leqq 2$.
If $\dim \mathfrak{p}' = 2$, then $\dim \mathfrak{k}' \geqq 1$ by (2.7) (i), and the desired result follows from (i).
(iii) By Lemma 2.4 (i)–(ii), $\mathfrak{k}' + \mathfrak{p}' + \mathfrak{m}_1 + \mathfrak{m}_2 (\subset \mathfrak{g})$ is a direct sum with $\dim \mathfrak{m}_i = \dim \mathfrak{m} (i = 1, 2)$. Hence $\dim \mathfrak{m} \leqq 3$, and $\mathfrak{k}' + \mathfrak{g}' = \{0\}$ if $\dim \mathfrak{m} = 3$.
Assume that $\dim \mathfrak{m} = 1$, and set $\mathfrak{m} = \langle X \rangle$. Let $Y \in \mathfrak{k}'$. Then $\mathrm{ad}(Y) X \in \mathfrak{m}$ is orthogonal to X by Lemmas 2.2 (ii) and 2.4 (iii), whence $\mathrm{ad}(Y) X = 0$. So

$$0 = \mathrm{ad}(Y) X = \mathrm{ad}(Y) X_1 + \mathrm{ad}(Y) X_2 \quad \text{for} \quad 0 \neq X_i = p_i(X) \in \mathfrak{m}_i \ (i = 1, 2).$$

Here $\mathrm{ad}(Y) X_i \in \mathfrak{m}_i$ by Lemma 2.4 (iii). Hence $[Y, X_1] = 0$, and further Y is orthogonal to $X_1 \neq 0$ by Lemma 2.4 (ii). These imply $Y = 0$ by (2.7) (i), and thus $\mathfrak{k}' = \{0\}$. From (ii) we see that $\dim \mathfrak{p}' \leqq 1$, and so $\dim \mathfrak{u} = \dim \mathfrak{p}' + \dim \mathfrak{m} \leqq 2$. This is contrary to (C1), and thus we obtain $\dim \mathfrak{m} \neq 1$. *q.e.d.*

Lemma 2.10. *If* $\dim \mathfrak{m} = 3$, *then* $\mathfrak{u}$ *is conjugate to* $\langle K_1 + rH_1, K_2 - H_3, K_3 + H_2 \rangle$ *for some* $0 \neq r \in R$.

Proof. By Lemma 2.9 (iii) and $\mathfrak{m}_1=\mathfrak{k}$, we may set

$$\mathfrak{n}=\mathfrak{m}=\langle X_1, X_2, X_3\rangle \quad \text{for} \quad X_j=K_j+\textstyle\sum_{i=1}^3 a_{ij}H_i \ (j=1,2,3).$$

Put $A=(a_{ij})$. Then $a=\det A\neq 0$, because $\mathfrak{k}'=\{0\}$. Also $\operatorname{rank}\begin{pmatrix} E & -E+\Delta \\ A & {}^tA-rE\end{pmatrix}=3$ by (2.5) and (2.7) (i), where $r=\operatorname{Trace} A$ and Δ is the matrix with ${}^t\Delta A=A{}^t\Delta=aE$. Then

$$(*) \qquad B+rE=A+{}^tA \quad \text{and} \quad B={}^tB \quad \text{for} \quad B=A\Delta.$$

Thus we get

$$(r-a)(\Delta-{}^t\Delta)=(B+rE)\Delta-{}^t\Delta(B+rE)=(A+{}^tA)\Delta-{}^t\Delta(A+{}^tA)=0.$$

If $\Delta={}^t\Delta$, then $A={}^tA$ and so $(a+r)E=2A$ by (*), which is contrary to $r=\operatorname{Trace} A$ and $0\neq a=\det A$. Therefore $r=a$.

Also by (*), it holds

$$r\Delta+r\,{}^t\Delta={}^t\Delta(B+rE)=rE+B \quad \text{and} \quad \operatorname{Trace} B=-r.$$

Then $\operatorname{Trace}\Delta=1$, and so the eigen-polynomial of A is

$$(**) \qquad \begin{aligned}\det(A-xE)&=-x^3+(\operatorname{Trace} A)x^2-(\operatorname{Trace}\Delta)x+\det A\\ &=(-x+r)(x^2+1).\end{aligned}$$

Put $C=\Delta A$. Then $BC=r^2E$ and $\operatorname{Trace} C=-r$. Similarly to (**), the eigen-polynomial of B is $\det(B-xE)=(-x+r)(x+r)^2$. This implies $B=C$, because $BC=r^2E$ and B is symmetric. Therefore $A\Delta=\Delta A$, and hence $A{}^tA={}^tAA$.

Since A is normal with eigen-polynomial (**), we have $PA{}^tP=\begin{pmatrix} r & 0\\ 0 & J\end{pmatrix}$ ($J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$) for some $P\in SO(3)$. Further $\operatorname{Ad}(g)=P\in GL(\mathfrak{p})$ for some $g\in K$, because the adjoint representation $\operatorname{Ad}\colon K\to SO(3)$ ($\subset GL(\mathfrak{p})$) is epimorphic; and so $\operatorname{Ad}(g)\langle X_1, X_2, X_3\rangle=\langle K_1+rH_1, K_2-H_3, K_3+H_2\rangle$ as desired. *q.e.d.*

Lemma 2.11. *If* $\dim\mathfrak{k}'=0$, $\dim\mathfrak{p}'=1$ *and* $\dim\mathfrak{m}=2$, *then* $\mathfrak{n}$ *is conjugate to* $\langle H_1, K_2-H_3, K_3+H_2\rangle$.

Proof. We may assume $\mathfrak{p}'=\langle H_1\rangle$ by (2.8). Then $\mathfrak{m}_2=\langle H_2, H_3\rangle$ by Lemma 2.4 (ii) and (2.7) (ii), and we set

$$\mathfrak{m}=\langle X_1, X_2\rangle \quad \text{for} \quad X_j=\textstyle\sum_{i=1}^3 a_{ij}K_i+H_{j+1} \quad (j=1,2).$$

Put $Y_j=[H_1, X_j]$ $(j=1,2)$. Then by (2.5)

$$\begin{aligned}&Y_j=a_{3j}H_2-a_{2j}H_3-(-1)^jK_{4-j} \quad \text{and}\\ &[H_1, Y_j]=\textstyle\sum_{i=2}^3 a_{ij}K_i+H_{j+1} \quad (j=1,2),\end{aligned}$$

which are in $\mathfrak{m}$ by Lemma 2.4 (iii). Then $a_{1j}=0$ by $X_j-[H_1, Y_j]\in\mathfrak{k}'=\{0\}$.

Put $A=(a_{i+1\,j})\,(i,j=1,2)$, and $a=\det A$, $r=\mathrm{Trace}\ A$. Then $a=1$, because $\mathfrak{u}\ni[Y_1, Y_2]=(a-1)K_1+rH_1$. Also we see rank $\begin{pmatrix} A & E \\ E & -{}^tA^{-1}\end{pmatrix}=2$. Thus $E+A{}^tA^{-1}=0$, and hence $r=0$. Therefore the eigen-polynomial of A is $\det(A-xE)=x^2+1$, and this implies the lemma by the same proof as that of the above lemma. *q.e.d.*

Proposition 2.12. *Let* $\mathfrak{u}$ *be a proper subalgebra with* $\dim\mathfrak{u}\geqq 3$. *If* $\mathfrak{k}'=\{0\}$, *then* $\mathfrak{u}$ *is conjugate to*

$$\mathfrak{w}_r=\langle rK_1+H_1, K_2-H_3, K_3+H_2\rangle \quad \textit{for some} \quad r\in R\,.$$

Proof. By Lemma 2.9 (ii) the assumption implies $\dim\mathfrak{p}'\leqq 1$ and $\dim\mathfrak{m}=\dim\mathfrak{u}-\dim\mathfrak{p}'\geqq 2$. Then, by Lemma 2.9 (iii) we see $\dim\mathfrak{m}=3$ or $\dim\mathfrak{m}=2$, $\dim\mathfrak{p}'=1$. Thus the proposition follows from Lemmas 2.10–11. *q.e.d.*

In the next place we prove the following lemma, when

(C2) $\mathfrak{u}$ *in* (C1) *satisfies* $\mathfrak{k}'=\langle K_1\rangle$.

Lemma 2.13. (i) $\mathfrak{u}=\langle K_1, H_2, H_3\rangle$ *if* $\dim\mathfrak{p}'=2$.

(ii) $\mathfrak{u}=\mathfrak{u}_r$ *for some* $0\neq r\in R$ *if* $\dim\mathfrak{p}'=0$, *and* $\mathfrak{v}_\varepsilon$ *for some* $\varepsilon=\pm 1$ *if* $\dim\mathfrak{p}'=1$, *where*

$$\mathfrak{u}_r=\langle K_1, rK_2-H_3, rK_3+H_2\rangle \quad \textit{and} \tag{2.14}$$
$$\mathfrak{v}_\varepsilon=\langle K_1, H_1, K_2-\varepsilon H_3, K_3+\varepsilon H_2\rangle\,.$$

Proof. (i) We see $\mathfrak{m}=\{0\}$ by Lemmas 2.4 (i) and 2.9 (iii). Put $\mathfrak{p}'=\langle X, Y\rangle$. Then $0\neq[X, Y]\in\mathfrak{k}'=\langle K_1\rangle$ by (2.7) (i). Also from (2.7) (i), this implies $X, Y\in\langle H_2, H_3\rangle$. Thus $\mathfrak{p}'=\langle H_2, H_3\rangle$, and (i) holds.

(ii) Assume $\dim\mathfrak{p}'\leqq 1$. Then $\dim\mathfrak{m}=\dim\mathfrak{u}-(\dim\mathfrak{k}'+\dim\mathfrak{p}')\geqq 1$, and Lemma 2.9 (iii) shows $\dim\mathfrak{m}=2$. Hence $\mathfrak{m}_1=\langle K_2, K_3\rangle$ by Lemma 2.4 (ii) and (2.7) (ii), and we set

$$\mathfrak{m}=\langle X, Y\rangle \quad \text{for} \quad X=K_2+X',\ Y=K_3+Y' \quad (0\neq X', Y'\in\mathfrak{m}_2\subset\mathfrak{p})\,.$$

Here we have $X'=[K_1, Y']$ and $Y'=-[K_1, X']$, since $[K_1, X]=-K_3+[K_1, X']$, $[K_1, Y]=K_2+[K_1, Y']\in\mathfrak{m}$ and $[K_1, X']$, $[K_1, Y']\in\mathfrak{m}_2$ by Lemma 2.4 (iii). By (2.5) these imply

$$X'=aH_2+bH_3 \quad \text{and} \quad Y'=-bH_2+aH_3 \quad \text{for some} \quad a, b\in R\,,$$

which are orthogonal by Lemma 2.2 (ii).

If $\dim\mathfrak{p}'=0$, then $aH_1\in\mathfrak{p}'=\{0\}$ from $[X, Y]=(a^2+b^2-1)K_1-2aH_1\in\mathfrak{u}$, and hence $a=0$.

If $\dim \mathfrak{p}'=1$, then $\mathfrak{m}_2=\langle H_2, H_3\rangle$ because X' and Y' are orthogonal in $\langle H_2, H_3\rangle$. Hence $\mathfrak{p}'=\langle H_1\rangle$ by Lemma 2.4 (ii) and (2.7) (ii). Thus $\langle X, Y\rangle=\mathfrak{m}\ni[H_1, X]=-bK_2+aK_3-H_3$, and so $-bX'+aY'=-H_3$. This shows $a=0$ and $b^2=1$, as desired. *q.e.d.*

Proposition 2.15. *Let* $\mathfrak{u}$ *be a proper subalgebra of* $\mathfrak{g}$ *with* $\dim \mathfrak{u}\geqq 3$. *If* $K_1 \in\mathfrak{u}$, *then* $\mathfrak{u}$ *is* $\mathfrak{k}=\langle K_1, K_2, K_3\rangle$, $\mathfrak{u}_r$ $(r\in R)$ *or* $\mathfrak{v}_\varepsilon$ $(\varepsilon=\pm 1)$ *given in* (2.14). *Here* $\mathfrak{u}_r$ *is conjugate to* $\mathfrak{k}$, $\mathfrak{u}_0$ *and* $\mathfrak{u}_1$ *if* $|r|>1$, $|r|<1$ *and* $|r|=1$, *respectively, and further* $\mathfrak{v}_1$ *is conjugate to* $\mathfrak{v}_{-1}$.

Proof. The first half follows from Lemmas 2.9 (i) and 2.13. By the elements $h=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ and $g_r=\begin{pmatrix} z & 0 \\ 0 & 1/z\end{pmatrix}$ $(z^4=(1+r)^2/|1-r^2|, r^2\neq 1)$ in G, we see that

$$\mathrm{Ad}(h)(\mathfrak{u}_1)=\mathfrak{u}_{-1},\quad \mathrm{Ad}(h)(\mathfrak{v}_1)=\mathfrak{v}_{-1}\quad\text{and}$$
$$\mathrm{Ad}(g_r)(\mathfrak{u}_r)=\mathfrak{k}\quad\text{if}\quad |r|>1,\quad\text{and}\quad \mathfrak{u}_0\quad\text{if}\quad |r|<1\,. \qquad q.e.d.$$

Corollary 2.16. *Any proper subalgebra* $\mathfrak{u}$ *of* $\mathfrak{g}$ *with* $\dim \mathfrak{u}\geqq 3$ *is conjugate to one of*

$$\mathfrak{v}_1=\langle K_1, H_1, K_2-H_3, K_3+H_2\rangle,\ \mathfrak{w}_r=\langle rK_1+H_1, K_2-H_3, K_3+H_2\rangle\ (r\in R)\,,$$
$$\mathfrak{k}=\langle K_1, K_2, K_3\rangle,\ \mathfrak{u}_0=\langle K_1, H_2, H_3\rangle\quad\textit{and}\quad \mathfrak{u}_1=\langle K_1, K_2-H_3, K_3+H_2\rangle\,,$$

and these subalgebras are not conjugate to each other in G.

Proof. The first half follows from Propositions 2.12 and 2.15. Consider the map d: $\mathfrak{g}\ni X\to\det X\in C$, which is $\mathrm{Ad}(G)$-invariant. By routine calculations we have

$$d(\mathfrak{w}_r)=R^+\langle-(1-ri)^2\rangle,\ d(\mathfrak{k})=d(\mathfrak{u}_1)=R^+\quad\text{and}\quad d(\mathfrak{u}_0)=R\,.$$

Furthermore the Killing form of $\mathfrak{g}$, which is also $\mathrm{Ad}(G)$-invariant, is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. These observations show the second half. *q.e.d.*

3. Subgroups and coset spaces of $SL(2, C)$

In this section we prepare some results on connected subgroups and coset spaces of $SL(2, C)$.

Consider the following subalgebras of $\mathfrak{g}=\mathfrak{sl}(2, C)$ and connected subgroups of $G=SL(2, C)$,

$$\mathfrak{v}(a)=\{\begin{pmatrix} ax & 0 \\ z & -ax\end{pmatrix};\ x\in R,\ z\in C\}\subset\mathfrak{g}\quad\text{and}$$

$$V(a)=\{\begin{pmatrix} \exp(ax) & 0 \\ z & \exp(-ax)\end{pmatrix};\ x\in R,\ z\in C\}\subset G\quad\text{for}\quad a\in C\,.$$

Then $\mathfrak{v}(a)$ is the Lie algebra of $V(a)$. Now we set

(3.1) $W_r=V(r\boldsymbol{i}-1)$ *and* $U_r=\{g\in G; gI_r g^*=I_r\}$ *for* $r\in R$, *where* $I_r=\begin{pmatrix} r-1 & 0 \\ 0 & r+1\end{pmatrix}$.
Here $U_r=\left\{\begin{pmatrix} z & (r-1)w \\ -(r+1)\bar{w} & \bar{z}\end{pmatrix}; |z|^2+(r^2-1)|w|^2=1\right\}$ *is connected, and* $U_1=V(\boldsymbol{i})$, $U_{-1}={}^tV(\boldsymbol{i})$.

Lemma 3.2. *The subalgebras* $\mathfrak{w}_r$ *in Proposition* 2.12 *and* $\mathfrak{u}_r$ *in* (2.14) *are the Lie algebras of* W_r *and* U_r, *respectively.*

Proof. The lemma holds for $\mathfrak{w}_r=\mathfrak{v}(r\boldsymbol{i}-1)$, $\mathfrak{u}_1=\mathfrak{v}(\boldsymbol{i})$ and $\mathfrak{u}_{-1}={}^t\mathfrak{v}(\boldsymbol{i})$. By definition we see that $X\in\mathfrak{u}_r$ if and only if Trace $X=0$ and $XI_r+I_rX^*=0$. When $r^2\neq 1$, these are equivalent to

$$\det(\exp tX)=1 \quad\text{and}\quad \exp(-tX^*)=I_r^{-1}(\exp tX)\,I_r \quad\text{for any}\quad t\in R\,.$$

This shows $\exp tX\in U_r$ for any $t\in R$, and thus the lemma for $\mathfrak{u}_r(r^2\neq 1)$ also holds. *q.e.d.*

Lemma 3.3. *The coset space* G/U_r *is homeomorphic to*

$$R^3 \quad \text{if} \quad |r|>1, \quad \text{and} \quad S^2\times R \quad \text{if} \quad |r|\leqq 1\,.$$

Proof. By Proposition 2.15 and Lemma 3.2, we see that U_r is conjugate to $K=SU(2)$ if $|r|>1$, U_0 if $|r|<1$, and U_1 if $|r|=1$. Thus it is sufficient to show

$$\text{(i) } G/K\approx R^3 \quad \text{and} \quad \text{(ii) } G/U_r\approx S^2\times R\,(r=0, 1)\,.$$

(i) holds, because K is a maximal compact subgroup of G with dim $G-$ dim $K=3$ (cf. [3; Ch. VI, Th. 2.2 (iii)]).

(ii) Consider the quotient space $N_1=C^2-\{0\}/S^1$ $(S^1\subset C)$ and the G-action

$$\phi_1\colon G\times N_1\to N_1,\ \phi_1(X, [P])=[XP]\ (X\in G,\ P\in C^2-\{0\})\,.$$

Then this action is transitive, and U_1 is the isotropy subgroup at $\begin{bmatrix} 0 \\ 1\end{bmatrix}\in N_1$. Hence $G/U_1\approx N_1\approx S^2\times R$.

Consider the quotient space $N_2=H^-/R^+$ of $H^-=\{P\in M(2, C); 0\neq P=P^*, \det P<0\}$ and the G-action

$$\phi_2\colon G\times N_2\to N_2,\ \phi_2(X, [P])=[XPX^*] \quad (X\in G,\ P\in H^-)\,.$$

Then we see that ϕ_2 is transitive, and U_0 is the isotropy subgroup at $[I_0]\in N_2$ $\left(I_0=\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\right)$. Hence $G/U_0\approx N_2$. Moreover the mapping $N_2\to N_3=((R\times C)-\{0\}/R^+)\times R\approx S^2\times R$, sending $\begin{bmatrix} x & z \\ \bar{z} & y\end{bmatrix}\in N_2$ to $([(x-y)/2, z], (x+y)/2s)\in N_3$ $(s=(|z|^2-xy)^{1/2})$, is homeomorphic. Thus $G/U_0\approx S^2\times R$. *q.e.d.*

Lemma 3.4. *$G=KLU_r=KLK$ $(r\in R)$ for* $L=\{\begin{pmatrix} x & 0 \\ 0 & 1/x\end{pmatrix};\ x>0\}$.

Proof. $G=KLU_{\pm 1}$ is obtained by the Gram-Schmidt orthonormalization process.

Assume $r^2\neq 1$. Then $gI_r g^*(g\in G)$ is a Hermitian matrix, and its eigenvalues are positive if $r>1$, negative if $r<-1$, and their product is negative if $r^2<1$. Hence we can find $k\in K$ such that $kgI_r g^*k^*$ is diagonal and $I_r^{-1}kgI_r g^* k^*\in L$. Put

$$I_r^{-1}kgI_r g^*k^* = l^{-2} \quad \text{for} \quad l\in L, \quad \text{and} \quad u = l\,kg\,.$$

Then $u\in U_r$, and so the decomposition $G=KLU_r$ holds.

Also we have $G=KLK$ by setting $I_r=E$ in the above proof. *q.e.d.*

4. Transitive actions

In this section we state an immediate consequence of the previous sections, and prove Theorem 1.2.

For each $r\in R$, consider the analytic $G=SL(2,C)$-action on $S^3=C^2-\{0\}/R^+$ defined by

$$(4.1) \quad \phi_r(X,[P]) = [\exp(ir\log(||XP||/||P||))\,XP] \quad (X\in G, P\in C^2-\{0\})\,.$$

Then we have the following lemma.

Lemma 4.2. *The action ϕ_r is transitive, and its isotropy subgroup is conjugate to W_r of* (3.1).

Proof. The restricted $SU(2)$-action of ϕ_r is transitive, and hence so is ϕ_r. Further W_r is the isotropy subgroup of ϕ_r at $\begin{bmatrix} 0 \\ 1\end{bmatrix}\in S^3$. Thus the lemma holds. *q.e.d.*

Proof of Theorem 1.2. The equivariant homeomorphism classes of transitive G-actions on S^3 is classified by the conjugacy classes of connected subgroups U of G with $G/U\approx S^3$. Therefore the theorem follows from Corollary 2.16 and Lemmas 3.2, 3.3 and 4.2. *q.e.d.*

5. Smooth actions

In this section let G be a Lie group, M be a smooth manifold, and assume that there is a smooth G-action $\phi\colon G\times M\to M$ on M. Denote by $\mathfrak{g}$ and $\mathfrak{X}(M)$ the Lie algebras of G and smooth vector fields on M, respectively. The following result is known (cf. [5; Ch. II, Th. II]).

(5.1) *The map* $\phi^+\colon \mathfrak{g}\to\mathfrak{X}(M)$, *given by*

$$\phi^+(X)_p h = \lim_{t\to 0} \{h(\phi(\exp(-tX), p)) - h(p)\}/t \quad (X \in \mathfrak{g})$$

for any smooth function h around $p \in M$, is a Lie algebra homomorphism.

In case that $\mathfrak{g}$ is simple and ϕ is non-trivial, ϕ^+ is monomorphic, and so $\mathfrak{g}$ may be regarded as a subalgebra of $\mathfrak{X}(M)$ by identifying $X = \phi^+(X)$.

We call $\mathfrak{g}_p = \{X \in \mathfrak{g};\ \phi^+(X)_p = 0\}$ the isotropy subalgebra of $\mathfrak{g}$ at $p \in M$. Clearly $p \in M$ is fixed under the subgroup $\{\exp tX;\ t \in R\}$ $(X \in \mathfrak{g})$ if and only if $\phi^+(X)_p = 0$. Therefore

(5.2) *The isotropy subalgebra $\mathfrak{g}_p$ is the Lie algebra of the isotropy subgroup G_p of G at p.*

6. Non-transitive actions

In the rest of this paper, we shall classify non-transitive smooth $SL(2, C)$-actions on S^3, and set

$$G = SL(2, C),\quad K = SU(2),\quad T = \{\begin{pmatrix} z & 0 \\ 0 & \bar{z}\end{pmatrix} \in K;\ |z| = 1\}\ (\cong U(1)),$$
$$\mathfrak{g} = \mathfrak{sl}(2, C) = \langle K_i, H_i;\ i = 1, 2, 3\rangle,\quad \mathfrak{k} = \mathfrak{su}(2)\quad \text{and}$$
$$S^3 = H/R^+ \quad \text{for} \quad H = \{P \in M(2, C);\ 0 \neq P = P^*\}.$$

To begin with we prepare some results on K-actions on S^3. The following (6.1) is known (cf. [1; Th. 1.3]).

(6.1) *Any non-transitive (and non-trivial) smooth K-action on S^3 is equivariantly diffeomorphic to*

$$\psi_0\colon K \times S^3 \to S^3,\quad \psi_0(X, [P]) = [XPX^*]\ (X \in K,\ P \in H).$$

The fixed point sets of ψ_0 under T and K are

$$(6.2) \qquad (C \supset) S^1 = F(T, S^3) \supset F(K, S^3) = \{\pm 1\},$$

by the diffeomorphism $S^1 \ni x + iy \to \begin{bmatrix} x+y & 0 \\ 0 & x-y\end{bmatrix} \in F(T, S^3)$ $(x, y \in R)$. Then the reflection J of S^1 is given by

$$(6.3) \qquad J(z) = \psi_0(j, z) \quad \textit{for} \quad z \in S^1 \quad \textit{and} \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \in K.$$

By Theorem 1.2, any smooth G-action on S^3 is transitive iff so is its restricted K-action (see Corollary 1.5 (i)). Thus, to classify non-transitive ones, we assume by (6.1)

(6.4) *ϕ is a smooth G-action on S^3 such that its restricted K-action coincides with ψ_0, i.e. $\phi | K \times S^3 = \psi_0$.*

Under this condition we prove the following lemmas.

Lemma 6.5. (i) *The map* φ: $R\times F(T, S^3) \to F(T, S^3)$, *given by*

$$(6.6)\qquad \varphi(t, z) = \phi(\exp(-tH_1), z)\ (t\in R,\ z\in F(T, S^3))\quad for \quad H_1\in\mathfrak{g}\,,$$

is a one-parameter transformation group on $F(T, S^3)$.

(ii) *There exists uniquely a real valued smooth function* f *on* $F(T, S^3)-F(K, S^3)$ *such that*

$$(6.7)\quad f(z)\,(K_2)_z = (H_3)_z\ (z\in F(T, S^3)-F(K, S^3))\quad for\quad K_2, H_3\in\mathfrak{g}\subset\mathfrak{X}(S^3)\,.$$

Proof. (i) is clear, because $\exp(-tH_1)\in N(T, G)$. (ii) The isotropy subalgebra $\mathfrak{g}_z$ at $z\in F(T, S^3)-F(K, S^3)$ satisfies

$$\dim \mathfrak{g}_z \geqq 3,\quad K_1\in\mathfrak{g}_z\quad \text{and}\quad \mathfrak{k}\not\subset\mathfrak{g}_z\,.$$

By Proposition 2.15 we can find $f(z)\in R$ such that $f(z)K_2-H_3\in\mathfrak{g}_z$, and hence $f(z)\,(K_2)_z=(H_3)_z$. Choose a Riemannian metric $\langle\!\langle\ ,\ \rangle\!\rangle$ on S^3. Then $f(z)\langle\!\langle K_2, K_2\rangle\!\rangle_z =\langle\!\langle K_2, H_3\rangle\!\rangle_z$ and $\langle\!\langle K_2, K_2\rangle\!\rangle_z\neq 0$. These show that f is unique and smooth on $F(T, S^3)-F(K, S^3)$. *q.e.d.*

(6.8) *Let* (φ, f) *be a pair of one-parameter transformation group* φ: $R\times S^1 \to S^1$ *and a smooth function* f: $S^1-\{\pm 1\}\to R$, *and consider the following conditions: For* $t\in R$ *and* $z=x+iy\in S^1$ $(x, y\in R)$,
(B1) $\varphi(t, J(z))=J\varphi(-t, z)$,
(B2) $f(\varphi(t, z))=(f(z)-\tanh t)/(1-f(z)\tanh t)$ *for* z, $\varphi(t, z)\neq\pm 1$,
(B3) $f(z)=-f(J(z))$, *and there is a smooth function* $F: S^1\to R$ *satisfying* $F(z)=yf(z)$ $(z\neq\pm 1)$ *and* $F(\pm 1)\neq 0$.

We say that pairs (φ, f) *and* (φ', f') *with* (B1–3) *are* B-*equivalent if there is a homeomorphism* Ψ *of* S^1 *onto itself such that* $\Psi J=J\Psi$ *and the following diagram commutes,*

$$\begin{tikzcd} R\times S^1 \arrow[r, "\varphi"] \arrow[d, "1\times\Psi"'] & S^1 \arrow[d, "\Psi"] & \supset & S^1-\{\pm 1\} \arrow[d, "\Psi"] \arrow[dr, "f"] & \\ & & & & R \\ R\times S^1 \arrow[r, "\varphi'"] & S^1 & \supset & S^1-\{\pm 1\} \arrow[ur, "f'"'] & \end{tikzcd}$$

where $\Psi(\{\pm 1\})=\{\pm 1\}$ *follows from* $\Psi J=J\Psi$.

Then we have the following lemma.

Lemma 6.9. *The pair* (φ, f) *in Lemma* 6.5 *satisfies* (B1–3) *under* (6.2).

Proof. (B1) follows from (6.3) and (6.6). Since $f(z)K_2-H_3\in\mathfrak{g}_z$, we have

$$\begin{aligned}\mathfrak{g}_{\varphi(t,z)} &= \mathrm{Ad}(\exp(-tH_1))\,\mathfrak{g}_z \ni \mathrm{Ad}(\exp(-tH_1))\,(f(z)\,K_2-H_3)\\ &= (f(z)\cosh t-\sinh t)\,K_2-(\cosh t-f(z)\sinh t)\,H_3 \quad \text{and}\\ \mathfrak{g}_{J(z)} &= \mathrm{Ad}(\boldsymbol{j})\,\mathfrak{g}_z \ni \mathrm{Ad}(\boldsymbol{j})\,(f(z)\,K_2-H_3) = -f(z)\,K_2-H_3\,.\end{aligned}$$

These show (B2) and the first half of (B3), respectively.

Consider the smooth function h: $S^3 \to R$, $h([P])=\mathrm{Trace}\,(\boldsymbol{j}P/\sqrt{2}\,||P||\,\boldsymbol{i})$ $(P\in H)$. Then $h(\phi(\exp(-tK_2), z))=y\sin t$ for $z=x+\boldsymbol{i}y\in S^1$, and hence

$$(K_2)_z\,h = \lim_{t\to 0}\,\{h(\phi(\exp(-tK_2), z))-h(z)\}/t = y\,.$$

Thus $F(z)=(H_3)_z\,h$ is smooth on S^1, and $F(z)=yf(z)$ if $z\neq\pm 1$.

Assume $F(a)=0$ for $a=1$ or -1. Then $\lim_{z\to a} f(z)=\lim_{z\to a} F(z)/y=0$, since F is smooth and $F\doteq FJ$. Hence we can find an open interval U in S^1 such that $a\in U$ and $|f|<1/2$ on $U-\{a\}$. Moreover $H_3\in\mathfrak{g}_a(\supset\mathfrak{k})$ by (6.7), and so $\mathfrak{g}_a=\mathfrak{g}$ by Lemma 2.9 (i). Thus a is stationary under ϕ, and hence $\lim_{t\to s}\varphi(t, z)\in U$ $(z\in U-\{a\})$ for $s=\infty$ or $-\infty$. Therefore $1/2\geqq|\lim_{t\to s} f(\varphi(t, z))|=1$ by (B2), and this leads a contradiction. *q.e.d.*

Proposition 6.10. *Let ϕ and ϕ' be smooth G-actions on S^3 with* (6.4). *Then the corresponding pairs with* (B1–3), *given by* (6.6–7), *are* B-*equivalent if ϕ is equivariantly homeomorphic to ϕ'.*

Proof. Denote by (φ, f) and (φ', f') the corresponding pairs of ϕ and ϕ', respectively. Let Φ: $S^3\to S^3$ be an equivariant homeomorphism between ϕ and ϕ', i.e. $\Phi(\phi(g, p))=\phi'(g, \Phi(p))$ $(g\in G, p\in S^3)$, and set $\Psi=\Phi|S^1$ for $S^1=F(T, S^3)$ by (6.2). Then

$$\Psi(\phi(\boldsymbol{j}, z)) = \phi'(\boldsymbol{j}, \Psi(z))\,,\quad \Psi(\phi(\exp(-tH_1), z)) = \phi'(\exp(-tH_1), \Psi(z))$$

for any $t\in R$, $z\in S^1$. These imply $\Psi J=J\Psi$ and $\Psi\varphi=\varphi'(1\times\Psi)$.

For a vector field $X\in\mathfrak{g}\subset\mathfrak{X}(S^3)$ and a smooth function h around $\Phi(z)$ $(z\in S^1)$, we have

$$(*)\qquad \begin{aligned}X_{\Phi(z)}\,h &= \lim_{t\to 0}\,\{h(\phi'(\exp(-tX), \Phi(z)))-h(\Phi(z))\}/t\\ &= \lim_{t\to 0}\,\{h\Phi(\phi(\exp(-tX), z))-h\Phi(z)\}/t\,.\end{aligned}$$

Suppose $(H_1)_z\neq 0$, i.e. $H_1\notin\mathfrak{g}_z$. Then $\dim\mathfrak{g}_z=3$ by Proposition 2.15, and the orbit Gz is of dimension 3. Hence Φ is locally diffeomorphic at z, and so (*) shows $X_{\Phi(z)}h=X_z(h\Phi)$. Applying this for $X=K_2$ and H_3, we get $f(z)\,(K_2)_{\Phi(z)}=(H_3)_{\Phi(z)}\,(z\neq\pm 1)$ by (6.7). Thus $f'(\Phi(z))=f(z)$. Next suppose $(H_1)_z=0$. Then $\varphi(t, z)=z$ for any $t\in R$, and thus $f(z)=\pm 1$ by (B2). Further (*) for $X=H_1$ shows $(H_1)_{\Phi(z)}=0$, whence $f'(\Phi(z))=\pm 1$. Therefore $f=f'\Phi$ follows from the continuity of f and f'. *q.e.d.*

7. Construction of $SL(2, C)$-actions

In this section we construct a smooth $G=SL(2, C)$-action on S^3 from a pair with (B1–3) so that the corresponding pair is the given one itself.

Let (φ, f) be a pair with (B1–3), and set

$$I(z) = \begin{pmatrix} F(z)-y & 0 \\ 0 & F(z)+y \end{pmatrix} \neq 0 \quad (z = x+iy \in S^1)$$

for the smooth function F in (B3). Consider the subgroup of G

$$U(z) = \{X \in G;\ XI(z)X^* = I(z)\},$$

which coincides with K if $z=\pm 1$ and $U_{f(z)}$ of (3.1) otherwise. Then $G=KLU(z)$ for any $z \in S^1$ by Lemma 3.4.

Take $(X, p) \in G \times S^3$. Let us choose

$$\begin{aligned} &g \in K,\ z \in S^1 = F(T, S^3) \text{ (by (6.2)) with } \psi_0(g, z) = p \quad \text{and} \\ &k \in K,\ t \in R,\ u \in U(z) \quad \text{with} \quad Xg = kl_t u \ (l_t = \exp(-tH_1) \in L), \end{aligned}$$

and set

$$\phi(X, p) = \psi_0(k, \varphi(t, z)) \in S^3. \tag{7.1}$$

Then we have the following proposition.

Proposition 7.2. *ϕ: $G \times S^3 \to S^3$ of* (7.1) *is a smooth G-action with* (6.4), *and the given pair (φ, f) satisfies* (6.6–7) *under* (6.2).

To prove this proposition we state some results on the pair (φ, f) with (B1–3). By (B2–3) we get

$$F(\varphi(t, z))\,(y - F(z) \tanh t) = y_t(F(z) - y \tanh t) \tag{7.3}$$

when $z=x+iy \neq \pm 1$ and $\varphi(t, z)=x_t+iy_t \neq \pm 1$. This also holds for any $(t, z) \in R \times S^1$, because F is continuous and $\{(t, z) \in R \times S^1;\ z \neq \pm 1,\ \varphi(t, z) \neq \pm 1\}$ is open dence in $R \times S^1$.

Lemma 7.4. *There is uniquely a smooth function α: $R \times S^1 \to R$ satisfying*

$$\begin{aligned} F(\varphi(t, z)) &= \alpha(t, z)\,(F(z) \cosh t - y \sinh t) \quad \textit{and} \\ y_t &= \alpha(t, z)\,(y \cosh t - F(z) \sinh t) \end{aligned} \tag{7.5}$$

for $t \in R$, $z=x+iy \in S^1$ *and* $\varphi(t, z)=x_t+iy_t$, *i.e.*

$$I(\varphi(t, z)) = \alpha(t, z)\, l_{2t}\, I(z). \tag{7.6}$$

Furthermore the following conditions are satisfied;

(7.7) $\alpha(0, z) = 1,\ \alpha(t+s, z) = \alpha(t, z)\,\alpha(s, \varphi(t, z))\,,$
$\alpha(t, z) = \alpha(-t, J(z))\,,\ \alpha(t, z)\,\alpha(-t, \varphi(t, z)) = 1\,,\ \alpha(t, z)>0\,.$

Proof. From $F(\pm 1)\neq 0$ it follows that $y-F(z)\tanh t$ and $F(z)-y\tanh t$ are not simultaneously equal to zero in (7.3). Then we obtain uniquely a smooth function α satisfying (7.5).

By (7.6) we get

$$I(z) = \alpha(0, z)\,I(z),\quad \alpha(t+s, z)\,l_{2(t+s)}\,I(z) = I(\varphi(s, \varphi(t, z))) =$$
$$\alpha(s, \varphi(t, z))\,l_{2s}\,I(\varphi(t, z)) = \alpha(s, \varphi(t, z))\,\alpha(t, z)\,l_{2(t+s)}\,I(z) \quad \text{and}$$
$$\alpha(t, z)\,l_{2t}\,I(z) = \boldsymbol{j}I(J\varphi(t, z))\,\boldsymbol{j}^* = \boldsymbol{j}I(\varphi(-t, J(z)))\,\boldsymbol{j}^* =$$
$$\alpha(-t, J(z))\,\boldsymbol{j}\,l_{-2t}\,I(J(z))\,\boldsymbol{j}^* = \alpha(-t, J(z))\,l_{2t}\,I(z)\,.$$

These show the first three equalities of (7.7) by $I(z)\neq 0$. Thus $\alpha(t, z)\,\alpha(-t, \varphi(t, z))=\alpha(0, z)=1$, and hence $\alpha(t, z)>0$ because α is continuous. *q.e.d.*

Lemma 7.8. *If $kl_{2t}\,I(z)\,k^*=l_{2s}\,I(z)$ for some $k\in K$, then $t=s$ or $\varphi(t, z)= J\varphi(s, z)$, and further $\alpha(t, z)=\alpha(s, z)$.*

Proof. From Trace $l_{2t}\,I(z)=2(F(z)\cosh t-y\sinh t)$, the assumption implies (1) $t=s$ or (2) $F(z)\tanh(t+s)/2=y$.

In case (1) the lemma is clear. In case (2), $\varphi((t+s)/2, z)=\pm 1$ by (7.5). Then $\varphi((t+s)/2, z)=J\varphi((t+s)/2, z)=\varphi(-(t+s)/2, J(z))$, and hence $\varphi(t, z)= \varphi(-s, J(z))=J\varphi(s, z)$. Furthermore, by (7.7),

$$\alpha(t+s, z)/\alpha(t, z) = \alpha(s, \varphi(t, z)) = \alpha(s, J\varphi(s, z)) = 1/\alpha(s, z)\,,$$

and similarly $\alpha(t+s, z)/\alpha(s, z)=1/\alpha(t, z)$. These show $\alpha(s, z)=\alpha(t, z)$ (>0). *q.e.d.*

Consider the smooth function β: $S^1\to R$, $\beta(z)=F(z)+x\,(z=x+\boldsymbol{i}y\in S^1)$. Then we get

(7.9) $\beta(z) = \beta(J(z))$ *and* $z = [\beta(z)\,E-I(z)]\in S^1 = F(T, S^3)\,.$

Lemma 7.10. *Let $k\in K$ and $z\in S^1$. Then*

(i) *$\psi_0(k, z)=w\in S^1$ if and only if $kI(z)k^*=I(w)$ and $w=z$ or $J(z)$.*

(ii) *If $kl_{2t}\,I(z)k^*=l_{2s}\,I(w)$ for $w=z$ or $J(z)$, then $\psi_0(k, \varphi(t, z))=\varphi(s, w)$.*

Proof. (i) Assume $\psi_0(k, z)=w$. When $z=\pm 1\in F(K, S^3)$, the result is clear. Suppose $z\neq\pm 1$. Then $k\in N(T, K)=T\cup\boldsymbol{j}\,T$, because T is the isotropy subgroup of ψ_0 at $S^1-\{\pm 1\}$. Therefore $w=z$, $kI(z)k^*=I(z)$ if $k\in T$, and $w= J(z)$, $kI(z)k^*=I(J(z))$ if $k\in\boldsymbol{j}T$.

Conversely, $\psi_0(k, z) = [\beta(z) E - kI(z)k^*] = w$ by (7.9).

(ii) When $w=z$, we see that $\alpha(t, z)=\alpha(s, z)$ and $\varphi(t, z)=\varphi(s, z)$ or $J\varphi(s, z)$ by Lemma 7.8. Thus $kI(\varphi(t, z))k^*=I(\varphi(s, z))$, and so the desired result follows from (i). In case $w=J(z)$, it holds $j^*kl_{2t}I(z)k^*j=j^*l_{2s}I(J(z))j=l_{-2s}I(z)$, and the above result implies

$$\psi_0(k, \varphi(t, z)) = \psi_0(j, \varphi(-s, z)) = \varphi(s, J(z)) . \qquad \textit{q.e.d.}$$

In (7.1) we obtain the following by (7.6).

$$(7.11) \qquad XgI(z)g^*X^* = kl_{2t}I(z)k^* = kI(\varphi(t, z))k^*/\alpha(t, z) .$$

Lemma 7.12. *ϕ of (7.1) defines a G-action on S^3 such that $\phi | K\times S^3=\psi_0$ and $\phi | L\times S^1=\varphi$.*

Proof. For $(X, p)\in G\times S^3$, let us choose as in (7.1);

$$(*) \qquad \psi_0(g, z) = p, Xg = kl_t u \quad \text{and} \quad \psi_0(g', z') = p, Xg' = k' l_{t'} u' .$$

Then $gI(z)g^*=g'I(z')g'^*$ and $z=z'$ or $J(z')$ by Lemma 7.10 (i). Hence $kl_{2t} I(z)k^*=k'l_{2t'}I(z')k'^*$ by (7.11). Thus $\psi_0(k, \varphi(t, z))=\psi_0(k', \varphi(t', z'))$ by Lemma 7.10 (ii), and this shows that ϕ is a mapping from $G\times S^3$ to S^3.

When $(X, p)\in K\times S^3$ (resp. $(X, p)\in L\times S^1$), we can choose $\psi_0(g, z)=p$, $Xg=k$ (resp. $z=p$, $X=l_t$) in (*). Thus

$$\phi(X, p) = \psi_0(k, z) = \psi_0(X, p) \quad (\text{resp. } \phi(X, p) = \varphi(t, z)) .$$

Therefore $\phi | K\times S^3=\psi_0$ (resp. $\phi | L\times S^1=\varphi$), and further $\phi(E, p)=p$.

Let $Y\in G$, and choose $m\in K$, $s\in R$, $v\in U(\varphi(t, z))$ with $Yk=ml_s v$. Then

$$\phi(Y, \phi(X, p)) = \psi_0(m, \varphi(s, \varphi(t, z))) = \psi_0(m, \varphi(t+s, z)) .$$

On the other hand $YXg=ml_{t+s}wu$ for $w=l_{-t}vl_t$, where $w\in U(z)$ by $wI(z)w^*=l_{-2t}I(\varphi(t, z))/\alpha(t, z)=I(z)/\alpha(t, z)\,\alpha(-t, \varphi(t, z))=I(z)$. Therefore

$$\phi(YX, p) = \psi_0(m, \varphi(t+s, z)) = \phi(Y, \phi(X, p)) . \qquad \textit{q.e.d.}$$

(7.13) (The standard G-action) Let ϕ_0: $G\times S^3\to S^3$,

$$\phi_0(X, [P]) = [XPX^*] \quad (X\in G, P\in H) .$$

Then ϕ_0 is an analytic G-action on S^3 with $\phi_0 | K\times S^3=\psi_0$. Denote by φ_0 the one-parameter transformation group on $S^1=F(T, S^3)$ induced from ϕ_0; $\varphi_0(t, z)=\phi_0(l_t, z)$ $(t\in R, z\in S^1)$.

Lemma 7.14. *Let ν: $S^1\to S^1$ be a smooth map given by $\nu(z)=[I(z)]$ $(z\in S^1)$. Then ν is locally diffeomorphic at $z\in S^1$ if* det $I(z)\neq 0$.

Proof. For each $z\in S^1$ we put

$$\varphi_0^{\nu(z)}(t) = \varphi_0(t, \nu(z)) \quad \text{and} \quad \varphi^z(t) = \varphi(t, z) \quad (t\in R).$$

Assume $\varphi_0^{\nu(z)}(t)=\nu(z)$ (resp. $\varphi^z(t)=z$). Then $l_{2t}\, I(z)=\lambda I(z)$ for some $\lambda>0$ (resp. $I(z)=\alpha(t, z)\, l_{2t}\, I(z)$ by (7.6)). If $\det I(z)\neq 0$, then $t=0$, whence $\varphi_0^{\nu(z)}$ and φ^z are locally diffeomorphic at $0\in R$. Therefore ν is also locally diffeomorphic at z, because $\varphi_0^{\nu(z)}=\nu\varphi^z$ by (7.6). *q.e.d.*

By using the Taylor developments, we see the following (cf. [2; Ch. VIII, §14, Problem 6-c]).

(7.15) *Let h be a smooth even function around $0\in R(h(t)=h(-t))$. Then $h(\|x\|)$ $(x\in R^n)$ is also smooth at the origin in R^n.*

Put $\psi_\pm=\psi_0|K\times S_\pm\colon K\times S_\pm\to S_0$ and its induced map $\tilde{\psi}_\pm\colon K/T\times S_\pm\to S_0$ for S_+ (resp. S_-)$=\{x+iy\in S^1;\ y>0$ (resp. $y<0$)$\}$ and $S_0=S^3-\{\pm 1\}$. Then $\tilde{\psi}_\pm$ are diffeomorphic and so $\psi_\pm$ are submersions.

Lemma 7.16. (i) *For each $p\in S^3$, let us choose $g\in K$, $z\in S^1$ with $p=\psi_0(g, z)$, and set $\xi(p)=gI(z)g^*$. Then $\xi\colon S^3\to H(\subset M(2, C))$ is a smooth mapping.*

(ii) *The composition $\tilde{\xi}\colon S^3 \xrightarrow{\xi} H \xrightarrow{\text{pr.}} S^3$ is locally diffeomorphic at $p\in S^3$ if $\det \xi(p)\neq 0$, and is equivariant between ϕ and ϕ_0, i.e. $\tilde{\xi}\phi(X, p)=\phi_0(X, \tilde{\xi}(p))$ $(X\in G, p\in S^3)$.*

Proof. (i) In the proof of Lemma 7.12, we have already seen that ξ gives a mapping from S^3 to H. By the commutative diagram

$$\begin{array}{ccc} S_0\subset S^3 & \xrightarrow{\xi} & H \\ {\scriptstyle\psi_+}\uparrow & \nearrow{\scriptstyle\xi_1} & \\ K\times S_+ & & \end{array} \qquad \text{for} \quad \xi_1(g, z) = gI(z)g^*,$$

we see that ξ is smooth on S_0 because ψ_+ is a submersion.

Put $p_\varepsilon=[\varepsilon E]$, $N_\varepsilon=\{[P]\in S^3;\ \varepsilon \operatorname{Trace} P>0\}$ for $\varepsilon=\pm 1$, and denote by $D\subset R\times C$ an open unit disk. Let $\rho_\varepsilon\colon D\to N_\varepsilon$,

$$\rho_\varepsilon(x, a) = \begin{bmatrix} \varepsilon(1-s^2)^{1/2}+x & a \\ \bar{a} & \varepsilon(1-s^2)^{1/2}-x \end{bmatrix} \qquad \text{for} \quad (x, a)\in D,$$

where $s^2=x^2+|a|^2<1$ $(s>0)$. Then $(N_\varepsilon, \rho_\varepsilon)$ is a local chart at p_ε, and

$$\xi\rho_\varepsilon(x, a) = \begin{pmatrix} F(z)-x & -a \\ -\bar{a} & F(z)+x \end{pmatrix} \qquad \text{for} \quad z = \varepsilon(1-s^2)^{1/2}+is.$$

Consider a smooth function $h(t)=F(\varepsilon(1-t^2)^{1/2}+it)$ $(|t|<1)$, which is an even function since $F=FJ$. From (7.15) it follows that $h(s)=F(z)$ is smooth on D, and hence so is $\xi\rho_\varepsilon$. Therefore ξ is also smooth at p_ε.

(ii) Let $\phi(X, p)=\psi_0(k, \varphi(t, z))$ for $p=\psi_0(g, z)$ as in (7.1). Then, by (7.11),

$$\xi\phi(X, p) = [kI(\varphi(t, z))k^*] = [XgI(z)g^*X^*] = \phi_0(X, \xi(p)).$$

By Lemma 7.14 and the commutative diagram

$$\begin{array}{ccc} K/T\times S_+ & \xrightarrow{\tilde{\psi}_+} & S_0 \\ {\scriptstyle 1\times(\nu|S_+)}\downarrow & & \downarrow{\scriptstyle \xi|S_0} \\ K/T\times S_- & \xrightarrow{\tilde{\psi}_-} & S_0 \end{array},$$

we see that ξ is locally diffeomorphic at $p\in S_0$ if $\det \xi(p)\neq 0$, because $\det \xi(p)=\det I(z)$ for $p=\psi_0(g, z)$ $(g\in K, z\in S_+)$. Furthermore, by using the local chart $(N_\varepsilon, \rho_\varepsilon)$ at p_ε, the routine calculation shows that the Jacobian of ξ at p_ε is non-zero. Thus ξ is also locally diffeomorphic at p_ε. *q.e.d.*

Consider the subsets

$$U=(G\times S_0)\cap\phi^{-1}(S_0), \quad V=(R\times S_+)\cap\varphi^{-1}(S_+) \quad \text{and}$$
$$W=(1\times\tilde{\psi}_+)^{-1}(U) \qquad \text{for} \quad 1\times\tilde{\psi}_+\colon G\times(K/T)\times S_+\to G\times S_0.$$

Clearly V is open in $R\times S^1$. Also, since $S_0=\xi^{-1}(S_0)$ and ξ is equivariant by Lemma 7.16, we see that U is open in $G\times S^3$, and hence so is W in $G\times(K/T)\times S^1$.

Lemma 7.17. *For any $w=(X, gT, z)\in W$, there exists uniquely $t\in R$ such that*

(*) $\qquad$ *$(t, z)\in V$ and $Xg=kl_t u$ for some $k\in K$, $u\in U(z)$.*

Furthermore $\delta\colon W\ni w\to(t, z)\in V$ is smooth.

Proof. (i) Choose $m\in K$, $s\in R$ and $v\in U(z)$ with $Xg=ml_s v$. If $\varphi(s, z)\in S_+$, then (*) is clear. Suppose $\varphi(s, z)\notin S_+$, whence $\varphi(s, z)\in S_-$ by $w\in W$. Then $(0<)\,y<F(z)\tanh s$ $(z=x+iy)$ by (7.5) and (7.7), and we can find $t\in R$ satisfying $y=F(z)\tanh((t+s)/2)$. By easy calculations this implies $l_{2(t+s)}I(z)=j^*I(z)j$, and hence $jl_{t+s}\in U(z)$. Also $\varphi((t+s)/2, z)=\pm 1\in F(K, S^3)$ by (7.5). Thus $\varphi((t+s)/2, z)=J\varphi((t+s)/2, z)=\varphi(-(t+s)/2, J(z))$, and so $\varphi(t, z)=\varphi(-s, J(z))=J\varphi(s, z)\in S_+$. This shows $(t, z)\in V$. Now we set

$$k=mj^*\in K \quad \text{and} \quad u=jl_{t+s}v\in U(z).$$

Then $kl_t u=mj^*l_t jl_{t+s}v=ml_s v=Xg$. Therefore (*) holds.

Assume $(s, z)\in V$ and $Xg=ml_s v$ for some $m\in K$, $v\in U(z)$. Then $kl_{2t}I(z)$

$k^*=ml_{2s}I(z)m^*$ by (7.11), and $t=s$ follows from Lemma 7.8.

(ii) Consider the smooth mappings

$$\delta_1\colon W\to R\times S_+,\quad \delta_1(X, gT, z)=((\mathrm{Trace}\ XgI(z)g^*X^*)/2, z)\quad\text{and}$$
$$\delta_2\colon V\to R\times S_+,\quad \delta_2(t, z)=(F(\varphi(t, z))/\alpha(t, z), z)\,.$$

Then $\delta_1=\delta_2\,\delta$ by (7.11). By (7.5), the routine calculation shows that the Jacobian of δ_2 is non-zero, and so δ_2 is locally diffeomorphic. Therefore δ is smooth. *q.e.d.*

Proof of Proposition 7.2. By Lemma 7.12, it is sufficient to show that ϕ of (7.1) is smooth and satisfies (6.7).

Let us set as in (7.1);

$$\phi(X, p)=\psi_0(k, \varphi(t, z))\qquad\text{for}\quad p=\psi_0(g, z)\quad\text{and}\quad Xg=kl_t\,u\,.$$

Assume p or $\phi(X, p)=\pm1\in F(X, S^3)$. Then $\det\xi(\phi(X, p))\neq0$, because $\xi(\pm1)=F(\pm1)E$ and $\xi(\phi(X, p))=\alpha(t, z)\,kl_{2t}\,I(z)k^*$ by (7.6). From Lemma 7.16 (ii) it follows that ϕ is smooth at (X, p). Next assume $(X, p)\in(G\times S_0)\cap\phi^{-1}(S_0)=U$. Consider the map

$$\phi_1\colon U\to H,\ \phi_1(X, p)=(\beta(\varphi(t, z))/\alpha(t, z))\,E-X\xi(p)\,X^*.$$

Then ϕ_1 is smooth, since $\xi(p)$ and $(t, z)=\delta(1\times\tilde{\psi}_+)^{-1}(X, p)$ are smooth by Lemmas 7.16 (i) and 7.17. Further the composition $(G\times S^3\supset)\ U\xrightarrow{\phi_1}H\xrightarrow{\mathrm{pr.}}S^3$ coincides with $\phi\,|\,U$ by (7.9) and (7.11). Thus ϕ is also smooth at $(X, p)\in U$.

Put $A=f(z)K_2-H_3\in\mathfrak{u}_{f(z)}(\subset\mathfrak{g})$ for $z\in S^1-\{\pm1\}$. Hence $\phi(\exp(-tA), z)=z$, and this implies $0=\phi^+(A)_z=f(z)\,(K_2)_z-(H_3)_z$. Therefore (6.7) holds.

The proof of the proposition is thus completed. *q.e.d.*

8. B-equivalence classes

In this section we show the following theorem.

Theorem 8.1. *There is a one-to-one correspondence between the equivariant homeomorphism classes of non-transitive smooth G-actions on S^3 and the* B*-equivalence classes of pairs with* (B1–3).

To prove this theorem we prepare the following Lemmas 8.2–3.

Lemma 8.2. *If pairs with* (B1–3) *are* B*-equivalent, then the corresponding G-actions on S^3, constructed by* (7.1), *are equivariantly homeomorphic to each other.*

Proof. Assume that pairs (φ, f) and (φ', f') are B-equivalent by a homeomorphism Ψ of S^1 onto itself, and let ϕ and ϕ' be respectively the G-

actions on S^3 by (7.1). Then there is a K-equivariant homeomorphism Φ of S^3 onto itself satisfying $\Phi(\psi_0(g, z))=\psi_0(g, \Psi(z))$ $(g\in K, z\in S^1)$, since $\Psi J=J\Psi$ and $\psi_0|K\times S^1$: $K\times S^1\to S^3$ is closed and surjective. Let $(X, p)\in G\times S^3$, and set as in (7.1); $\phi(X, p)=\psi_0(k, \varphi(t, z))$ for $p=\psi_0(g, z)$, $Xg=kl_t u$ $(u\in U(z))$, where $U(z)=K$ if $z=\pm 1$, and $U_{f(z)}=U_{f'\Psi(z)}$ if $z\neq\pm 1$. Thus

$$\begin{aligned}\phi'(X, \Phi(p)) &= \phi'(X, \psi_0(g, \Psi(z))) = \psi_0(k, \varphi'(t, \Psi(z))\\ &= \psi_0(k, \Psi\varphi(t, z)) = \Phi\psi_0(k, \varphi(t, z)) = \Phi\phi(X, p)\,.\end{aligned}$$

This shows that Φ is G-equivariant, and hence the lemma holds. *q.e.d.*

Lemma 8.3. *Let (φ, f) be a pair with* (B1–3) *defined from ϕ of* (6.4) *by* (6.6–7). *Then the G-action on S^3, constructed from (φ, f) by* (7.1), *coincides with the given one ϕ.*

Proof. Let $(X, p)\in G\times S^3$, and set $\phi'(X, p)=\psi_0(k, \varphi(t, z))$ for $p=\psi_0(g, z)$ and $Xg=kl_t u$ as in (7.1).

Now we show $\phi(u, z)=z$. Clearly this holds when $z=\pm 1$, since $u\in U(z)$ $=K$ and $z\in F(K, S^3)$. Suppose $z\neq\pm 1$. Then $f(z)K_2-H_3\in\mathfrak{g}_z$ by (6.7), $K_1\in$ $\mathfrak{g}_z$, and so $f(z)K_3+H_2=[f(z)K_2-H_3, K_1]\in\mathfrak{g}_z$. These show $\mathfrak{u}_{f(z)}\subset\mathfrak{g}_z$ by (2.14). Thus $U(z)=U_{f(z)}\subset G_z$, and hence $\phi(u, z)=z$ holds. Therefore

$$\phi(X, p) = \phi(kl_t ug^{-1}, \psi_0(g, z)) = \phi(kl_t, z) = \psi_0(k, \varphi(t, z)) = \phi'(X, p)\,.$$

q.e.d.

Proof of Theorem 8.1. By Propositions 6.10, 7.2 and Lemmas 8.2–3, we see that the correspondence defined by (6.6–7) and (7.1) is one-to-one between the equivariant homeomorphism classes of (6.4) and the B-equivalence classes. Moreover, by (6.1), the former coincides with the equivariant homeomorphism classes of non-transitive smooth G-actions on S^3. *q.e.d.*

9. Pairs with (B1–3)

In this section we restate pairs with (B1–3), and show that a triad of subsets with (A1–2) in (1.3) is obtained from a pair with (B1–3).

It is well-known that a one-parameter transformation group φ on S^1 is regarded as a vector field on S^1, and so a smooth function g on S^1 as follows;

(9.1) $g(z)L_z h=[dh\varphi(t, z)/dt]_{t=0}$ *for any smooth function h around* $z\in S^1$.

Here L is the unit vector field on S^1, $L_z=-y(\partial/\partial x)_z+x(\partial/\partial y)_z$ $(z=x+iy\in S^1\subset C)$.

(9.2) *Let g and f be respectively smooth functions on S^1 and $S^1-\{\pm 1\}$, and consider the following conditions;*

(B1)′ $g(z)=g(J(z))$, (B2)′ $g(z)L_z f=f(z)^2-1$ $(z\neq\pm 1)$.

Then, for φ and g in (9.1), we have the following lemma.

Lemma 9.3. (i) (B1)′ *is equivalent to* (B1).
(ii) *If f satisfies* (B3), *then* (B2)′ *is equivalent to* (B2).

Proof. (i) By definition we get

(*) $L_z h=-L_{J(z)}(hJ)$ for any smooth function h around $z\in S^1$.

If φ satisfies (B1), then (B1)′ follows from

$$\begin{aligned} g(z)\,L_z h &= [dh\,\varphi(t,z)/dt]_{t=0}=[dh\,J\varphi(-t,J(z))/dt]_{t=0}\\ &= -g(J(z))\,L_{J(z)}(hJ)=g(J(z))\,L_z h\,. \end{aligned}$$

Suppose that g satisfies (B1)′. Then $J_*(gL)=-gL$, because $J_*(gL)_z h=g(J(z))\,L_{J(z)}(hJ)=-g(z)\,L_z h$. Hence

$$\begin{aligned} J\varphi(t,J(z)) &= J(\mathrm{Exp}\,t(gL))\,J(z)=(\mathrm{Exp}\,tJ_*(gL))\,(z)\\ &= (\mathrm{Exp}\,(-t)\,(gL))\,(z)=\varphi(-t,z)\,. \end{aligned}$$

(ii) By routine calculations, (B2)′ follows immediately from (B2). To see the converse, let us fix $z\in S^1-\{\pm 1\}$ and set

$$H(t)=f(\varphi(t,z)) \quad \text{for any} \quad t\in R \quad \text{with} \quad \varphi(t,z)\neq\pm 1\,.$$

Then $H(t)$ satisfies the differential equation

(**) $$dH(t)/dt=H(t)^2-1 \quad \text{by (B2)}'$$

and the initial condition $H(0)=f(z)$. Clearly $H_c(t)=(c-\tanh t)/(1-c\tanh t)$ $(c\in R)$ are solutions of (**), and their maximal interval of existence are

$$R \ \text{ if } \ |c|\leqq 1,\ (-\infty,a) \ \text{ if } \ c>1, \ \text{ and } \ (a,\infty) \ \text{ if } \ c<-1\,,$$

where $c\tanh a=1$ and $\lim_{t\to a}|H_c(t)|=\infty$.

Let $N=\{t\in R;\ \varphi(t,z)=\pm 1\}$. Then $H(t)$ is smooth on $R-N$, $\lim_{t\to s}|H(t)|=\infty$ for $s\in N$ by (B3), and N is discrete. Therefore $H(t)=H_{f(z)}(t)$ follows from the initial condition, and thus (B2) holds. *q.e.d.*

In the rest of this section we assume that a pair (φ, f) with (B1–3) is given, and hence so is a smooth function g with (B1–2)′. Set

$$\begin{aligned} A_i &= \{z\in S^1;\ f(z)=(-1)^{i-1}\}\ (i=1,2)\,, \quad A_0=A_1\cup A_2\,,\\ C_i &= \{z\in A_0;\ (-1)^{i-1}g(z)>0\}\ (i=1,2) \quad \text{and} \quad C_0=A_0-(C_1\cup C_2)\,. \end{aligned}$$

Then we have the following Lemmas 9.4–6.

Lemma 9.4. (i) *A_i ($i=1, 2$) and C_0 are φ-invariant closed subsets of S^1. In particular C_0 is the fixed point set of φ.*

(ii) *$J(A_i)=A_{3-i}(i=1, 2)$ and $J(C_j)=C_j$ ($j=1, 2, 3$).*

Proof. (i) If $f(z)=\pm 1$, then (B2) shows that $f(\varphi(t, z))=\pm 1$ for any $t\in R$. Hence $A_i(i=1, 2)$ are φ-invariant and closed in S^1.

The fixed point set of φ is given by $F=\{z\in S^1;\ g(z)=0\}$. Then $C_0=A_0\cap F$, and further $F\subset A_0$ by (B2)$'$. Thus $C_0=F$.

(ii) follows immediately from $fJ=-f$ and $gJ=g$. *q.e.d.*

Lemma 9.5. (i) *$A_1\cap A_2=\phi$, $A_0\cap\{\pm 1\}=\phi$ and $A_0\neq\phi$.*

(ii) *A_0 is a finite union of closed intervals.*

(iii) *The components of A_1 alternate with those of A_2.*

Proof. (i) The first result is clear, and the second follows from (B3). From (7.5) it follows that ± 1 are not in the same orbit of φ, and hence φ is not transitive on S^1. Then $\lim_{t\to\infty}\varphi(t, z)=a\in S^1$ ($z\in S^1$), and $\pm 1=\lim_{t\to\infty}f(\varphi(t, z))=f(a)$ ($z\neq\pm 1$) by (B2). These imply $A_0\neq\phi$.

(ii) Regard $S_+=\{x+iy\in S^1;\ y>0\}$ as a bounded open interval. Since $A_0\cap\{\pm 1\}=\phi$ and A_0 is closed, we see that $A_0\cap S_+$ is also closed in S^1. Thus, by $J(A_0)=A_0$, it is sufficient to show that the components of $A_0\cap S_+$ is finite.

Assume the contrary. Then we can find a monotone increasing sequence $\{z_n\}$ of $A_0\cap S_+$ such that z_n and z_m ($n\neq m$) are not in the same component. Moreover, let us choose $v_n\in S_+$ satisfying $v_n\notin A_0\cap S_+$ and $z_n<v_n<z_{n+1}$. Then $|f(v_n)|<1$ by (B2), and so $f(v_n)=\tanh s$ for some $s\in R$. Put $w_n=\varphi(s, v_n)$. Then

$$f(w_n)=0 \quad \text{by (B2),} \quad \text{and} \quad z_n<w_n<z_{n+1}\,.$$

Hence $z=\lim_{n\to\infty} z_n=\lim_{n\to\infty} w_n\in A_0\cap S_+$, since $A_0\cap S_+$ is closed and bounded. Therefore $f(z)=\lim_{n\to\infty}f(z_n)=\lim_{n\to\infty}f(w_n)$, and this leads a contradiction.

(iii) By (B2) we have $\lim_{t\to\infty}f(\varphi(t, z))=-1$ and $\lim_{t\to-\infty}f(\varphi(t, z))=1$ for any $z\in S^1-(A_0\cup\{\pm 1\})$, whence $\lim_{t\to\infty}\varphi(t, z)\in A_2$ and $\lim_{t\to-\infty}\varphi(t, z)\in A_1$. Clearly these show our desired result. *q.e.d.*

Lemma 9.6. *$C_i(i=1, 2)$ are open in S^1, and $C_0\supset\partial A_0$.*

Proof. By Lemmas 9.4 (i) and 9.5 (ii), ∂A_0 is a φ-invariant finite subset, whence is fixed by φ. Thus $\partial A_0\subset C_0$ follows from Lemma 9.4 (i). Since C_i ($i=1, 2$) are open in A_0 and $C_i\subset A_0-\partial A_0$, they are also open in S^1. *q.e.d.*

Proposition 9.7. *Let (φ, f) be a pair with* (B1–3), *and set*

(9.8) $$A=A_1 \quad \text{and} \quad B_i=A_1\cap C_i\ (i=1, 2)\,.$$

Then the triad (A, B_i) *satisfies* (A1–2). *If pairs with* (B1–3) *are* B-*equivalent, then the corresponding triads with* (A1–2) *are* A-*equivalent.*

Proof. The first half follows from Lemmas 9.4–6.

Let Ψ be a homeomorphism of S^1 onto itself, which gives B-equivalence between (φ, f) and (φ', f'), i.e.

$$\Psi J = J\Psi,\ f = f'\,\Psi \quad \text{and} \quad \Psi\varphi(t, z) = \varphi'(t, \Psi(z)) \ \ (t\in R,\ z\in S^1)\,,$$

and let (A, B_i) and (A', B_i') be the corresponding triads with (A1–2) respectively. Then the above last two equalities imply

$$\Psi(A) = A',\ \Psi(C_0) = C_0' \quad \text{and hence} \quad \Psi(B_1 \cup B_2) = B_1' \cup B_2'\,,$$

where C_0 and C_0' are the fixed point sets of φ and φ' respectively.

Fix $z\in B_1\cup B_2$. Then $g(z)\neq 0$ and $g'(\Psi(z))\neq 0$ for the smooth functions g and g' of (9.1) by φ and φ' respectively. Hence $\varphi^z(t)=\varphi(t, z)$ and $\varphi'^{\Psi(z)}(t)=\varphi'(t, \Psi(z))$ are locally diffeomorphic at $t=0$, whence so is Ψ at z. Thus

$$\begin{aligned} g'(\Psi(z))\, L_{\Psi(z)}h &= [dh\,\varphi'(t, \Psi(z))/dt]_{t=0} = [dh\,\Psi\varphi(t, z)/dt]_{t=0} = g(z)\, L_z(h\Psi) \\ &= g(z)\, \Psi_*(L_z)\, h \quad \text{for any smooth function } h \text{ around } \Psi(z)\,. \end{aligned}$$

Therefore $g(z)g'(\Psi(z))>0$ (resp. <0) if Ψ is orientation preserving (resp. reversing), and hence $\Psi(B_i)=B_i'$ (resp. $\Psi(B_i)=B_{3-i}'$) $(i=1, 2)$. Thus, by Lemma 9.4 (ii),

$$\Phi = \begin{cases} \Psi & \text{if } \Psi \text{ is orientation preserving}\,, \\ \Psi J & \text{otherwise} \end{cases}$$

gives the A-equivalence between (A, B_i) and (A', B_i'). *q.e.d.*

10. Construction of smooth functions

In this section we construct smooth functions with (B1–2)$'$ and (B3) from a triad of subsets of S^1 with (A1–2).

Lemma 10.1. *There exist smooth functions* α *and* β *on* R *satisfying the following conditions;*

(1) $\beta(x)\,(d\alpha(x)/dx)=\alpha(x)^2-1$.

(2) $|\alpha(x)|<1$ *and* $d\alpha(x)/dx>0$ *if* $|x|<1$, $\alpha(x)=-1$ *if* $x\leqq -1$, $\alpha(x)=1$ *if* $x\geqq 1$, *and* α *is an odd function.*

(3) $(\sin x)/\alpha(x)$ $(x\neq 0)$ *can be extended to a smooth function* $\tilde{\alpha}$ *on* R *with* $\tilde{\alpha}(0)\neq 0$.

(4) $\beta(x)=0$ *if* $|x|\geqq 1$, *and* β *is an even function.*

Proof. Put

$$\rho(x)=\begin{cases}\exp(-1/x^2) & \text{if } x>0\,,\\ 0 & \text{if } x\leqq 0\,,\end{cases} \quad \text{and} \quad \eta(x)=\rho(\rho(x))\,.$$

Then, by routine calculations,

$$\alpha(x)=(\eta(x_1)-\eta(x_2))/(\eta(x_1)+\eta(x_2)) \quad \text{and}$$
$$\beta(x)=-x_1^3\,x_2^3\,\rho(x_1)^2\,\rho(x_2)^2/(x_1^3\,\rho(x_1)^2+x_2^3\,\rho(x_2)^2)$$

for $x_1=(1+x)/2$ and $x_2=(1-x)/2$, satisfy (1)–(4). *q.e.d.*

Lemma 10.2. *Let α and β be as in the above lemma, and set $\gamma(x)=1/\alpha(x)$ $(x\neq 0)$. Then*
(1) $\beta(x)\,(d\gamma(x)/dx)=\gamma(x)^2-1$ $(x\neq 0)$.
(2) $|\gamma(x)|>1$ and $d\gamma(x)/dx<0$ if $0<|x|<1$, $\gamma(x)=-1$ if $x\leqq -1$, $\gamma(x)=1$ if $x\geqq 1$, and γ is an odd function.
(3) $\gamma(x)\sin x$ can be extended to a smooth function $\tilde{\gamma}(x)$ on R with $\tilde{\gamma}(0)\neq 0$.

Proof. The lemma follows from Lemma 10.1. *q.e.d.*

Lemma 10.3. *Let $N_i(i=1,2)$ be disjoint open subsets of R. Then there exists a smooth function μ on R satisfying $N_i=\{x\in R;\,(-1)^{i-1}\mu(x)>0\}$ $(i=1,2)$.*

Proof. It is well-known that there are smooth functions $\mu_i(i=1,2)$ on R such that $R-N_i=\{x\in R;\,\mu_i(x)=0\}$ (cf. [4; Ch. 1, Th. 1.5]). Then the desired function is obtained by setting $\mu(x)=\mu_1(x)^2-\mu_2(x)^2$. *q.e.d.*

Let (A, B_i) be a triad of subsets of S^1 with (A1–2), and put

$$S_+\cap A_0=\cup_{l=1}^{k}[r_l, s_l]\;(0<r_l\leqq s_l<r_{l+1}<\pi) \qquad \text{for} \quad A_0=A\cup J(A)\,.$$

Here S_+ is regarded as the open interval $(0,\pi)$ by sending $e^{i\theta}\in S_+$ to $\theta\in(0,\pi)$. Also put $u_l=(r_{l+1}+s_l)/2$, $v_l=(r_{l+1}-s_l)/2$ and $\omega_l(\theta)=(\theta-u_l)/v_l$ $(0\leqq l\leqq k)$ for $s_0=-r_1$ and $r_{k+1}=2\pi-s_k$. By using α, β and γ in Lemmas 10.1–2, consider the smooth functions

$$a(\theta)=\varepsilon\gamma(\omega_0(\theta))\,\gamma(\omega_k(\theta))\,\textstyle\prod_{l=1}^{k-1}\alpha(\omega_l(\theta)) \quad (0<\theta<\pi) \quad \text{and}$$
$$b(\theta)=\varepsilon\,\textstyle\sum_{l=0}^{k}(-1)^{k+l}\,v_l\,\beta(\omega_l(\theta)) \qquad (0\leqq\theta\leqq\pi)\,,$$

where $\varepsilon=-1$ if $[r_k, s_k]\subset A$, $=1$ if $[r_k, s_k]\subset J(A)$. Then

$$\begin{aligned}(10.4)\quad & S_+\cap A=\{e^{i\theta};\,a(\theta)=1\}\,,\quad S_+\cap J(A)=\{e^{i\theta};\,a(\theta)=-1\}\,,\\ & S_+\cap A_0=\{e^{i\theta};\,b(\theta)=0\} \quad \textit{and} \quad b(\theta)\,(da/d\theta)=a(\theta)^2-1\;(0<\theta<\pi)\,.\end{aligned}$$

By Lemma 10.3 there is a smooth function c on $[0,\pi]$ such that

$$(10.5)\quad S_+\cap C_i=\{e^{i\theta};\,(-1)^{i-1}c(\theta)>0\}\;(i=1,2) \quad \textit{for} \quad C_i=B_i\cup J(B_i)\,.$$

Then $c=0$ on a neighbourhood of $\{0, \pi\}$, since $A_0 \cap \{\pm 1\}=\phi$ and $C_1 \cup C_2 \subset A_0 -\partial A_0$.

(10.6) *Let us set*

$$f(e^{i\theta}) = \begin{cases} a(\theta) & \text{if} \quad 0<\theta<\pi\,, \\ -a(-\theta) & \text{if} \quad -\pi<\theta<0\,, \end{cases}$$

$$g(e^{i\theta}) = \begin{cases} b(\theta)+c(\theta) & \text{if} \quad 0\leqq\theta\leqq\pi\,, \\ b(-\theta)+c(-\theta) & \text{if} \quad -\pi<\theta<0\,. \end{cases}$$

Then we have the following proposition.

Proposition 10.7. *Let* (A, B_i) *be a triad with* (A1–2). *Then the smooth functions* f *and* g *of* (10.6) *satisfy* (B1–2)′ *and* (B3). *Furthermore the triad of* (9.8) *by* f *and* g *coincides with the given one* (A, B_i).

Proof. We notice that there is a smooth function F on S^1 satisfying (B3) by Lemma 10.2(3). Therefore (B1)′ and (B3) follows immediately from definition (10.6).

For $z=e^{i\theta}\in S_+$, $a(\theta)=f(z)$ and $da/d\theta=L_z f$, whence $b(\theta)L_z f=f(z)^2-1$ by (10.4). Here $f(z)=\pm 1$ if $z\in C_1\cup C_2 (C_i=B_i\cup J(B_i))$ by (10.4) (hence $L_z f=0$), and $g(z)=b(\theta)$ if $z\notin C_1\cup C_2$ by (10.5). Thus

$$g(z)\, L_z f = f(z)^2-1 \quad \text{for any} \quad z\in S_+\,.$$

Also, by (*) in the proof of Lemma 9.3, we get

$$\begin{aligned} g(z)\, L_z f &= -g(z)\, L_{J(z)}(fJ) = g(J(z))\, L_{J(z)}(f) = f(J(z))^2-1 \\ &= f(z)^2-1 \quad \text{for any} \quad z\in J(S_+)\,. \end{aligned}$$

Therefore (B2)′ holds.

The latter half of the proposition follows from (10.4–6). *q.e.d.*

11. A-equivalence classes

In this section we show the following theorem, and prove Theorem 1.4.

Theorem 11.1. *There is a one-to-one correspondence between* A- *and* B-*equivalence classes, induced from* (9.8) *and* (10.6).

Let φ be a one-parameter transformation group on a closed interval I, and regard this as a smooth function $g(x)=[d\varphi(t, x)/dt]_{t=0} (x\in I)$. Consider the subsets of I,

$$\begin{aligned} C_i(\varphi) &= \{x\in I;\, (-1)^{i-1} g(x)>0\} \ (i = 1, 2) \quad \text{and} \\ C_0(\varphi) &= I-(C_1(\varphi)\cup C_2(\varphi))\,. \end{aligned}$$

Then $C_0(\varphi)$ ($\supset\partial I$) is the fixed point set of φ. For each $x\in C_i(\varphi)$ ($i=1, 2$), the mapping φ^x: $R\to I$, $\varphi^x(t)=\varphi(t, x)$ ($t\in R$), is diffeomorphic onto the component of $C_i(\varphi)$ containing x.

Also, let φ' be a one-parameter transformation group on a closed interval I', and assume that

(11.2) *There is an increasing homeomorphism* Φ: $I\to I'$ *such that* $\Phi(C_i(\varphi))=C_i(\varphi')$ ($i=1, 2$).

Then we have the following Lemmas 11.3–4.

Lemma 11.3. *Suppose* $I-\partial I=C_i(\varphi)$ ($i=1$ *or* 2).

(i) *Choose base points* $m\in I-\partial I$ *and* $m'\in I'-\partial I'$. *Then there exists an equivariant map* Ψ: $I\to I'$ *such that* $\Psi(m)=m'$ *and* $\Psi=\Phi$ *on* ∂I.

(ii) *Any equivariant map* Ψ: $I\to I'$ *is an increasing homeomorphism if* $\Psi=\Phi$ *on* ∂I.

Proof. (i) Since φ^m is diffeomorphic onto $I-\partial I$, the desired equivariant map is obtained by

$$\Psi(x)=\begin{cases}\Phi(x) & \text{if} \quad x\in\partial I\,,\\ \varphi'(t, m') & \text{if} \quad x=\varphi(t, m)\in I-\partial I\,.\end{cases}$$

(ii) Fix $m\in I-\partial I$, and put $m'=\Psi(m)\in I'-\partial I'$. Since Ψ is equivariant, it follows that Ψ is diffeomorphic on $I-\partial I=C_i(\varphi)$, and

$$g'(\Psi(x))=g(x)\,(d\Psi(x)/dx) \quad \text{for} \quad x\in I-\partial I\,,$$

where g and g' are the corresponding smooth functions of φ and φ' respectively. Then $d\Psi(x)/dx>0$ by (11.2). Thus Ψ is an increasing homeomorphism on $I-\partial I$, whence so on I because $\Psi=\Phi$ on ∂I and Φ is an increasing homeomorphism. *q.e.d.*

Lemma 11.4. *There exists an equivariant homeomorphism*

$$\Psi\colon I\to I' \quad \textit{such that} \quad \Psi=\Phi \quad \textit{on } C_0(\varphi)\,.$$

Proof. Put $C_1(\varphi)\cup C_2(\varphi)=\cup_{k=1}^{\infty} I_k$, the disjoint union of open intervals I_k, and let $m_k\in I_k$ and $m_k'\in\Phi(I_k)$ be the middle points. Then, by the same method as in the above lemma, we obtain an increasing equivariant bijection Ψ with $\Psi=\Phi$ on $C_0(\varphi)\cup\{m_k\}$.

We show that

(*) Ψ is right continuous at $a\in I$.

If $I_\varepsilon(a)\cap C_0(\varphi)=\phi$ for some $\varepsilon>0$, where $I_\varepsilon(a)=\{x\in I;\ a<x<a+\varepsilon\}$, then $I_\varepsilon(a)$

$\subset I_k$ for some k, and thus (*) follows from Lemma 11.3 (ii). Suppose $I_\varepsilon(a)\cap C_0(\varphi)\neq\phi$ for any $\varepsilon>0$. Then $a\in C_0(\varphi)$ and $\Psi(a)=\Phi(a)$. Choose $y_\varepsilon\in I_\varepsilon(a)\cap C_0(\varphi)$. Then, for any $a<x<y_\varepsilon$, $\Psi(x)=\Phi(x)<\Phi(y_\varepsilon)$ if $x\in C_0(\varphi)$ and $=\varphi'(t, m'_k)<\Phi(y_\varepsilon)$ if $x=\varphi(t, m_k)\in I_k$. Thus

$$0<\Psi(x)-\Psi(a)<\Phi(y_\varepsilon)-\Phi(a)<\Phi(a+\varepsilon)-\Phi(a),$$

and so (*) holds.

By the same method as above, we see that Ψ is left continuous. Therefore Ψ is continuous on I, and similarly so is Ψ^{-1} on I'. *q.e.d.*

Let (φ, f) be a pair with (B1–3), and (A, B_i) be the corresponding triad with (A1–2). By (B1), put $S^1-A_0=\cup_{l=1}^k N_l$ the disjoint union of φ-invariant open intervals with $(-1)^{j-1}\in N_j$ $(j=1, 2)$ for $A_0=A\cup J(A)$. Hence, for the middle points $m_l\in N_l$,

$$J(\{m_l\})=\{m_l\} \quad\text{and}\quad m_j=(-1)^{j-1}\ (j=1, 2). \tag{11.5}$$

Let g be the smooth function by φ, and set

$$C_i(\varphi)=\{z\in S^1;\ (-1)^{i-1}g(z)>0\} \quad\text{and so}\quad B_i=A\cap C_i(\varphi)\ (i=1, 2).$$

Also, let (φ', f') be a pair with (B1–3), (A', B'_i) be the corresponding triad with (A1–2), and assume that

(11.6) *There is an orientation preserving homeomorphism Φ of S^1 onto itself such that*

$$\Phi J=J\Phi,\ \Phi(A)=A' \quad\textit{and}\quad \Phi(B_i)=B'_i\ (i=1, 2).$$

Then we have the following Lemmas 11.7–8.

Lemma 11.7. (i) *For each $l\geqq 3$, there exists $m'_l\in\Phi(N_l)$ such that $f(m_l)=f'(m'_l)$. Furthermore $J(m_l)'=J(m'_l)$ for $1\leqq l\leqq k$, where $m'_j=\Phi(m_j)$ $(j=1, 2)$.*
(ii) $\Phi(C_i(\varphi))=C_i(\varphi')$ $(i=1, 2)$.

Proof. (i) If $l\geqq 3$, then $f(m_l)\tanh t\neq 1$ and $f'(\Phi(m_l))\tanh t\neq 1$ for any $t\in R$ by (B2). Thus $|f(m_l)|<1$ and $|f'(\Phi(m_l))|<1$, whence we can find $s\in R$ such that $\tanh s=\{f(m_l)-f'(\Phi(m_l))\}/\{f(m_l)f'(\Phi(m_l))-1\}$. Set $m'_l=\varphi'(s, \Phi(m_l))$. Then

$$m'_l\in\Phi(N_l) \quad\text{and}\quad f(m_l)=f'(m'_l) \quad\text{by (B2)}.$$

The latter half is clear for $l=1, 2$, and

$$J(m_l)'=\varphi'(-s, \Phi J(m_l))=J\varphi'(s, \Phi(m_l))=J(m'_l) \quad\text{for}\quad l\geqq 3.$$

(ii) The sign of the smooth function g (resp. g') by φ(resp. φ') is invariant on each orbit. Then, by (11.6), it is sufficient to show $g(z)g'(w)>0$ for some $z\in N_l$ and $w\in\Phi(N_l)$. Put $z=m_l$, $w=m'_l$ if $l\geqq 3$, and $z=\varphi(t, m_l)$, $w=\varphi'(t, m'_l)$ for some $t\neq 0$ if $l=1, 2$. Then $f(z)=f'(w)$, and hence

$$g(z)\, L_z f = g'(w)\, L_w f' \quad \text{by (B2)}' .$$

Here $(L_z f)\,(L_w f')>0$ follows from $\Phi(A)=A'$. Therefore (ii) holds. *q.e.d.*

Lemma 11.8. *(φ, f) is* B-*equivalent to (φ', f').*

Proof. By Lemma 11.4, there is an orientation preserving equivariant homeomorphism

$$\Psi_1\colon A \to A' \quad \text{such that} \quad \Psi_1 = \Phi \quad \text{on} \quad \partial A .$$

Also, by Lemmas 11.3 and 11.7 (ii), there is an orientation preserving equivariant homeomorphism

$$\begin{aligned} &\Psi_2\colon S^1-(A_0-\partial A_0) \to S^1-(A'_0-\partial A'_0) \quad \text{such that} \quad \Psi_2(m_l) = m'_l \quad \text{and} \\ &\Psi_2 = \Phi \ \text{ on } \partial A_0, \quad \text{for} \quad A_0 = A \cup J(A) \quad \text{and} \quad A'_0 = A' \cup J(A') . \end{aligned}$$

Then, for $z=\varphi(t, m_l)\in N_l\subset S^1-A_0$, we get

$$\begin{aligned} J\Psi_2(z) &= J\varphi'(t, \Psi_2(m_l)) = \varphi'(-t, J(m'_l)) = \varphi'(-t, J(m_l)') \\ &= \varphi'(-t, \Psi_2 J(m_l)) = \Psi_2\, \varphi(-t, J(m_l)) = \Psi_2 J(z) \quad \text{and} \\ f(z) &= f'(\varphi'(t, m'_l)) = f'\,\Psi_2(t, m_l) = f'\,\Psi_2(z) \ \text{ when } z\neq\pm 1 . \end{aligned}$$

Therefore the homeomorphism Ψ of S^1 onto itself,

$$\Psi = \Psi_1 \text{ on } A\,, \quad = J\Psi_1 J \text{ on } J(A) \quad \text{and} \quad = \Psi_2 \text{ on } S^1-A_0\,,$$

gives a B-equivalence between (φ, f) and (φ', f'). *q.e.d.*

Proposition 11.9. *Pairs with* (B1–3) *are* B-*equivalent if the corresponding triads with* (A1–2) *are* A-*equivalent.*

Proof. Let (φ, f) and (φ', f') be pairs with (B1–3), and assume that the corresponding triads (A, B_i) and (A', B'_i) are A-equivalent by an orientation preserving homeomorphism Φ of S^1 onto itself; $\Phi J=J\Phi$ and

(1) $\Phi(A) = A'$, $\Phi(B_i) = B'_i$ or (2) $\Phi(A) = J(A')$, $\Phi(B_i) = J(B'_{3-i})$ $(i = 1, 2)$.

For the case (1), the proposition follows from Lemma 11.8.

In case (2), consider the one-parameter transformation group on S^1, $\varphi''(t, z)=\varphi'(-t, z)$ $(t\in R, z\in S^1)$. Then $(\varphi'', -f')$ satisfies (B1–3), and is B-equivalent to (φ', f') by the reflection J. Furthermore its corresponding triad is given by $(J(A'), J(B'_{3-i}))$. Therefore $(\varphi'', -f')$ is B-equivalent to (φ, f) by Lemma 11.8, and hence so is (φ', f'). *q.e.d.*

Proof of Theorem 11.1. By Proposition 9.7, there is a mapping from B-equivalence classes to A-equivalence ones, which is surjective by Proposition 10.7 and injective by Proposition 11.9. Therefore the theorem holds. *q.e.d.*

Proof of Theorem 1.4. The theorem follows from Theorems 8.1 and 11.1. *q.e.d.*

Proof of Corollary 1.5. (i) Since the $G=SL(2, C)$-action of (7.1) has no fixed points by (7.3), the first half holds. The latter half follows from Theorem 1.2.

(ii–iii) For a non-transitive smooth G-action ϕ on S^3, there corresponds a pair (φ, f) with (B1–3), and so a triad (A, B_i) with (A1–2). If ϕ is real analytic, then so is f, and hence A is finite and $B_i=\phi(i=1, 2)$. Thus Theorem 1.4 shows that the equivariant homeomorphism class of ϕ is determined by the order $|A|$ of A. Moreover, the action of (7.1) constructed from (φ, f) has precisely $2|A|+1$ orbits. These imply (ii) and (iii). *q.e.d.*

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