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SOME SYMPLECTIC GEOMETRY ON COMPACT KÄHLER MANIFOLDS (I)

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0. Introduction

In real Riemannian geometry, the space of all Riemannian metrics of a given compact differentiable manifold admits a Riemannian structure*) to provide us with several nice theories. In this paper, we shall seek its complex analogue. Namely, in view of the fact that all Kähler manifolds are symplectic, we shall define a very natural Riemannian structure (slightly different from classical ones) on the space of all Kähler metrics in a fixed cohomology class of a given compact Kähler manifold (see also [10] for more algebraic geometric treatments).

Throughout this paper, we fix an n-dimensional compact complex connected manifold X with a cohomology class $h \in H^{1,1}(X)_{\boldsymbol{R}}$ such that

$$\mathcal{K}:=\{\omega \mid \omega \text{ is a Kähler form on } X \text{ in the class } h\}$$

is nonempty. Let $\omega_0 \in \mathcal{K}$ and consider the $\mathcal{K}$-energy map $\mu\colon \mathcal{K} \to \boldsymbol{R}$ of the Kähler manifold (X, ω_0) introduced in [9]. Now the main purpose of this paper is to define a natural Riemannian structure on $\mathcal{K}$ such that

(0.1) *μ is a convex function on $\mathcal{K}$*, i.e., Hess *μ is positive semidefinite everywhere on $\mathcal{K}$* (cf. §5);

(0.2) *sectional curvature of $\mathcal{K}$ is explicitly written in terms of Poisson brackets of functions and moreover it is always nonpositive* (cf. §4).

We next assume that

$$\mathcal{E}:=\{\omega \in \mathcal{K} \mid \omega \text{ has a constant scalar curvature}\}$$

is nonempty. Recall that the Albanese map $\alpha\colon X \to \mathrm{Alb}(X)$ of X naturally induces the Lie group homomorphism $\tilde{\alpha}\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(\mathrm{Alb}(X))$ $(\cong \mathrm{Alb}(X))$, where $\mathrm{Aut}^0(X)$ (resp. $\mathrm{Aut}^0(\mathrm{Alb}(X))$) denotes the identity component of the

*) See, for instance, Ebin's article "The manifold of Riemannian metrics" in Global Analysis (Proc. Symp. Pure Math.) 15 (1968), 11–40.

group of holomorphic automorphisms of X (resp. Alb(X)). Then by a theorem of Fujiki [5], the identity component G of Ker $\tilde{\alpha}$ has a natural structure of a linear algebraic group. Let K be a maximal compact subgroup of G, and we decompose $\mathcal{E}$ into G-orbits:

$$\mathcal{E} = \cup_{i\in I}\mathcal{E}_i \quad \text{(disjoint union)}.$$

In view of a theorem of Lichnerowicz [8], one sees that:

i) *G is a reductive algebraic group,*
ii) *each $\mathcal{E}_i$ is an $Aut^0(X)$-orbit, and*
iii) *there exist $\theta_i\in\mathcal{E}_i$, $i\in I$, such that the isotropy subgroup of G at each θ_i coincides with K.*

Then a combination of ii) with a result of Calabi [4] shows that each $\mathcal{E}_i$ is a connected component of $\mathcal{E}$ in terms of a suitable topology of $\mathcal{E}$. Furthermore by iii), such a connected component $\mathcal{E}_i$ of $\mathcal{E}$ is G-equivariantly diffeomorphic to the Riemannian symmetric space G/K. Now, restricting our Riemannian structure of $\mathcal{K}$ to $\mathcal{E}$, we obtain:

(0.3) *each $\mathcal{E}_i$ is isometric to the Riemannian symmetric space G/K endowed with a suitable metric, and furthermore, $Aut^0(X)$ acts isometrically on $\mathcal{E}_i$* (cf. §6).

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1. Notation, convention and preliminaries

(1.1) Fix an element ω_0 of $\mathcal{K}$ once for all and express it as

$$\omega_0 = \sqrt{-1}\sum g_{\alpha\bar\beta}\, dz^\alpha \wedge dz^{\bar\beta}$$

in terms of holomorphic local coordinates $(z^1, \cdots, z^n)$ of X. For each real-valued C^∞ function $\varphi\in C^\infty(X)_{\mathbf{R}}$ on X, we put $\omega_0(\varphi):=\omega_0+\sqrt{-1}\,\partial\bar\partial\varphi$, and write it in the form

$$\omega_0(\varphi) = \sqrt{-1}\sum g(\varphi)_{\alpha\bar\beta}\, dz^\alpha \wedge dz^{\bar\beta},$$

where $g_{\alpha\bar\beta}(\varphi)=g_{\alpha\bar\beta}+\varphi_{\alpha\bar\beta}$, $\varphi_{\alpha\bar\beta}$ being $\partial_\alpha\partial_{\bar\beta}\,\varphi(=\partial^2\varphi/\partial z^\alpha\partial z^{\bar\beta})$. We furthermore denote by $\sum \mathrm{Ric}(\varphi)_{\alpha\bar\beta}\, dz^\alpha\otimes dz^{\bar\beta}$ the Ricci tensor of the Kahler form $\omega_0(\varphi)$. Put

$$\mathrm{Ric}(\varphi) := \sqrt{-1}\sum \mathrm{Ric}(\varphi)_{\alpha\bar\beta}\, dz^\alpha \wedge dz^{\bar\beta}.$$

Then $\mathrm{Ric}(\varphi)/2\pi$ represents $c_1(X)_{\boldsymbol{R}}$ and we have $\mathrm{Ric}(\varphi)=\sqrt{-1}\,\bar\partial\partial\log\det(g_{\alpha\bar\beta}(\varphi))$. Let $\sigma(\varphi)$ (resp. $\square_\varphi$) be the corresponding scalar curvature (resp. Laplacian on functions):

$$\sigma(\varphi):=\sum g(\varphi)^{\bar\beta\alpha}\,\mathrm{Ric}(\varphi)_{\alpha\bar\beta}\,,$$
$$\square_\varphi\ :=\sum g(\varphi)^{\bar\beta\alpha}\,\partial^2/\partial z^\alpha\partial z^{\bar\beta}\,,$$

where $(g(\varphi)^{\bar\beta\alpha})$ is the inverse matrix of $(g(\varphi)_{\alpha\bar\beta})$. We now put

$$\mathcal{H}:=\{\varphi\in C^\infty(X)_{\boldsymbol{R}}\,|\,\omega_0(\varphi)\in\mathcal{K}\}\,.$$

Note that the natural map

$$\mathcal{H}\to\mathcal{K}\,,\quad \varphi\mapsto\omega_0(\varphi)$$

is surjective.

(1.2) (i) A mapping Φ: $t\in[a,b]\mapsto\varphi_t\in C^\infty(X)_{\boldsymbol{R}}$ (often denoted by $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$) is said to be *smooth* (or a *smooth path*) if the mapping φ: $[a,b]\times X\mapsto\boldsymbol{R}$ defined by

$$\varphi(t,x):=\varphi_t(x)\,,\qquad (t,x)\in[a,b]\times X\,,$$

is a C^∞ map. For such a smooth path $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$, we put $\dot\varphi_t:=\partial\varphi_t/\partial t$ and $\ddot\varphi_t:=\partial^2\varphi_t/\partial t^2$. Then the corresponding paths $\{\dot\varphi_t\,|\,a\leqq t\leqq b\}$, $\{\ddot\varphi_t\,|\,a\leqq t\leqq b\}$ in $C^\infty(X)_{\boldsymbol{R}}$ are again smooth. We furthermore define $\dot\varphi\in C^\infty([a,b]\times X)_{\boldsymbol{R}}$ by

$$\dot\varphi(t,x):=\dot\varphi_t(x)=(\partial\varphi/\partial t)\,(t,x)\,,\quad (t,x)\in[a,b]\times X\,.$$

If there is no fear of confusion, φ and φ_t (resp. $\dot\varphi$ and $\dot\varphi_t$) are used interchangeably. (To be precise, $\varphi_t=\varphi_{|\{t\}\times X}\in C^\infty(X)_{\boldsymbol{R}}$ and $\dot\varphi_t=\dot\varphi_{|\{t\}\times X}\in C^\infty(X)_{\boldsymbol{R}}$ via the identification of $\{t\}\times X$ with X.)

(ii) A mapping θ: $t\in[a,b]\mapsto\theta_t\in\mathcal{K}$ (often denoted by $\Theta=\{\theta_t\,|\,a\leqq t\leqq b\}$) is said to be *smooth* (or a *smooth path*) if there exists a smooth path $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$ in $\mathcal{H}$ such that $\theta_t=\omega_0(\varphi_t)$. Note that the concept of smoothness of paths in $\mathcal{K}$ doesn't depend on the choice of ω_0. To each such smooth path $\Theta=\{\theta_t\,|\,a\leqq t\leqq b\}$, we associate a C^∞ (1,1)-form θ on $[a,b]\times X$ by

$$\theta(t,x)=\theta_t(x)\,,\quad (t,x)\in[a,b]\times X\,.$$

We put $\dot\theta_t=\partial\theta_t/\partial t$, and let $\dot\theta$ be the C^∞ (1,1)-form on $[a,b]\times X$ defined by

$$\dot\theta(t,x):=\dot\theta_t(x)\,,\quad (t,x)\in[a,b]\times X\,.$$

(1.3) (cf. [9]). For each $\varphi\in\mathcal{H}$, we set $\Omega_0(\varphi):=\omega_0(\varphi)^n/n!$. We then define the real constants $\mathrm{Vol}(X)$ and σ_0 (which depend only on the class h) as follows:

$$\mathrm{Vol}(X) := \int_X \Omega_0(0)\,,$$
$$\sigma_0 \quad := 2\pi\int_X c_1(X)\omega_0^{n-1}/((n-1)!\mathrm{Vol}(X))\,.$$

To each pair $(\varphi', \varphi'')\in\mathcal{H}\times\mathcal{H}$ (resp. $(\varphi', \varphi'')\in C^\infty(X)_{\boldsymbol{R}}\times C^\infty(X)_{\boldsymbol{R}}$), we associate a real number $M(\varphi', \varphi'')$ (resp. $L(\varphi', \varphi'')$) by

$$M(\varphi', \varphi'') := -\int_a^b\{\int_X \dot\varphi_t(\sigma(\varphi_t)-\sigma_0)\Omega_0(\varphi_t)/\mathrm{Vol}(X)\}\,dt\,, \tag{1.3.1}$$

$$(\text{resp. } L(\varphi', \varphi'') := \int_a^b\{\int_X \dot\varphi_t\Omega_0(\varphi_t)/\mathrm{Vol}(X)\}\,dt)\,, \tag{1.3.2}$$

where $\{\varphi_t\,|\,a\leqq t\leqq b\}$ is an arbitrary piecewise smooth path in $\mathcal{H}$ (resp. $C^\infty(X)_{\boldsymbol{R}}$) such that $\varphi_a=\varphi'$ and $\varphi_b=\varphi''$. Then $L(\varphi', \varphi'')$ (resp. $M(\varphi, \varphi'')$) is independent of the choice of the path $\{\varphi_t\,|\,a\leqq t\leqq b\}$ and therefore well-defined. Recall that M (resp. L) satisfies the 1-cocycle condition. Furthermore,

$$M(\varphi_1+C_1, \varphi_2+C_2) = M(\varphi_1, \varphi_2)\,, \tag{1.3.3}$$

$$(\text{resp. } L(\varphi_1, \varphi_2+C) = L(\varphi_1-C, \varphi_2) = L(\varphi_1, \varphi_2)+C)\,, \tag{1.3.4}$$

for all $\varphi_1, \varphi_2\in\mathcal{H}$ (resp. $\varphi_1, \varphi_2\in C^\infty(X)_{\boldsymbol{R}}$) and all $C_1, C_2\in\boldsymbol{R}$ (resp. $C\in\boldsymbol{R}$). In view of (1.3.3) above, M: $\mathcal{H}\times\mathcal{H}\to\boldsymbol{R}$ factors through $\mathcal{K}\times\mathcal{K}$. Hence we can define the mapping M: $\mathcal{K}\times\mathcal{K}\to\boldsymbol{R}$ (denoted by the same M) by

$$M(\omega', \omega'') := M(\varphi', \varphi'') \quad (\omega', \omega''\in\mathcal{K})\,,$$

where φ', φ'' are elements of $\mathcal{H}$ such that $\omega_0(\varphi')=\omega'$ and $\omega_0(\varphi'')=\omega''$. Then the mapping

$$\mu\colon \mathcal{K}\to\boldsymbol{R}\,,\quad \omega\mapsto\mu(\omega) := M(\omega_0, \omega)$$

is called the $\mathcal{K}$-energy map of the Kähler manifold (X, ω_0). Moreover we put

$$\tilde{\mathcal{H}} := \{\varphi\in\mathcal{H}\,|\,L(0, \varphi)=0\}\,.$$

We now have the following identifications:

$$\begin{aligned} \tilde{\mathcal{H}} &\cong \mathcal{K} \\ \varphi &\leftrightarrow \omega_0(\varphi)\,, \end{aligned} \tag{1.3.5}$$

$$\begin{array}{ccccc} \mathcal{H} \cong & \mathcal{K}\times\boldsymbol{R} & \cong & \tilde{\mathcal{H}}\times\boldsymbol{R} \\ \varphi\leftrightarrow & (\omega_0(\varphi), L(0, \varphi)) & \leftrightarrow & (\varphi-L(0, \varphi), L(0, \varphi))\,. \end{array} \tag{1.3.6}$$

(1.4) At each point ξ of $\mathcal{H}$, we can identify $C^\infty(X)_{\boldsymbol{R}}$ with the tangent space $T\mathcal{H}_\xi$ of $\mathcal{H}$ at ξ via the isomorphism

(1.4.1)
$$C^\infty(X)_{\boldsymbol{R}} \cong T\mathcal{H}_\xi$$
$$\eta \leftrightarrow \frac{\partial}{\partial s}|_{s=0}\,(\xi+s\eta)$$

where $s\in[-\varepsilon,\varepsilon]\mapsto\xi+s\eta\in\mathcal{H}$ is a smooth path in $\mathcal{H}$ with a sufficiently small $\varepsilon>0$. In terms of this identification, and also by (1.3.5), we have

(1.4.2)
$$T\mathcal{K}_{\omega_0(\xi)} \cong T\tilde{\mathcal{H}}_\xi = \{\eta\in C^\infty(X)_{\boldsymbol{R}} \mid \int_X \eta\Omega_0(\xi)=0\}$$
$$\sqrt{-1}\ \partial\bar{\partial}\eta \leftrightarrow \eta\,,$$

whenever $\xi\in\tilde{\mathcal{H}}$. Note here that
$$\sqrt{-1}\ \partial\bar{\partial}\eta = \frac{\partial}{\partial t}|_{t=0}\,\omega_0(\xi+t\eta)\,.$$

Let $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ be a smooth path in $\mathcal{H}$. We denote by $\Gamma_{\mathrm{diff}}([a,b],\Phi^*T\mathcal{H})$ the space of (real) C^∞ sections of the induced bundle $\phi^*T\mathcal{H}$ of the tangent bundle $T\mathcal{H}$ of $\mathcal{H}$. Then $\Gamma_{\mathrm{diff}}([a,b],\Phi^*T\mathcal{H})$ is naturally identified with $C^\infty([a,b]\times X)_{\boldsymbol{R}}$ via the isomorphism

(1.4.3)
$$C^\infty([a,b]\times X)_{\boldsymbol{R}} \cong \Gamma_{\mathrm{diff}}([a,b],\Phi^*T\mathcal{H})$$
$$\psi \leftrightarrow \Psi = \{\psi_t \mid a\leqq t\leqq b\}\,,$$

where ψ_t denotes, for each t, the function in $C^\infty(X)_{\boldsymbol{R}}$ defined by
$$\psi_t(x) := \psi(t,x) \quad (x\in X)\,,$$
and is regarded as an element of $T\mathcal{H}_{\varphi_t}$ in terms of the isomorphism of (1.4.1).

(1.5) Let $\xi\in\mathcal{H}$. We then define the linear maps $V_\xi\colon C^\infty(X)_{\boldsymbol{C}}\to\Gamma_{\mathrm{diff}}(X,TX)_{\boldsymbol{C}}$ and $W_\xi\colon C^\infty(X)_{\boldsymbol{R}}\to\Gamma_{\mathrm{diff}}(X,TX)_{\boldsymbol{R}}$ by
$$V_\xi(\eta) := (1/2)\sum g(\xi)^{\bar{\beta}\alpha}\eta_{\bar{\beta}}\,\partial/\partial z^\alpha \qquad (\eta\in C^\infty(X)_{\boldsymbol{C}})\,,$$
$$W_\xi(\eta) := (\sqrt{-1}/2)\sum g(\xi)^{\bar{\beta}\alpha}(\eta_\alpha\partial/\partial z^{\bar{\beta}} - \eta_{\bar{\beta}}\partial/\partial z^\alpha) \quad (\eta\in C^\infty(X)_{\boldsymbol{R}})\,,$$
in terms of holomorphic local coordinates $(z^1,\cdots,z^n)$ on X, where $\eta_\alpha := \partial_\alpha\eta = \partial\eta/\partial z^\alpha$ and $\eta_{\bar{\beta}} := \partial_{\bar{\beta}}\eta = \partial\eta/\partial z^{\bar{\beta}}$. To each pair $(\eta',\eta'')\in C^\infty(X)_{\boldsymbol{R}}\times C^\infty(X)_{\boldsymbol{R}}$, we associate a function $[\eta',\eta'']_\xi\in C^\infty(X)_{\boldsymbol{R}}$ by

(1.5.1)
$$[\eta',\eta'']_\xi := (W_\xi(\eta'))\,(\eta'')\,.$$

Recall that $[\ ,\ ]_\xi$ is nothing but the Poisson bracket of C^∞ functions on the symplectic manifold $(X,\omega_0(\xi))$, and the mapping $W_\xi\colon C^\infty(X)_{\boldsymbol{R}}\to\Gamma_{\mathrm{diff}}(X,TX)_{\boldsymbol{R}}$ is a Lie algebra homomorphism. Hence for all $\eta,\eta',\eta''\in C^\infty(X)_{\boldsymbol{R}}$, we have:

(1.5.2)
$$W_\xi([\eta',\eta'']_\xi) = [W_\xi(\eta'),W_\xi(\eta'')]\,,$$

(1.5.3)
$$\int_X [\eta,\eta']_\xi\,\eta''\Omega_0(\xi) = \int_X \eta[\eta',\eta'']_\xi\Omega_0(\xi)\,.$$

(1.6) (See Calabi [4] and also Bando [1]), For each $\xi\in\mathcal{H}$, let $\langle\ ,\ \rangle_\xi$: {p-forms on X} $\times$ {p-forms on X} $\to C^\infty(X)_{\boldsymbol{R}}$, $p=1,2,\cdots,n$, be the natural Hermitian pairings induced from the Kähler metric $\omega_0(\xi)$. We now consider the operator $\boldsymbol{L}_\xi$: $C^\infty(X)_{\boldsymbol{R}}\to C^\infty(X)_{\boldsymbol{R}}$ of Lichnerowicz [8] defined by

$$(1.6.1)\qquad \boldsymbol{L}_\xi\psi := (\square_\xi)^2\psi+\langle\sqrt{-1}\,\partial\bar\partial\psi,\ \mathrm{Ric}(\xi)\rangle_\xi+\langle\bar\partial\psi,\ \bar\partial\sigma(\xi)\rangle_\xi\,,$$

with $\psi\in C^\infty(X)_{\boldsymbol{R}}$. Recall that, in view of Calabi [4; p. 100],

$$4\|\bar\partial V_\xi(\psi)\|^2_{L^2(X,\omega_0(\xi))}=\int_X\psi(\boldsymbol{L}_\xi\psi)\Omega_0(\xi)\,.$$

Then, taking the real parts of both sides, we obtain

$$(1.6.2)\qquad 4\|\bar\partial V_\xi(\psi)\|^2_{L^2(X,\omega_0(\xi))}=\int_X\psi(Re\,\boldsymbol{L}_\xi\psi)\Omega_0(\xi)$$

for all $\psi\in C^\infty(X)_{\boldsymbol{R}}$ and $\xi\in\mathcal{H}$, where

$$(1.6.3)\qquad \mathrm{Re}\,\boldsymbol{L}_\xi\psi=\frac{1}{2}(\boldsymbol{L}_\xi\psi+\overline{\boldsymbol{L}_\xi\psi})\,.$$

(1.7) A *Euclidean lattice* is, by abuse of terminology, a triple $(\mathfrak{t}, \Lambda, ((\ ,\ )))$ of an $\boldsymbol{R}$-vector space $\mathfrak{t}$, its lattice Λ (so that $\mathfrak{t}=\Lambda\otimes_{\boldsymbol{Z}}\boldsymbol{R}$), and a positive definite symmetric $\boldsymbol{R}$-bilinear form $((\ ,\ ))$ on $\mathfrak{t}$. Two Euclidean lattices $(\mathfrak{t}', \Lambda', ((\ ,\ ))')$, $(\mathfrak{t}'', \Lambda'', ((\ ,\ ))'')$ are called *isometric* if there exists a bijective $\boldsymbol{R}$-linear map j: $\mathfrak{t}'\cong\mathfrak{t}''$ such that

i) $j(\Lambda')=\Lambda''$, and

ii) $((j(\alpha), j(\beta)))''=((\alpha,\beta))'$ for all $\alpha, \beta, \in\mathfrak{t}'$.

For Euclidean lattices $(\mathfrak{t}_\nu, \Lambda_\nu, ((\ ,\ ))_\nu)$, $\nu=1, 2, \cdots, r$, we have their *direct sum* $\oplus_{\nu=0}^r(\mathfrak{t}_\nu, \Lambda_\nu, ((\ ,\ ))_\nu)$ which is just the Euclidean lattice $(\oplus_{\nu=0}^r\mathfrak{t}_\nu, \oplus_{\nu=0}^r\Lambda_\nu, \oplus_{\nu=0}^r((\ ,\ ))_\nu)$.

2. Natural Riemannian structure of $\mathcal{H}$ and $\mathcal{K}$

This section is crucial in our later study of the geometry of $\mathcal{H}$ and $\mathcal{K}$. Especially, a natural Riemannian "metric" on $\mathcal{H}$ (and also on $\mathcal{K}$) together with the compatible connection will be defined.

(2.1) We regard $\mathcal{H}$ as a "Riemannian manifold" by defining the bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle_\xi$: $T\mathcal{H}_\xi\times T\mathcal{H}_\xi$ $(=C^\infty(X)_{\boldsymbol{R}}\times C^\infty(X)_{\boldsymbol{R}})\to\boldsymbol{R}$ for each $\xi\in\mathcal{H}$ as follows:

$$(2.1.1)\qquad \langle\!\langle\eta_1,\eta_2\rangle\!\rangle_\xi:=\int_X\eta_1\eta_2\Omega_0(\xi)/\mathrm{Vol}(X)\,,\quad \eta_1,\eta_2\in C^\infty(X)_{\boldsymbol{R}}\,,$$

(see (1.4.1) for the identification of $T\mathcal{H}_\xi$ with $C^\infty(X)_{\boldsymbol{R}}$). The restriction of this pairing $\langle\!\langle\ ,\ \rangle\!\rangle_\xi$ (where $\xi\in\tilde{\mathcal{H}}$) to $T\tilde{\mathcal{H}}_\xi$ (cf. (1.4.2)) is again denoted by the same

$\langle\!\langle\ ,\ \rangle\!\rangle_{\xi}$, and in terms of this, $\tilde{\mathcal{H}}$ is also a "Riemannian manifold". We furthermore endow $\boldsymbol{R}$ with the Euclidean metric $\langle\!\langle\ ,\ \rangle\!\rangle$ by the formula

$$\langle\!\langle a, b\rangle\!\rangle = ab \quad \text{for all } a, b\in\boldsymbol{R}\,.$$

Then the isomorphism (cf. (1.3.6))

$$\begin{aligned}\mathcal{H} &\cong \tilde{\mathcal{H}}\times\boldsymbol{R}\\ \varphi &\leftrightarrow (\varphi-L(0,\varphi),\ L(0,\varphi))\end{aligned}$$

is an "isometry of Riemannian manifolds". Now, in view of (1.3.5), $\mathcal{K}$ is also a "Riemannian manifold". Namely, for each $\omega\in\mathcal{K}$, we define the bilinear pairing $\langle\!\langle\ ,\ \rangle\!\rangle_{\omega}$: $T\mathcal{K}_{\omega}\times T\mathcal{K}_{\omega}\to\boldsymbol{R}$ by

$$\text{(2.1.2)}\qquad \langle\!\langle\sqrt{-1}\,\partial\bar\partial\eta_1,\ \sqrt{-1}\,\partial\bar\partial\eta_2\rangle\!\rangle_{\omega} := \int_X \eta_1\eta_2\omega^n/(n!\,\mathrm{Vol}(X))\,,$$

where $\eta_1, \eta_2\in\{\eta\in C^{\infty}(X)_{\boldsymbol{R}}\,|\int_X\eta\omega^n=0\}$ $(\cong T\mathcal{K}_{\omega})$ (cf. (1.4.2)). (Note that this pairing is independent of the choice of ω_0).

(2.2) Let $\{\varphi_t\,|\,a\leqq t\leqq b\}$ be a smooth path in $\mathcal{H}$. Recall that we have the function $\varphi\in C^{\infty}([a,b]\times X)_{\boldsymbol{R}}$ defined by

$$\text{(2.2.1)}\qquad \varphi(t,x)=\varphi_t(x)\,,\qquad (t,x)\in[a,b]\times X\,.$$

To elements $\psi=\psi(t,x)$, $\eta=\eta(t,x)$ in $C^{\infty}([a,b]\times X)_{\boldsymbol{R}}$, we associate $\langle\!\langle\psi,\eta\rangle\!\rangle_{\varphi}\in C^{\infty}([a,b])_{\boldsymbol{R}}$ by

$$\text{(2.2.2)}\qquad \langle\!\langle\psi,\eta\rangle\!\rangle_{\varphi}(t) := \int_X\psi_t\eta_t\Omega_0(\varphi_t)/\mathrm{Vol}(X) = \langle\!\langle\psi_t,\eta_t\rangle\!\rangle_{\varphi_t}\,,$$

where for each $t\in[a,b]$, ψ_t and η_t are the functions in $C^{\infty}(X)_{\boldsymbol{R}}$ defined by

$$\text{(2.2.3)}\qquad \psi_t := \psi\,|_{\{t\}\times X} \text{ and } \eta_t := \eta\,|_{\{t\}\times X}$$

via the identification of $\{t\}\times X$ with X.

(2.3) (i) For a piecewise smooth path $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$ in $\mathcal{H}$, we define its *arclength* $\mathrm{Lgth}(\Phi)$ and *energy* $\mathrm{Engy}(\Phi)$ as follows:

$$\text{(2.3.1)}\qquad \mathrm{Lgth}(\Phi) := \int_a^b\Big(\int_X(\dot\varphi_t)^2\Omega_0(\varphi_t)/\mathrm{Vol}(X)\Big)^{1/2}dt = \int_a^b(\langle\!\langle\dot\varphi,\dot\varphi\rangle\!\rangle_{\varphi})^{1/2}dt\,,$$

$$\text{(2.3.2)}\qquad \mathrm{Engy}(\Phi) := \int_a^b\Big(\int_X(\dot\varphi_t)^2\Omega_0(\varphi_t)/\mathrm{Vol}(X)\Big)dt = \int_a^b\langle\!\langle\dot\varphi,\dot\varphi\rangle\!\rangle_{\varphi}dt\,.$$

(ii) Let $\Theta=\{\theta_t\,|\,a\leqq t\leqq b\}$ be a piecewise smooth path in $\mathcal{K}$. We then define the real numbers $\mathrm{Lgth}(\Theta)$, $\mathrm{Engy}(\Theta)$ by

$$\text{(2.3.3)}\qquad \mathrm{Lgth}(\Theta) := \mathrm{Lgth}(\Phi) \text{ and } \mathrm{Engy}(\Theta) := \mathrm{Engy}(\Phi)\,,$$

where $\Phi=\{\varphi_t | a\leqq t\leqq b\}$ is the unique piecewise smooth path in $\tilde{\mathcal{H}}$ such that $\omega_0(\varphi_t)=\theta_t$ for all t.

(2.4) We shall next define the corresponding "Riemannian connection" of $\mathcal{H}$. Fix an arbitrary smooth path $\Phi=\{\varphi_t | a\leqq t\leqq b\}$ in $\mathcal{H}$. Using the notation of (i) of (1.2), we define the real vector field $\frac{D}{\partial t}$ on $[a, b]\times X$ by

$$\frac{D}{\partial t}:=\partial/\partial t-\frac{1}{2}\sum g(\varphi)^{\bar{\beta}\alpha}(\dot{\varphi}_{\alpha}\partial/\partial\bar{z}^{\beta}+\dot{\varphi}_{\bar{\beta}}\partial/\partial z^{\alpha}). \tag{2.4.1}$$

Note here that, though φ is in $C^{\infty}([a, b]\times X)_{\boldsymbol{R}}$ (instead of $C^{\infty}(X)_{\boldsymbol{R}}$), we can still define $(g(\varphi)^{\bar{\beta}\alpha})$ as the inverse matrix of $(g_{\alpha\bar{\beta}}+\varphi_{\alpha\bar{\beta}})$. Now via the identification of $C^{\infty}([a, b]\times X)_{\boldsymbol{R}}$ with $\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H})$ (cf. (1.4.3)), we define

$$\nabla_{\dot{\varphi}}\colon \Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H})\to\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H}) \tag{2.4.2}$$

as the operator induced by $\frac{D}{\partial t}$ from the following commutative diagram:

$$\begin{array}{ccc}\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H}) & \cong & C^{\infty}([a, b]\times X)_{\boldsymbol{R}} \\ \nabla_{\dot{\varphi}}\downarrow & & \downarrow\frac{D}{\partial t} \\ \Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H}) & \cong & C^{\infty}([a, b]\times X)_{\boldsymbol{R}}\,. \end{array} \tag{2.4.3}$$

The operator $\nabla_{\dot{\varphi}}$ (resp. $\frac{D}{\partial t}$) is called the *covariant differentiation* on $\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H})$ (resp. $C^{\infty}([a, b]\times X)_{\boldsymbol{R}}$) along the path $\Phi=\{\varphi_t | a\leqq t\leqq b\}$.

DEFINITION 2.4.4. $\psi\in C^{\infty}([a, b]\times X)_{\boldsymbol{R}}$ is said to be *parallel* along $\Phi=\{\varphi_t | a\leqq t\leqq b\}$ if $\frac{D}{\partial t}\psi=0$ in $C^{\infty}([a, b]\times X)_{\boldsymbol{R}}$.

DEFINITION 2.4.5. Let $\xi\in C^{\infty}(X)_{\boldsymbol{R}}(\cong T\mathcal{H}_{\varphi_a})$. Then $\psi=\psi(t, x)\in C^{\infty}([a, b]\times X)_{\boldsymbol{R}}$ is said to be a *parallel translation* of ξ along $\Phi=\{\varphi_t | a\leqq t\leqq b\}$ if the following conditions are satisfied:

i) $\psi|_{t=a}=\xi$;
ii) ψ is parallel along Φ.

Note that, for each $\xi\in C^{\infty}(X)_{\boldsymbol{R}}$, there exists a unique parallel translation of ξ along Φ. In fact, denoting by $g_s:=\exp(s\frac{D}{\partial t})$ $(a-t\leqq s\leqq b-t)$ the local 1-parameter group of $[a, b]\times X$ generated by $\frac{D}{\partial t}$, one can easily see that

$$\psi(t, x):=\xi(pr_2(g_{a-t}\cdot(t, x))),\quad (t, x)\in[a, b]\times X,$$

is the unique parallel translation of ξ along Φ (where $pr_2\colon [a, b]\times X\to X$ denotes the projection to the second factor).

We shall now show that our connection is compatible with the "Riemannian metric" defined in (2.1) and (2.2).

Theorem 2.5. *Let $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ be a smooth path in $\mathcal{H}$. Then in terms of the notation in* (2.2) *and* (2.4), *we have*

$$\frac{d}{dt}\langle\!\langle\psi, \eta\rangle\!\rangle_\varphi = \langle\!\langle\frac{D}{\partial t}\psi, \eta\rangle\!\rangle_\varphi+\langle\!\langle\psi, \frac{D}{\partial t}\eta\rangle\!\rangle_\varphi$$

for all $\psi, \eta\in C^\infty([a, b]\times X)_{\boldsymbol{R}}$.

Proof. We first observe that, though φ is in $C^\infty([a, b]\times X)_{\boldsymbol{R}}$ (instead of $C^\infty(X)_{\boldsymbol{R}}$), we still have the following notational analogue of (1.1):

$$\Omega_0(\varphi) = \omega_0(\varphi)^n/n! := (\omega_0+\sqrt{-1}\,\partial\bar\partial\,\varphi)^n/n!\,,$$
$$\square_\varphi := \sum g(\varphi)^{\bar\beta\alpha}\partial^2/\partial z^\alpha\partial \bar z^\beta, \text{ (cf. (2.4))}\,,$$

where $\Omega_0(\varphi)$ (resp. $\square_\varphi$) is regarded as a C^∞ $2n$-form on $[a, b]\times X$ (resp. an operator on $C^\infty([a, b]\times X)_{\boldsymbol{R}}$). Then

$$\begin{aligned}
&\langle\!\langle\frac{D}{\partial t}\psi, \eta\rangle\!\rangle_\varphi+\langle\!\langle\psi, \frac{D}{\partial t}\eta\rangle\!\rangle_\varphi\\
&=\int_X\{\dot\psi-2^{-1}\sum g(\varphi)^{\bar\beta\alpha}(\dot\varphi_\alpha\psi_{\bar\beta}+\dot\varphi_{\bar\beta}\psi_\alpha)\}\,\eta\Omega_0(\varphi)/\mathrm{Vol}(X)\\
&+\int_X\{\dot\eta-2^{-1}\sum g(\varphi)^{\bar\beta\alpha}(\dot\varphi_\alpha\eta_{\bar\beta}+\dot\varphi_{\bar\beta}\eta_\alpha)\}\,\psi\Omega_0(\varphi)/\mathrm{Vol}(X)\\
&=\int_X\{\dot\psi-2^{-1}(\square_\varphi(\dot\varphi\psi)-(\square_\varphi\dot\varphi)\psi-(\square_\varphi\psi)\dot\varphi)\}\,\eta\Omega_0(\varphi)/\mathrm{Vol}(X)\\
&+\int_X\{\dot\eta-2^{-1}(\square_\varphi(\dot\varphi\eta)-(\square_\varphi\dot\varphi)\eta-(\square_\varphi\eta)\dot\varphi)\}\,\psi\Omega_0(\varphi)/\mathrm{Vol}(X)\\
&=\int_X\{\frac{\partial}{\partial t}(\psi\eta)+(\square_\varphi\dot\varphi)\psi\eta\}\,\Omega_0(\varphi)/\mathrm{Vol}(X)\\
&=\frac{d}{dt}\Big(\int_X\psi\eta\Omega_0(\varphi)/\mathrm{Vol}(X)\Big)=\frac{d}{dt}\langle\!\langle\psi, \eta\rangle\!\rangle_\varphi\,. \qquad\text{Q.E.D.}
\end{aligned}$$

(2.6) In concluding this section, we define the natural "Riemannian connection" of $\mathcal{K}$. First consider a smooth path $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ in $\mathcal{H}$. Note that, for an element $\Psi=\{\psi_t \mid a\leqq t\leqq b\}$ of $\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H})$, the following are equivalent:

i) $\Psi\in\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\tilde{\mathcal{H}})$;

ii) the corresponding $\psi\in C^\infty([a, b]\times X)_{\boldsymbol{R}}$ (cf. (1.4.3)) satisfies $\langle\!\langle\psi, 1\rangle\!\rangle_\varphi=0$ in $C^\infty([a, b])_{\boldsymbol{R}}$ (see (2.2.1) for the definition of φ);

iii) $\psi_t\in T\tilde{\mathcal{H}}_{\varphi_t}$ for all $t\in[a, b]$.

Now, the next observation is crucial to our definition of the connection on $\mathcal{K}$.

Proposition 2.6.1. *Let Φ be as above, and suppose that $\Psi=\{\psi_t \mid a\leqq t\leqq b\}$ $\in\Gamma_{\text{diff}}([a, b], \Phi^*T\tilde{\mathcal{H}})$. Then $\nabla_{\dot{\varphi}}\Psi\in\Gamma_{\text{diff}}([a, b], \Phi^*T\tilde{\mathcal{H}})$.*

Proof. In view of the commutative diagram (2.4.3), it suffices to show $\langle\langle \frac{D}{\partial t}\psi, 1\rangle\rangle_{\varphi}=0$ in $C^\infty([a, b])_{\boldsymbol{R}}$. Obviously, $\frac{D}{\partial t}1=0$ and $\langle\langle\psi, 1\rangle\rangle_{\varphi}=0$. Then by Theorem 2.5,

$$\langle\langle \frac{D}{\partial t}\psi, 1\rangle\rangle_{\varphi} = \frac{d}{dt}\langle\langle\psi, 1\rangle\rangle_{\varphi} = 0\,,$$

as required. Q.E.D.

Fix an arbitrary smooth path $\Theta=\{\theta_t \mid a\leqq t\leqq b\}$ in $\mathcal{K}$. Recall that there exists a unique smooth path $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ in $\tilde{\mathcal{H}}$ such that $\theta_t=\omega_0(\varphi_t)$ for all t. Now, via the identification of $\tilde{\mathcal{H}}$ with $\mathcal{K}$ (cf. (1.3.5)), we have the operator

$$\nabla_{\dot{\theta}}\colon \Gamma_{\text{diff}}([a, b], \Theta^*T\mathcal{K}) \to \Gamma_{\text{diff}}([a, b], \Theta^*T\mathcal{K})$$

induced by $\nabla_{\dot{\varphi}}$ from the following commutative diagram:

$$\text{(2.6.2)}\qquad \begin{array}{ccc} \Gamma_{\text{diff}}([a, b], \Theta^*T\mathcal{K}) & \cong & \Gamma_{\text{diff}}([a, b], \Phi^*T\tilde{\mathcal{H}}) \\ \nabla_{\dot{\theta}}\big\downarrow & & \big\downarrow\nabla_{\dot{\varphi}} \\ \Gamma_{\text{diff}}([a, b], \Theta^*T\mathcal{K}) & \cong & \Gamma_{\text{diff}}([a, b], \Phi^*T\tilde{\mathcal{H}})\,. \end{array}$$

Then one immediately sees that this operator $\nabla_{\dot{\theta}}$ does not depend on the choice of ω_0 (and depends only on $\{\theta_t \mid a\leqq t\leqq b\}$).

3. Geodesics in $\mathcal{H}$ and $\mathcal{K}$

In this section, we shall define the concept of geodesics in $\mathcal{H}$ (and also in $\mathcal{K}$) in terms of the "Riemannian connection" of §2, and then prove Theorem 3.5 which provides us with a typical example of an infinitely extensible geodesic in $\mathcal{K}$.

(3.1) (i) Let $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ be a smooth path in $\mathcal{H}$. We denote by $\dot{\Phi}$ the element of $\Gamma_{\text{diff}}([a, b], \Phi^*T\mathcal{H})$ which sends each $t\in[a, b]$ to $\dot{\varphi}_t$. Then Φ is called a *geodesic* in $\mathcal{H}$ if one of the following equivalent conditions is satisfied:

(i–1) $\nabla_{\dot{\varphi}}\dot{\Phi}=0$ in $\Gamma_{\text{diff}}([a, b], \Phi^*T\mathcal{H})$;

(i–2) $\frac{D}{\partial t}\dot{\varphi}=0$ in $C^\infty([a, b]\times X)_{\boldsymbol{R}}$ (i.e., $\dot{\varphi}$ is parallel along Φ);

(i–3) $\ddot{\varphi}=\sum g(\varphi)^{\bar{\beta}\alpha}\dot{\varphi}_{\alpha}\dot{\varphi}_{\bar{\beta}}$ on $[a, b]\times X$.

(ii) Let $\Theta=\{\theta_t \mid a\leqq t\leqq b\}$ be a smooth path in $\mathcal{K}$. We denote by $\dot{\Theta}$ the element of $\Gamma_{\mathrm{diff}}([a, b], \Theta^*T\mathcal{K})$ which sends each $t\in[a, b]$ to $\dot{\theta}_t$. Recall that there exists a unique smooth path $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ in $\tilde{\mathcal{H}}$ such that $\theta_t=\omega_0(\varphi_t)$ for all t. Now, Θ is called a *geodesic* in $\mathcal{K}$ if one of the following equivalent conditions is satisfied:

(ii–1) $\nabla_{\dot{\theta}}\dot{\Theta}=0$ in $\Gamma_{\mathrm{diff}}([a, b], \Theta^*T\mathcal{K})$;

(ii–2) Φ is a geodesic in $\mathcal{H}$.

(3.2) Fix an arbitrary subset $\mathcal{N}$ of $\mathcal{H}$. Let $\nu\in\mathcal{N}$. Then a function η in $C^\infty(X)_{\boldsymbol{R}}$ is said to be *tangent* to $\mathcal{N}$ at ν if there exists a smooth path $\{\varphi_t \mid -\varepsilon\leqq t\leqq\varepsilon\}$ in $\mathcal{H}$, for some $\varepsilon>0$, with the following properties:

i) $\varphi_0=\nu$;

ii) $\dot{\varphi}_t|_{t=0}=\eta$;

iii) $\varphi_t\in\mathcal{N}$ for all $t\in[-\varepsilon, \varepsilon]$.

As a generalization of $T\mathcal{H}_\xi$ and $T\tilde{\mathcal{H}}_\xi$ in (1.4), we now put

$$T\mathcal{N}_\nu := \{\eta\in C^\infty(X)_{\boldsymbol{R}} \mid \eta \text{ is tangent to } \mathcal{N} \text{ at } \nu\} .$$

Let $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ be a smooth path in $\mathcal{H}$ satisfying $\varphi_t\in\mathcal{N}$ for all t. Then $\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{N})$ denotes the set of all

$$\Psi(=\{\psi_t \mid a\leqq t\leqq b\})\in\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H})$$

such that $\psi_t\in T\mathcal{N}_{\varphi_t}$ for all t.

(3.2.1) $\mathcal{N}$ is said to be *totally convex* in $\mathcal{H}$ if every geodesic $\{\varphi_t \mid a\leqq t\leqq b\}$ in $\mathcal{H}$ with $\varphi_a, \varphi_b\in\mathcal{N}$ always lies in $\mathcal{N}$.

(3.2.2) $\mathcal{N}$ is said to be *totally geodesic* in $\mathcal{H}$ if for every smooth path $\{\varphi_t \mid a\leqq t\leqq b\}$ in $\mathcal{H}$ sitting in $\mathcal{N}$, the operator $\nabla_{\dot{\varphi}}$ preserves the subset $\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{N})$ of $\Gamma_{\mathrm{diff}}([a, b], \Phi^*T\mathcal{H})$. If $\mathcal{N}$ is a finite-dimensional Riemannian C^∞ manifold (in terms of the metric and the smooth structure induced from those of $\mathcal{H}$), then one can easily show that $\mathcal{N}$ is totally geodesic in $\mathcal{H}$ if and only if every geodesic of the Riemannian manifold $\mathcal{N}$ is at the same time a geodesic of $\mathcal{H}$.

REMARK 3.3. (i) By Proposition 2.6.1, *$\tilde{\mathcal{H}}$ is totally geodesic.* We shall now show that *$\tilde{\mathcal{H}}$ is totally convex*: Let $\{\varphi_t \mid a\leqq t\leqq b\}$ be a geodesic in $\mathcal{H}$ such that $\varphi_a, \varphi_b\in\tilde{\mathcal{H}}$. Then for every $t\in[a, b]$,

$$\frac{d^2}{dt^2} L(0, \varphi_t) = \left(\frac{d}{dt} \langle\!\langle \dot{\varphi}, 1\rangle\!\rangle_\varphi\right)(t) = \left(\langle\!\langle \frac{D}{\partial t} \dot{\varphi}, 1\rangle\!\rangle_\varphi\right)(t) = 0 .$$

Furthermore $L(0, \varphi_a)=L(0, \varphi_b)=0$. Hence $L(0, \varphi_t)=0$ (i.e., $\varphi_t\in\tilde{\mathcal{H}}$) for all $t\in[a, b]$.

(ii) *Let $\xi\in\mathcal{H}$. Suppose that both ω_0 (cf. §1) and ξ are C^ω in terms of the natural real analytic structure of X. Then for every $\eta\in C^\omega(X)_{\boldsymbol{R}}$, there exists a real analytic function $\varphi=\varphi(t, x)\in C^\omega([-\varepsilon, \varepsilon]\times X)_{\boldsymbol{R}}$, with $\varepsilon>0$ sufficiently small, such that*

a) $\varphi|_{t=0}=\xi$,

b) $\dot{\varphi}|_{t=0}=\eta$, *and*

c) $\{\varphi_t|-\varepsilon\leqq t\leqq\varepsilon\}$ *is a geodesic in* $\mathcal{H}$,

where $\varphi_t:=\varphi|_{\{t\}\times X}\in C^\omega(X)_{\boldsymbol{R}}$ $(t\in[-\varepsilon, \varepsilon])$.

This is actually an immediate consequence of the fact that by Cauchy-Kovalevskaja existence theorem, the equation

$$\ddot{\varphi} = \sum g(\varphi)^{\bar{\beta}\alpha}\dot{\varphi}_\alpha\dot{\varphi}_{\bar{\beta}}$$

with the initial conditions $\varphi|_{t=0}=\xi$ and $\dot{\varphi}|_{t=0}=\eta$ has a unique solution $\varphi=\varphi(t,x)$ in $C^\omega([-\varepsilon, \varepsilon]\times X)_{\boldsymbol{R}}$ for some $\varepsilon>0$.

NOTATION 3.4. To each holomorphic vector field $Y\in\Gamma(X, \mathcal{O}(TX))$ on X, we associate a real vector field $Y_{\boldsymbol{R}}:=Y+\bar{Y}$. Recall that $(\sqrt{-1}\,Y)_{\boldsymbol{R}}=J\cdot Y_{\boldsymbol{R}}$, where J is the complex structure of X. Let $\mathfrak{g}$ be the Lie subalgebra of $\Gamma(X, \mathcal{O}(TX))$ corresponding to the Lie subgroup G of $\mathrm{Aut}^0(X)$ (see Introduction for the definition of G). For each $\omega\in\mathcal{K}$, we put

$$\mathfrak{k}_\omega := \{Y\in\mathfrak{g}\,|\,L_{Y_{\boldsymbol{R}}}(\omega) = 0\}\ ,$$
$$\mathfrak{p}_\omega := \sqrt{-1}\,\mathfrak{k}_\omega = \{\sqrt{-1}\,Y\,|\,Y\in\mathfrak{k}_\omega\}\ ,$$

where $L_{Y_{\boldsymbol{R}}}(\omega)$ denotes the Lie derivative of ω with respect to the vector field $Y_{\boldsymbol{R}}$. By writing ω as $\omega_0(\xi)$ (for some $\xi\in\mathcal{H}$), we have

$$\mathfrak{k}_\omega = \Gamma(X, \mathcal{O}(TX))\cap\{V_\xi(f)\,|\,f\in\sqrt{-1}\,C^\infty(X)_{\boldsymbol{R}}\}\ ,$$
$$\mathfrak{p}_\omega = \Gamma(X, \mathcal{O}(TX))\cap\{V_\xi(f)\,|\,f\in C^\infty(X)_{\boldsymbol{R}}\}\ \text{(cf. (1.5))}\ ,$$

(see for instance, Kobayashi [7; p. 94]).

Theorem 3.5. *Let $\omega\in\mathcal{K}$ and $0\neq Y\in\mathfrak{p}_\omega$. Put $g_t:=\exp(tY_{\boldsymbol{R}})$ $(t\in\boldsymbol{R})$. Then $\{g_t^*\omega\,|\,t\in\boldsymbol{R}\}$ is a geodesic in $\mathcal{K}$.*

Proof. Fix $\xi\in\mathcal{H}$. Note that there exists a unique smooth path $\{\varphi_t|\ a\leqq t\leqq b\}$ in $\tilde{\mathcal{H}}$ such that $g_t^*\omega=\omega_0(\varphi_t)$ for all t. By setting $Z_t:=V_{\varphi_t}(\dot{\varphi}_t)$, we have

$$\text{(3.5.1)}\qquad L_{(Z_t)_{\boldsymbol{R}}}(\omega_0(\varphi_t)) = d\circ i_{(Z_t)_{\boldsymbol{R}}}(\omega_0(\varphi_t)) = \sqrt{-1}\,\partial\bar{\partial}\dot{\varphi}_t$$
$$= \frac{\partial}{\partial t}(\omega_0(\varphi_t))\left(=\frac{\partial}{\partial t}(g_t^*\omega)\right) = L_{Y_{\boldsymbol{R}}}(\omega_0(\varphi_t))\ .$$

In particular, $Z_t-Y\in\Gamma(X, \mathcal{O}(TX))$ (cf. Kobayashi [7; p. 93]). Note that $\mathfrak{p}_{\omega_0(\varphi_t)}=\mathfrak{p}_{g_t^*\omega}=\mathrm{Ad}(g_t^{-1})\mathfrak{p}_\omega$. Hence for all t,

$$Y=(\mathrm{Ad}(g_t^{-1}))Y\in\mathfrak{p}_{\omega_0(\varphi_t)}\subset\{V_{\varphi_t}(f)\,|\,f\in C^\infty(X)_{\boldsymbol{R}}\}\ ,$$

and in particular

$$Z_t-Y\in\{V_{\varphi_t}(f)\,|\,f\in C^\infty(X)_{\boldsymbol{R}}\}\cap\Gamma(X,\mathcal{O}(TX))=\mathfrak{p}_{\omega_0(\varphi_t)}\,.$$

In view of (3.5.1), it follows that

$$(3.5.2)\qquad Z_t-Y\in\mathfrak{k}_{\omega_0(\varphi_t)}\cap\mathfrak{p}_{\omega_0(\varphi_t)}=\{0\},\ (t\in[a,b])\,.$$

We now obtain

$$\begin{aligned}\sqrt{-1}\,\partial\bar\partial\ddot\varphi_t(&=\frac{\partial}{\partial t}(\sqrt{-1}\,\partial\bar\partial\dot\varphi_t))=\frac{\partial}{\partial t}(L_{Y_{\boldsymbol{R}}}(\omega_0(\varphi_t)))\ (\text{cf. }(3.5.1))\\&=L_{Y_{\boldsymbol{R}}}(\sqrt{-1}\,\partial\bar\partial\dot\varphi_t)=\sqrt{-1}\,\partial\bar\partial(Y_{\boldsymbol{R}}\dot\varphi_t)\\&=\sqrt{-1}\,\partial\bar\partial((Z_t)_{\boldsymbol{R}}\dot\varphi_t)\ (\text{cf. }(3.5.2))\,.\end{aligned}$$

Hence $\left(\frac{D}{\partial t}\dot\varphi\right)(t)=\ddot\varphi_t-(Z_t)_{\boldsymbol{R}}\dot\varphi_t=C(t)$ for some function $C(t)\in C^\infty([a,b])_{\boldsymbol{R}}$. Then by $\varphi_t\in\tilde{\mathcal{H}}$, we finally obtain

$$\begin{aligned}0&=\frac{d}{dt}L(0,\varphi_t)=\left(\frac{d}{dt}\langle\!\langle\dot\varphi,1\rangle\!\rangle_\varphi\right)(t)\\&=\left(\langle\!\langle\frac{D}{\partial t}\dot\varphi,1\rangle\!\rangle_\varphi\right)(t)=C(t),\ (t\in\boldsymbol{R})\,.\end{aligned}$$

Q.E.D.

Theorem 3.6. *Let $Y\in H^0(X,\mathcal{O}(TX))$ be such that $L_{Y_{\boldsymbol{R}}}(\omega_0)=0$. Furthermore suppose that $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$ is a smooth path in $\mathcal{H}$ such that $Y_{\boldsymbol{R}}\varphi_t=0$ in $C^\infty(X)_{\boldsymbol{R}}$ for all t. Then*

$$Y_{\boldsymbol{R}}\left(\frac{D}{\partial t}\psi\right)=\frac{D}{\partial t}(Y_{\boldsymbol{R}}\psi)$$

for all $\psi\in C^\infty([a,b]\times X)_{\boldsymbol{R}}$.

Proof. We here use the notation of (1.2), (2.4) and (2.5). On $[a,b]\times X$, we put

$$\omega^\vee:=(1/\sqrt{-1})\sum g(\varphi)^{\bar\beta\alpha}(\partial/\partial z^\alpha)\wedge(\partial/\partial\bar z^\beta)$$

and by $(\partial\dot\varphi,\omega^\vee)$ (resp. $(\bar\partial\dot\varphi,\omega^\vee)$), we mean the contraction of $\partial\dot\varphi$ (resp. $\bar\partial\dot\varphi$) with $\omega^\vee$. Then the covariant differentiation $\frac{D}{\partial t}$ along the path $\{\varphi_t\,|\,a\leqq t\leqq b\}$ is written as

$$\frac{D}{\partial t}=\frac{\partial}{\partial t}+\frac{\sqrt{-1}}{2}((\bar\partial\dot\varphi,\omega^\vee)-(\partial\dot\varphi,\omega^\vee))\,.$$

Since $\left[Y_{\boldsymbol{R}},\frac{\partial}{\partial t}\right]=0$, the proof is reduced to showing

$$[Y_{\mathbf{R}}, (\bar{\partial}\dot{\varphi}, \omega^{\vee})] = 0 = [Y_{\mathbf{R}}, (\partial\dot{\varphi}, \omega^{\vee})] .$$

Note here that

$$L_{Y_{\mathbf{R}}}(\omega_0(\varphi)) = L_{Y_{\mathbf{R}}}(\omega_0) + \sqrt{-1}\,\partial\bar{\partial}(Y_{\mathbf{R}}\varphi) = 0 .$$

Hence $L_{Y_{\mathbf{R}}}(\omega^{\vee})=0$. Therefore

$$[Y_{\mathbf{R}}, (\bar{\partial}\dot{\varphi}, \omega^{\vee})] = L_{Y_{\mathbf{R}}}((\bar{\partial}\dot{\varphi}, \omega^{\vee})) = (L_{Y_{\mathbf{R}}}(\bar{\partial}\dot{\varphi}), \omega^{\vee})$$
$$= (\frac{\partial}{\partial t}\{\bar{\partial}(Y_{\mathbf{R}}\varphi)\}, \omega^{\vee}) = 0 .$$

Similarly, we have $[Y_{\mathbf{R}}, (\partial\dot{\varphi}, \omega^{\vee})]=0$. Q.E.D.

REMARK 3.6.1. Let $K(\subset G)$ be the same as in Introduction, and assume that we have chosen a K-invariant ω_0. Let $\mathcal{H}^K$, $\tilde{\mathcal{H}}^K$, $\mathcal{K}^K$ be respectively the set of all K-invariant elements of $\mathcal{H}$, $\tilde{\mathcal{H}}$, $\mathcal{K}$. Then almost all results in this paper are reformulated in terms of these K-invariant objects. Theorem 3.6 above assures the validity of such K-invariant versions of our results.

4. Torsion and curvature of the natural connection

The main purpose of this section is to prove (0.2) of Introduction. We shall also show that the torsion of our connection is zero.

(4.1) Fix an arbitrary point ξ of $\mathcal{H}$. Let $\eta_1, \eta_2 \in C^\infty(X)_{\mathbf{R}}$ ($\cong T\mathcal{H}_\xi$). Consider a function

$$\varphi = \varphi(s, t, x) \in C^\infty([-\varepsilon, \varepsilon]\times[-\varepsilon, \varepsilon]\times X)_{\mathbf{R}}$$

such that the following conditions are satisfied:

i) $\varphi_{s_1,t_1} := \varphi|_{(s,t)=(s_1,t_1)}$ belongs to $\mathcal{H}$ whenever $s_1, t_1 \in [-\varepsilon, \varepsilon]$.

ii) $\varphi_{0,0}=\xi$.

iii) $\dfrac{\partial\varphi}{\partial s}|_{(s,t)=(0,0)}=\eta_1$ and $\dfrac{\partial\varphi}{\partial t}|_{(s,t)=(0,0)}=\eta_2$.

(Such a φ always exists, because we can choose $\varphi(s, t, x)=\xi(x)+s\eta_1(x)+t\eta_2(x)$, $(s, t, x)\in[-\varepsilon, \varepsilon]\times[-\varepsilon, \varepsilon]\times X$, with $0<\varepsilon\ll 1$.) Note that, by formal definitions, the *torsion*

$$\tau\colon T\mathcal{H}\otimes T\mathcal{H} \to T\mathcal{H}$$

and the *curvature*

$$R\colon T\mathcal{H}\otimes T\mathcal{H} \to \mathrm{Hom}\,(T\mathcal{H}, T\mathcal{H})$$

of our connection on $\mathcal{H}$ are given by

$$\tau(\eta_1, \eta_2)_\xi = \left\{\frac{D}{\partial s}\left(\frac{\partial\varphi}{\partial t}\right) - \frac{D}{\partial t}\left(\frac{\partial\varphi}{\partial s}\right)\right\}|_{(s,t)=(0,0)} ,$$

$$R(\eta_1, \eta_2)_\xi = \left(\frac{D}{\partial s}\frac{D}{\partial t} - \frac{D}{\partial t}\frac{D}{\partial s}\right)\Big|_{(s,t)=(0,0)},$$

at the point ξ. Since $\mathcal{K}$ is identified with the totally geodesic "submanifold" $\tilde{\mathcal{H}}$ in $\mathcal{H}$ (cf. (i) of (3.3)), the torsion and the curvature of our connection on $\mathcal{K}$ is just the restriction of τ and R above to $T\mathcal{K}\otimes T\mathcal{K}(=T\tilde{\mathcal{H}}\otimes T\tilde{\mathcal{H}})$. We now have:

Theorem 4.2. $\tau=0$, *i.e.*, $\tau(\eta_1, \eta_2)_\xi=0$ *for all* $\eta_1, \eta_2\in C^\infty(X)_R$ *and* $\xi\in\mathcal{H}$. (*Thus our connection on* $\mathcal{H}$ *is the unique one satisfying both* $\tau=0$ *and* (2.5).)

Theorem 4.3. $R(\eta_1, \eta_2)_\xi(\eta_3)=[[\eta_1, \eta_2]_\xi, \eta_3]_\xi$ *for all* $\eta_1, \eta_2, \eta_3\in C^\infty(X)_R$ *and all* $\xi\in\mathcal{H}$.

Proof of (4.2). We denote by $(\cdots)|_0$ the restriction of $(\cdots)$ to $(s, t)=(0, 0)$. Then

$$\begin{aligned}\tau(\eta_1, \eta_2)_\xi &= \left\{\frac{D}{\partial s}\left(\frac{\partial\varphi}{\partial t}\right) - \frac{D}{\partial t}\left(\frac{\partial\varphi}{\partial s}\right)\right\}\Big|_0 \\ &= \{\partial/\partial s - 2^{-1}\textstyle\sum g(\varphi)^{\bar\beta\alpha}((\partial\varphi/\partial s)_\alpha\partial/\partial z^{\bar\beta} + (\partial\varphi/\partial s)_{\bar\beta}\partial/\partial z^\alpha)\}(\partial\varphi/\partial t)|_0 \\ &\quad - \{\partial/\partial t - 2^{-1}\textstyle\sum g(\varphi)^{\bar\beta\alpha}((\partial\varphi/\partial t)_\alpha\partial/\partial z^{\bar\beta} + (\partial\varphi/\partial t)_{\bar\beta}\partial/\partial z^\alpha)\}(\partial\varphi/\partial s)|_0 \\ &= 0. \qquad\qquad \text{Q.E.D.}\end{aligned}$$

Proof of (4.3). Fix $\xi\in\mathcal{H}$ and $x_0\in X$ arbitrarily. We then choose holomorphic local coordinates $(z^1, \cdots, z^n)$ centered at x_0 such that

$$g(\xi)_{\alpha\bar\beta}(x_0) = \delta_{\alpha\beta} \text{ and } d(g(\xi)_{\alpha\bar\beta})(x_0) = 0$$

for all $\alpha, \beta\in\{1, 2, \cdots, n\}$. Note that, when evaluated at x_0,

$$\begin{cases} \dfrac{\partial g(\varphi)^{\bar\delta\gamma}}{\partial s}\Big|_{(s,t)=(0,0)} = -\left(\dfrac{\partial\varphi}{\partial s}\right)_{\delta\bar\gamma|(s,t)=(0,0)} = -(\eta_1)_{\delta\bar\gamma}, \\ \dfrac{\partial g(\varphi)^{\bar\delta\gamma}}{\partial t}\Big|_{(s,t)=(0,0)} = -(\eta_2)_{\delta\bar\gamma}. \end{cases}$$

Hence, at the point x_0,

$$\begin{aligned}R(\eta_1, \eta_2)_\xi &= \left(\frac{D}{\partial s}\frac{D}{\partial t} - \frac{D}{\partial t}\frac{D}{\partial s}\right)\Big|_{(s,t)=(0,0)} \\ &= 4^{-1}\textstyle\sum_{\alpha,\gamma}\{(\eta_2)_\gamma(\eta_1)_{\alpha\bar\gamma}\partial/\partial z^{\bar\alpha} + (\eta_2)_{\bar\gamma}(\eta_1)_{\gamma\bar\alpha}\partial/\partial z^\alpha\} \\ &\quad - 4^{-1}\textstyle\sum_{\alpha,\gamma}\{(\eta_1)_\alpha(\eta_2)_{\gamma\bar\alpha}\partial/\partial z^{\bar\gamma} + (\eta_1)_{\bar\alpha}(\eta_2)_{\alpha\bar\gamma}\partial/\partial z^\gamma\} \\ &\quad + [2^{-1}\textstyle\sum g(\xi)^{\bar\beta\alpha}(\eta_1)_\alpha\partial/\partial z^{\bar\beta}, 2^{-1}\sum g(\xi)^{\bar\delta\gamma}(\eta_2)_{\bar\delta}\partial/\partial z^\gamma] \\ &\quad + [2^{-1}\textstyle\sum g(\xi)^{\bar\beta\alpha}(\eta_1)_{\bar\beta}\partial/\partial z^\alpha, 2^{-1}\sum g(\xi)^{\bar\delta\gamma}(\eta_2)_\gamma\partial/\partial z^{\bar\delta}] \\ &= [W_\xi(\eta_1), W_\xi(\eta_2)] = W_\xi([\eta_1, \eta_2]_\xi) \text{ (cf. (1.5))}.\end{aligned}$$

Thus we have

$$R(\eta_1, \eta_2)_\xi(\eta_3) = \{W_\xi([\eta_1, \eta_2]_\xi)\}(\eta_3)$$
$$= [[\eta_1, \eta_2]_\xi, \eta_3]_\xi \quad \text{(cf. (1.5.1))}. \qquad \text{Q.E.D.}$$

DEFINITION 4.4. Fix an arbitrary point ξ of $\mathcal{H}$. Let η_1, η_2 be $\boldsymbol{R}$-linearly independent elements of $C^\infty(X)_{\boldsymbol{R}}(\cong T\mathcal{H}_\xi)$. Regard $P:=\boldsymbol{R}\psi_1+\boldsymbol{R}\psi_2$ as a 2-plane in $T\mathcal{H}_\xi$. We then define the sectional curvature $K(P)_\xi$ of $\mathcal{H}$ at ξ along the plane section P by

$$\text{(4.4.1)} \qquad K(P)_\xi := \frac{-\langle\!\langle R(\eta_1, \eta_2)_\xi(\eta_1), \eta_2\rangle\!\rangle_\xi}{\langle\!\langle \eta_1, \eta_1\rangle\!\rangle_\xi\langle\!\langle \eta_2, \eta_2\rangle\!\rangle_\xi-(\langle\!\langle \eta_1, \eta_2\rangle\!\rangle_\xi)^2}.$$

Theorem 4.5. *At each point ξ of $\mathcal{H}$,*

$$K(P)_\xi \leqq 0$$

for every 2-plane $P=\boldsymbol{R}\eta_1+\boldsymbol{R}\eta_2$ in $T\mathcal{H}_\xi$ (where $\eta_1, \eta_2\in C^\infty(X)_{\boldsymbol{R}}$ are $\boldsymbol{R}$-linearly independent). Furthermore, $K(P)_\xi=0$ if and only if $[\eta_1, \eta_2]_\xi=0$.

Proof. In view of (4.3) and (4.4.1),

$$K(P)_\xi = \frac{-\langle\!\langle [[\eta_1, \eta_2]_\xi, \eta_1]_\xi, \eta_2\rangle\!\rangle_\xi}{\langle\!\langle \eta_1, \eta_1\rangle\!\rangle_\xi\langle\!\langle \eta_2, \eta_2\rangle\!\rangle_\xi-(\langle\!\langle \eta_1, \eta_2\rangle\!\rangle_\xi)^2}$$
$$= \frac{-\langle\!\langle [\eta_1, \eta_2]_\xi, [\eta_1, \eta_2]_\xi\rangle\!\rangle_\xi}{\langle\!\langle \eta_1, \eta_1\rangle\!\rangle_\xi\langle\!\langle \eta_2, \eta_2\rangle\!\rangle_\xi-(\langle\!\langle \eta_1, \eta_2\rangle\!\rangle_\xi)^2} \quad \text{(cf. (1.5.3))}.$$

Since the denominator is always positive, we have $K(P)_\xi \leqq 0$. Clearly, $K(P)_\xi =0$ if and only if $[\eta_1, \eta_2]_\xi=0$. Q.E.D.

REMARK 4.5.1. Recall that $\mathcal{K}$ is identified with the totally geodesic "submanifold" $\tilde{\mathcal{H}}$ of $\mathcal{H}$. Hence by (4.5) above, the nonpositivity of sectional curvature is true also for $\mathcal{K}$.

5. Convexity of the $\mathcal{K}$-energy map μ

In this section, several facts related to the convexity of μ will be given. We begin by showing:

Lemma 5.1. *For every smooth path $\{\varphi_t \mid a\leqq t\leqq b\}$ in $\mathcal{H}$, we have, for all t, the following:*

$$\text{(5.1.1)} \qquad \langle\!\langle \dot{\varphi}_t, \frac{D}{\partial t}\sigma(\varphi_t)\rangle\!\rangle_{\varphi_t} = -(4/\mathrm{Vol}(X))\|\bar{\partial} V_{\varphi_t}(\dot{\varphi}_t)\|^2_{L^2(X,\omega_0(\varphi_t))},$$

$$\text{(5.1.2)} \qquad \frac{d^2}{dt^2}\mu(\omega_0(\varphi_t)) = -\langle\!\langle \frac{D}{\partial t}\dot{\varphi}_t, \sigma(\varphi_t)-\sigma_0\rangle\!\rangle_{\varphi_t}$$
$$+(4/\mathrm{Vol}(X))\|\bar{\partial} V_{\varphi_t}(\dot{\varphi}_t)\|^2_{L^2(X,\omega_0(\varphi_t))},$$

where $\frac{D}{\partial t}\dot{\varphi}_t := \left(\frac{D}{\partial t}\dot{\varphi}\right)(t) = \ddot{\varphi}_t - \sum g(\varphi_t)^{\bar{\beta}\alpha}(\dot{\varphi}_t)_\alpha(\dot{\varphi}_t)_{\bar{\beta}}$.

Proof. From a straightforward computation, one obtains

$$\frac{\partial}{\partial t}\sigma(\varphi_t) = -(\square_{\varphi_t})^2\dot{\varphi}_t - \langle\sqrt{-1}\,\partial\bar{\partial}\dot{\varphi}_t,\ \mathrm{Ric}(\varphi_t)\rangle_{\varphi_t}$$

in terms of the notation of (1.6). Hence

$$\frac{D}{\partial t}\sigma(\varphi_t) = -Re\,\boldsymbol{L}_{\varphi_t}\dot{\varphi}_t\,.$$

Now by (1.6.2) applied to $\xi=\varphi_t$ and $\psi=\dot{\varphi}_t$, the required identity (5.1.1) immediately follows. For (5.1.2), recall that

$$\frac{d}{dt}\mu(\omega_0(\varphi_t)) = -\langle\!\langle\dot{\varphi}_t,\ \sigma(\varphi_t)-\sigma_0\rangle\!\rangle_{\varphi_t}\quad \text{(cf. (1.3.1))}\,.$$

Then by Theorem 2.5,

$$\frac{d^2}{dt^2}\mu(\omega_0(\varphi_t)) = -\langle\!\langle\frac{D}{\partial t}\dot{\varphi}_t,\ \sigma(\varphi_t)-\sigma_0\rangle\!\rangle_{\varphi_t} - \langle\!\langle\dot{\varphi}_t,\ \frac{D}{\partial t}\sigma(\varphi_t)\rangle\!\rangle_{\varphi_t}\,.$$

Together with (5.1.1), we finally obtain (5.1.2). Q.E.D.

In view of [9; (3.2)], the following is an immediate consequence of (5.1.2):

Corollary 5.1.3 (cf. [9; (6.3)]). *If* ω *is a critical point of* $\mu\colon \mathcal{K}\to\boldsymbol{R}$, *then the inequality*

$$\frac{d^2}{dt^2}\mu(\theta_t)\,|_{t=0} \geqq 0$$

holds for every smooth path $\{\theta_t\,|\,-\varepsilon\leqq t\leqq\varepsilon\}$ *in* $\mathcal{K}$ *such that* $\theta_0=\omega$.

We shall now show that $\mathcal{E}$ is totally convex in $\mathcal{K}$ (see Introduction for the definition of $\mathcal{E}$):

Corollary 5.1.4. *Let* $\{\theta_t\,|\,a\leqq t\leqq b\}$ *be a geodesic in* $\mathcal{K}$ *such that both* θ_a *and* θ_b *belong to* $\mathcal{E}$. *Then there exists a* C^∞ *map*

$$[a,\ b]\ni t\mapsto g_t\in \mathrm{Aut}^0(X)$$

such that $\theta_t = g_t^*\theta_a$ *for all* t, *and hence all* θ_t *belong to* $\mathcal{E}$. *In particular,* $\mathcal{E}$ *is totally convex in* $\mathcal{K}$.

Proof. For each $t\in[a,\ b]$, we write θ_t as $\omega_0(\varphi_t)$ for some unique $\varphi_t\in\tilde{\mathcal{H}}$. Then by (5.1.3),

$$\frac{d^2}{dt^2}\mu(\theta_t)\geqq 0\quad (t\in[a,\ b])\,.$$

On the other hand, by θ_a, $\theta_b \in \mathcal{E}$, we have

$$\frac{d}{dt}\mu(\theta_t)_{|t=a} = \frac{d}{dt}\mu(\theta_t)_{|t=b} = 0\,.$$

Hence $\mu(\theta_t)=C$ for some $C \in \boldsymbol{R}$ independent of t. In particular, in view of (5.1.2),

$$0 = \frac{d^2}{dt^2}\mu(\theta_t) = \frac{4}{\mathrm{Vol}(X)}\,||\bar{\partial}V_t||^2_{L^2(X,\omega_0(\varphi_t))} \quad (\text{where } V_t := V_{\varphi_t}(\dot{\varphi}_t))$$

and therefore $V_t \in \Gamma(X, \mathcal{O}(TX))$ for all $t \in [a, b]$. Note that we have the unique smooth solution $\{y_t \in \mathrm{Aut}^0(X) \,|\, a \leqq t \leqq b\}$ of the equation

$$\dot{y}_t y_t^{-1} = -2V_t \quad (a \leqq t \leqq b)$$

with the initial condition $y_a = id_X$. Now, for all t,

$$\begin{aligned}\frac{\partial}{\partial t}(y_t^*\theta_t) &= y_t^*(y_t^{-1})^*\lim_{\varepsilon\to 0}\frac{y_{t+\varepsilon}{}^*\theta_t - y_t^*\theta_t}{\varepsilon} + \lim_{\varepsilon\to 0} y_{t+\varepsilon}{}^*\left(\frac{\theta_{t+\varepsilon}-\theta_t}{\varepsilon}\right)\\ &= y_t^*(-2L_{V_t}(\theta_t)) + y_t^*(\dot{\theta}_t) = y_t^*(-2L_{V_t}(\theta_t) + \sqrt{-1}\,\partial\bar{\partial}\dot{\varphi}_t)\\ &= y_t^*\{-2d(i_{V_t}\theta_t) + \sqrt{-1}\,\partial\bar{\partial}\dot{\varphi}_t\} = 0\,.\end{aligned}$$

Hence $y_t^*\theta_t = y_a^*\theta_a = \theta_a$ ($t \in [a, b]$). Q.E.D.

REMARK 5.2 (see also Calabi [4] and Bourguignon [3]): Suppose that there exists a critical point $\xi \in \mathcal{H}$ of the functional

$$\text{(5.2.1)} \qquad C\colon \mathcal{H} \to \boldsymbol{R},\quad \psi \mapsto C(\psi) := \langle\!\langle \sigma(\psi), \sigma(\psi) \rangle\!\rangle_\psi\,.$$

We now put $\varphi_t := \xi + t\sigma(\xi)$ $(-\varepsilon \leqq t \leqq \varepsilon)$. Then by (5.1.1),

$$\begin{aligned}&(4/\mathrm{Vol}(X))||\bar{\partial}V_\xi(\sigma(\xi))||^2_{L^2(X,\omega_0(\xi))}\\ &= \langle\!\langle \dot{\varphi}_t, \frac{D}{\partial t}\sigma(\varphi_t)\rangle\!\rangle_{\varphi_t|t=0} = \langle\!\langle \sigma(\varphi_t), \frac{D}{\partial t}\sigma(\varphi_t)\rangle\!\rangle_{\varphi_t|t=0}\\ &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\langle\!\langle \sigma(\varphi_t), \sigma(\varphi_t)\rangle\!\rangle_{\varphi_t} = 0\,.\end{aligned}$$

Thus $V_\xi(\sigma(\xi)) \in \Gamma(X, \mathcal{O}(TX))$. Hence we obtain the following well-known fact: either $\sigma(\xi)$ is constant on X or $0 \neq V_\xi(\sigma(\xi)) \in \Gamma(X, \mathcal{O}(TX))$.

Until the end of this section for simplicity, we write $\mu(\omega_0(\varphi))$ as $\mu(\varphi)$ for all $\varphi \in \mathcal{H}$. Note that $\mu(\varphi) = M(0, \varphi)$ in terms of the notation of (1.3). Let $d\mu$ be the "1-form" on $\mathcal{H}$ defined by

$$d\mu(\dot{\varphi}_t) = \frac{d}{dt}\mu(\varphi_t)\,,$$

where $\{\varphi_t \mid a \leqq t \leqq b\}$ is an arbitrary smooth path in $\mathcal{H}$. We shall now show the convexity of μ.

Theorem 5.3. *$\mu\colon \mathcal{H} \to \boldsymbol{R}$ is a convex function, i.e.,* Hess μ *is positive semi-definite everywhere on $\mathcal{H}$.*

Proof. Fix an arbitrary point ξ of $\mathcal{H}$. Let $\eta \in C^\infty(X)_{\boldsymbol{R}} (\cong T\mathcal{H}_\xi)$. Choose a smooth path $\{\varphi_t \mid -\varepsilon \leqq t \leqq \varepsilon\}$ in $\mathcal{H}$ such that $\dot\varphi_t|_{t=0} = \eta$ (say, let $\varphi_t = \xi + t\eta$). Then in view of (5.1.2), we obtain

$$\text{(5.3.1)} \qquad \begin{aligned}(\text{Hess}\ \mu)_\xi(\eta, \eta) &= \frac{d^2}{dt^2}\mu(\varphi_t)_{|t=0} - (d\mu)_\xi\Big(\Big(\frac{D}{\partial t}\dot\varphi_t\Big)_{|t=0}\Big) \\ &= (4/\text{Vol}(X))\|\bar\partial V_\xi(\eta)\|^2_{L^2(X,\omega_0(\xi))} \geqq 0\,. \end{aligned} \qquad \text{Q.E.D.}$$

REMARK 5.3.2. Since $\mathcal{K}$ is identified with the totally geodesic "submanifold" $\tilde{\mathcal{H}}$ of $\mathcal{H}$ (cf. (3.3)), the above theorem shows that $\mu\colon \mathcal{K} \to \boldsymbol{R}$ is also a convex function.

REMARK 5.3.3. Fix an arbitrary point ξ of $\mathcal{H}$. Let $\eta_1, \eta_2 \in C^\infty(X)_{\boldsymbol{R}} (\cong T\mathcal{H}_\xi)$. Then from (5.3.1), one easily obtains

$$(\text{Hess}\ \mu)_\xi(\eta_1, \eta_2) = (4/\text{Vol}(X))\ \text{Re}(\bar\partial V_\xi(\eta_1), \bar\partial V_\xi(\eta_2))_{L^2(X,\omega_0(\xi))}\,.$$

REMARK 5.3.4. (cf. Bando [1]): Let $\{\theta_t \mid -\varepsilon_1 \leqq t \leqq \varepsilon_2\}$ be an arbitrary smooth path in $\mathcal{K}$ such that $\theta_0 \in \mathcal{E}$. Then (5.3.1) shows that $(d^2/dt^2)(\mu(\theta_t))_{|t=0} \geqq 0$ and is >0 whenever the path $\{\theta_t\}$ is transversal to the orbit $\{g^*\theta_0 \mid g \in G\}$ $(=\{a^*\theta_0 \mid a \in \text{Aut}^0(X)\})$ at $t=0$ (see Introduction for the definition of G). *Therefore each connected component of $\mathcal{E}$ forms a single G-orbit in $\mathcal{E}$.* (This fact was first obtained by Calabi [4] with an effective use of the functional $C\colon \mathcal{K} \to \boldsymbol{R}$ instead of that of μ).

6. Natural Riemannian structure of $\mathcal{E}$

Throughout this section, we assume $\mathcal{E} \neq \phi$ and fix an arbitrary component $\mathcal{E}_0$ of $\mathcal{E}$. Recall that $\mathcal{E}_0 = \{g^*\theta \mid g \in G\}$ for all $\theta \in \mathcal{E}_0$ (cf. (5.3.4)). We now endow $\mathcal{E}$ (and hence $\mathcal{E}_0$) with the natural Riemannian metric induced from the one on $\mathcal{K}$ (cf. (2.1)). $\mathcal{E}_0$ is then a finite-dimensional Riemannian manifold diffeomorphic to G/K, and for every smooth path $\Theta = \{\theta_t \mid a \leqq t \leqq b\}$ in $\mathcal{E}_0$, its arclength Lgth(Θ) (both in terms of the metric of $\mathcal{E}_0$ and of $\mathcal{K}$) is given by

$$\text{Lgth}(\Theta) = \int_a^b \Big\{\int_X (\dot\varphi_t)^2 \Omega_0(\varphi_t)/\text{Vol}(X)\Big\}^{1/2} dt\,,$$

where each φ_t is the unique element of $\tilde{\mathcal{H}}$ such that $\theta_t = \omega_0(\varphi_t)$. Note that Lgth($\Theta$) does not depend on the choice of ω_0 (and depends only on Θ).

DEFINITION 6.1. A bijection λ: $\mathcal{K}\rightarrow\mathcal{K}$ is called an *isometry* of $\mathcal{K}$ if for every smooth path $\Theta=\{\theta_t\,|\,a\leqq t\leqq b\}$ in $\mathcal{K}$, $\lambda(\Theta):=\{\lambda(\theta_t)\,|\,a\leqq t\leqq b\}$ is again a smooth path in $\mathcal{K}$ with $\mathrm{Lgth}(\lambda(\Theta))=\mathrm{Lgth}(\Theta)$.

Theorem 6.2. *For each* $g\in\mathrm{Aut}^0(X)$, *the mapping* $\mathcal{K}\ni\omega\mapsto g^*\omega\in\mathcal{K}$ *is an isometry of* $\mathcal{K}$.

Proof. Let $\Theta=\{\theta_t\,|\,a\leqq t\leqq b\}$ be a smooth path in $\mathcal{K}$ and we write θ_t as $\omega_0(\varphi_t)(\varphi_t\in\tilde{\mathcal{H}})$ for all t. Consider the function η_g in $\mathcal{H}$ uniquely determined by the properties $g^*\omega_0=\omega_0(\eta_g)$ and $\eta_g+g^*\varphi_a\in\tilde{\mathcal{H}}$. We put $\psi_t:=\eta_g+g^*\varphi_t$ $(a\leqq t\leqq b)$. Then $g^*\theta_t=\omega_0(\psi_t)$. Furthermore

$$L(0,\psi_t)=\int_a^b\{\int_X\dot{\psi}_t\,\Omega_0(\psi_t)/\mathrm{Vol}(X)\}\,dt=\int_a^b\int_X\{(g^*\dot{\varphi}_t)g^*(\Omega_0(\varphi_t))/\mathrm{Vol}(X)\}\,dt$$
$$=\int_a^b\int_X\{\dot{\varphi}_t\Omega_0(\varphi_t)/\mathrm{Vol}(X)\}\,dt=L(0,\varphi_t)=0\,.$$

Hence $\psi_t\in\tilde{\mathcal{H}}$. We finally obtain

$$\mathrm{Lgth}(g^*\Theta)=\int_a^b\{\int_X(\dot{\psi}_t)^2\Omega_0(\psi_t)/\mathrm{Vol}(X)\}^{1/2}\,dt$$
$$=\int_a^b\{\int_X(g^*\dot{\varphi}_t)^2g^*(\Omega_0(\varphi_t))/\mathrm{Vol}(X)\}^{1/2}\,dt=\mathrm{Lgth}(\Theta)\,.$$

Q.E.D.

Corollary 6.2.1. *The Riemannian manifold* $\mathcal{E}_0$ *is G-equivariantly isometric to the Riemannian symmetric space* G/K *endowed with a suitable metric, and furthermore,* $\mathrm{Aut}^0(X)$ *acts isometrically on* $\mathcal{E}_0$.

Proof. Since K is a maximal compact subgroup of the reductive algebraic group G (cf. Introduction), it follows that (G, K) is a Riemannian symmetric pair (cf. Helgason [6; p. 209]). Then in view of (6.2), the homogeneous space $\mathcal{E}_0$ has a natural structure of a Riemannian symmetric space (cf. Helgason [6; Proposition 3.4, p. 209]). Q.E.D.

Theorem 6.3. $\mathcal{E}_0$ *(and hence* $\mathcal{E}$*) is totally geodesic in* $\mathcal{K}$.

Proof. Fix an arbitrary element ω of $\mathcal{E}_0$, and let $\mathfrak{p}_\omega$ be as in (3.4). Since $\mathcal{E}_0$ is Riemannian symmetric, every geodesic $\gamma(t)$ in $\mathcal{E}_0$ through ω $(=\gamma(0))$ is written in the form

$$\gamma(t)=(\exp(tY_{\boldsymbol{R}}))^*\omega\qquad(t\in\boldsymbol{R})$$

for some $0\neq Y\in\mathfrak{p}_\omega$. This is at the same time a geodesic in $\mathcal{K}$ by Theorem (3.5). Q.E.D.

(6.4) Recall the decomposition of $\mathcal{E}$ into connected components:

$$\mathcal{E} = \bigcup_{i \in I} \mathcal{E}_i \text{ (disjoint union)}.$$

We shall now associate, to each $\mathcal{E}_i$, a Euclidean lattice $(\mathfrak{t}, \Lambda, ((\ ,\ ))_i)$ (which is uniquely determined by $\mathcal{E}_i$ up to isometry) as follows:

Let ω be a point of $\mathcal{E}_i$, and T an arbitrary maximal torus in the isotropy subgroup K_ω of G at ω. Put $\mathfrak{t}:=\text{Lie } T$. We then have the exponential map

$$\exp: \mathfrak{t} \to T(=\mathfrak{t}/\Lambda)$$

where T is written as $\mathfrak{t}/\Lambda$ for some lattice Λ in $\mathfrak{t}$. In view of the natural inclusions

$$\sqrt{-1}\,\mathfrak{t} \subset \sqrt{-1}\,\mathfrak{k}_\omega = \mathfrak{p}_\omega = T(G/K_\omega)_{[e]} = (T\mathcal{E}_i)_\omega \subset T\mathcal{K}_\omega,$$

the pairing $\langle\!\langle\ ,\ \rangle\!\rangle_\omega$: $T\mathcal{K}_\omega \times T\mathcal{K}_\omega \to \boldsymbol{R}$ (cf. (2.1.2)) induces the positive definite bilinear form $((\ ,\ ))_i$: $\mathfrak{t}\times\mathfrak{t} \to \boldsymbol{R}$ by

$$((\tau_1, \tau_2))_i := \langle\!\langle \sqrt{-1}\,\tau_1, \sqrt{-1}\,\tau_2 \rangle\!\rangle_\omega \qquad (\tau_1, \tau_2 \in \mathfrak{t}).$$

We shall now show that, up to isometry, our Euclidean lattice $(\mathfrak{t}, \Lambda, ((\ ,\ ))_i)$ does not depend on the choice of ω and T. Let $\omega' \in \mathcal{E}_i$, and T' be a maximal torus in $K_{\omega'}$. Put $\mathfrak{t}':=\text{Lie } T'$. Then there exists an element g of G such that

i) $g^*\omega = \omega'$ (i.e., $R_g(\omega)=\omega'$), and

ii) $T'=g^{-1}Tg$,

where R_g denotes the isometry of $\mathcal{K}$ sending each $\theta \in \mathcal{K}$ to $R_g(\theta):=g^*\theta \in \mathcal{K}$. In view of the commutative diagram

$$\begin{array}{ccccc} T\mathcal{K}_\omega \supset \sqrt{-1}\,\mathfrak{t} & \cong & \mathfrak{t} & \xrightarrow{\exp} & T\ (=\mathfrak{t}/\Lambda) \\ \downarrow (R_g)_* & & \downarrow \text{Ad}(g^{-1}) & & \downarrow \text{Ad}(g^{-1}) \\ T\mathcal{K}_{\omega'} \supset \sqrt{-1}\,\mathfrak{t}' & \cong & \mathfrak{t}' & \xrightarrow{\exp} & T'\ (=\mathfrak{t}'/\Lambda'), \end{array}$$

we easily conclude that $\text{Ad}(g^{-1})$ induces an isometry of the Euclidean lattices $(\mathfrak{t}, \Lambda, ((\ ,\ ))_i)$, $(\mathfrak{t}, \Lambda', ((\ ,\ ))'_i)$ considered.

REMARK 6.4.1. Let K be the same as in Introduction, and choose a $\omega \in \mathcal{E}_i$ such that $K_\omega = K$. Recall that G/K is diffeomorphic to $\boldsymbol{R}^N$ (for some N) and hence simply connected. Consequently, $\mathcal{E}_i (=G/K)$ is written as a product

$$M_\infty \times M_1 \times \cdots \times M_r$$

of Riemannian symmetric spaces, where M_∞ is of Euclidean type, and M_ν ($1 \leqq \nu \leqq r$) are irreducible and of noncompact type. Now the K-actions on M_ν's induce the natural group homomorphism

$$\gamma: K \to \textstyle\prod_{\nu=1}^{r} \text{Isometry}(M_\nu).$$

Note that the identity component of Ker γ is the compact torus which is maximal in the center $Z(G)$ of G. Using the notation of (6.4), we consider

$$\gamma_{|T}\colon T \to T_0\,(:=\gamma(T))\subset \textstyle\prod_{\nu=1}^{r} \text{Isometry}(M_\nu)\,.$$

Then for each $\nu \geqq 1$, there exists a toral subgroup T_ν of Isometry(M_ν) such that $T_0 = T_1\times T_2\times \cdots \times T_r \subset \prod_{\nu=1}^{r}$ Isometry(M_ν). Put $\mathfrak{t}_\nu := \text{Lie } T_\nu$ and $((\ ,\ ))_{i,\nu} := ((\ ,\ ))_{i|\mathfrak{t}_\nu}$ $(0\leqq \nu \leqq r)$, where we regard each $\mathfrak{t}_\nu$ as a Lie subalgebra of $\mathfrak{t}$ via the natural isomorphism

$$\mathfrak{t} \cong (\text{Lie}(\text{Ker }\gamma))\oplus \mathfrak{t}_0 = (\text{Lie}(\text{Ker }\gamma))\oplus\mathfrak{t}_1\oplus\mathfrak{t}_2\oplus\cdots\oplus\mathfrak{t}_r\,.$$

Let $\bar{\Lambda}$ be the lattice of $\mathfrak{t}_0$ obtained as the image of Λ under the natural linear map $\mathfrak{t}\to\mathfrak{t}_0$ $(=\mathfrak{t}/\text{Lie}(\text{Ker }\gamma))$. For each $\nu \geqq 0$, corresponding to the exponential map

$$\exp\colon \mathfrak{t}_\nu \to T_\nu\ (=\mathfrak{t}_\nu/\Lambda_\nu)\,,$$

we obtain the lattice Λ_ν in $\mathfrak{t}_\nu$ as its kernel. It then follows that

i) $\bar{\Lambda}$ is a sublattice of Λ_0 of finite index, and

ii) $(\mathfrak{t}_0, \Lambda_0, ((\ ,\ ))_{i,0}) = \oplus_{\nu=1}^{r} (\mathfrak{t}_\nu, \Lambda_\nu, ((\ ,\ ))_{i,\nu})$.

REMARK 6.4.2. In a forthcoming paper [10], we shall make some studies and computations of these Euclidean lattices.

7. Appendix

As in the case of "finite-dimensional Riemannian geometry", Theorems (2.5) and (4.2) allow us to obtain variational formulas for energies of paths in $\mathcal{H}$ (cf. (7.2), (7.3)). We shall also show that there are no conjugate points on any geodesic of $\mathcal{H}$. (In this appendix, we restrict ourselves to a very simple situation in order to avoid tedious routine works caused by going too much general.)

DEFINITION 7.1. Let $\Phi = \{\varphi_t \mid a \leqq t \leqq b\}$ be a smooth path in $\mathcal{H}$.

(i) $\psi = \psi(u; t, x) \in C^\infty([-\varepsilon, \varepsilon]\times[a, b]\times X)_{\mathbf{R}}$ is called a *1-parameter variation* of Φ if the following conditions are satisfied:

(i–1) $\psi_{0,t} = \varphi_t$ for all $t\in[a, b]$,

(i–2) $\psi_{u,a} = \varphi_a$ and $\psi_{u,b} = \varphi_b$ for all $u\in[-\varepsilon, \varepsilon]$,

where $\psi_{u,t}\in C^\infty(X)_{\mathbf{R}}$ denotes the function defined by

$$\psi_{u,t}(x) = \psi(u; t, x) \qquad (x\in X)\,.$$

(ii) $\psi = \psi(u, v; t, x) \in C^\infty([-\varepsilon, \varepsilon]\times[-\varepsilon, \varepsilon]\times[a, b]\times X)_{\mathbf{R}}$ is called a *2-parameter variation* of Φ if the following conditions are satisfied:

(ii–1) $\psi_{0,0,t} = \varphi_t$ for all $t\in[a, b]$,

(ii–2) $\psi_{u,v,a}=\varphi_a$ and $\psi_{u,v,b}=\varphi_b$ for all $u, v\in[-\varepsilon, \varepsilon]$,
where $\psi_{u,v,t}\in C^\infty(X)_R$ denotes the function defined by

$$\psi_{u,v,t}(x)=\psi(u, v; t, x) \qquad (x\in X).$$

Theorem 7.2 (First variational formula). *Let $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$ be a smooth path in $\mathcal{H}$, and let $\psi=\psi(u; t, x)\in C^\infty([-\varepsilon, \varepsilon]\times[a, b]\times X)_R$ be a 1-parameter variation of Φ. Then*

$$(7.2.1)\qquad \frac{1}{2}\frac{d}{du}(\mathrm{Engy}(\{\psi_{u,t}\,|\,a\leqq t\leqq b\}))_{|u=0}=-\int_a^b\left\langle\!\left\langle\frac{\partial\psi}{\partial u}_{|u=0},\ \frac{D}{\partial t}\dot{\varphi}\right\rangle\!\right\rangle_{\varphi}dt\,.$$

Proof. In view of Theorems (2.5) and (4.2), it follows that

$$(7.2.2)\qquad -\int_a^b\left\langle\!\left\langle\frac{\partial\psi}{\partial u},\ \frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}dt$$
$$=\int_a^b\left\{-\frac{\partial}{\partial t}\left(\left\langle\!\left\langle\frac{\partial\psi}{\partial u},\ \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}\right)+\left\langle\!\left\langle\frac{D}{\partial t}\left(\frac{\partial\psi}{\partial u}\right),\ \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}\right\}dt$$
$$=-\left\langle\!\left\langle\frac{\partial\psi}{\partial u},\ \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}\Bigg|_{t=a}^{t=b}+\int_a^b\left\langle\!\left\langle\frac{D}{\partial u}\left(\frac{\partial\psi}{\partial t}\right),\ \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}dt$$
$$=\int_a^b\left\langle\!\left\langle\frac{D}{\partial u}\left(\frac{\partial\psi}{\partial t}\right),\ \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}dt=\frac{1}{2}\frac{d}{du}\left(\int_a^b\left\langle\!\left\langle\frac{\partial\psi}{\partial t},\ \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,t}}dt\right).$$

Evaluating this at $u=0$, we obtain (7.2.1). Q.E.D.

Theorem 7.3 (Second variational formula). *Let $\Phi=\{\varphi_t\,|\,a\leqq t\leqq b\}$ be a geodesic in $\mathcal{H}$, and let*

$$\psi=\psi(u, v; t, x)\in C^\infty([-\varepsilon, \varepsilon]\times[-\varepsilon, \varepsilon]\times[a, b]\times X)_R$$

be a 2-parameter variation of Φ. Then

$$(7.3.1)\qquad \frac{1}{2}\frac{\partial^2}{\partial u\partial v}(\mathrm{Engy}(\{\psi_{u,v,t}\,|\,a\leqq t\leqq b\}))_{|(u,v)=(0,0)}$$
$$=-\int_a^b\left\langle\!\left\langle W_2,\ \frac{D^2}{\partial t^2}W_1-R(\dot{\varphi}, W_1)_{\varphi}(\dot{\varphi})\right\rangle\!\right\rangle_{\varphi}dt\,,$$

where $W_1:=\dfrac{\partial\psi}{\partial u}_{|(u,v)=(0,0)}\in C^\infty([a, b]\times X)_R\,,$

$W_2:=\dfrac{\partial\psi}{\partial v}_{|(u,v)=(0,0)}\in C^\infty([a, b]\times X)_R\,,$

and $\dfrac{D^2}{\partial t^2}W_1:=\dfrac{D}{\partial t}\left(\dfrac{D}{\partial t}W_1\right).$

Proof. Since $\dfrac{D^2}{\partial t^2}\left(\dfrac{\partial\psi}{\partial u}\right)=\dfrac{D}{\partial t}\dfrac{D}{\partial u}\dfrac{\partial\psi}{\partial t}$ (cf. (4.2)), and since $R\left(\dfrac{\partial\psi}{\partial t},\ \dfrac{\partial\psi}{\partial u}\right)=$

$\frac{D}{\partial t}\frac{D}{\partial u}-\frac{D}{\partial u}\frac{D}{\partial t}$, it follows that

$$
\begin{aligned}
&-\int_a^b \left\langle\!\left\langle \frac{\partial\psi}{\partial v}, \frac{D^2}{\partial t^2}\left(\frac{\partial\psi}{\partial u}\right) - R\left(\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial u}\right)\left(\frac{\partial\psi}{\partial t}\right)\right\rangle\!\right\rangle_{\psi_{u,v,t}} dt \\
&= -\int_a^b \left\langle\!\left\langle \frac{\partial\psi}{\partial v}, \frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}} dt\,.
\end{aligned}
$$

Evaluating this at $(u, v)=(0, 0)$, we obtain

$$
\begin{aligned}
(7.3.2)\qquad &-\int_a^b \left\langle\!\left\langle W_2, \frac{D^2}{\partial t^2}W_1 - R(\dot{\varphi}, W_1)_\varphi(\dot{\varphi})\right\rangle\!\right\rangle_\varphi dt \\
&= -\int_a^b \left\langle\!\left\langle W_2, \left(\frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right)_{|(u,v)=(0,0)}\right\rangle\!\right\rangle_\varphi dt\,.
\end{aligned}
$$

On the other hand, in view of (7.2.2),

$$
\frac{1}{2}\frac{\partial}{\partial v}\left(\int_a^b \left\langle\!\left\langle \frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}} dt\right) = -\int_a^b \left\langle\!\left\langle \frac{\partial\psi}{\partial v}, \frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}} dt
$$

and therefore, by (2.5),

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial u\partial v}\left(\int_a^b \left\langle\!\left\langle \frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}} dt\right) \\
&= -\int_a^b \left(\left\langle\!\left\langle \frac{D}{\partial u}\frac{\partial\psi}{\partial v}, \frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}} + \left\langle\!\left\langle \frac{\partial\psi}{\partial v}, \frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}}\right) dt\,.
\end{aligned}
$$

We evaluate this at $(u, v)=(0, 0)$. Note that Φ is a geodesic. Then

$$
\begin{aligned}
(7.3.3)\qquad &\frac{1}{2}\frac{\partial^2}{\partial u\partial v}\left(\int_a^b \left\langle\!\left\langle \frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial t}\right\rangle\!\right\rangle_{\psi_{u,v,t}} dt\right)_{|(u,v)=(0,0)} \\
&= -\int_a^b \left\langle\!\left\langle W_2, \left(\frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial\psi}{\partial t}\right)_{|(u,v)=(0,0)}\right\rangle\!\right\rangle_\varphi dt\,.
\end{aligned}
$$

Comparing (7.3.2) and (7.3.3), we obtain (7.3.1). Q.E.D.

DEFINITION 7.4. Let $\Phi=\{\varphi_t \mid a\leqq t\leqq b\}$ be a geodesic in $\mathcal{H}$ and let a', $b'\in\boldsymbol{R}$ be such that $a\leqq a'<b'\leqq b$. Then $\psi=\psi(t, x)\in C^\infty([a', b']\times X)_{\boldsymbol{R}}$ is called a *Jacobi field* along Φ if

$$
\frac{D^2}{\partial t^2}\psi = R(\dot{\varphi}, \psi)_\varphi\dot{\varphi}\,.
$$

Two elements $\varphi_{a'}$, $\varphi_{b'}$ of $\mathcal{H}$ are called *conjugate* along Φ if there exists a non-trivial Jacobi field $\psi=\psi(t, x)\in C^\infty([a', b']\times X)_{\boldsymbol{R}}$ along Φ such that $\psi_{|t=a'}=0$ and $\psi_{|t=b'}=0$ in $C^\infty(X)_{\boldsymbol{R}}$.

Theorem 7.5. *Let $\Phi=\{\varphi_t \mid a \leqq t \leqq b\}$ be an arbitrary geodesic in $\mathcal{H}$. For any a', b' with $a \leqq a' < b' \leqq b$, $\varphi_{a'}$ and $\varphi_{b'}$ cannot be conjugate along Φ.*

Proof. Suppose that $\psi=\psi(t, x) \in C^\infty([a', b'] \times X)_{\boldsymbol{R}}$ satisfies the equation

$$\frac{D^2}{\partial t^2}\psi = R(\dot{\varphi}, \psi)_\varphi \dot{\varphi}$$

with the boundary condition

$$\psi_{|t=a'} = 0 \quad \text{and} \quad \psi_{|t=b'} = 0 \qquad \text{in } C^\infty(X)_{\boldsymbol{R}}\,.$$

Then $\left\langle\!\left\langle \frac{D}{\partial t}\psi, \psi \right\rangle\!\right\rangle_\varphi \in C^\infty([a', b'])_{\boldsymbol{R}}$ is a monotone increasing function in t, because

$$\begin{aligned} \frac{d}{dt}\left\langle\!\left\langle \frac{D}{\partial t}\psi, \psi \right\rangle\!\right\rangle_\varphi &= \left\langle\!\left\langle \frac{D^2}{\partial t^2}\psi, \psi \right\rangle\!\right\rangle_\varphi + \left\langle\!\left\langle \frac{D}{\partial t}\psi, \frac{D}{\partial t}\psi \right\rangle\!\right\rangle_\varphi \\ &= \langle\!\langle R(\dot{\varphi}, \psi)_\varphi \dot{\varphi}, \psi \rangle\!\rangle_\varphi + \left\langle\!\left\langle \frac{D}{\partial t}\psi, \frac{D}{\partial t}\psi \right\rangle\!\right\rangle_\varphi \\ &\geqq \left\langle\!\left\langle \frac{D}{\partial t}\psi, \frac{D}{\partial t}\psi \right\rangle\!\right\rangle_\varphi \geqq 0 \qquad \text{(cf. (4.5))}\,. \end{aligned}$$

Note that

$$\left\langle\!\left\langle \frac{D}{\partial t}\psi, \psi \right\rangle\!\right\rangle_{\varphi|t=a'} = 0 = \left\langle\!\left\langle \frac{D}{\partial t}\psi, \psi \right\rangle\!\right\rangle_{\varphi|t=b'}\,.$$

Now, on the whole $[a', b']$,

$$\frac{d}{dt}\langle\!\langle \psi, \psi \rangle\!\rangle_\varphi = 2\left\langle\!\left\langle \frac{D}{\partial t}\psi, \psi \right\rangle\!\right\rangle_\varphi = 0\,.$$

Thus we conclude that

$$\langle\!\langle \psi, \psi \rangle\!\rangle_\varphi = \langle\!\langle \psi, \psi \rangle\!\rangle_{\varphi|t=a'} = 0\,,$$

i.e., $\psi=0$ in $C^\infty([a, b] \times X)_{\boldsymbol{R}}$. Q.E.D.

8. Conclusion

As a final remark to the whole paper, we should say that some of the ideas given above are valid also for Riemannian analogues in conformal differential geometry. In fact, Bourguignon [2] independently studied similar topics ("generalizations of Kazdan-Warner's invariant") from different viewpoints.

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