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THE NINOMIYA OPERATORS AND THE GENERALIZED DIRICHLET PROBLEM IN POTENTIAL THEORY

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Introduction

Investigating various aspects of Keldyš theorem [11], N. Ninomiya introduced and studied special operators related to the generalized Dirichlet problem in classical potential theory.

To recall his interesting result from [18], let us introduce the following notation.

Let $\{B_j\}$ be the sequence of balls in the Euclidean space $\boldsymbol{R}^m$ of dimension $m>2$ having a rational center and rational radius. Denote by λ_j the normalized surface measure on ∂B_j and fix positive numbers a_j such that the Newtonian potential q of the measure $\sum a_j \lambda_j$ is continuous on $\boldsymbol{R}^m$. (The potential q will be called the Cartan potential here.)

Suppose that $U \subset \boldsymbol{R}^m$ is a bounded open set and denote by $C(\partial U)$ the set of all continuous functions on ∂U. Let $\mathcal{F}(U)$ stand for the set of all real-valued functions defined on U. As usual, H^U denotes the operator of *the Perron-Wiener-Brelot solution of the generalized Dirichlet problem on* U.

To state the Ninomiya uniqueness result, suppose that $A: C(\partial U) \to \mathcal{F}(U)$ is an operator having the following properties:

(i) A is linear and positive;

(ii) $\sup Af(U) \leqq \sup f(\partial U)$ whenever $f \in C(\partial U)$;

(iii) $A(p_{|\partial U}) = p_{|U}$ whenever p is a continuous Newtonian potential of a measure supported by the complement CU of the set U;

(iv) $A(q_{|\partial U})$ is harmonic (or subharmonic) on U for the Cartan potential q.

(Obviously, the operator $A=H^U$ enjoys (i)-(iv), thus no existence problem arises.)

N. Ninomiya [18] was able to prove that such an operator A is uniquely determined by conditions (i)-(iv). This remarkable result improves the statement of Keldyš uniqueness theorem for the generalized Dirichlet problem (see Theorem 1 below; note that conditions (4) and (6) stated there are automatically satisfied in classical potential theory; cf. also [6] and [16]).

The proof of uniqueness given by Ninomiya makes use of potentials of finite energy and of Lebesgue measure. Hence it does not admit a straightforward modification to a more general situation as in the case of other partial differential equations or of abstract potential theory.

In this paper, in the context of harmonic spaces, necessary and sufficient conditions for uniqueness of the Ninomiya type operators are established and relations to previous investigations of Keldyš operators are shown. The validity of the theorems given below requires that the relevant function spaces are rich enough, as illustrated by a counterexample. Also, a density result for harmonic functions is proved. This result does not seem to be known even in classical potential theory where the corresponding statement can be formulated as follows: Every function continuous on $\bar{U}$ and harmonic on U is a uniform limit of continuous potentials of signed measures supported by CU.

1. The Ninomiya and Keldyš operators

In what follows, let X be a $\mathfrak{P}$-*harmonic space* with countable base in the sense of axiomatics of Constantinescu and Cornea [7]. (All notions not recalled here are to be found in this monograph.) The corresponding harmonic sheaf is denoted by $\mathcal{H}$ and the cone of continuous potentials on X is denoted by $\mathcal{P}$. For a potential p, the symbol $C(p)$ stands for its superharmonic carrier.

Suppose that $U\subset X$ is a nonempty, relatively compact open set. Denote by U_r and U_i the set of *regular* and *irregular* points of U, respectively. A Borel set $M\subset\partial U$ is said to be *negligible,* if M has harmonic measure zero at every point of U, i.e. $\varepsilon_x^{CU}(M)=0$ whenever $x\in U$.

As above, $\mathcal{F}(U)$ stands for the space of real-valued functions on U, $\mathcal{S}(U)$ is the cone of (not necessarily continuous) superharmonic functions on U. Of course, $\mathcal{H}(U)=\mathcal{S}(U)\cap(-\mathcal{S}(U))$.

Let us introduce the following notations:

$$H(U)=\{h\in C(\bar{U});\ h_{|U}\in\mathcal{H}(U)\},\quad H(\partial U)=H(U)_{|\partial U},$$

$$P(U)=\{p_{|\bar{U}};\ p\in\mathcal{P},\quad C(p)\subset CU\},\quad Q(U)=P(U)-P(U).$$

Clearly, $Q(U)$, the space of differences of functions of $P(U)$, is a subspace of $H(U)$.

The following definition is a slight modification of that proposed by N. Ninomiya. (It should be noticed that the above mentioned condition (ii) is a consequence of conditions (i) and (iii) and so does not appear in our definition.)

An operator $A\colon C(\partial U)\to\mathcal{F}(U)$ is said to be a *Ninomiya operator* on U, if

(1) A is linear and positive;

(2) $A(p_{|\partial U})=p_{|U}$ whenever $p\in P(U)$;

(3) there is a strict potential $q\in\mathcal{P}$ such that $A(q_{|\partial U})\in\mathcal{H}(U)$.

If (1) and (2) hold and (3) is replaced by

(3*) there is a strict potential $q \in \mathcal{P}$ such that $A(q_{|\partial U}) \in -\mathcal{S}(U)$,
then A will be termed a *weak Ninomiya operator.*

Recall that an operator $A: C(\partial U) \to \mathcal{F}(U)$ is said to be a *Keldyš operator,* provided (1) holds and the following conditions are fulfilled:

(2') $A(h_{|\partial U}) = h_{|U}$ whenever $h \in H(U)$;

(3') $A(C(\partial U)) \subset \mathcal{H}(U)$.

Clearly, every Keldyš operator is a Ninomiya operator and every Ninomiya operator is a weak Ninomiya operator. Of course, the operator H^U of the Perron-Wiener-Brelot solution of the generalized Dirichlet problem is a Keldyš operator.

The main question studied in this paper is to investigate under what circumstances a Ninomiya operator is uniquely determined.

The uniqueness problem for Keldyš operators has been extensively studied; for references see [15], [16], [17]. Recall the following result; for a proof see [12] or [21]; cf. also [15].

Theorem 1. *Suppose that*

(4) *the space $H(U)$ linearly separates the points of $\bar{U}$ and contains a strictly positive function.*

Then the following conditions are equivalent:

(5) *there is exactly one Keldyš operator on U;*

(6) *the set U_i is negligible.*

The main result of this paper reads as follows:

Theorem 2. *Suppose that*

(7) *the space $Q(U)$ linearly separates the points of $\bar{U}$ and contains a strictly positive function.*

Then the following conditions are equivalent:

(8) *there is exactly one weak Ninomiya operator on U;*

(9) *there is exactly one Ninomiya operator on U;*

(10) *the set U_i is negligible.*

The proof of Theorem 2 will be postponed until after having proved several auxiliary results.

2. Simpliciality and density of a space of harmonic functions

First recall that, for $x \in \bar{U}$, the set M_x of *representing measures* (with respect to $Q(U)$) is the set of all positive Radon measures μ on $\bar{U}$ such that $\mu(q) = q(x)$ whenever $q \in Q(U)$. The *Choquet boundary* of $\bar{U}$ with respect to $Q(U)$ is defined by

$$Ch_{Q(U)}\,\bar{U} = \{x \in \bar{U};\quad M_x = \{\varepsilon_x\}\}\ .$$

Note that $Ch_{Q(U)}\,\bar{U}$ is a G_δ set. If (7) holds, then by the Choquet theorem (cf.

e.g. [19], p. 43), for every $x \in \bar{U}$, there is a $\mu_x \in M_x$ carried by the Choquet boundary. This means that $\mu_x(\bar{U} \setminus Ch_{Q(U)} \bar{U})=0$.

Let us denote by F the essential base of CU; see [4], p. 94.

Theorem 3. *Suppose that condition* (7) *holds. Then* $Ch_{Q(U)} \bar{U}=F \cap \bar{U}$ *and the space* $Q(U)$ *is simplicial* (*i.e. for every* $x \in \bar{U}$ *there is a unique* $\mu_x \in M_x$ *carried by* $Ch_{Q(U)} \bar{U}$).

Proof. Since $Q(U) \subset H(U)$, we have $\varepsilon_x^F \in M_x$ for every $x \in \bar{U}$ by [4], p. 103. If $x \in Ch_{Q(U)} \bar{U}$, then $x \in \bar{U}$ and $\varepsilon_x^F=\varepsilon_x$. Thus $x \in F \cap \bar{U}$ by [4], p. 102. Consequently, $Ch_{Q(U)} \bar{U} \subset F \cap \bar{U}$.

For $p \in \mathcal{P}$ put

$$\mathcal{L}(p) = \{q \in \mathcal{P};\ q \leqq p,\quad R_q^F = q\} .$$

Then $\mathcal{L}(p)$ is increasingly filtered by [10], p. 500 and by [4], p. 98,

$$R_p^F = \sup \mathcal{L}(p) .$$

Let $x \in F \cap \bar{U}$ and $\nu_x \in M_x$ be a measure carried by $Ch_{Q(U)} \bar{U}$. We are going to prove that $\nu_x=\varepsilon_x$, which, in turn, shows that $x \in Ch_{Q(U)} \bar{U}$.

To this end, fix $p \in \mathcal{P}$ and put $P_1=\mathcal{L}(p)_{|\bar{U}}$. Then P_1 is increasingly filtered, $P_1 \subset Q(U)$ and $p=\sup P_1$ on $F \cap \bar{U}$ because $R_p^F=p$ on F. Since ν_x is carried by $Ch_{Q(U)} \bar{U} \subset F \cap \bar{U}$, we have

$$\int p \, d\nu_x = \int \sup P_1 \, d\nu_x = \sup \{\int h \, d\nu_x;\ h \in P_1\}$$
$$= \sup \{h(x);\ h \in P_1\} = p(x) .$$

Consequently, $\nu_x(p)=p(x)$ whenever $p \in \mathcal{P}$ and $\nu_x=\varepsilon_x$ by [7], p. 45. It follows that $Ch_{Q(U)} \bar{U}=F \cap \bar{U}$.

Let $x \in \bar{U}$ and ρ_x, τ_x be elements of M_x carried by $Ch_{Q(U)} \bar{U}$ and, as above, let $p \in \mathcal{P}$ be a fixed potential. Then

$$\int p \, d\rho_x = \int \sup P_1 \, d\rho_x = \sup \{\int h \, d\rho_x;\ h \in P_1\}$$
$$= \sup \{h(x);\ h \in P_1\} = R_p^F(x)$$

and, similarly,

$$\int p \, d\tau_x = R_p^F(x) .$$

Consequently, $\rho_x(p)=\tau_x(p)$ for every $p \in \mathcal{P}$ and $\rho_x=\tau_x$. Hence $Q(U)$ is simplicial.

Theorem 4. *Suppose that* (7) *holds. Then* $Q(U)$ *is uniformly dense in* $H(U)$.

Proof. Clearly, $\overline{Q(U)}\subset H(U)$. By Theorem 2, we have $Ch_{\overline{Q(U)}}\bar{U}=Ch_{Q(U)}\bar{U}=F\cap\bar{U}$ and it follows from [4], pp. 102, 103, that $Ch_{H(U)}\bar{U}=F\cap\bar{U}$ and $H(U)$ is simplicial. Thus $\overline{Q(U)}$ and $H(U)$ have the same annihilating measures; cf. [9], p. 20 or [15], p. 240. Consequently, $\overline{Q(U)}=H(U)$.

REMARK. One also could use the Stone-Weierstrass type theorem established in [8]. For classical potential theory in $\boldsymbol{R}^3$, the result of Theorem 4 for regular sets is mentioned in [1].

3. Uniqueness of Ninomiya operators

In this section, we are going to prove Theorem 2. To this end, some auxiliary results are needed.

Lemma 1. *For* $x\in U$ *and* $f\in C(\partial U)$, *put* $D^U f(x)=\varepsilon_x^F(f)$. *Let an operator* $A: C(\partial U)\to\mathcal{F}(U)$ *satisfy* (1) *and* (2). *Then*

$$A(p_{|\partial U})\geqq D^U(p_{|\partial U}),$$

whenever $p\in\mathcal{P}$.

Proof. For $x\in U$,

$$\alpha_x: f\mapsto Af(x), \quad f\in C(\partial U),$$

is a positive Radon measure on ∂U. Let $p\in\mathcal{P}$ and $q\in\mathcal{L}(p)$. Since $q_{|\bar{U}}\in P(U)$, we have $\alpha_x(q)=q(x)$. Thus, for any $x\in U$,

$$\begin{aligned} A(p_{|\partial U})(x) &= \alpha_x(p)\geqq\sup\{\alpha_x(q);\ q\in\mathcal{L}(p)\} \\ &= \sup\mathcal{L}(p)(x)=R_p^F(x)=\varepsilon_x^F(p)=D^U(p_{|\partial U})(x). \end{aligned}$$

REMARK. For a similar result, cf. [13], p. 175.

Lemma 2. *Let* (7) *hold and let an operator* $A: C(\partial U)\to\mathcal{F}(U)$ *satisfy* (1) *and* (2). *If* $z\in Ch_{Q(U)}\bar{U}$, *then*

$$A(p_{|\partial U})(x)\to p(z) \qquad \textit{for} \quad x\to z,$$

whenever $p\in\mathcal{P}$.

Proof. By (7), there is $p_0\in P(U)$ with $\inf p_0(\bar{U})\geqq 1$. If α_x has the same meaning as in the proof of Lemma 1, then, for every $x\in U$,

$$\alpha_x(\partial U)\leqq\alpha_x(p_0)=p_0(x)\leqq\sup p_0(\bar{U})<\infty.$$

So, for a given $z \in Ch_{Q(U)}\bar{U}$, it is sufficient to prove the following statement:

If $x_n \in U$, $x_n \to z$ and $\alpha_{x_n} \to \alpha$ weakly, then $\alpha = \varepsilon_z$.

To show this, fix $p \in P(U)$. Then $\alpha_{x_n}(p) = p(x_n)$ and, consequently, $\alpha(s) = s(x)$ for every $s \in Q(U)$. Hence $\alpha \in M_z$ and $\alpha = \varepsilon_z$, since $z \in Ch_{Q(U)}\bar{U}$.

Lemma 3. *Suppose* (6) *and* (7). *Let* $A: C(\partial U) \to \mathcal{F}(U)$ *be an operator satisfying* (1) *and* (2). *If* $q \in \mathcal{P}$ *and* $A(q_{|\partial U}) \in -\mathcal{S}(U)$, *then*

$$A(q_{|\partial U}) = H^U(q_{|\partial U}).$$

Proof. Let q satisfy the hypothesis. By (7), there is $p \in P(U)$ such that $q_{|\partial U} \leqq p_{|\partial U}$. Thus

$$A(q_{|\partial U}) \leqq A(p_{|\partial U}) = p_{|U}$$

and the function $s = H^U(q_{|\partial U}) - A(q_{|\partial U})$ is superharmonic and lower bounded on U. Condition (6) and Theorem 3 imply that $D^U(q_{|\partial U}) = H^U(q_{|\partial U})$ and $U_r = Ch_{Q(U)}\bar{U}$; see [4], p. 106. By Lemma 2,

$$\lim_{x \to z} s(x) = 0$$

for every $z \in U_r$. Since the set U_i is negligible by (6), the minimum principle (cf. [2], p. 145) yields $s \geqq 0$ on U. But $s \leqq 0$ by Lemma 1. Consequently, $A(q_{|\partial U}) = H^U(q_{|\partial U})$.

Proof of Theorem 2. Obviously, (8) implies (9). Since (4) follows from (7), (9) implies (10) by Theorem 1. It remains to prove that (10) implies (8).

Let A be a weak Ninomiya operator. By definition, there is a strict potential $q \in \mathcal{P}$ such that $A(q_{|\partial U}) \in -\mathcal{S}(U)$.

Fix $x \in U$. By (10) and [4], p. 106, $\varepsilon_x^{CU} = \varepsilon_x^F$, thus

$$\alpha_x(p) \geqq \varepsilon_x^{CU}(p), \quad p \in \mathcal{P},$$

by Lemma 1. We have

$$\alpha_x(q) = \varepsilon_x^{CU}(q)$$

by Lemma 3. Since q is a strict potential, $\alpha_x = \varepsilon_x^{CU}$ by [6], pp. 166, 43.

We conclude that $A = H^U$ and (8) is verified.

REMARK. The implications (5)⇒(6), and also (8)⇒(9)⇒(10) are true without hypotheses (4) and (7); cf. [15], pp. 239, 244. The following theorem shows, however, that the implication (6)⇒(5), and, consequently, the implication (10)⇒(8) do not hold in general.

4. Non-uniqueness of Keldyš operators and the negligible set of irregular points

We shall construct a harmonic space Y and a relatively compact open set $U\subset Y$ for which the Keldyš theorem fails despite of the fact that U_i is negligible. It is also shown that the uniqueness of a Keldyš operator implies that the space $H(\partial U)$ is in a sense rich enough.

Theorem 5. *There is a $\mathfrak{P}$-harmonic Bauer space Y with countable base and a nonempty, relatively compact open set $U\subset Y$ such that U_i is negligible and there exist two distinct Keldyš operators on U.*

Proof. If $V\subset \boldsymbol{R}$ is an interval (possibly degenerated), denote by $\mathcal{L}(V)$ and $\mathcal{K}(V)$ the set of all affine and constant functions on V, respectively.

Define $Y=[0, 1[$ (endowed with the relative topology from $\boldsymbol{R}$) and $M=\{1/(n+1); n\in\boldsymbol{N}\}$. Every open set $V'\subset Y$ not containing 0 is a union of a disjoint system of intervals which are open in $\boldsymbol{R}$.

Suppose that $V\subset Y$ is a nonempty open connected set. If $0\in V$, put $\mathcal{H}(V)=\mathcal{K}(V)$. If $0\notin V$ and $M\cap V=\emptyset$, put $\mathcal{H}(V)=\mathcal{L}(V)$. If, finally, $0\notin V$ and $M\cap V\neq\emptyset$, define $\alpha_V=\inf (M\cap V)$, $V_1=]\inf V, \alpha_V[$, $V_2=]\alpha_V, \sup V[$. Notice that $V_1=\emptyset$, if and only if $\inf V=0$. A function $h\in C(V)$ is said to belong to $\mathcal{H}(V)$, if $h_{|V_1}\in\mathcal{L}(V_1)$ and $h_{|V_2}\in\mathcal{K}(V_2)$. If $V'\subset Y$ is an open set, then h is said to belong to $\mathcal{H}(V')$, provided $h_{|V}\in\mathcal{H}(V)$ for every component V of the set V'. Then $\mathcal{H}$ is a harmonic sheaf possessing the Doob convergence property and containing constant functions. It is easily seen that $]a,b[$ and $[0,a[$ are regular sets, if $0<a<b<1$. Hence there is a strong base of regular sets in Y. Notice that $]0,a[$ is not regular.

One easily verifies that $x\mapsto 1-x$ is a superharmonic function on Y, thus Y is a $\mathfrak{P}$-harmonic space by [7], p. 44.

Put $U=]0, \frac{1}{2}[$. Then $\mathcal{H}(U)=\mathcal{K}(U)$ and also $H(U)=\mathcal{K}(\bar{U})$. We have $H^U f=f\left(\frac{1}{2}\right)$ on U for $f\in C(\partial U)$. Obviously, $\frac{1}{2}$ is a regular point, $0\in U_i$ and the set $\{0\}$ is negligible, since $H^U f=0$, provided $f(0)=1, f\left(\frac{1}{2}\right)=0$. (Note that for any $a\in M$ and $\beta\in\boldsymbol{R}$, the function $x\mapsto(1-a^{-1}x)^+ +\beta$ is superharmonic.)

For $f\in C(\partial U)$, define

$$Af(x)=\frac{1}{2}\left(f(0)+f\left(\frac{1}{2}\right)\right),\quad x\in U.$$

Then A is a Keldyš operator on U and $Af=H^U f$ if and only if $f(0)=f\left(\frac{1}{2}\right)$.

REMARK. The above example is a modification of a construction given in [7], p. 71; cf. also [20], [21]. A similar example for a Brelot space is shown in

[6] but U is not relatively compact there.

In the example described above, $H(U)$ is the space of constant functions. Thus $H(U)$ does not separate points of $\bar{U}$.

The following theorem shows that, for nontrivial $H(U)$, the space $H(\partial U)$ contains a strictly positive function if Keldyš uniqueness theorem holds. The proof uses an idea from [2].

Theorem 6. *Suppose that $H(U)\neq\{0\}$. If there is a unique Keldyš operator on U, then $H(\partial U)$ contains a strictly positive function.*

Proof. Let j denote the function which is equal to 1 on ∂U. Suppose that there is a unique Keldyš operator on U and there is no strictly positive function in $H(\partial U)$. We deduce from these hypotheses that $H(U)=\{0\}$.

Notice that for $f_1, f_2\in H(\partial U)$, the function f_1-f_2 is constant, if and only if $f_1=f_2$.

Define

$$H_1(\partial U) = \{g+c\cdot j;\ g\in H(\partial U),\ c\in\boldsymbol{R}\}\ .$$

Then $H_1(\partial U)$ is a majorizing subspace of $C(\partial U)$. If $h\in H_1(\partial U)$, then there is exactly one $g\in H(U)$ and exactly one $c\in\boldsymbol{R}$ such that $h=g_{|\partial U}+c\cdot j$. If $h\geqq 0$, then $c\geqq 0$, since otherwise $g_{|\partial U}$ would be strictly positive.

The mapping

$$A_k\colon h\mapsto g_{|U}+c\cdot H^U(k\cdot j)\,,\quad k=1,2\,,$$

is obviously linear on $H_1(\partial U)$ and

$$A_k(g_{|\partial U})=g_{|U}$$

for every $g\in H(U)$. Suppose now that $h\geqq 0$ and $z\in U_r$. Then $c\geqq 0$ and

$$\lim_{x\to z}\ (g_{|U}(x)+c\cdot H^U(k\cdot j)\,(x)) = g(z)+ck\geqq g(z)+c = h(z)\geqq 0\,.$$

Notice that $H^U j$, being majorized by a continuous potential, is bounded and U_i is negligible since there is a unique Keldyš operator on U; cf. the remark following the proof of Theorem 2. Thus $A_k h\geqq 0$ by the minimum principle; see [2], p. 145. One can extend A_k to a Keldyš operator; cf. [15], p. 253. Thus uniqueness gives, in particular, $A_1 j=A_2 j$ or $H^U j=H^U(2\cdot j)$. Consequently, $H^U j=0$ on U. It follows that $H(U)=\{0\}$.

Theorem 7. *Let $V\subset X$ be an open relatively compact set, $S\subset X$ be a closed semipolar set such that $S\cap V$ is not polar. Then there are at least two distinct Keldyš operators on $U=V\setminus S$.*

Proof. Denote $S_1=V\cap S$. Then $\partial U=\partial V\cup S_1$, because S is nowhere dense by [7], pp. 118, 153. It is easily seen that there is a non-polar compact set $K \subset S_1$; see [7], p. 144. For $f \in C(\partial U)$, define

$$A_1 f = H^U f, \quad A_2 f = (H^V(f_{|\partial V}))_{|U}.$$

Then A_1, A_2 are clearly positive linear operators from $C(\partial U)$ into $\mathcal{H}(U)$. Let $h \in H(U)$. Then $h \in H(V)$ by [14], p. 121, and $A_1(h_{|\partial U}) = A_2(h_{|\partial U}) = h_{|U}$. Thus A_1, A_2 are Keldyš operators. Let $f \in C(\partial U)$, $f=0$ on ∂V and $f \geqq 1$ on K. Then $A_2 f = 0$ but $A_1 f$ is not identically zero on U, since K is not polar; see [7], p. 147. Thus A_1, A_2 are distinct Keldyš operators on U.

Corollary. *If there is exactly one Keldyš operator on every nonempty open relatively compact subset of a harmonic space Y, then the axiom of polarity holds in Y.*

Proof. Suppose that the axiom of polarity does not hold in Y. Then there is a nonempty open subset X of Y such that X is a $\mathfrak{P}$-harmonic space in which the axiom of polarity does not hold; see [7], pp. 225, 48. By [7], p. 219, there is a compact non-polar totally thin subset of X. The rest follows from Theorem 7.

REMARKS. The result stated in the corollary is known; see [12], [15] for another proof. Theorem 7 gives a method to construct sets on which the Keldyš uniqueness theorem does not hold. The special case for the harmonic space associated to the heat equation was investigated in [12].

The main results of this paper were presented in a talk at the Conference on Potential Theory, Oberwolfach, July 1984.

During the Conference, Professor W. Hansen communicated to me another proof of Theorem 4 using continuous dilations studied in [5].

In September 1984, Professor L.I. Hedberg informed me that a similar density result for classical potential theory was proved in his manuscript concerning approximation by harmonic functions. Instead of $Q(U)$, however, potentials of measures of finite energy are considered.

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