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THE DUALITY CONJECTURE IN FORMAL KNOT THEORY

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0. Introduction

This paper deals with the formal knot theory of Kauffman [2]. In the first three sections we prove his Duality Conjecture ([2], p. 57); in the fourth we give a new and simpler proof of Kauffman's key combinatorial result, his Clock Theorem ([2], Theorem 2.5). This is based on a reformulation of several of the notions of formal knot theory, and in these terms the theorem can be expressed as follows: the set of maximal trees in a connected plane graph has a naturally defined partial order, which gives it the structure of a distributive lattice.

We assume that the reader is familiar with the basic definitions of [2] (universe, state, black and white holes...), most of which are to be found in the introduction. One piece of this terminology seems worthy of explicit mention here: the word "knot" means a universe with a specification of over- and under-crossings; in standard terminology this would be a knot (or link) diagram. Also, we make a minor departure from Kauffman's usage in that we regard a universe as a subset of S^2 rather than $\boldsymbol{R}^2$.

We now outline the proof of the Duality Conjecture. A polynomial $f(B, W)\in\boldsymbol{Z}[B, W]$ will be called *good* if $f(-W, -B)=f(B, W)$. (Note that since $f(B, W)\mapsto f(-W, -B)$ is an automorphism, the good polynomials form a subring of $\boldsymbol{Z}[B, W]$.) It is easy to see that the Duality Conjecture asserts that the state polynomial $\langle U\rangle$ of any universe is good. In fact we prove that $\langle K\rangle$ is good for any knot K. We start from the observation that if we try to prove this by induction on the number of crossings of K, we may change crossings at will. This is because, for knots K and $\bar{K}$ differing at a single crossing, the exchange identity ([2], Theorem 4.1) expresses $\langle K\rangle-\langle\bar{K}\rangle$ as the state polynomial of a knot with fewer crossings. In § 1 we show that a similar situation obtains for knots related by a Reidemeister move. Then in § 2 we show, in effect, that one can use Reidemeister moves and crossing changes to reduce the number of crossings of a knot without ever increasing this number. These

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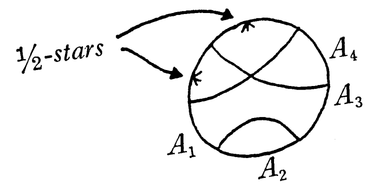
results are put together to complete the proof in § 3.

We remark that it is not hard to see that a polynomial in $\boldsymbol{Z}[B, W]$ is good iff it can be written as a polynomial in BW and $W-B$. This makes it plain that the image of a good polynomial in $\boldsymbol{Z}[B, W]/(BW-1)$ is a polynomial in $z=W-B$; cf. [2], p. 72.

1. Reidemeister moves and state polynomials

Kauffman is somewhat ambiguous about disconnected universes; his definition of a universe ([2], p. 13) excludes such objects, but they make fleeting appearances later in his book. In §§ 1–3 we shall allow a universe to be disconnected; in order to avoid the consideration of several special cases we also adopt in these sections a slightly different definition of a state. Thus, a *marker assignment* for a universe U will mean an assignment to each vertex of U of one of the adjacent regions (indicated pictorially by markers), and a *state* of U will mean a marker assignment with exactly two empty regions. If U is connected, this agrees with the previous definition. If not, U has no states, and therefore has state polynomial zero, as it should. At this point we mention the fact (whose proof we leave to the reader) that in any marker assignment for a disconnected universe there are at least two non-adjacent, simply-connected empty regions.

We shall have cause to consider a universe split into two pieces by a transverse simple closed curve. This leads to the following definitions. A *tangle universe* in a 2-disc D is a graph U embedded in D such that every vertex in int D is 4-valent, while each vertex on ∂D (called an *end* of U) is 1-valent. Moreover, U is required to have ends, and to be oriented, in the sense that each edge is oriented so that at each interior vertex (or crossing) the orientations look like . Note that the number of ends must be even. We define a *marker assignment* for a tangle universe as above. If the tangle universe U has $2n+2$ ends and $U\cap\partial D$ has c components, an Euler characteristic —Alexander duality argument shows that in any marker assignment for which each region has at most one marker there are $n+c+1$ empty regions. We therefore define a *state* of U to be a marker assignment with exactly $n+2$ empty regions; U has a state iff $c=1$. We shall only consider states in which the empty regions are all adjacent to ∂D. Such a state will be further decorated as follows; in two of the empty regions we place half a star on an arc of ∂D next to the region; in the others we place similarly an arrow pointing into the region; see Fig. 1. Now the ends of U divide ∂D into $2n+2$ arcs. Fix two adjacent arcs, and consider only states of U with 1/2-stars on these arcs. Let $\mathcal{A}$ be the set of the remaining arcs. For $X\subseteq\mathcal{A}$ with $|X|=n$, let $\mathcal{S}_X=\mathcal{S}_X(U)$ denote the set of states of U with arrows on the arcs of X; see Fig. 1.



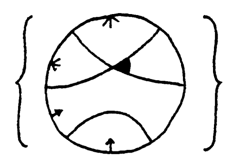
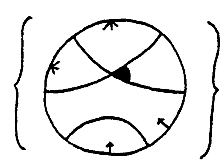
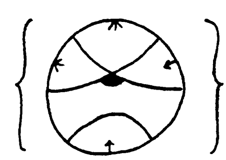
X:	$\mathcal{S}_X$
$\{A_1, A_2\}$	
$\{A_1, A_3\}$	$\emptyset$
$\{A_1, A_4\}$	$\emptyset$
$\{A_2, A_3\}$	
$\{A_2, A_4\}$	
$\{A_3, A_4\}$	$\emptyset$

Fig. 1.

A *tangle* is, for us, a tangle universe with a specification of over- and under-crossings (equivalently, with a standard or reverse labelling at each vertex; see [2] p. 67). If K is a tangle and S is a state of (the underlying universe of) K, $\langle K \mid S\rangle$ is defined just as for knots ([2], pp. 53, 67). If S is a state without markers, $\langle K \mid S\rangle = 1$. For any set $\mathcal{S}$ of states we set

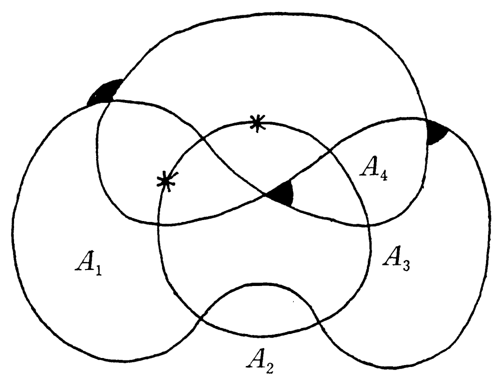
$$\langle K \mid \mathcal{S}\rangle = \Sigma_{S \in \mathcal{S}} \langle K \mid S\rangle,$$

and regard this as zero if $\mathcal{S} = \emptyset$.

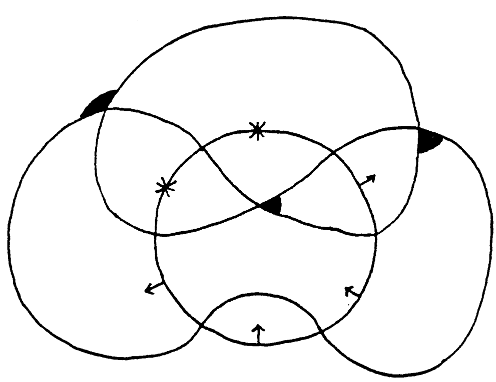
Now let K be a knot and C a simple closed curve in S^2 meeting K transversely in $2n+2$ points, $n>0$. Let D' and D'' be the discs into which C divides S^2, and let $K'\subset D'$ and $K''\cap D''$ be the tangles into which C divides K. We choose the stars of K to lie on two adjacent arcs of C, so that each contributes half a star to K' and to K''. Let $\mathcal{S}$ be the set of all states of K with these stars. As above, $\mathcal{A}$ is the set of the other $2n$ arcs of C; for $X\subseteq\mathcal{A}$, $X'=\mathcal{A}-X$.

Lemma 1. *In the above situation,*

$$\langle K\,|\,\mathcal{S}\rangle=\sum_X\langle K'\,|\,\mathcal{S}_X(K')\rangle\langle K''\,|\,\mathcal{S}_{X'}(K'')\rangle\,,$$



A state S of K.



With arrows added:

$$S\in\mathcal{S}_{(A_2,A_3)}(K')\times\mathcal{S}_{(A_1,A_4)}(K'').$$

Fig. 2.

the sum extending over all $X \subseteq \mathcal{A}$ *with* $|X|=n$.

REMARK. The case $n=0$ says that for knots K' and K'',

$$\langle K' \# K'' \rangle = \langle K' \rangle \langle K'' \rangle .$$

Proof. We may identify a marker assignment for K with an ordered pair of marker assignments, one for K' and one for K''. With this identification, we claim that

$$\mathcal{S} = \bigcup_X \mathcal{S}_X(K') \times \mathcal{S}_{X'}(K''),$$

and that this is a disjoint union; the lemma will follow immediately.

First suppose that $S \in \mathcal{S}$. If $A \in \mathcal{A}$ then A cuts some region R of K. Note that the assumption that K has a state forces R to be a disc. If R contains a marker, put an arrow on A pointing away from the marker. If R contains a star, put an arrow on A pointing away from the star. Now each region of K' or K'' contains exactly one marker, 1/2-star or arrow; see Fig. 2. Let $X \subseteq \mathcal{A}$ be the set of arcs with arrows pointing into D'. Then K' (resp. K'') has $|X|+2$ (resp. $|X'|+2$) empty regions, so $|X| \geqslant n$ and $|X'| \geqslant n$. Since $|X|+|X'|=2n$ we have $|X|=|X'|=n$, and we see that $S \in \mathcal{S}_X(K') \times \mathcal{S}_{X'}(K'')$. We remark that we have also given an algorithm leading from the state S of K to a suitable subset X of $\mathcal{A}$.

Conversely, let $S \in \mathcal{S}_X(K') \times \mathcal{S}_{X'}(K'')$ for some X, $|X|=n$. Let R be a region of K. There are three possibilities:

(i) R is not cut by C;

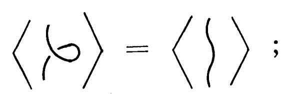
(ii) R contains a star;

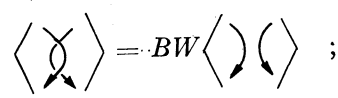
(iii) R is cut by C, and does not contain a star.

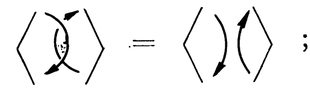
In case (i), R contains a single marker. In case (ii), let the number of arcs $A \in \mathcal{A}$ cutting R be a, and let the number of regions of K' or K'' into which R is cut by these arcs and the starred arc be b. Since K' and K'' have states, $\partial D' \cup K'$ and $\partial D'' \cup K''$ are connected and so the regions of K' and K'' are discs. Therefore $\chi(R)=b-a-1$. Since each region of K' or K'' contains just one marker, 1/2-star or arrow, $b \geqslant 2+a$, so $\chi(R) \geqslant 1$. Since any planar region has Euler characteristic at most one, R is a disc and contains no markers. Case (iii) is similar; here R is cut by a arcs of $\mathcal{A}$ into b regions, $b \geqslant a$, and $\chi(R)= b-a \geqslant 0$. It follows that R is either a disc containing just one marker, or an empty annulus. The latter possibility cannot occur, for if it did, K would be disconnected, while its only empty, simply-connected regions would be the adjacent, starred ones, contrary to an earlier remark. This establishes that $S \in \mathcal{S}$. Finally, one can check that the algorithm of the first part of the proof, applied to S, leads back to the given set X, showing that the union is indeed disjoint.

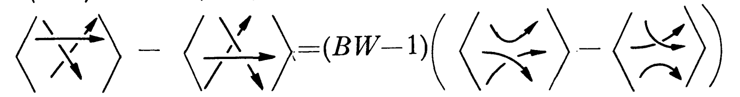
We now state the main result of this section.

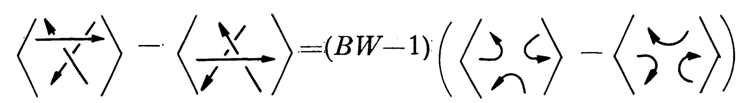
Lemma 2. *The following identities between state polynomials of knots are valid.*

(I) 

(IIa) 

(IIb) 

(IIIa) 

(IIIb) 

the notation means that in each case the identity holds between the polynomials of knots which differ only as indicated.

REMARK. We have not listed identities covering every case of the Reidemeister moves; although similar identities hold in the remaining cases we shall not need them.

Proof. Parts (I), (IIa) and (IIb) are contained in the proof of [2], Theorem 4.3. The others are obtained by examining the remainder of that proof more closely. In the case of either (IIIa) or (IIIb) let the knots involved be K, $\bar{K}$, L and $\bar{L}$, in that order. We take a simple closed curve C cutting each of these into two tangles of which one is common to all; this is the part of the knots not shown above. Let the other tangles be K', $\bar{K}'$, L' and $\bar{L}'$ respectively. Put stars on two adjacent arcs of C, and let $\mathcal{A}$ be the set of the other arcs. In view of Lemma 1, it is enough to show that

$$\begin{aligned}&\langle K' \mid \mathcal{S}_X(K')\rangle - \langle \bar{K}' \mid \mathcal{S}_X(\bar{K}')\rangle \\ &\qquad = (BW-1)(\langle L' \mid \mathcal{S}_X(L')\rangle - \langle \bar{L}' \mid \mathcal{S}_X(\bar{L}')\rangle) \qquad (1)\end{aligned}$$

for each $X \subseteq \mathcal{A}$ with $|X|=2$. Computation of these polynomials involves the simple but tedious listing of the states of the tangles; we omit this, and simply tabulate below the results, for the labelling of the arcs of C shown in Fig. 3. In the tables, the polynomial given opposite $X \subseteq \mathcal{A}$ and $J=K'$, $\bar{K}'$, L' or $\bar{L}'$ is $\langle J \mid \mathcal{S}_X(J)\rangle$. For the reader who wishes to check the results, we remark that the states of $\bar{L}'$ in case (IIIa) appear in Fig. 1, and those of K' and $\bar{K}'$ in the same case are in Fig. 23 of [2].

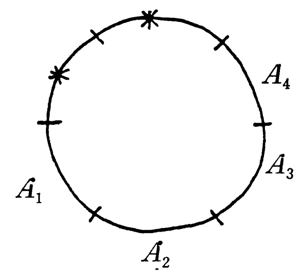


Fig. 3.

Case (IIIa).

	K'	$\bar{K}'$	L'	$\bar{L}'$
$\{A_1, A_2\}$	BW^2	BW^2	W	W
$\{A_1, A_3\}$	BW	1	1	0
$\{A_1, A_4\}$	B	B	0	0
$\{A_2, A_3\}$	$W-B^2W$	BW^2-B	$-B$	W
$\{A_2, A_4\}$	$1-B^2-BW$	$-B^2$	0	1
$\{A_3, A_4\}$	$-B$	$-B$	0	0

Case (IIIb)

	K'	$\bar{K}'$	L'	$\bar{L}'$
$\{A_1, A_2\}$	$-B$	$-B$	0	0
$\{A_1, A_3\}$	1	BW	0	1
$\{A_1, A_4\}$	B	B	0	0
$\{A_2, A_3\}$	$W-B$	$W-B$	0	0
$\{A_2, A_4\}$	BW	1	1	0
$\{A_3, A_4\}$	$-B$	$-B$	0	0

It is evident that the equation (1) holds in each case.

REMARK. Proofs of I, IIa, IIb and the exchange identity may be obtained using the same formalism.

2. Reidemeister moves on universes

Let U_1 and U_2 be universes. We write $U_1 \downarrow U_2$ if U_2 is obtained by a single move of type I or II in Fig. 4, and $U_1 \sim U_2$ if U_2 is obtained from U_1 by

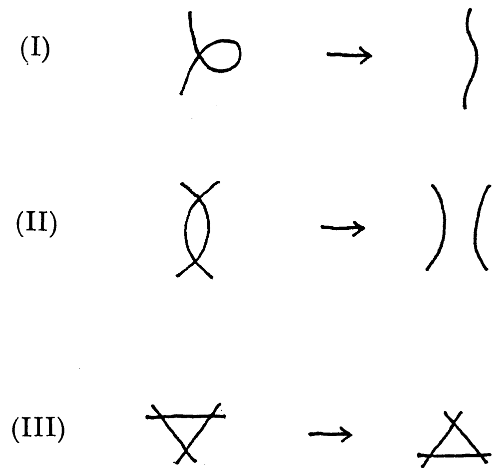


Fig. 4.

a finite sequence of moves of type III in the same figure. If U_i has n_i crossing, note that $U_1 \downarrow U_2$ implies that $n_1 > n_2$, while $U_1 \sim U_2$ implies that $n_1 = n_2$.

Lemma 3. *If a connected universe U has at least one crossing, there exist universes U' and U'' with $U \sim U' \downarrow U''$.*

REMARK. A similar result has been proved by Randall Weiss (unpublished).

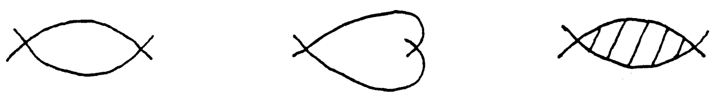
Proof. Consider U as the image of an immersion of a disjoint union of circles. An *arc* of U will mean the image of a closed interval on one of these circles. A *monogon* in U is an arc of U which crosses itself just once. A monogon A contains a unique simple closed curve C; the *inside* of A is $cl(X)$ where X is the component of $S^2 - C$ such that $X \cap A = \emptyset$ (Fig. 5).

A monogon. The inside of a monogon.

Fig. 5.

Now consider two embedded arcs A_1 and A_2 of U which cross each other twice. Then $A_1 \cup A_2$ contains a unique simple closed curve C; $A_1 \cup A_2$ is a *digon* if there is a component X of $S^2 - C$ such that $X \cap (A_1 \cup A_2) = \emptyset$, and then $cl(X)$ is the *inside* of $A_1 \cup A_2$ (Fig. 6).

Since U has crossings, it contains either a monogon or a digon. Let A be one such which is innermost in the sense that its inside R contains the inside



A digon. Not a digon. The inside of a digon.

Fig. 6.

of no monogon or digon. We show that

a) if A is a monogon then $U \downarrow U'$ by a move of type I;

b) if A is a digon and there are 2 crossing points in R then $U \downarrow U'$ by a move of type II;

c) if A is a digon and there are $n>2$ crossing points in R then $U \sim U'$ where U' has an innermost digon with fewer than n crossing points in its inside.

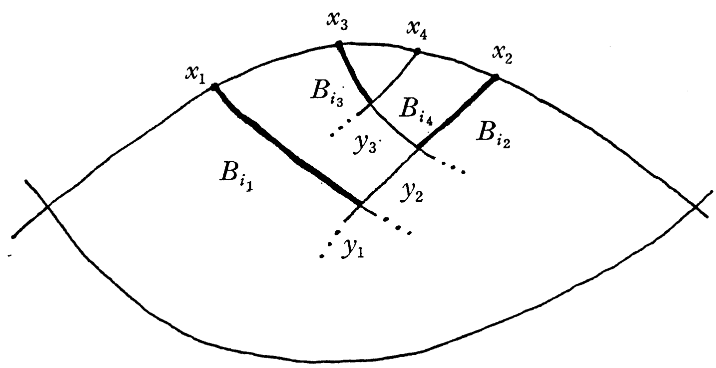
This will serve to establish the lemma.

The assertion b) is the easiest, since in this case $\mathrm{int}(R) \cap U = \emptyset$. (Note that the two crossing points are the crossings of the digon.)

In both the remaining cases, let $A' = cl(U-A)$. Then $R \cap A'$ consists of arcs $B_1, \cdots, B_n$ of U with their endpoints on A. Each B_i must be embedded, or else it contains a monogon which contradicts the assumption that A is innermost. Now in case a) it follows that $k=0$ (which proves the assertion), for otherwise B_1 (extended slightly) together with a subarc of A would be a digon, again contradicting the innermost nature of A.

It remains to prove c). Let the arcs of A be A_1 and A_2. Similar reasoning to that in case a) shows that each B_i has an endpoint on each of A_1 and A_2, and that B_i and B_j meet in at most one point for $i \neq j$. Suppose first that $\mathrm{int}(R)$ contains a crossing. We construct a sequence of arcs $B_{i_1}, \cdots, B_{i_{p+1}}$ and two sequences of points $x_1, \cdots, x_{p+1}$ and $y_1, \cdots, y_q$ such that

i) $x_j = B_{i_j} \cap A_1$;



The heavy arcs contain no crossings.

Fig. 7

ii) x_{j+2} lies between x_j and x_{j+1} on A_1;
iii) B_{i_j} and $B_{i_{j+1}}$ cross at y_j;
iv) The open subarc (x_j, y_j) of B_{i_j} contains no crossing points;
v) The open subarc (y_p, x_{p+1}) of B_{i_p} contains no crossing points;

see Fig. 7. To start the construction, let B_{i_1} be any arc which crosses some other B_i, let $x_1=B_{i_1}\cap A_1$, let y_1 be the crossing point on B_{i_1} such that (x_1, y_1) contains no crossing points, let B_{i_2} be the arc crossing B_{i_1} at y_1, and let $x_2=B_{i_2}\cap A_1$. If (y_1, x_2) contains no crossing points then we are done (with $p=1$). Otherwise, let y_2 be the crossing point on (y_1, x_2) such that (x_2, y_2) contains none, let B_{i_3} be the arc crossing B_{i_2} at y_2, and let $x_3=B_{i_3}\cap A_1$. Note that x_3 lies between x_1 and x_2 because B_{i_3} cannot cross B_{i_2} again, and cannot cross the subarc (x_1, y_1) of B_{i_1} by choice of y_1. If (y_2, x_3) contains no crossing points, we are done (with $p=2$); otherwise we continue in the same way. The construction must eventually terminate. Now a type III move involving the crossings x_p, y_p and x_{p+1} achieves the stated objective (Fig. 8).

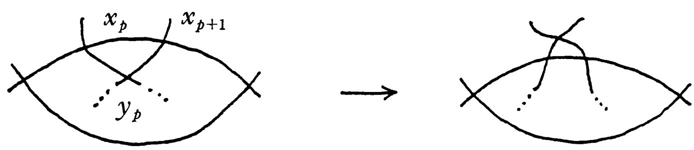


Fig. 8.

Finally, suppose that int(R) does not contain a crossing (i.e. that $B_i\cap B_j=\emptyset$ for $i\neq j$). Let x be one point of $A_1\cap A_2$. Let B_i be the arc such that, for $y=B_i\cap A_1$, the arc $(x, y)\subset A_1$ contains no crossings. Then if $z=B_i\cap A_2$, the arc $(x, z)\subset A_2$ also contains no crossings, and a type III move involving x, y and z completes the proof (Fig. 9).

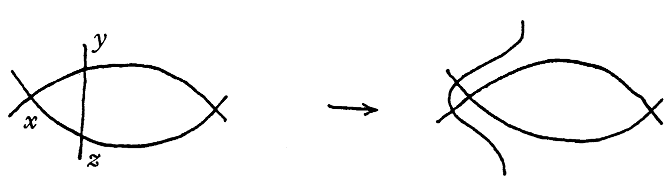
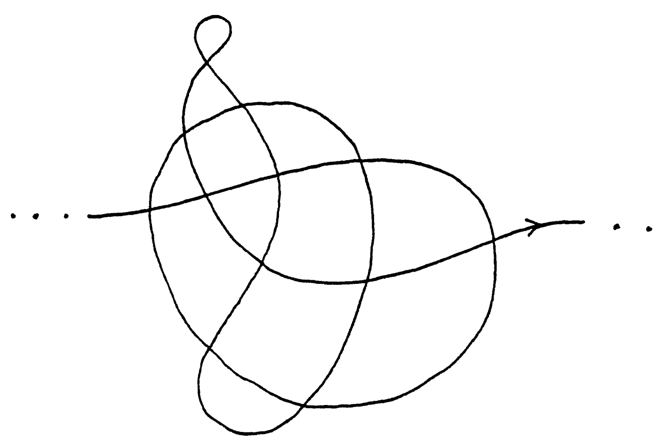
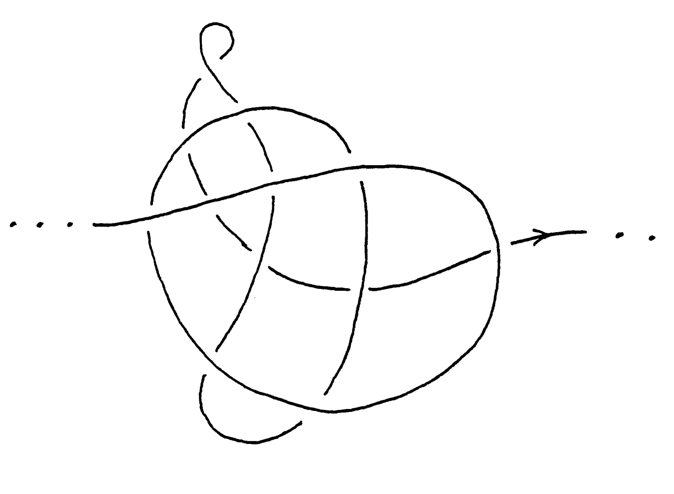


Fig. 9.

We remark that the above technique can be used to give a reasonably simple direct proof that one can change crossings on a diagram of a (1-component) knot so that the resulting knot diagram can be trivialised via Reidemeister moves. For this purpose, it is easier to make the following modifications. Fix a point at infinity in S^2, consider only universes passing through ∞ (but not at a crossing), and restrict the moves I, II and III above to take place in $\boldsymbol{R}^2=S^2-\{\infty\}$. The above lemma holds, with the same proof, in this context. Now if a universe U can be continuously traced (i.e. corresponds to the diagram of a 1-component knot) it follows that U can be reduced to the trivial universe $\cdots \longrightarrow \cdots$ by the moves I, II and III. Define the *standard trivialisation* of U to be the choice of crossings which corresponds to a monotonically decreasing height function (Fig. 10; cf. [2] pp. 79, 80). Then any of the moves $U \to U'$ can be realised by the corresponding Reidemeister move on the standard trivialisation of U, and moreover the result is the standard trivialisation of U'. Note that it is important that we do not use moves I^{-1} or II^{-1}.



U



The standard trivialisation of U.

Fig. 10.

3. Proof of the duality conjecture

Theorem. *For any knot K, the state polynomial $\langle K\rangle$ is good.*

REMARKS. For the definition of a good polynomial, see the introduction. Also, it is pointed out there that this theorem includes the Duality Conjecture as a special case.

Proof. This is by induction on the number, n, of crossings, the case $n=0$ being trivial. So let $n>0$ and suppose that the result holds for all knots of fewer than n crossings. First we show that if K and $\bar{K}$ are knots of n crossings with the same underlying universe then $\langle K\rangle-\langle\bar{K}\rangle$ is good. It suffices to consider the case where K and $\bar{K}$ differ at a single crossing. But then the exchange identity shows that $\langle K\rangle-\langle\bar{K}\rangle=\pm\langle L\rangle$ where L is a knot of fewer than n crossings, and $\langle L\rangle$ is good by assumption. It follows that if U is an n-crossing universe, the polynomials $\langle K\rangle$ for the various knots K underlain by U are either all good or all bad; in the former case we say that U is good. Since any disconnected universe is good (having state polynomial zero for any choice of crossings) we need only consider connected universes. In view of Lemma 3 it will be enough to show, for an n-crossing universe U, that

a) if $U\downarrow\bar{U}$ then U is good;

b) if $\bar{U}$ is obtained from U by a move of type III and $\bar{U}$ is good then so is U.

As for a), we can choose crossings on U and $\bar{U}$ to yield knots K and $\bar{K}$ such that Lemma 2 applies to show that $\langle K\rangle=\langle\bar{K}\rangle$ or $\langle K\rangle=BW\langle\bar{K}\rangle$; since $\langle\bar{K}\rangle$ is good by the inductive hypothesis, so is $\langle K\rangle$. In the situation of b), we can choose the knots K and $\bar{K}$ for U and $\bar{U}$ so that Lemma 2 gives

$$\langle K\rangle-\langle\bar{K}\rangle=(BW-1)(\langle L\rangle-\langle\bar{L}\rangle)$$

for certain knots L and $\bar{L}$ of fewer than n crossings. Then $\langle L\rangle$ and $\langle\bar{L}\rangle$ are good, and $\langle\bar{K}\rangle$ is good since $\bar{U}$ is, so $\langle K\rangle$ is also.

4. The Clock Theorem

The clock theorem as formulated by Kauffman concerns "states" of "universes" (to be discussed briefly below). There is a parallel interpetation in terms of spanning trees of graphs, which we will give first. Our proof will make use of both points of view as some facts are more easily seen one way and some the other.

Recall that a lattice is a partially ordered set where any two elements X, Y have a meet (greatest lower bound) $X\wedge Y$ and a join (least upper bound) $X\vee Y$. A lattice is called distributive if meet distributes over join i.e. $X\wedge(Y\vee Z)=(X\wedge Y)\vee(X\wedge Z)$ for all X, Y, Z. It turns out that this is equivalent to saying that join distributes over meet. See Birkhoff-MacLane [1] chapter XIV, a good elementary reference for lattices.

By a *plane graph* we will mean an embedded 1-complex in the plane. Thus we allow multiple edges between vertices as well as edges both of whose endpoints are the same vertex. We will be considering connected plane graphs where one exterior vertex (i.e. in the closure of the unbounded component of the complement) has been singled out. We will put a star at this vertex and call it the *base* of the graph. A *spanning tree* of a graph is a 1-connected subcomplex which includes each vertex or equivalently a maximal 1-connected subcomplex. Since we will only be discussing spanning trees from now on *tree* will mean spanning tree. Occasionally we will still say spanning tree for emphasis. A *face* of a plane graph is a bounded region of the complement of the graph.

We now wish to describe a move that can be made on the trees of a plane graph based at an exterior vertex. Suppose we are given a tree. Orient all the edges in the tree with arrows pointing away from the base. Draw an arrow across each edge which does not belong to the tree pointing into the bounded region of the complement of the graph obtained by adjoining this edge to the tree. Suppose a tree and a nontree edge share a common vertex and a common face such that the tree edge points toward the vertex and the non-tree arrow points toward the face. We may obtain a new subgraph by deleting the tree edge from the tree and adding the non-tree edge. If the non-tree edge swings in a (counter) clockwise direction across the common face, we call this a (counter) *clockwise move*. A clockwise move is illustrated in Figure 11. The new subgraph will include every vertex. We will show it is connected. Then a simple Euler characteristic argument shows that it is 1-connected. Suppose the path joining c to the starred vertex in the original tree passes through b. Then the transverse arrow to the non tree edge forces a (and thus the base) to be interior to the simple closed curve formed by adding the non-tree edge to this path. Since the base is exterior, this can't happen. Thus the path joining c to the starred vertex does not pass through the deleted edge. So there is a path in the new subgraph joining b to the starred vertex via c. It follows that the new subgraph is connected and thus is a spanning tree. Notice

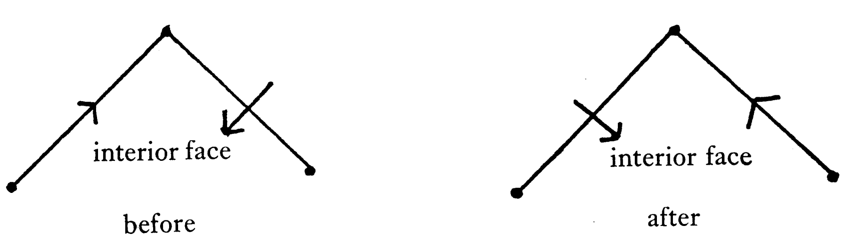


Fig. 11.

that the arrows after a move are forced and are as illustrated.

If T and T' denote trees, we will write $T \geqslant T'$ if there is a sequence of clockwise moves leading from T to T'. We can now state:

The Clock Theorem. *The collection of spanning trees of a connected plane graph based at an exterior vertex becomes a graded distributive lattice under this relation.*

The proof occupies most of the rest of this section. At this point we must ask that the reader become familiar with Kauffman's original formulation of his clock theorem (through page 19 [2]). In particular, the proof of (2.4) and the remark following indicate a correspondence between states of a universe and trees in the graph that corresponds to the shaded regions of the universe.

Explicitly, given a connected oriented universe with choice of fixed adjacent stars (in S^2) delete a point in one of the starred regions and obtain a universe U in the plane with the unbounded region starred as well as an adjacent region. Shade each region which is separated from the unbounded region by an odd number of edges. Form a new plane graph G, with one vertex for each shaded region and one edge for each crossing shared by these shaded regions. The faces of G correspond to unshaded faces of U. Pick the base of G to be the exterior vertex which corresponds to the starred shaded region of U. Then states of U with this choice of stars correspond bijectively to spanning trees of G and a clockwise move of markers in U corresponds precisely to a clockwise move among trees in G.

Note also that any plane graph G arises in this way. Begin by drawing a crossing at the midpoint of each edge and join these crossings up by arcs that parallel the boundaries of the faces of G. The result is a universe U whose crossings correspond to the edges of G. It is immediate that a clockwise move takes one state to another. Thus using the above correspondence, it follows that a clockwise move on a spanning tree leads to a new spanning tree. Thus we could have avoided making the involved argument earlier.

Lemma 4. *In a universe one may not perform an infinite number of clockwise moves in sequence.*

Proof. First consider an exterior vertex of a universe U. Since a marker may never be placed in the unbounded region, a marker at such a vertex may only move once from a given position and it cannot make a complete circuit. If a vertex is one edge away from an exterior vertex one of the marker moves at this vertex must be paired to a marker move at the exterior vertex. Thus a marker located at this vertex can move at most twice from a given location. It can make a complete circuit of this vertex but cannot make two circuits.

Continuing in this way one sees that if a vertex is n edges away from an exterior vertex, the number of complete circuits a marker can make is less than 2^n. The result follows.

REMARK. There exist universes where one can repeat a clockwise move at the same location twice. The move indicated by a in figure 12 can be repeated for a second time after eight other moves have been made.

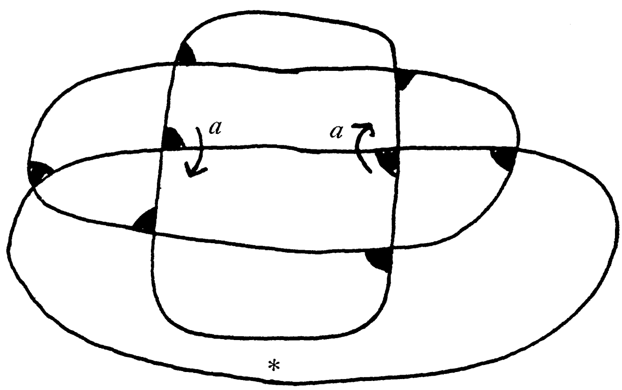


Fig. 12.

Given two states S and S', write $S \geqslant S'$ if a sequence of clockwise moves leads from S to S'. Say a state or tree is (un) *clocked* if only (counter) clockwise moves are available.

Proposition 1. "$\geqslant$" *defines a partial order on the collection of states. Clocked and unclocked states exist.*

Proof. The reflexive and transitive properties are clear. Suppose $S \geqslant S'$, $S' \geqslant S$ and $S \neq S'$. Then we could perform an infinite sequence of clockwise moves, first going from S to S', then S' to S, then S' to S etc. This contradicts Lemma 4. Thus antisymmetry holds. To find an unclocked state just pick a state and start turning clockwise until no clockwise moves are available. To find a clocked state turn the other way.

We wish to show that there is only one clocked state. It is here that the graph model has advantages over the universe model. Suppose we have a tree in a graph with arrows drawn as above. We claim that if within the diagram one can find a subdiagram isotopic to Figure 13 where the curved line indicates a path in the tree, then a counterclockwise move is available somewhere in the graph. If the arrowed edges share a face, we are done. Suppose not, then there must be a simple path in the graph lieing within the figure joining v to the tree path. Consider the first such path one reaches starting at the non-tree edge and traveling counterclockwise around v. This path must

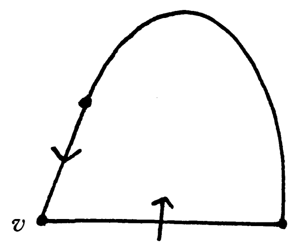


Fig. 13.

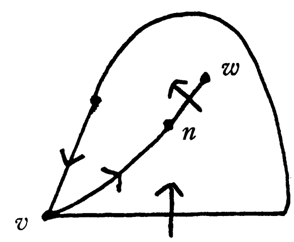


Fig. 14.

include a non-tree edge (otherwise the complete tree would not be simply connected). Consider the first non-tree edge n one reaches traveling on this path from v and the vertex w at the far end of n from v. See figure 14. Note that the arrows must be as indicated. Moreover the tree must pass through w and therefore we can find a simple path from w to some part of the tree showing in the figure. In this way, we find nested within the figure an identical subdiagram of the same form. Taking an innermost such subdiagram, we find an allowable counterclockwise move.

We know clocked trees exist. We consider what choices are available to such a tree as it grows from the base. We will say a connected graph separates into pieces if it disconnects when we delete the base. Its *pieces* are the components of the complement of the base considered separately with the base added back to each one. There is one edge leading from the base of a nonseparating graph which we call the *clockwise edge.* Consider a small circle centered at the base. Only one segment of this circle lies in the unbounded region. The first edge one reaches traveling clockwise on this segment is the clockwise edge.

Lemma 5. *A clocked tree includes the clockwise edge of each of the pieces of the graph except those pieces with only one vertex, and includes no other edge which meets the base.*

Before we begin the proof we note that there is only one nonseparating graph with one vertex. It has only one tree which consists of just the base.

Proof of Lemma 5. If a tree does not contain the clockwise edge of a piece with more than one vertex then the other vertex of the clockwise edge must be connected by a path in the tree to the base. Thus we can find a subdiagram of the type pictured in Figure 13. Thus a clocked tree must contain each such clockwise edge. Now suppose we have a tree which includes several edges in the same piece which meet the base. The tree then disconnects if we delete the base. We will call the components branches. The non-tree edges fall into two classes: those whose two endpoints lie in the same branch; and those that join two different branches. The latter kind of edge leads to a diagram of the type of figure 13 and thus a clockwise move. On the other hand if there are two branches in the same piece, this kind of edge must occur.

According to the lemma 5, there is no choice in which collection of edges meeting the base can be in a clocked tree. Let T be a clocked tree in G. Suppose n edges meeting the base are in T. Let v_i be the n vertices of these edges which are not the base. If we delete from G all edges (in T or not) which meet the base and put the v_i back in we obtain n disjoint graphs G_i based at v_i and $T_i = G_i \cap T$ is a clocked tree in each one. The lemma applies again to each component and we see there is no choice in which collection of edges meeting v_i can be in a clocked tree. Continuing in this way, there is no choice at any vertex. So we have proved:

Proposition 2. *There is only one clocked tree (in a given graph).*

Notice that our proof actually gives an easy algorithm for drawing the clocked tree. We now know there is a unique maximal element for the partial order. Thus there is a sequence of counterclockwise and clockwise moves joining any two states. This is because one can reach *a* (and therefore the) clocked state from any state by performing counterclockwise moves until none is available. This is the only part of the clock theorem that Kauffman makes use of in his development of formal knot theory.

We now proceed to prove that this partial order is a distributive lattice. Here we follow Kauffman's general scheme [2] pages 51 and 52. However we fill in many of the missing details.

We first observe (using the universe model) that a marker may participate in at most one clockwise move at a given time. Thus we have

Lemma 6. *If a clockwise move is available, it remains available after any sequence of other clockwise moves.*

Lemma 7. *If two clockwise moves are available, then they can be performed in sequence with the same effect in either order.*

At this point, we point out that dual lemmas hold replacing clockwise by counterclockwise. Also an unclocked tree exists and is unique.

Assign a letter to each pair of edges that share a vertex (not the base) and a face. Let that letter stand for the associated clockwise move, performed for the first time. Let a_n, where $n \geqslant 2$, denote this move performed for the nth time. In future, we will also refer to a_n as a letter. A word in these letters is called an *allowable word* if these operations can be performed in sequence from the clocked tree. Notice every tree is described by an allowable word (probably many).

Proposition 3. *If two words describe the same tree, they are related through a sequence of allowable words each differing from the next by a commutation.*

Proof. The proof is by induction on the length of the longer word. This is clear for words of length one. Assume that it is true for pairs of words of length less than n. For convenience we will use W_i to denote words and we will write $W_i \sim W_j$ if the conclusion of the proposition holds. Let W_1 and W_2 be two words with n or fewer letters that describe the same tree. If W_1 and W_2 both end with the same letter say a, then $W_1 = W_3 a$ and $W_2 = W_4 a$. Moreover W_3 and W_4 have less than n letters and both describe the same tree. So $W_3 \sim W_4$ and so $W_1 \sim W_2$.

Now suppose W_1 and W_2 end with different letters. We have $W_1 = W_3 a$ and $W_2 = W_4 b$. Since W_1 and W_2 describe the same tree, by lemma 7 (the counterclockwise case) there is a state described by say W_5 such that W_3 describes the same state as $W_5 b$ and W_4 describes the same state as $W_5 a$. By induction, $W_3 \sim W_5 b$ and $W_4 \sim W_5 a$ so $W_1 = W_3 a \sim W_5 ba$. By lemma 7 again $W_5 ba \sim W_5 ab$. By the above $W_5 ab \sim W_4 b = W_2$. Thus $W_1 \sim W_2$.

The set of letters in an allowable word will be called an *allowable set of letters*. Note that trees correspond bijectively with allowable sets of letters. Also note given an allowable set of letters, one does not need to order them into an allowable word to find the state that corresponds to it. Using the universe model start with the clocked state and turn the appropriate pair of markers clockwise for each letter in the set. One arrives at the state which corresponds to the given set of letters. The following proposition follows easily from lemma 6.

Proposition 4. *The set of allowable sets of letters is closed under unions.*

Let L denote the union of all allowable sets of letters. L is the allowable set of letters corresponding to the unclocked tree. Let L_1 be an allowable set of letters corresponding to the tree T_1. Then the complement of L_1 in L describes the set of counterclockwise moves needed to reach the tree T from the unclocked tree. By duality, the complements are closed under unions so by De'Morgan's laws we have:

Proposition 5. *The set of allowable subsets of L is closed under intersections as well as unions.*

Finally notice that if T_1 corresponds to L_1 and T_2 corresponds to L_2 we have $T_1 \geqslant T_2$ if and only if $L_1 \subseteq L_2$. This also requires use of lemma 6. Thus the set of trees forms a distributive lattice with $T_1 \wedge T_2$ corresponding to $L_1 \cup L_2$ and $T_1 \vee T_2$ corresponding to $L_1 \cap L_2$. *This completes the proof of the clock theorem.*

One may place a hierarchy (a partial order) on the set of letters L as follows: If $a \in L$ define I_a to be the intersection of all allowable subsets of L which include a. Define $a \geqslant b$ if $a \in I_b$. This defines a partial order on L. Then one may show using Proposition 4 that a subset $S \subset L$ is allowable if and only if S satisfies the property $a \in S$ and $b \geqslant a$ implies $b \in S$.

Thus the hierarchy on the letters encodes the essential information contained in the lattice of trees. The lattice of trees (named by the corresponding allowable subsets) can be recovered formally from the hierarchy.

By way of illustration we include the graph corresponding to the universe of Figure 12 with the operations labelled and the hierarchy of operations. See Figure 15.

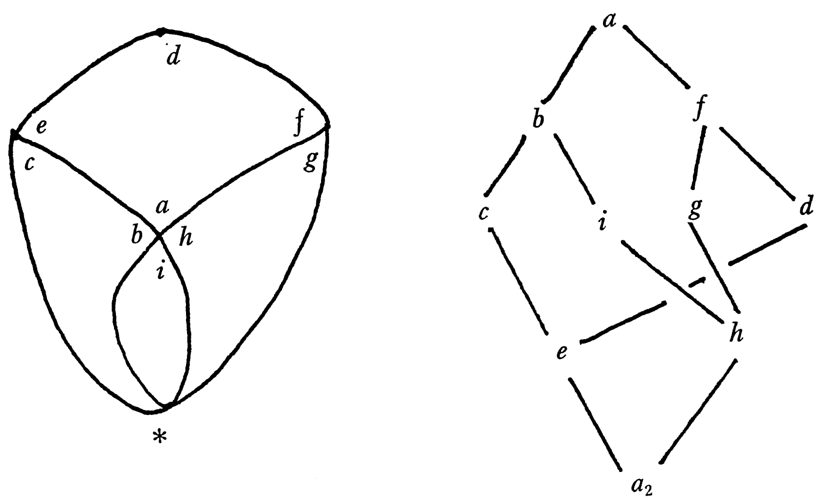


Fig.15.

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