Harada, M.
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GENERALIZATIONS OF NAKAYAMA RING I

MANABU HARADA

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T. Nakayama found a very important ring in ring theory, which we call a generalized uniserial ring [6]. He showed that a left and right artinian ring R is a generalized uniserial ring if and only if every (finitely generated) left (resp. right) R-module is a direct sum of uniserial modules. We shall generalize further such a ring from this point of view.

We shall define Conditions (*, 3) and (**, 3) (see §1) for a direct sum $D(3)$ of three hollow modules. If R is a generalized uniserial ring, (*, 3) and (**, 3) are satisfied ([2] and [3]). In §2 we shall give a characterization of a right artinian ring R which satisfies (**, 3) for any $D(3)$. In §3 we shall expose several examples related to the results in the previous section.

We shall study Condition (*, 3) in a forthcoming paper.

1. Definitions. Let R be throughout a right artinian ring with identity. Modules in this note are unitary right R-modules with finite length. Let e be a primitive idempotent in R. If $eR\supset eJ\supset eJ^2\supset\cdots\supset eJ^n=0$ is a unique chain of the submodules of eR for each e, R is called a *right* (*generalized uni-*) *serial ring* (*Nakayama ring*), where $J=\mathrm{J}(R)$ is the Jacobson radical of R.

As a generalization of a serial ring, we have considered the following two conditions [1]:

(*, n) *Every* (*non-zero*) *maximal submodule of a direct sum $D(n)$ of n non-zero hollow modules is also a direct sum of hollow modules,* and

(**, n) *Every* (*non-zero*) *maximal submodule of the $D(n)$ above contains a non-trivial direct summand of $D(n)$.*

By Nakayama [6], if R is a right and left serial ring, (*, n) holds for any $D(n)$ and any n as right (resp. left) R-modules. Further R is a right serial ring if and only if (*, n), replaced hollow by uniserial, holds for any $D(n)$ and any n as right R-modules [5].

In general, if $J^2=0$, (*, 2) holds for any $D(2)$ by [3], Proposition 3. Let $\{N_i\}_{i=1}^n$ be a set of hollow modules, and put $D(n)=\sum_{i=1}^{n}\oplus N_i$. Let M_1 be a maximal submodule of $D(n-1)=\sum_{i=1}^{m-1}\oplus N_i$. Then $M=M_1\oplus N_n$ is a maximal submodule of D. If D satisfies (*, n), M_1 is also a direct sum of hollow modules by Krull-Remak-Schmidt's theorem. Hence $D(n-1)$ satisfies $(*, n-1)$. Contrarily, if $D(n-1)$ satisfies $(**, n-1)$, $D(n)$ does $(**, n)$ by [2], Lemma 1.

DEFINITION. *A ring R is called a right US-n ring if $(**, n)$ is satisfied for any $D(n)$* (US is an abreviation of uni-serial).

We have obtained the following theorem [2]:

Theorem 1. *Let R be a right artinian ring. Then R is a right US-1 ring if and only if R is a semi-simple ring. R is a right US-2 ring if and only if R is a right generalized uni-serial ring.*

Hence the next problem concerning $(**, n)$ is to study the structure of right US-3 rings.

2. US-3 rings. Let N_1 and N_2 be two hollow modules. Assume that $N_1 \approx eR/eA_1$ and $N_2 \approx eR/A_2$, where e is a primitive idempotent and the A_i are submodules of eR. If there exists an epimorphism of N_1 to N_2, then there exists a unit element x in eRe such that $xA_1 \subset A_2$. If there exists an epimorphism of one to another between N_1 and N_2, we indicate it by $N_1 \sim N_2$ or $A_1 \sim A_2$, namely there exists a unit element y in eRe such that $yA_1 \subset A_2$ or $yA_2 \subset A_1$. Since $y(eR/A_1) = eR/yA_1$, we may assume, in this case, that $A_1 \subset A_2$ or $A_2 \subset A_1$.

Now we put $\Delta = eRe/eJe$, a division ring, and $\Delta(A_1) = \{\bar{x} \mid \in \Delta$, there exists x' in eRe such that $x'A_1 \subset A_1$, and $x - x' \in eJe\}$. If we put $\mathrm{S}(A_1) = \{x \mid \in eRe, xA_1 \subset A_1\}$, then $\mathrm{S}(A_1)$ is a subring of eRe. Let ν be the natural epimorphism of eRe onto $eRe/eJe = \Delta$. Then $\Delta(A_1) = \nu(\mathrm{S}(A_1))$, and $\Delta(A_1)$ is a sub-division ring of Δ. We may regard Δ as a right $\Delta(A_1)$-module, and hence we denote the dimension of Δ over $\Delta(A_1)$ by $[\Delta: \Delta(A_1)]$ (see [1]). $|A_1|$ means the length of A_1.

Lemma 1. *Let $\{A_i\}_{i=1}^{3}$ be a set of three submodules in eR. If $A_i = eJ$ for some i, $D = \sum_{i=1}^{3} \oplus eR/A_i$ satisfies $(**, 3)$. Conversely, if D satisfies $(**, 3)$, $A_i \sim A_j$ for some pair (i, j).*

Proof. If $A_1 = eJ$, eR/A_1 is simple. Let M be a maximal submodule of D. Then $M \supset eR/A_1$ or $D = M \oplus eR/A_1$. Hence $(**, 3)$ holds. If $A_i = eJ$ for some i, then $A_i \sim A_j$ for certain j, since eJ is a unique maximal submodule of eR. Assume that $A_i \neq eJ$ for all i. Then any eR/A_i does not satisfy $(**, 1)$. Hence $A_{i'} \sim A_{j'}$ for some pair (i', j') by [4], Corollary 2.

Proposition 1. *Assume that R is a US-3 ring. Then* 1) *$[\Delta: \Delta(A)] \leqslant 2$ for any submodule A in eR.* 2) *If there exists a submodule B in eR such that $[\Delta: \Delta(B)] = 2$, then $B \sim C$ for any submodule C in eR.* 3) *$t = |eJ^i/eJ^{i+1}| \leqslant 2$ for all i.* 4) *Assume that eJ^i contains a maximal submodule A_k $(\supset eJ^{i+1})$ with $\Delta(A_k) = \Delta$. Then* i) *eJ^i contains at most two maximal submodules A_1 and A_2.* ii) $\Delta(A_1)$

$=\Delta(A_2)=\Delta$. iii) A_1 *and* A_2 *are characteristic in eR.* iv) *Either* A_1 *or* A_2 *is hollow, provided* $t=2$.

Proof. 1) and 2). They are immediate consequences of [2], Theorem 2 and [4], Corollary 1, respectively.
3) Let $eJ^i/eJ^{i+1}=\bar{C}_1\oplus\bar{C}_2\oplus\cdots\oplus\bar{C}_t$, where the $\bar{C}_i$ are simple modules and $eJ^i\supsetneqq C_k\supsetneqq eJ^{i+1}$. It is clear that $|C_1|=|C_k|=1+|eJ^{i+1}|$ for all k. Assume that $\Delta(C_i)=\Delta$ and $C_1\sim C_2$. Then there exists a unit x in eRe such that $xC_1=C_2$. Since $\Delta(C_1)=\Delta$, put $x=x_1+j$, where $x_1 C_1=C_1$ and $j\in eJe$. Then $C_2=xC_1=(x_1+j)C_1\subset xC_1+jC_1\subset C_1+eJ^{i+1}=C_1$. Hence, if $\Delta(C_i)=\Delta$ for all i, $t\leqslant 2$ by Lemma 1. Next assume that $\Delta(C_1)\neq\Delta$. Then $\Delta(C_i)\neq\Delta$ for all i by 2) and the above proof. There exists, from 2), a unit x_2 in eRe with $C_2=x_2C_1$. Then $\bar{x}_2\notin\Delta(C_1)$. Since $[\Delta:\Delta(C_1)]=2$, $\Delta=\Delta(C_1)+\bar{x}_2\Delta(C_1)$. From the observation above, $eJ^i/eJ^{i+1}=\Delta\,\bar{C}_1=\Delta(C_1)\bar{C}_1+\bar{x}_2\,\Delta(C_1)\,\bar{C}_1=\bar{C}_1+\bar{C}_2$. Hence $t\leqslant 2$.
4), ii) and iii). They are clear from the first part of the proof of 3).
4), i). Assume that A_1 is a maximal submodule of eJ^i with $\Delta(A_1)=\Delta$. Then A_1 is characteristic from iii). Let A_2 be another maximal submodule of eJ^i. Then $\Delta(A_2)=\Delta$ from 1) and 2). Hence $A_1\not\sim A_2$, so eJ^i contains at most two maximal submodules A_1 and A_2.
4), iv). Assume that $A_1\neq A_2$. Then $A_1\cap A_2=eJ^{i+1}$. Let B be a maximal submodule in A_1. If $B\sim eJ^{i+1}$, $B=eJ^{i+1}$ since eJ^{i+1} is characteristic. Hence, if $|A_1/A_1J|\geqslant 2$, A_1 conatins a maximal submodule B_1 such that $B_1\not\sim eJ^{i+1}$. Let B' be a maximal submodule of A_2. If $B'\neq eJ^{i+1}$ and $|A_1/A_1J|\geqslant 2$, $B'\sim B_1$. Hence there exists a unit element x in eRe such that $B'=xB_1\subset xA_1$, and so $B'\subset A_1\cap A_2=eJ^{i+1}$, which is a contradiction. Hence A_2 is hollow.

Corollary. *Let R be a right artinian ring. Then R is a right US-3 ring if and only if we have the following properties for each primitive idempotent e:* 1) *For any three submodules* A_i *of eR such that* $\Delta(A_i)=\Delta$, $A_i\sim A_j$ *for some pair* (i, j). 2) $[\Delta:\Delta(A)]\leqslant 2$ *for any submodule A of eR.* 3) *If* $[\Delta:\Delta(B)]=2$ *for a submodule B of eR,* $B\sim A$.

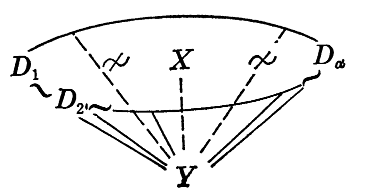
Proof. "Only if" part is clear from Lemma 1 and Proposition 1. Assume 1)$\sim$3) and put $D=eR/A_1\oplus eR/A_2\oplus eR/A_3$. If the A_i satisfy 1), then D satisfies (**, 2) and hence (**, 3) by [4], Corollary 3. Assume that $[\Delta:\Delta(A_1)]=2$. Then we may assume that $A_1\subset A_2$ or $A_2\subset A_1$ from 3). If $\Delta=\Delta(A_2)$, then $\Delta(A_1, A_2)=\{\bar{x}\,|\,x\in eRe,\ xA_1\subset A_2\}=\Delta$ or $\Delta(A_2, A_1)=\Delta$, respectively. Hence D satisfies (**, 2) by [4], Theorem 2. Finally assume that $[\Delta:\Delta(A_i)]=2$ for all i. Then D satisfies (**, 3) by 3) and [4], Corollary 4.

In Corollary to Proposition 1, we have given the condition under which (**, 3) holds for any $D(3)$. Using this corollary, we shall give the complete form of the lattice of submodules of eR, provided R is a right US-3 ring.

First we consider the following situation:

$X \supset Y$ are characteristic submodule of eJ, X/Y is simple and Y is hollow.

Let $\{D_i\}$ be the set of submodules of eR containing Y such that $|D_i|=|X|$ and $D_i \neq X$. Then $D_i \cap X=Y$ and $D_i \not\sim X$, so $D_1 \sim D_2$ provided $i \geqslant 2$. Since $D_1 \cap D_3=Y$, D_1 and D_2 can not contain a common maximal submodule except Y. Let B_i $(\neq Y)$ be a maximal submodule of D_i for each i (note that $D_i \approx D_1$). Then $B_1 \neq B_2$, and put $E=B_1+B_2$. $D_1/B_1=(B_1+Y)/B_1 \approx Y/(B_1 \cap Y)$ is simple and Y is hollow by assumption, and so $B_1 \cap Y=\mathrm{J}(Y)$ $(=Z)$. Similarly, $B_2 \cap Y=Z$. Since $D_1/Y \approx B_1/Z$ and $B_1 \cap B_2=Z$, $|E|=|D_1|=|D_2|$, and $E \sim D_i$, for $E \neq X$. From the fact: $D_i \supset Y$ and Y is characteristic, $E \supset Y$, and so $E=D_j$ for some j. However, $D_j \cap D_1 \supset B_1$ implies $D_j=D_1$, and similarly $D_j=D_2$, which is a contradiction. Hence, whenever $i \geqslant 2$, each D_i is hollow. Thus we obtain

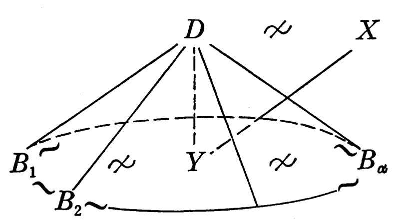
i)–1 

$\alpha \geqslant 2$

(X is characteristic and D is hollow)

(Y is hollow and characteristic)

or

i)–2 

Now following Proposition 1, we divide the situations into the following cases:

I'

$$\begin{array}{cl} eJ^i & (eJ^i \text{ is hollow}) \\ | & \\ eJ^{i+1} & (eJ^{i+1} \text{ is hollow}) \\ | & \\ eJ^{i+2} & \end{array}$$

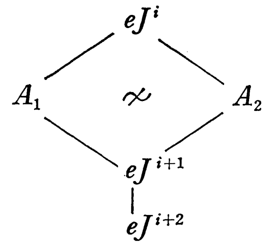
II'

$$\begin{tikzcd} & eJ^i \arrow[d, dash] & \\ & eJ^{i+1} \arrow[dl, dash] \arrow[dr, dash] & \\ C_1 \arrow[rr, phantom, "\not\sim"] \arrow[dr, dash] & & C_2 \arrow[dl, dash] \\ & eJ^{i+2} & \end{tikzcd}$$

(eJ^i is hollow)

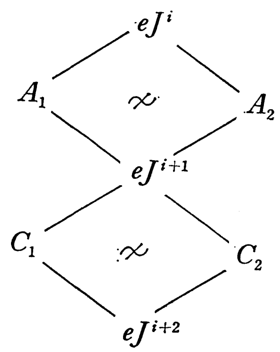
(C_1 and C_2 are characteristic and either C_1 or C_2 is hollow and $\Delta(C_i)=\Delta$ $(i=1, 2)$)

III′



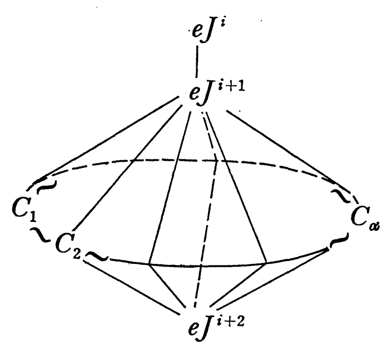
(A_1 and A_2 are chacteraristic, either A_1 or A_2 is hollow and $\Delta(A_i)=\Delta$ ($i=1, 2$))
(eJ^{i+1} is hollow)

IV′



(C_1 and C_2 are characteristic, either C_1 or C_2 is hollow and $\Delta(C_i)=\Delta$ ($i=1, 2$))

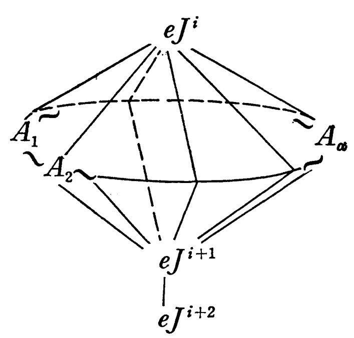
V′



(eJ^i is hollow)

$\alpha\geqslant 2$ ([Δ: $\Delta(C_i)$]$=2$ for all i)

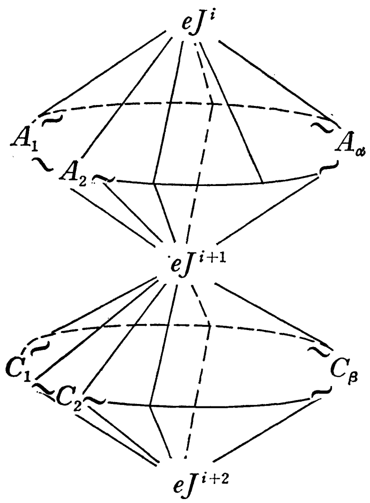
VI′



$\alpha\geqslant 2$ ([Δ: $\Delta(A_i)$]$=2$ for all i)

(eJ^{i+1} is hollow)

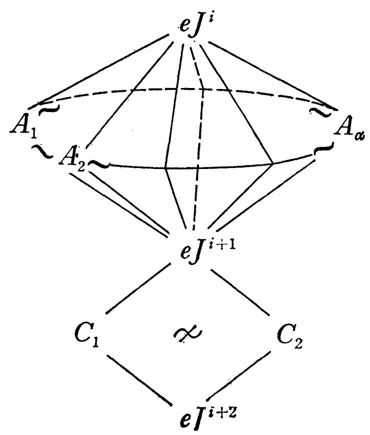
VII′



$\alpha \geqslant 2$ ($[\Delta\colon \Delta(A_i)]=2$ for all i)

$\beta \geqslant 2$ ($[\Delta\colon \Delta(C_i)]=2$ for all i)

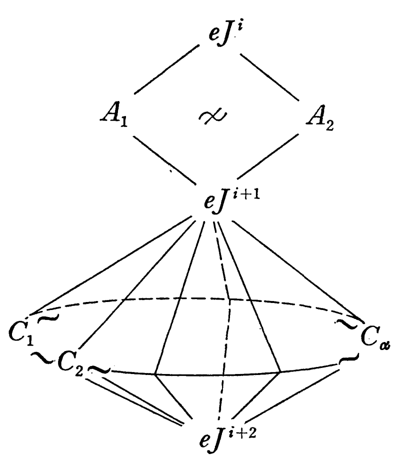
VIII′



$\alpha \geqslant 2$ ($[\Delta\colon \Delta(A_i)]=2$ for all i)

(C_1 and C_2 are characteristic, either C_1 or C_2 is hollow and $\Delta=\Delta(C_i)$ ($i=1, 2$))

IX′



(A_1 and A_2 are characteristic, either A_1 or A_2 is hollow and $\Delta=\Delta(A_i)$ ($i=1, 2$))

$\alpha \geqslant 2$ ($[\Delta\colon \Delta(C_i)]=2$ for all i)

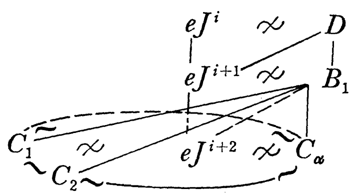
In the above and following observations, every chain of a diagram means a composition series, all modules located on the same horizontal line in the diagram have the same length, and all modules with same length appear in the diagrams below. It may happen that some modules in the diagrams do not appear. Further we always consider a case where eJ^i is a waist (every composition series contains eJ^i) or D_1 in the diagram exists.

I′ Let D' $(\neq eJ^i)$ be a submodule of eR with $|D'|=|eJ^i|$. Since there exists D_1 containing eJ^{i+1} with $|D_1|=|eJ^i|$ by assumption, $D'\sim D_1$ by Lemma 1. Hence $D_1\supset eJ^{i+1}$ implies $D'\supset eJ^{i+1}$. Thus we obtain from i)

I)-1

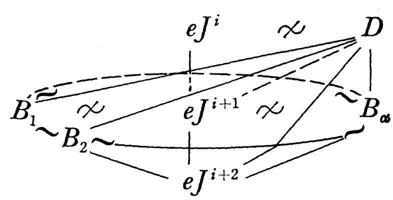
$$\begin{array}{ll} eJ^i & (eJ^i \text{ is hollow and a waist}) \\ | & \\ eJ^{i+1} & (eJ^{i+1} \text{ is hollow and a waist}) \\ | & \\ eJ^{i+2} & \end{array}$$

I)-2



$\alpha\geqslant 2$

I)-3



$\alpha\geqslant 2$

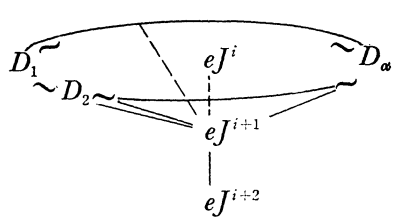
$(\Delta=\Delta(D))$

(B_i is hollow and $\Delta=\Delta(B_i)$ for all i)

(eJ^{i+2} is a waist)

(We shall know later that the B_i are hollow)

I)-4



$\alpha\geqslant 2$

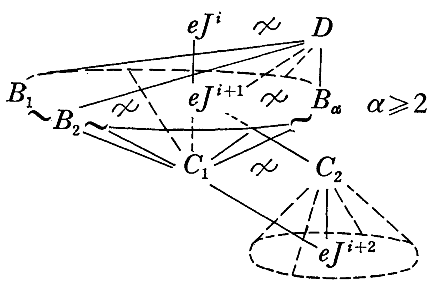
(D_i is hollow and $\Delta=\Delta(D_i)$ for all i)

(eJ^{i+1} is a waist)

(eJ^{i+2} is waist)

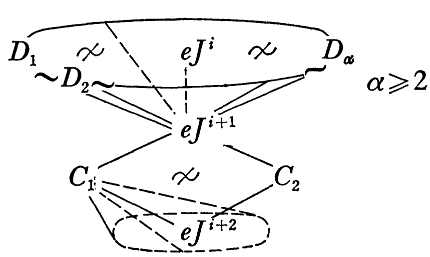
II′ Let $\{D_p\ (\neq eJ^i)\}_{p=1}^{n}$ be a set of submodules of eJ such that $D_p\supset eJ^{i+1}$ and $|D_p|=|eJ^i|$ (see the initial part of I′). Let B_k $(k=1, 2, \cdots)$ be maximal submodules of D_1 different from eJ^{i+1}. Since D_1/B_1 is simple and eJ^{i+1} contains at most two maximal submodules C_j, we may assume that $B_1\cap eJ^{i+1}=C_1$. Then since $B_2=xB_1$ for a unit x, $B_2\cap eJ^{i+1}=x(B_1\cap eJ^{i+1})=xC_1=C_1$. Let C be a maximal submodule of B_1. Since $B_1 \cap eJ^{i+1}=C_1 \not\sim C_2$, $C\sim C_1$ *or* $C\sim C_2$. However, since C_1 and C_2 are characteristic and $|C|=|C_1|=|C_2|$, $C=C_1$ or $C=C_2$. Therefore C_1 is a unique maximal submodule of B_1. Next assume that $n \geqslant 2$, i.e. $D_1 \neq D_2$. If D_i is not hollow, D_2 contains a maximal submodule B_1' ($\neq eJ^{i+1}$) which contains also a unique maximal submodule C_k from the above argument ($k=1$ or 2). Since $B_1' \sim B_1$ by Lemma 1 and $C_1 \not\sim C_2$, $k=1$. Therefore $B_1 \cap B_1' = C_1$. If we replace $\mathrm{J}(Y)$ by C_1 in the proof of i), we obtain the same situation (put $E=B_1+B_1'$) and the D_i are hollow. Thus we have

II)–1



(B_i is hollow and $\Delta=\Delta(B_i)$)

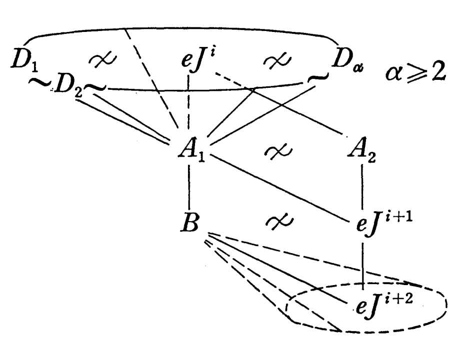
II)–2



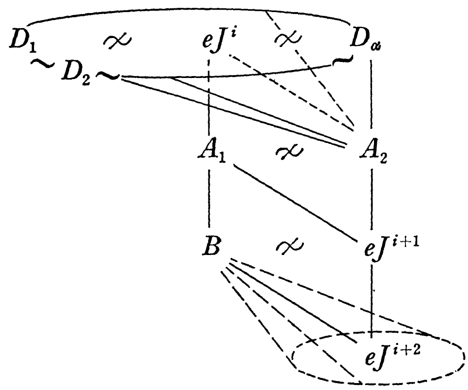
(D_i is hollow and $\Delta=\Delta(D_i)$ for all i))
(eJ^{i+1} is a waist)

III$'$ The A_i are characteristic. Replacing C_i in II) by A_i, we obtain from i)

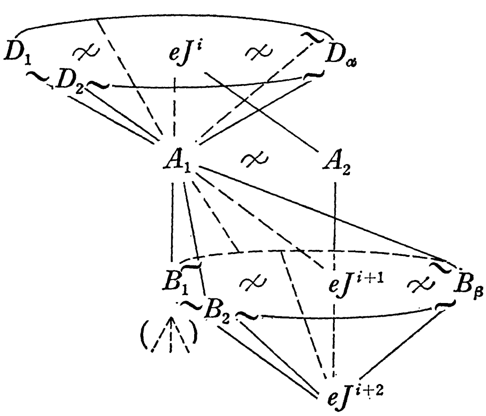
III)–1



or



III)–2

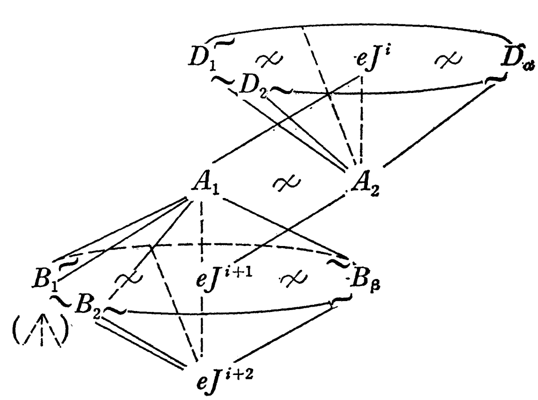


$\beta \geqslant 2$

(D_i is hollow and $\Delta = \Delta(D_i)$ for all i)

(B_i is hollow and $\Delta = \Delta(B_i)$ for all i)

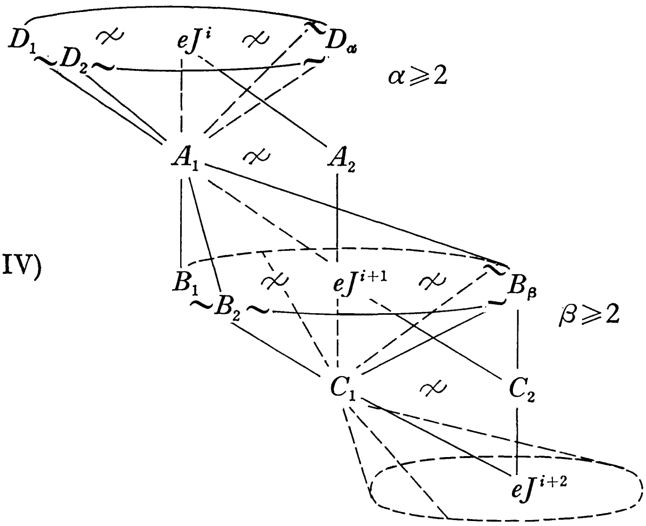
or



(We shall know later that the B_i are hollow.)

IV′ Similarly to II′ and the above, we obtain

IV)



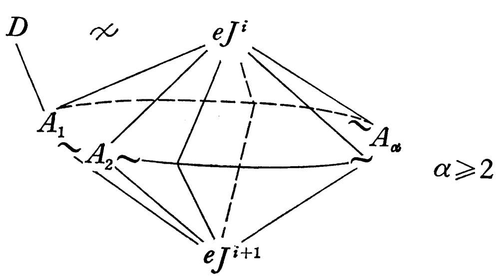
(D_i is hollow and $\Delta = \Delta(D_i)$ for all i)

(B_i is hollow and $\Delta = \Delta(B_i)$ for all i)

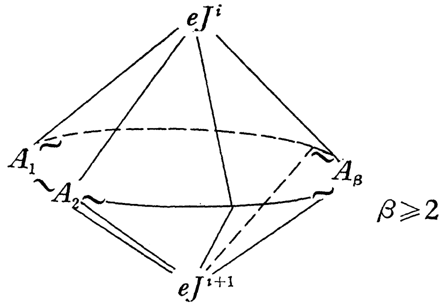
In order to study the remaining cases, we consider the following:

(We omit other forms similar to the second form in III)

ii′



Since $\Delta(A_i) \neq \Delta$ by Proposition 1, 4), for any submodule E of eR, $E \sim A_1$ by Proposition 1, 2). Hence $E \supset eJ^{i+1}$ provided $|E| = |A_1|$, since $A_1 \supset eJ^{i+1}$ and eJ^{i+1} is characteristic, and so maximal submodules of D consist of a subset of $\{A_i\}$. Therefore D is hollow. On the other hand, $\Delta = \Delta(D)$ and $\Delta \neq \Delta(A_i)$ by assumption and Proposition 1, 2). Therefore there exists a unit x in eRe such that $(x+j)A_1 \neq A_1$ for any j in eJe. Since $\Delta = \Delta(D)$, there exists j' in eJe with $(x+j')D = D$. Hence D contains A_1 and $(x+j')A_1$, and so $D \supset A_1 + (x+j')A_1 = eJ^i$, which is a contradiction. Hence $D = eJ^i$. Therefore eJ^i is a waist from the assumption of this observation. If further A_1 is not hollow, there exists a maximal submodule B_1 ($\neq eJ^{i+1}$) of A_1. Then $\Delta(B_1) = \Delta$ and we obtain the same situation as above. Hence $B_1 \subset A_1 \cap (x+j')A_1 = eJ^{i+1}$, a contradiction. Therefore eJ^{i+1} is also a waist. Thus from the argument above and the consideration of I)$\sim$IV) we obtain ii)

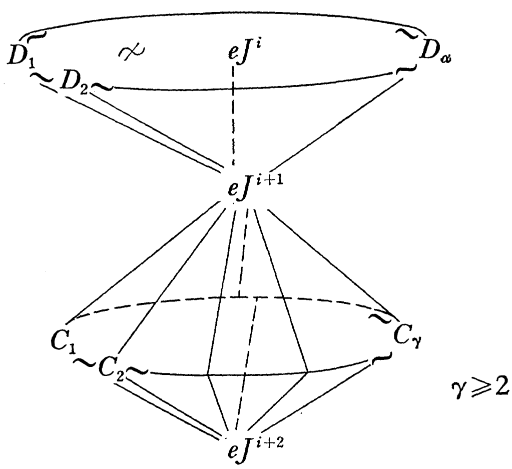


(eJ^i is a waist)

(A_i is hollow and $[\Delta\colon \Delta(A_i)]=2$ for all i)

(eJ^{i+1} is a waist)

V)



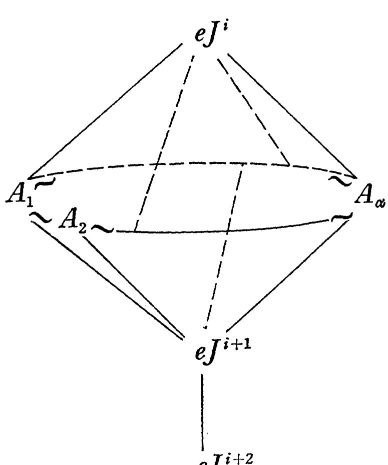
(D_i is hollow and $\Delta=\Delta(D_i)$ for all i)

(eJ^{i+1} is a waist)

(C_i is hollow and $[\Delta\colon \Delta(C_i)]=2$ for all i)

(eJ^{i+2} is a waist)

VI)



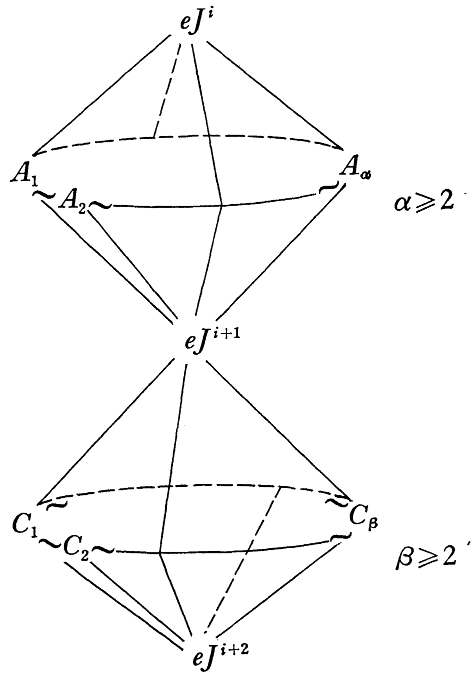
(eJ^i is a waist)

(A_i is hollow and $[\Delta\colon \Delta(A_i)]=2$ for all i)

(eJ^{i+1} is a waist)

(eJ^{i+2} is a waist)

VII)



(eJ^i is a waist)

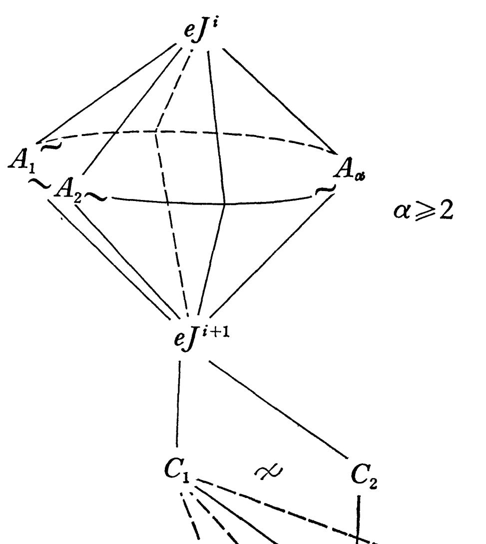
(A_i is hollow and $[\Delta\colon \Delta(A_i)]=2$ for all i)

(eJ^{i+1} is a waist)

(C_i is hollow and $[\Delta\colon \Delta(C_i)]=2$ for all i)

(eJ^{i+2} is a waist)

VIII)

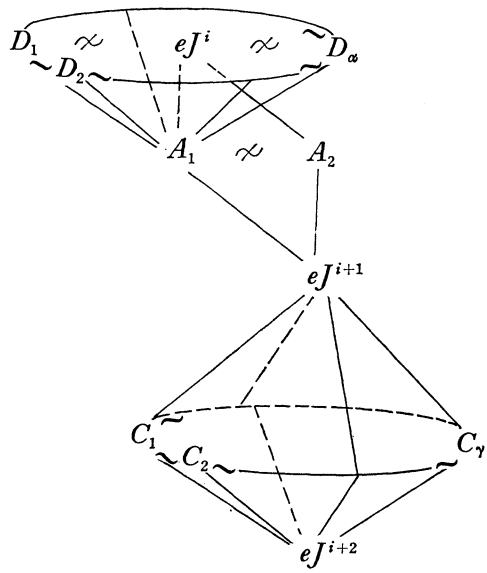


(eJ^i is a waist)

(A_i is hollow and $[\Delta\colon \Delta(A_i)]=2$ for all i)

(eJ^{i+1} is a waist)

IX)



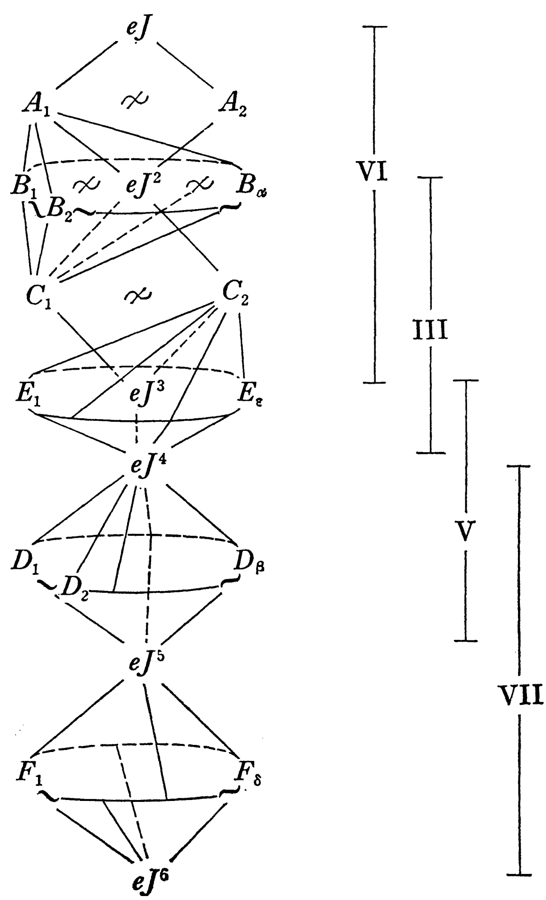
(D_i is hollow and $\Delta = \Delta(D_i)$ for all i)

(eJ^{i+1} is a waist)

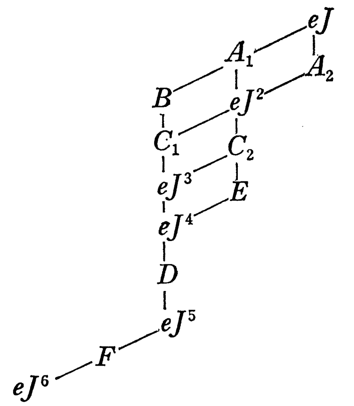
(C_i is hollow and $[\Delta: \Delta(C_i)] = 2$ for all i)

(eJ^{i+2} is a waist)

X)



XI)

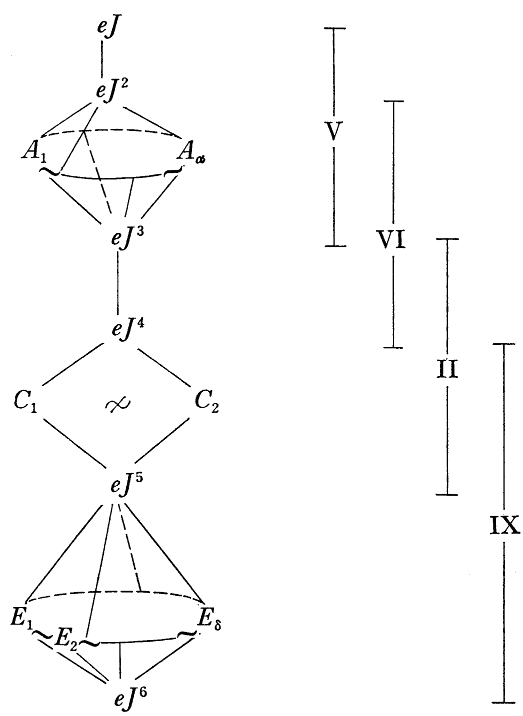


(isomorphism classes)

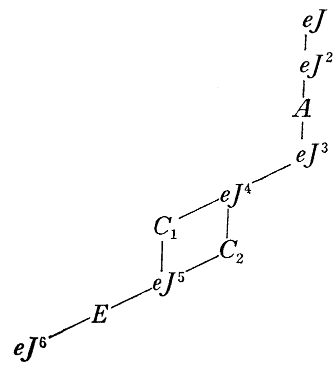
If the B_i in I)–3 or III)–2 are not hollow, we should obtain a circle around eJ^{i+2} (note that $\beta \geqslant 2$), which is directly connected to the circle around eJ^{i+1} like a cylinder. However, there are no circles directly connected in the diagrams. Hence B_i is hollow.

If we start from eJ and use the diagrams and induction on the nilpotency of J, we know that either eJ^k is a waist or D_1 in each diagram exists. Thus we obtain the lattice of all submodules in eR by connecting the diagrams I)$\sim$ IX). For example see X), XI) and

XII)



XIII)



(isomorphism classes)

If we consider the lattice of isomorphism classes given by the left-sided multiplication of unit elements in *eRe*, we obtain Diagrams XI) and XIII). Checking each diagram, we know that $|A/AJ|\leqslant 2$ for any submodule A of *eR* and that the A satisfy the conditions in Corollary to Proposition 1.

Theorem 2. *Let R be a right artinian ring. Then the following conditions are equivalent:*

1) *R is a right* US-3 *ring.*

2) *Let e be a primitive idempotent in R. For any submodule A of eR,* $A/AJ=\bar{A}_1\oplus\bar{A}_2$, *where* $A\supset A_i\supset AJ$ *and* $\bar{A}_i$ *is simple or zero, and one of the following situations occurs:*

i) $\bar{A}_2=0$, *i.e., A is hollow.*

ii) $\bar{A}_1\not\approx\bar{A}_2$ *and* $A_1J=AJ$.

iii) *There exists a unique characteristic and maximal submodule C in A, and for any maximal submodule* $B_i\neq C$, *there exists a unit element* x_i *in eRe such that* $x_iB_1=B_i$, *and further* $B_iJ=AJ$ *and* $\Delta(B_i)=\Delta$.

iv) *There are no characteristic and maximal submodules in A, and for any maximal submodules* B_i *in A, there exists a unit element* x_i *in eRe such that* $x_iB_1=B_i$, *and further* $B_iJ=AJ$ *and* $[\Delta:\Delta(B_i)]=2$.

In case of iii) *and* iv) $\bar{A}_1\approx\bar{A}_2$.

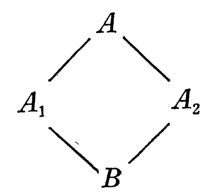
3) *The lattice of the submodules of eR is obtained by connecting Diagrams I)$\sim$IX).*

Proof. We note that A contains exactly two maximal submodules if and only if $A/AJ=\bar{A}_1\oplus\bar{A}_2$ and $\bar{A}_1\not\approx\bar{A}_2$. Hence 1)$\leftrightarrow$3) and 3)$\rightarrow$2) are clear from the argument before Theorem 2. 2)$\rightarrow$3). We can easily see by induction on eJ^i that D in Diagram ii′) is hollow and $(D+eJ^i)/D$ is simple. Hence we obtain this implication from the same argument.

Finally we consider a case of $\Delta=\Delta(A)$ for any submodule A of *eR*. If *eR* is uniserial, every submodule of *eR* is characteristic. From this pcint of view, we consider

Condition II$'$. *Every hollow module is quasi-projective* [3].
If R is a US-3 ring and Condition II$'$ fulfils, every submodule A of eR contains at most two maximal submodules A_1 and A_2 by Lemma 1. Put $B=A_1\cap A_2$. If A_1/B is isomorphic to A_2/B via f, $A_3=\{a_1+a_2;\ f(a_1+B)=a_2+B\}$ is a submodule of A and $A_3/B\approx A_1/B$. It is clear that $A_1\not\supset A_3$ and $A_2\not\supset A_3$ which contradicts the assumption. Hence $A/A_1\not\approx A/A_2$ provided $A_1\neq A_2$. Thus we obtain the following diagrams:

a) 

b) 

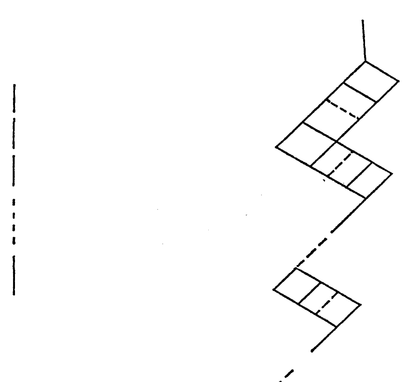
where the A/A_i are simple and $A/A_1\not\approx A/A_2$, and $\{A_i\}$ is the set of maximal submodules of A. Conversely, assume that A is characteristic in the diagrams *a*) and *b*). Then $A_1=\mathrm{J}(A)$ for *a*). Hence A_1 is also characteristic. Consider the diagram *b*). Let x be any element in eRe. Since $\{A_i\}$ is the set of maximal submodules of A, $xA_1\subset A_1$ or $xA_1\subset A_2$. Assume $xA_1\not\subset A_1$. Since $(e+x)A_1\subset A$, $(e+x)A_1\subset A_1$ or $\subset A_2$. $xA_1\not\subset A_1$ implies $(e+x)A_1\subset A_2$. On the other hand, $xA_1\subset A_2$ for $xA_1\not\subset A_1$. Hence $A_1\subset A_2$, which is a contradiction. Therefore the A_i are also characteristic.

Theorem 3. *Let R be as in Theorem 2. Then the following conditions are equivalent*:

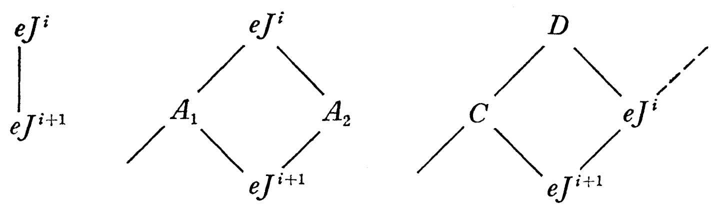
1) *R is a right* US-3 *ring and Condition II$'$ holds.*

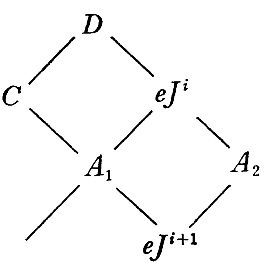
2) *For each primitive idempotent e, any submodule A of eR contains at most two maximal submodules A_1 and A_2, and either A_1 or A_2 is hollow, provided $A_1\neq 0$ and $A_2\neq 0$.*

3) *The lattice of the submodules of eR is of Diagram XIV) below, where each parallelogram is of Diagram b).*

XIV) 

Proof. The first two conditions are equivalent from the argument before Theorem 3.
2)→3) is trivial from Diagram XIV).
3)→2). It is clear from the same argument that we obtain Diagram a) or b). We can show by induction on eJ^i that there exists one of the following situations:

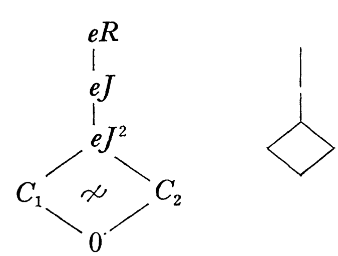




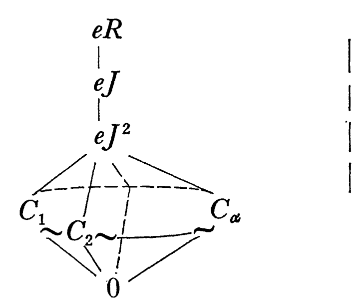
Hence we obtain Diagram XIV).

3. Examples. Let R be a ring with $J^3=0$. We shall give the complete list of the lattice of submodules of eR, when R is US-3.

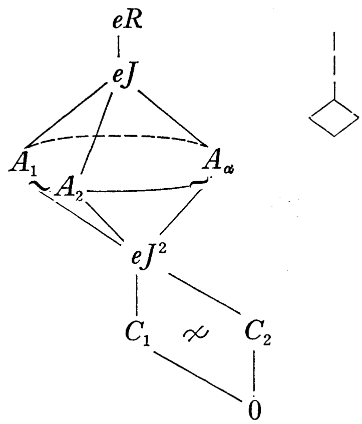
a$_1$)



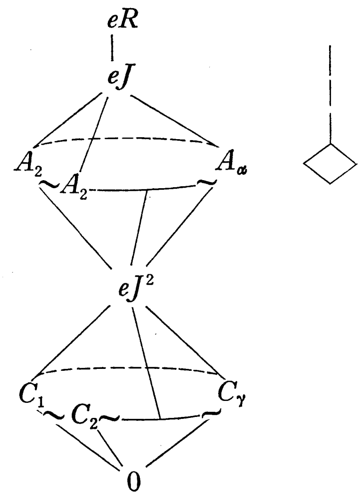
a$_2$)



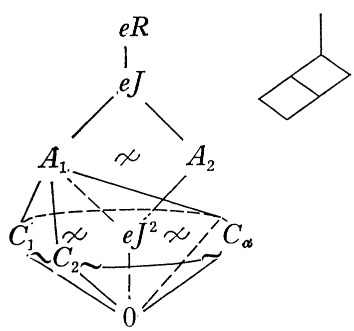
b$_1$)



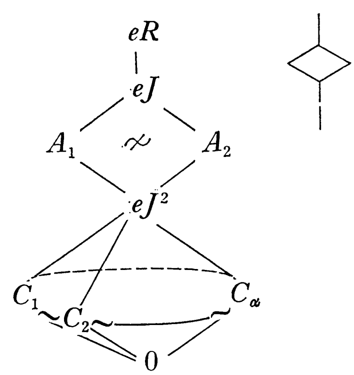
b$_2$)



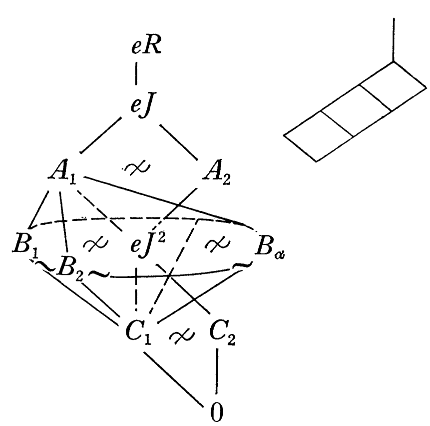
c$_1$)



c$_2$)



c$_3$)



(It may happen that some modules do not appear)

We shall construct a ring for each case. Let $L \supset K$ be fields with $[L: K] = 2$.

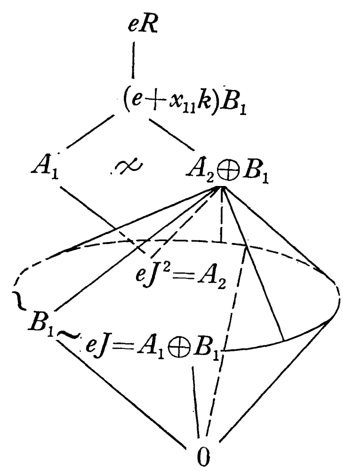
a$_1$)
$$R = \begin{pmatrix} K & K & K & K \\ 0 & K & K & K \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix}$$

a$_2$)
$$R = \begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix}$$

b$_1$)
$$R = \begin{pmatrix} L & L & L & L \\ 0 & K & L & L \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

b$_2$)
$$R = \begin{pmatrix} L & L & L \\ 0 & K & L \\ 0 & 0 & K \end{pmatrix}$$

c_1) ([3], Example 2). Let R be a vector space over K with basis $\{e_1, x_{11}, y_{12}, x_{12}, e_2, x_{22}, y_{21}, x_{21}\}$. Define $e_i e_j = e_i \delta_{ij}$, $e_i x_{jk} e_p = x_{jk} \delta_{ij} \delta_{kp}$, $e_i y_{jk} e_p = y_{jk} \delta_{ij} \delta_{kp}$, $x_{11} x_{12} = y_{12}$ and $x_{22} x_{21} = y_{21}$. Putting other multiplications to be zero, we see that R is a ring with $J^3 = 0$. Put $e = e_1$, $A_1 = x_{11}K + y_{12}K$, $A_2 = y_{12}K$ and $B_1 = x_{12}K$. Then



where k are in K.

c$_2$)
$$R = \begin{pmatrix} L & L & L & L \\ 0 & L & 0 & L \\ 0 & 0 & L & L \\ 0 & 0 & 0 & K \end{pmatrix}$$

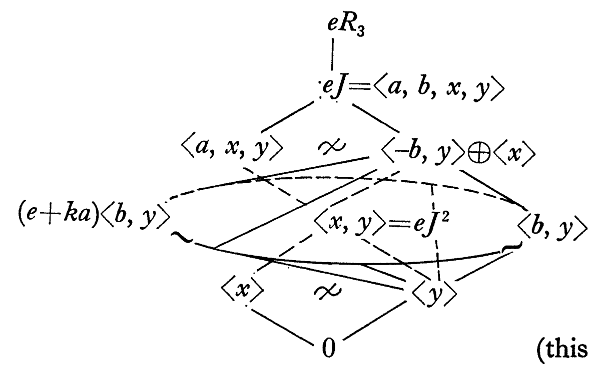
c$_3$)
$$R_1 = \begin{pmatrix} K & K & K & K & K \\ 0 & K & 0 & 0 & K \\ 0 & 0 & K & K & K \\ 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & K \end{pmatrix} \quad \text{(this is a case } \alpha = 1\text{)}$$

However if

$$R_2 = \begin{pmatrix} L & L & L & L & L \\ 0 & K & 0 & 0 & L \\ 0 & 0 & L & L & L \\ 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & L \end{pmatrix},$$

then the lattice of submodules of $e_{11}R_2$ has the same form as c_3), but $[\Delta: \Delta(B_1)] = 2$. Hence R_2 is not US-3.

Let R_3 be a vector space over K with basis $\{e, f, a, b, x, y, u\}$. Define the multiplication of elements in the basis as follows: $e^2=e$, $f^2=f$, $eae=a$, $ebf=b$, $exf=x$, $eye=y$, $fue=u$, $ab=x$, $bu=y$ *and* $aa=y$. Other multiplications are zero. Then R_3 is a ring and the lattice of the submodules of eR_3 is



(this is a case $\alpha>1$)

If $B_1/C_1 \not\approx C_2$ in c_3), then $B_2=B_3=\cdots=0$. Because, $B_2=xB_1$ for some x in eRe. Since $\Delta(B_1)=\Delta$, $x=x_1+j$, where $x_1B_1=B_1$ and $j\in eJe$. Then $B_2\subset B_1+jB_1$. If $jB_1\subset B_1$, $B_2=0$. $jB_1\subset ejJ\subset eJ^2$ and $|B_1|>|jB_1|$. Hence $jB_1=C_2$ provided $jB_1\not\subset B_1$, which implies that $B_1/C_1\approx C_2$. Therefore, if $B_1/C_1\not\approx C_2$, $jB_1\subset B_1$, so $B_2=0$.

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Department of Mathematics
Osaka City University
Osaka 558, Japan