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ON THE UNIQUENESS OF MARKOVIAN SELF-ADJOINT EXTENSION OF DIFFUSION OPERATORS ON INFINITE DIMENSIONAL SPACES

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1. Introduction

Let $(\mathcal{S}(R^d), L^2(R^d), \mathcal{S}'(R^d))$ be a rigged Hilbert space, where $\mathcal{S}(R^d)$ is the Schwartz space of test functions and $\mathcal{S}'(R^d)$ is its dual space. Letting $\{e_i\}_{i=1}^{\infty} \subset \mathcal{S}(R^d)$ be a complete orthonormal basis of $L_2(R^d)$, we put $FC_0^\infty=\{f;\ f$ is a function on $\mathcal{S}'(R^d)$ of the form $f(\xi)=\tilde{f}(\langle \xi, e_{i_1}\rangle, \cdots, \langle \xi, e_{i_n}\rangle)$ for some n and a real $C_0^\infty(R^n)$-function $\tilde{f}\}$, where $\langle\ ,\ \rangle$ is the dualization between $\mathcal{S}'(R^d)$ and $\mathcal{S}(R^d)$. Let ν be a quasi-invariant measure on $\mathcal{S}'(R^d)$ with respect to $\mathcal{S}(R^d)$. We call the measure ν *admissible* if the symmetric bilinear form $\mathcal{E}_\nu(u, v)=\frac{1}{2}(Du, Dv)_{L^2(R^d)\otimes L^2(\nu)}$, $u, v\in FC_0^\infty$, is closable. Its closed extension $(\mathcal{E}_\nu, \mathcal{F}_\nu)$ is said to be the *energy form* associated with the quasi-invariant admissible measure ν. Here, $Du=\sum_{i=1}^{\infty} e_i\otimes D_i u\in L^2(R^d)\otimes L^2(\nu)$ and D_i is a derivative in the direction of e_i. Furthermore, a self-adjoint operator H_ν representing the energy form $(\mathcal{E}_\nu, \mathcal{F}_\nu)$ is said to be a *diffusion operator*. For example, the probability measure μ_0 on $\mathcal{S}'(R^d)$ defined by the following formula is quasi-invariant and admissible:

$$\int_{\mathcal{S}'(R^d)} e^{i\langle \xi,\phi\rangle}\, d\mu_0(\xi) = e^{-1/4(\phi,(\Delta+m^2)^{-1/2}\phi)},\ \phi\in\mathcal{S}(R^d)\,,$$

where $(\ ,\ )$ is the scalar product in $L^2(R^d)$.

Let μ_0^* be the Euclidian random field $\langle \xi^*, \psi\rangle$ over R^{d+1}, defined by

$$\int_{\mathcal{S}'(R^{d+1})} e^{i\langle \xi^*,\psi\rangle}\, d\mu_0^*(\xi^*) = e^{-1/2(\psi,(\Delta+m^2)^{-1}\psi)},\ \psi\in\mathcal{S}(R^{d+1})\,.$$

The random field $\langle \xi^*, \psi\rangle$ can be regard as the restriction to $\mathcal{S}(R^{d+1})$ of the generalized random field indexed by the Sobolev space H_{-1} the completion of $\mathcal{S}(R^{d+1})$ with respect to the norm $||(-\Delta+m^2)^{-1/2}\psi||$. We denote by Σ_0 the σ-field generated by random variable $\{\langle \xi^*, \delta_0\otimes\phi\rangle;\ \phi\in\mathcal{S}(R^d)\}$, and regard the restriction of μ_0^* to Σ_0 as the measure on $\mathcal{S}'(R^d)$ by the natural identification of $\langle \xi, \phi \rangle$ with $\langle \xi^*, \delta_0 \otimes \phi \rangle$. Then, it coincides with μ_0 and the diffusion operator H_{μ_0} corresponding to μ_0 is nothing but the energy operator H of free Euclidean field model μ_0^*. To see this, it is enough to show that H_{μ_0} and H are the same operators on FC_0^∞ and that the symmetric operator $S = H_{\mu_0} \uparrow FC_0^\infty$ has a unique Markovian self-adjoint extension, where the notation $H_{\mu_0} \uparrow FC_0^\infty$ indicates the restriction of H_{μ_0} to FC_0^∞. In fact, the operator S is known to be an essentially self-adjoint operator.

Albeverio and Høegh-Krohn have raised a question in [3] whether the diffusion operator associated with $\mu^* \uparrow \Sigma_0$ is identical with energy operator of the Euclidean field model μ^* with trigonometric (or exponential) interaction and have shown that these operators are the same on FC_0^∞ when $d=1$. Thus, we now cope with the question: what kind of quasi-invariant admissible measure ν induces the symmetric operator $S_\nu = H_\nu \uparrow FC_0^\infty$ with a unique Markovian self-adjoint extension?

In this paper, we consider this problem in a simpler case that ν is an absolutely continuous measure with respect to the Wiener measure on the abstract Wiener space (H, B, μ). We conclude the uniqueness of Markovian self-adjoint extension of S_ν under the condition that the Radon-Nikodym derivative ρ^2 is strictly positive and belongs to the space D_∞ $(= \bigcap_{p \geq 1, r \in R^1} D_p^r)$, where D_p^r is the Sobolev space of order r and degree p on the Wiener space. In the proof, we use the Malliavin's calculus and in particular the hypoellipticity of the Ornstein-Uhlenbeck generator. We note at the end of this paper that, if ρ happens to be a tame function, then Wielens' method [8] applies and S_ν becomes essentially self-adjoint.

2. Notations and the closability of a symmetric form

Let (H, B, μ) be an abstract Wiener space and $\{e_i\}_{i=1}^\infty \subset B^*$ (dual space of B) be a complete orthonormal basis of H. We set $FC_0^\infty = \{f; f$ is a function on B of the form $f(x) = \tilde{f}(\langle e_{i_1}, x \rangle, \cdots, \langle e_{i_n}, x \rangle)$ for some n and $\tilde{f} \in C_0^\infty(R^n)\}$ and $FC_0^\infty(H) = \{F; F$ is a H-valued function on B which is of form $F(x) = \sum_{i=1}^n e_i \otimes f_i(x)$ for some $n, f_i \in FC_0^\infty\}$. We denote by D_p^r the completion of FC_0^∞ with respect to the norm $||f||_p^r = ||f||_p + ||D^r f||_p$, where $D^r f$, $f \in FC_0^\infty$, is the r-times iteration of the Fréchet derivative which is an element in $L_p(B \to \overbrace{H \otimes \cdots \otimes H}^{r})$. Note that

$$||D^r f||_p = ||\{\sum_{(n_1, \cdots, n_r) \in N^r} (D_{n_1}(D_{n_2} \cdots (D_{n_r} f)))^2\}^{1/2}||_p$$

where D_i is the derivative in the direction of e_i. It is convienient to use two different expressions of D_p^1 $(1 < p < \infty)$ according to Sugita [7] and Kusuoka [6]:

$$D_p^1 = \left\{ u \in L_p(\mu);\ \begin{array}{l} \text{There exists some } g \in L_p(B \to H) \\ \text{such that } (u, D^*v) = (g, v), \text{ for} \\ \text{any } v \in FC_0^\infty(H) \end{array} \right\} \tag{2.1}$$

and

$$D_p^1 = \left\{ u \in L_p(\mu);\ \begin{array}{l} u \text{ is stochastic } H \text{ Gateaux differentiable} \\ (SGD) \text{ with respect to } \mu, \text{ ray absolutely} \\ \text{continuous } (RAC) \text{ and the stochastic} \\ \text{Gateaux derivative } Du \text{ of } u \text{ satisfies that} \\ \|Du(x)\|_H \in L_p(\mu) \end{array} \right\} \tag{2.2}$$

Here, a function u is called SGD, if there exists a measurable map Du; $B \to H$ such that for any $k \in B^*$, the convergence $\frac{1}{t}[u(x+tk)-u(x)-t(Du(x), k)_H] \to 0$, $t \to 0$, take place in probability with respect to μ, and u is called RAC, if for any $k \in B^*$, there exists a measurable function u_k such that

1) $\tilde{u}_k(x) = u(x)$ for μ-a.e.
2) $\tilde{u}_k(x+tk)$ is absolutely continuous in t for each $x \in B$ (See [6; Definition 1,1 and Definition 1,2]). Then, we have for $u \in D_p^1$, $\|Du(x)\|_H = \sqrt{\sum_{i=1}^{\infty} (Du(x), e_i)^2}$ $= \sqrt{\sum_{i=1}^{\infty} (D_i u(x))^2}$, where $D_i u(x) = \lim_{t \downarrow 0} \frac{1}{t} (\tilde{u}_{e_i}(x+te_i) - \tilde{u}_{e_i}(x))$.

We fix a function ρ on B satisfying

$$\text{i) } \rho > 0 \qquad \text{ii) } \rho \in D_\infty \tag{2.3}$$

where $D_\infty = \bigcap_{\substack{p \geq 1 \\ r \in R^1}} D_p^r$. We define the symmetric bilinear form $(\mathcal{E}_\rho, FC_0^\infty)$ by

$$\mathcal{E}_\rho(u, v) = \frac{1}{2} \int_B (Du(x), Dv(x))_H \rho^2(x)\, d\mu, \quad u, v \in FC_0^\infty. \tag{2.4}$$

Lemma 1. *$(\mathcal{E}_\rho, FC_0^\infty)$ is closable on $L_2(\rho^2\mu)$.*

Proof. We follows the argument of [1; Theorem 2.3]. Since $D_i^* \rho^2 = -2\rho D_i \rho + \langle e_i, x \rangle \rho^2(x)$, we have for $g \in FC_0^\infty$,

$$\begin{aligned} (Dg, e_i \otimes 1)_{H \otimes L^2(\rho^2\mu)} &= (D_i g, \rho^2)_{L^2(\mu)} \\ &= (g, -2\rho D_i \rho + \langle e_i, x \rangle \rho^2)_{L^2(\mu)} \\ &= \left(g, -2\frac{D_i \rho}{\rho} + \langle e_i, x \rangle\right)_{L^2(\rho^2\mu)}. \end{aligned}$$

By noting that $\int \left(\frac{D_i \rho}{\rho}\right)^2 \rho^2\, d\mu \leqq \|\rho\|_2^1 < \infty$, we see that $-2\frac{D_i \rho}{\rho} + \langle e_i, x \rangle \in L_2(\rho^2\mu)$

and $e_i\otimes 1\in[D_\rho^*]$, where D_ρ^* denote the adjoint operator of D which is an operator from $L_2(\rho^2\mu)$ to $H\otimes L_2(\rho^2\mu)$. Put $\beta(e_i)=D_\rho^*(e_i\otimes 1)$. Then, we see that for $g, f\in FC_0^\infty$

$$\begin{aligned}(Dg, e_i\otimes f)_{H\otimes L^2(\rho^2\mu)} &= (D_i g\cdot f, 1)_{\rho^2\mu} \\ &= (D_i (g\cdot f)-gD_i f,1)_{\rho^2\mu} \\ &= (g, \beta(e_i)f-D_i f)_{\rho^2\mu}\,.\end{aligned}$$

Because the function $\beta(e_i)f-D_i f$ belongs to $L_2(\rho^2\mu)$, it holds that $e_i\otimes f\in\mathcal{D}[D_\rho^*]$ and consequently $FC_0^\infty(H)$ is contained in $\mathcal{D}[D_\rho^*]$. Since $FC_0^\infty(H)$ is dense in $H\otimes L_2(\rho^2\mu)$, the closure $\bar{D}=(D_\rho^*)^*$ is well defined and hence $(\mathcal{E}_\rho, FC_0^\infty)$ is closable. q.e.d.

We denote by $(\mathcal{E}_\rho, \mathcal{F})$ the closed extension of $(\mathcal{E}_\rho, FC_0^\infty)$.

3. The uniqueness of Markovian self-adjoint extension

Let H_ρ be a self-adjoint operator associated with the closed form $(\mathcal{E}_\rho, \mathcal{F})$ and S be a symmetric operator defined by $S=H_\rho\upharpoonright FC_0^\infty$. S can be represented as

$$Su = \frac{1}{2}\mathcal{L}u+\frac{1}{\rho}\langle D\rho, Du\rangle_H,\ u\in FC_0^\infty\,, \tag{3.1}$$

where $\mathcal{L}$ is a Ornstein-Uhlenbeck generator. We denote by $\mathcal{A}_M(S)$ the totality of Markovian self-adjoint extensions: $A\in\mathcal{A}_M(S)$ means that A is a self-adjoint extension of S which generates a strongly continuous contraction Markovian semi-group on $L_2(\rho^2\mu)$. H_ρ is called the Friedrichs extension of S and is an element of $\mathcal{A}_M(S)$. Then, the following theorem holds.

Theorem 1. *Under the condition* (2.3), $\mathcal{A}_M(S)$ *has only one element* H_ρ, *namely, S has a unique Markovian self-adjoint extension.*

For any $A\in\mathcal{A}_M(S)$, the form domain $\mathcal{D}[\sqrt{-A}]$ is orthogonally decomposed with respect to $\mathcal{E}_{A,\alpha}$ $(=(\sqrt{-A}\,\cdot,\sqrt{-A}\,\cdot)_{\rho^2\mu}+\alpha(\ ,\ )_{\rho^2\mu})$ as

$$\mathcal{D}[\sqrt{-A}] = \mathcal{F}\oplus(\mathcal{N}_\alpha\cap\mathcal{D}[\sqrt{-A}])\,, \tag{3.2}$$

where $\mathcal{N}_\alpha=\{u\in L_2(\rho^2\mu);\ (\alpha I-S^*)\,u=0\}$ ([4; Theorem 2.3.2]). Hence, for the proof of Theorem 1 we must show that

$$\mathcal{N}_\alpha\cap\mathcal{D}[\sqrt{-A}]\subset\mathcal{F}\,. \tag{3.3}$$

In order to prove (3.3) we introduce the intermediate space $\mathcal{H}$ by (3.4) and prove that $\mathcal{N}_\alpha\cap\mathcal{D}[\sqrt{-A}]\subset\mathcal{H}\subset\mathcal{F}$ (Lemma 2 and Lemma 4).

Let $\{a_l(t)\}_{l=1}^\infty$ be a sequence of $C_0^\infty(R^1)$-functions satisfying that

i) $0 \leqq a_l(t) \leqq 1$ ii) $a_l(t) = \begin{cases} 1 & \text{on } \frac{1}{2^l} < t < 2^l \\ 0 & \text{on } t \leqq \frac{1}{2^{l+1}},\ t \geqq 2^{l+1} \end{cases}$

iii) $|a_l'(t)| \leqq \begin{cases} c\, 2^{l+1} & \text{on } t \leqq \frac{1}{2^l} \\ c' & \text{otherwise} \end{cases}$ for some constants c and c'.

We put $\phi_l(x) = a_l \circ \rho(x)$. If a function u satisfies $\phi_l \cdot u \in \bigcup_{1<p<2} D_p^1$, for any l, then we have $D(\phi_{l+1} \cdot u) = D(\phi_l \cdot u)$ μ-a.e. on $\mathcal{M}_l = \left\{\frac{1}{2^l} < \rho < 2^l\right\}$, because $D(\phi_l \cdot u) = D(\phi_l \cdot \phi_{l+1} u) = \phi_l \cdot D(\phi_{l+1} \cdot u) + \phi_{l+1} u \cdot D\phi_l$. Therefore, we can well define Du by

$$Du = D(\phi_l \cdot u) \quad \text{on } \mathcal{M}_l\,.$$

Let we consider the function space

$$\mathcal{H} = \left\{ u \in L_2(\rho^2\mu);\ \begin{matrix} \phi_l \cdot u \in \bigcup_{1<p<2} D_p^1 \text{ for any } l \text{ and} \\ \int \langle Du, Du \rangle_H \rho^2 d\mu < \infty \end{matrix} \right\} \tag{3.4}$$

Then, we have the following lemma.

Lemma 2. *It holds that*

$$\mathcal{H} \subset \mathcal{F}\,. \tag{3.5}$$

Proof. For any $u \in \mathcal{H}$, we see that $u_{(N)} = (-N \vee u) \wedge N \in \mathcal{H}$ since $\phi_l \cdot u_{(N)} = ((-N\phi_l) \vee \phi_l u) \wedge N\phi_l \in D_p^1$ by (2.2), $u_{(N)}$ converges to u in $\mathcal{E}_{\rho,1}$. Furthermore $u_{(N)}$ can be approximated by $\phi_l \cdot u_{(N)} \in \mathcal{H}$. In fact, we have

$$\begin{aligned} &\int \|Du_{(N)} - D(\phi_l \cdot u_{(N)})\|_H^2\, \rho^2 d\mu \\ &\leqq 2\left[\int |1-\phi_l|^2 \|Du_{(N)}\|_H^2\, \rho^2 d\mu + \int u_{(N)}^2 \|D\phi_l\|_H^2\, \rho^2 d\mu\right]. \end{aligned} \tag{3.6}$$

The second term of the right hand side is equal to $\int u_{(N)}^2 (a_l'(\rho))^2 \|D\rho\|_H^2\, \rho^2 d\mu$ and is not greater than $\int_{\{\rho \leqq 1/2^l\}} u_{(N)}^2 \cdot (c2^{l+1})^2\, \|D\rho\|_H^2\, \rho^2 d\mu + \int_{\{\rho \geqq 2^l\}} u_{(N)}^2\, c'^2\, \|D\rho\|_H^2\, \rho^2 d\mu$ $\leqq 4c^2 N^2 \int_{\{\rho \leqq 1/2^l\}} \|D\rho\|_H^2\, d\mu + c'^2 N^2 \int_{\{\rho \geqq 2^l\}} \|D\rho\|_H^2\, \rho^2 d\mu$, which tends to zero as $l \to \infty$ by the assumption (2.3). Hence the left hand side of (3.6) tends to zero as $l \to \infty$.

Next we show that there exists a sequence $\{f_m\}_{m=1}^{\infty} \subset FC_0^{\infty}$ such that

$$\phi_{l+1} f_m \to \phi_l\, u_{(N)}\ (m \to \infty) \text{ in } \mathcal{E}_{\rho,1}\,. \tag{3.7}$$

Since we see $\phi_l u_{(N)} \in D^1_{2,b}$ by

$$\begin{aligned}\int \|D(\phi_l \cdot u_{(N)})\|_H^2 d\mu &\leqq 2[\int \phi_l^2 \|Du_{(N)}\|_H^2 d\mu + \int u_{(N)}^2 \|D\phi_l\|_H^2 d\mu] \\ &\leqq 2[\int_{\{1/2^{l+1}<\rho<2^{l+1}\}} \phi_l^2 \|Du_{(N)}\|_H^2 d\mu + N^2 \int \|D\phi_l\|_H^2 d\mu] \\ &\leqq 2^{2l+3} \int \|Du_{(N)}\|_H^2 \rho^2 d\mu + 2N^2 \int \|D\phi_l\|_H^2 d\mu \\ &< \infty ,\end{aligned}$$

there exists a sequence $\{f_m\}_{m=1}^\infty \subset FC_0^\infty$ such that 1) $|f_m| \leqq N$, 2) $f_m \to \phi_l u_{(N)}$, μ-a.e., 3) $f_m \to \phi_l u_{(N)}$ in D^1_2. Then, (3.7) follows because

$$\begin{aligned}&\int \|D(\phi_{l+1} f_m) - D(\phi_l u_{(N)})\|_H^2 \rho^2 d\mu = \int \|D(\phi_{l+1}(f_m - \phi_l u_{(N)}))\|_H^2 \rho^2 d\mu \\ &\leqq 2\left[\int \phi_{l+1}^2 \|Df_m - D(\phi_l u_{(N)})\|_H^2 \rho^2 d\mu + \int (f_m - \phi_l u_{(N)})^2 \|D\phi_{l+1}\|_H^2 \rho^2 d\mu\right] \\ &\leqq 2^{2l+5} \int \|Df_m - D(\phi_l u_{(N)})\|_H^2 d\mu + 2\int (f_m - \phi_l u_{(N)})^2 \|D\phi_{l+1}\|_H^2 \rho^2 d\mu \\ &\to 0 \quad (m \to \infty) .\end{aligned}$$

Finally we take a sequence $\{g_n\}_{n=1}^\infty \subset FC_0^\infty$ satisfying that $g_n \to \phi_{l+1} f_m$ in D^1_4. Then, we see that

$$g_n \to \phi_{l+1} f_m \quad \text{in} \quad \mathcal{E}_{\rho,1} , \tag{3.8}$$

since $\int \|D(\phi_{l+1} f_m) - Dg_n\|_H^2 \rho^2 d\mu \leqq (\int \|D(\phi_{l+1} f_m) - Dg_n\|_H^4 d\mu)^{1/2} \cdot (\int \rho^4 d\mu)^{1/2}$. q.e.d.

Denote by $\bar{S}^{(p)}$, $1<p$, the closure of S in $L_p(\rho^2\mu)$. We need the following lemma in the proof of Lemma 4.

Lemma 3. *If* $w \in D^2_p$, $p>1$, *then for any* l, $\phi_l w \in \bigcap_{\substack{1<p'<p \\ p' \leqq 2}} \mathcal{D}[\bar{S}^{(p')}]$ *and*

$$\bar{S}^{(p')}(\phi_l w) = \frac{1}{2} \mathcal{L}(\phi_l w) + \frac{1}{\rho} \langle D\rho, D(\phi_l w) \rangle_H . \tag{3.9}$$

Proof. First of all we show that $\phi_l \psi \in \mathcal{D}[\bar{S}^{(2)}]$, for $\psi \in FC_0^\infty$. Take a sequence $\{g_k\}_{k=1}^\infty \subset FC_0^\infty$ such that g_k converges to ϕ_l with respect to $\|\ \ \|_4^2$. Then, we obtain

$$\begin{aligned}S(g_k \psi) &= \frac{1}{2} \mathcal{L} g_k \psi + \frac{1}{2} g_k \mathcal{L}\psi + \frac{1}{2}\langle Dg_k, D\psi\rangle_H + \frac{g_k}{\rho}\langle D\rho, D\psi\rangle_H \\ &\quad + \frac{\psi}{\rho}\langle D\rho, Dg_k\rangle_H \\ &\xrightarrow[k\to\infty]{} \frac{1}{2} \mathcal{L}\phi_l \psi + \frac{1}{2} \phi_l \mathcal{L}\psi + \frac{1}{2}\langle D\phi_l, D\psi\rangle_H + \frac{\phi_l}{\rho}\langle D\rho, D\psi\rangle_H \\ &\quad + \frac{\psi}{\rho}\langle D\rho, D\phi_l\rangle_H\end{aligned} \tag{3.10}$$

$$=\frac{1}{2}\mathcal{L}(\phi_l\psi)+\frac{1}{\rho}\langle D\rho,\ D(\phi_l\psi)\rangle_H\ ,$$

the convergence being in $L_2(\rho^2\mu)$. In fact, by Schwartz inequality we have

$$\int|\mathcal{L}g_k\psi-\mathcal{L}\phi_l\psi|^2\rho^2\,d\mu\leqq\left(\int|\mathcal{L}g_k-\mathcal{L}\phi_l|^4\,d\mu\right)^{1/2}\left(\int(\psi\rho)^4\,d\mu\right)^{1/2}\to 0\quad(k\to\infty)\,,$$

and in the same way we can show the convergence of other terms of (3.10). Next, if $\{h_m\}_{m=1}^{\infty}\subset FC_0^{\infty}$ converges to w with respect to $\|\ \|_p^2$, we have

$$(3.11)\quad \bar{S}^{(2)}(\phi_l h_m)=\frac{1}{2}\mathcal{L}\phi_l\,h_m+\frac{1}{2}\phi_l\,\mathcal{L}h_m+\frac{1}{2}\langle D\phi_l,\ Dh_m\rangle_H+\frac{1}{2}\phi_l\,\mathcal{L}h_m$$
$$+\frac{1}{2}\langle D\phi_l,\ Dh_m\rangle_H$$
$$\xrightarrow[m\to\infty]{}\ \frac{1}{2}\mathcal{L}\phi_l\,w+\frac{1}{2}\phi_l\,\mathcal{L}w+\frac{1}{2}\langle D\phi_l,\ Dw\rangle_H+\frac{w}{\rho}\langle D\rho,\ D\phi_l\rangle_H$$
$$+\frac{\phi_l}{\rho}\langle D\rho,\ Dw\rangle_H$$
$$=\frac{1}{2}\mathcal{L}(\phi_l\,w)+\frac{1}{\rho}\langle D\rho,\ D(\phi_l\,w)\rangle_H\ ,$$

the convergence being in $L_{p'}(\rho^2\mu)$. In fact, by Hölder inequality, we get

$$\int|\mathcal{L}\phi_l\,w-\mathcal{L}\phi_l\,h_m|^{p'}\rho^2\,d\mu\leqq\left(\int|w-h_m|^p\,d\mu\right)^{p'/p}\left(\int(|\mathcal{L}\phi_l|^{p'}\rho^2)^{p/p-p'}\,d\mu\right)^{p-p'/p}$$
$$\to 0\qquad(m\to\infty)\,,$$

and the second and the third terms of (3.11) also converge to the corresponding terms in $L_{p'}(\rho^2\mu)$. Furthermore,

$$\int\left|\frac{w}{\rho}\langle D\rho,\ D\phi_l\rangle_H-\frac{h_m}{\rho}\langle D\rho,\ D\phi_l\rangle_H\right|^{p'}\rho^2 d\mu=\int|w-h_m|^{p'}|\langle D\rho,\ D\phi_l\rangle_H|^{p'}\rho^{2-p'}d\mu$$
$$\leqq\left(\int|w-h_m|^p d\mu\right)^{p/p'}\left(\int|\langle D\rho,\ D\phi_l\rangle_H^{p'}\,\rho^{2-p'}|^{p/p-p'}d\mu\right)^{p-p'/p}$$
$$\to 0\quad(m\to\infty)\,,$$

and the last term in (3.11) also tends to $\frac{\phi_l}{\rho}\langle D\rho,\ Dw\rangle_H$. q.e.d.

Take any element $A\in\mathcal{A}_M(S)$ and let $\{T_t\}_{t\geqq 0}$ be a semi-group on $L_2(\rho^2\mu)$ corresponding to A. Then, by the contractivity and symmetry, we can extend $\{T_t\}_{t\geqq 0}$ to a strongly continuous semi-group $\{T_t^{(p)}\}_{t\geqq 0}$ on $L_p(\rho^2\mu)$. We denote by $\{G_\alpha^{(p)}\}_{\alpha>0}$ the corresponding resolvent.

Lemma 4. *It hold that*

$$(3.12)\qquad \mathcal{N}_\alpha\cap\mathcal{D}[\sqrt{-A}]\subset\mathcal{H}\quad \textit{for}\quad A\in\mathcal{A}_M(S)\,.$$

Proof. Take any element $v\in\mathcal{N}_\alpha\cap\mathcal{D}[\sqrt{-A}]$. We first show that $\phi_l v\in$

$\bigcap_{1<p<2} D_p^1$ for any l. Let $w=\frac{\phi_l}{\rho^2}\in D_\infty$. Then, by Lemma 3, we see $w\psi$ $\left(=\phi_{l+1}\frac{\phi_l\psi}{\rho^2}\right)\in\mathcal{D}[\bar{S}^{(2)}]$, for any $\psi\in FC_0^\infty$. Then, by the definition

$$(v\rho^2, \alpha\phi-\bar{S}^{(2)}\phi)_\mu=0, \quad \text{for} \quad \phi\in\mathcal{D}[\bar{S}^{(2)}].$$

Hence, we obtain for $\psi\in FC_0^\infty$

$$(v\rho^2, \mathcal{L}(w\psi))_\mu=2\alpha(v\rho^2, w\psi)_\mu-2(v\rho, \langle D\rho, D(w\psi)\rangle_H)_\mu. \tag{3.13}$$

Now, we have for $\psi\in FC_0^\infty$

$$(\phi_l v, \mathcal{L}\psi)_\mu=(g, \psi)_\mu$$

where $g=2\alpha v\rho^2 w-2D^*(vw\rho D\rho)-2\rho\langle D\rho, Dw\rangle_H-v\rho^2\mathcal{L}w-D^*(v\rho^2 Dw)$. Now, we use the hypoellipticity of $\mathcal{L}$ ([5]) as follows: since $vw\rho D\rho$ and $v\rho^2 Dw$ belong to $\bigcap_{1<p<2} L_p(B\to H)$, we have $g\in\bigcap_{1<p<2} D_p^{-1}$. By [5], $\phi_l v$ belongs to the domain of extended $\mathcal{L}$, $\mathcal{L}(\phi_l v)\in\bigcap_{1<p<2} D_p^{-1}$ and $\phi_l v=R(\mathcal{L}(\phi_l v))\in\bigcap_{1<p<2} D_p^1$, where R is the resolvent of $\mathcal{L}$. Using this property of $\phi_l v$ and repeating the same procedure as above, we get $g\in\bigcap_{1<p<2} D_p^0$ and consequently $\phi_l v\in\bigcap_{1<p<2} D_p^2$ as was to be proved.

We next prove that $\int\langle Dv, Dv\rangle_H\rho^2 d\mu$ is finite. To this end, let $\{b_{(n)}(t)\}_{n=1}^\infty\subset C_b^\infty(R^1)$ be a sequence satisfying that
i) $b_{(n)}(t)=t$ on $-n\leqq t\leqq n$ ii) $b_{(n)}(t)-b_{(n)}(s)\leqq t-s$, $t>s$ iii) $|b_{(n)}(t)|\leqq n+1$. Then, $v_{(n)}=b_{(n)}(v)\in\mathcal{D}[\sqrt{-A}]$ by virtue of the Markovian property of Dirichlet space $\mathcal{D}[\sqrt{-A}]$. According to [4; (2.3.24)], we get

$$\begin{aligned}\mathcal{E}_A(v_{(n)}, v_{(n)})&=(\sqrt{-A}v_{(n)}, \sqrt{-A}v_{(n)})\,\rho^2\mu\\&=\lim_{\beta\to\infty}\mathcal{E}_A^{(\beta)}(v_{(n)}, v_{(n)})\\&=\lim_{\beta\to\infty}\frac{1}{2}(f_\beta, 1)\,\rho^2\mu,\end{aligned}$$

where $f_\beta=-\beta(v_{(n)}^2-\beta G_\beta^{(2)}v_{(n)}^2)+2\beta v_{(n)}(v_{(n)}-\beta G_\beta^{(2)}v_{(n)})+v_{(n)}^2(1-\beta G_\beta^{(2)}1)$. First, we see that

$$\begin{aligned}\lim_{\beta\to\infty}-\beta(v_{(n)}^2-\beta G_\beta^{(2)}v_{(n)}^2, \phi_l)\,\rho^2\mu&=\lim_{\beta\to\infty}-\beta(v_{(n)}^2, \phi_l-\beta G_\beta^{(2)}\phi_l)\,\rho^2\mu\\&=\lim_{\beta\to\infty}-\beta(v_{(n)}^2, \beta G_\beta^{(2)}\bar{S}^{(2)}\phi_l)\,\rho^2\mu\\&=(v_{(n)}^2, \bar{S}^{(2)}\phi_l)\,\rho^2\mu.\end{aligned}\tag{3.14}$$

But, since $\phi_l v_{(n)}$ belongs to D_p^2 for any l, we see that the right hand side of (3.14) is equal to $(v_{(n)}(\mathcal{L}v_{(n)}+\frac{2}{\rho}\langle D\rho, Dv_{(n)}\rangle_H)+\langle Dv_{(n)}, Dv_{(n)}\rangle_H, \phi_l)\,\rho^2\mu$. On the other hand, $\phi_l v_{(n)}\in\mathcal{D}[\bar{S}^{(p)}]$, $1<p<2$, by Lemma 3. Hence

$$\begin{aligned}\lim_{\beta\to\infty}\beta(v_{(n)}-\beta G_\beta^{(2)}v_{(n)},\ \phi_l v_{(n)})\,\rho^2\mu &= \lim_{\beta\to\infty}\beta(v_{(n)},\ \phi_l v_{(n)}-\beta G_\beta^{(p)}(\phi_l v_{(n)}))\,\rho^2\mu\\ &= \lim_{\beta\to\infty}\beta(v_{(n)},\ \phi_l v_{(n)}-\beta G_\beta^{(p)}(\phi_l v_{(n)}))\,\rho^2\mu\\ &= (v_{(n)},\ -\bar{S}^{(p)}(\phi_l v_{(n)}))\,\rho^2\mu\\ &= (-\frac{\mathcal{L}}{2}v_{(n)}-\frac{1}{\rho}\langle D\rho,\ Dv_{(n)}\rangle_H,\ \phi_l v_{(n)})\,\rho^2\mu\,.\end{aligned}$$

By noting $1\in\mathcal{D}[\sqrt{-A}]$, we see that $\beta G_\beta^{(2)}1=1$. Hence,

$$\begin{aligned}\mathcal{E}_A(v,v)&\geqq\mathcal{E}_A(v_{(n)},v_{(n)})\\ &\geqq\lim_{\beta\to\infty}\frac{1}{2}(f_\beta,\ \phi_l)\,\rho^2\mu\\ &=\frac{1}{2}\int\langle Dv_{(n)},\ Dv_{(n)}\rangle_H\,\phi_l\rho^2\,d\mu\\ &=\frac{1}{2}\int(b'_{(n)}(v))^2\langle Dv,\ Dv\rangle_H\,\phi_l\rho^2\,d\mu\,.\end{aligned}$$

therefore, we can conclude that the function v belongs to $\mathcal{H}$ by letting $l,\ n\to\infty$.

q.e.d.

REMARK. If ρ is a tame function represented as $\rho(x)=\tilde{\rho}(\langle e_1, x\rangle, \cdots, \langle e_n, x\rangle)$, $\tilde{\rho}>0$ $C^2(R^n)$, and $\int\rho^2d\mu<\infty$, we can show that S is an essentially self-adjoint operator by using Wielens' idea. In fact, let $\psi_l(t)$ be a C_b^∞-function satisfying that i) $0\leqq\psi_l(t)\leqq1$ ii) $\psi_l=\begin{cases}1 \text{ on } t\leqq l\\ 0 \text{ on } t\geqq l+1\end{cases}$ iii) $|\psi_l'(t)|,\ |\psi_l''(t)|<M$, and $\tilde{\psi}_l(r)=\psi_l(|r|)$, $r\in R^n$. Put $\phi_l(x)=\tilde{\psi}_l(\langle e_1, x\rangle, \cdots, \langle e_n, x\rangle)$ and $\mathcal{M}_l=\{x\in B;\ (\langle e_1, x\rangle, \cdots, \langle e_n, x\rangle)\in B_l(=\{r\in R^n;\ |r|<l\})\}$. Then, it holds that $\phi_l^2v\in\mathcal{D}[\bar{S}]$, $v\in\mathcal{N}_\alpha$, and that

$$(v,(\alpha-\bar{S})(\phi_l^2v))\rho^2\mu=\alpha(\phi_lv,\ \phi_lv)\,\rho^2\mu+\int\phi_lv\langle D\phi_l,\ Dv\rangle_H\,\rho^2d\mu$$
$$+\frac{1}{2}\int\phi_l^2\langle Dv,\ Dv\rangle_H\,\rho^2d\mu=0\,.$$

On the other hand, since

$$\begin{aligned}(\phi_lv,(\alpha-\bar{S})\,\phi_lv)\,\rho^2\mu&=\alpha(\phi_lv,\ \phi_lv)\,\rho^2\mu+\int\phi_lv\langle D\phi_l,\ Dv\rangle_H\,\rho^2d\mu\\ &\quad+\frac{1}{2}\int v^2\langle D\phi_l,\ D\phi_l\rangle_H\cdot\rho^2d\mu+\frac{1}{2}\int\phi_l^2\langle Dv,\ Dv\rangle_H\,\rho^2d\mu\\ &=\frac{1}{2}\int v^2\langle D\phi_l,\ D\phi_l\rangle_H\,\rho^2d\mu\,,\end{aligned}$$

we have

$$\frac{1}{2}\int v^2\langle D\phi_l,\ D\phi_l\rangle_H\,\rho^2d\mu\geqq\alpha\int\phi_l^2\,v^2\,\rho^2d\mu\,.$$

therefore, $\frac{1}{2}M^2\cdot n\int_{\mathcal{M}_l^c}\rho^2 d\mu \geqq \alpha\int_{\mathcal{M}_{l+1}} v^2\rho^2 d\mu$ and by letting $l\to\infty$, we obtain $v=0$.

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