Kikuchi, K.
Osaka J. Math.
22 (1985), 575-635

ASYMPTOTIC BEHAVIOR AT INFINITY OF THE GREEN FUNCTION OF A CLASS OF SYSTEMS INCLUDING WAVE PROPAGATION IN CRYSTALS

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(Received July 31, 1984)

0. Introduction

Many phenomena of wave propagation problems for example acoustic, electromagnetic and elastic waves, can be written in first order symmetric hyperbolic system. According to C.H. Wilcox [10] they can be represented in general as

$$E(x)D_t u - \sum_{j=1}^{n} A_j D_j u = f(t, x)\,. \tag{0.1}$$

where $t \in \boldsymbol{R}^1$ (time), $x \in \boldsymbol{R}^n$ (space), $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Here $u = (u_1(t, x), \cdots, u_m(t, x))$ is a $\boldsymbol{C}^m$-valued function which describes the state of the media at position x and time t, $E(x)$ is a positive definite hermitian matrix valued function of x, A_j's are $m \times m$ constant hermitian matrices and $f(t, x) = (f_1(t, x), \cdots, f_m(t, x))$ is a prescribed function which specifies the sources acting in the medium. If we write

$$\Lambda = E(x)^{-1} \sum_{j=1}^{n} A_j(x) D_j\,,$$

(0.1) can be written as

$$D_t u - \Lambda u = f(t, x)\,. \tag{0.1$'$}$$

When $E(x) = I$ (identity matrix) the equation (0.1)$'$ is represented as

$$D_t u - \Lambda^0 u = f(t, x)\,, \tag{0.2}$$

where

$$\Lambda^0 = \sum_{j=1}^{n} A_j D_j\,.$$

Now if we assume that f has the form

$$-f(t, x) = e^{i\lambda t} f(x) \qquad \lambda \in \boldsymbol{R}^1 \backslash \{0\}$$

and that the solution of (0.1)$'$ has the same form

$$u(t, x) = e^{i\lambda t}v(x, \lambda)\,,$$

then $v(x, \lambda)$ must satisfy

$$\Lambda v - \lambda v = f(x), \qquad x \in \boldsymbol{R}^n\,. \tag{0.3}$$

Steady-state wave propagation problem is the problem of deciding the solution of (0.3). In this paper we consider the asymptotic behavior at infinity of the Green function $G_{\pm}(x, \lambda)$ of steady-state wave propagation problem corresponding to (0.2), that is,

$$\Lambda^0 v - \lambda v = f(x), \quad x \in \boldsymbol{R}^n\,. \tag{0.4}$$

The Green function is defined as the following: for $\zeta \in \boldsymbol{C} \setminus \boldsymbol{R}$ the Green function for $\Lambda^0 - \zeta I$ is defined by inverse Fourier transformation

$$G(x, \zeta) = \mathcal{F}^{-1}[(\Lambda^0(\cdot) - \zeta I)^{-1}] \text{ in } \mathcal{S}'\,, \tag{0.5}$$

where $\Lambda^0(\xi) = \sum_{i=1}^{n} \xi_j A_j$, $\xi = (\xi_1, \cdots, \xi_n)$ (symbol of Λ^0). Then for $\lambda \in \boldsymbol{R}^1 \setminus \{0\}$ the Green function is defined by the limit

$$\lim_{\varepsilon \downarrow 0} G(x, \lambda \pm i\varepsilon) \equiv G_{\pm}(x, \lambda) \tag{0.6}$$

if it exists. Our final purpose is to show the existence of $G_{\pm}(x, \lambda)$ and give its asymptotic estimate at infinity under some suitable conditions. Remark that the Green function is a fundamental solution of $\Lambda^0 - \lambda I$:

$$(\Lambda^0 - \lambda I)G_{\pm}(x, \lambda) = \delta(x)I\,.$$

The asymptotic behavior at infinity of the Green function is useful to develop the scattering theory for the system Λ, that is, the Rellich uniqueness theorem, limiting absorption principle and eigenfunction expansion for Λ. Especially the Green function takes an important role in the proof of the Rellich uniqueness theorem for steady-state wave propagation problem (0.3) under suitable radiation condition (condition at infinity).

Properties of the Green function are much effected by the geometrical properties of the slowness surface which is defined by

$$S = \{\xi \in \boldsymbol{R}^n;\ \det(I - \Lambda^0(\xi)) = 0\}\,. \tag{0.7}$$

In this paper we assume that for some integer l

$$\operatorname{rank} \Lambda^0(\xi) = m - l \qquad \text{for any } \xi \in \boldsymbol{R}^n \setminus \{0\}\,. \tag{0.8}$$

C.H. Wilcox [12] called the system with (0.8) strongly propagative system. (0.8) is equivalent to that S is bounded. Thus there exists some constant C_S such that

$$S \subset \{\xi;\ C_S^{-1} \geq |\xi| \leqq C_S\}\ . \tag{0.9}$$

If S consists of concentric spheres concerned at origin the system is called isotropic, and if S has no algebraic singularities the system is called uniformly propagative.

The important properties of the Green function for scattering theory are the following expansion formula and the estimate of the remainder term:

$$\begin{aligned} &G(x, \lambda) \\ &= \sum_{\gamma=1}^{\rho} (2\pi)^{-(n-1)/2} e^{i\lambda x\cdot s} e^{\pm(\pi i/4)\mathrm{sign}\,\lambda} |x|^{-(n-1)/2} \\ &\quad \cdot |\lambda|^{(n-1)/2} |T(s)|^{-1} |K(s)|^{-1/2} \hat{P}(s)|_{s=s^{(\gamma)}(\pm\eta)} \\ &\quad + q_{\pm}(x, \lambda)\,, \end{aligned} \tag{0.10}$$

where

$$|q(x, \lambda)| \leqq C|x|^{-n/2} \tag{0.11}$$

for some constant C independent of $\eta = x/|x|$. Here $s^{(\gamma)}$'s are maps from S^{n-1} to S, $\hat{P}(s)$ is the projection onto the eigenspace for the eigenvalue $\lambda=1$ of $\Lambda^0(s)$ ($s \in S$), $K(s)$ is the Gaussian curvature of S at s and $T(s)$ is the polar reciprocal map whose definition will be given later. For isotropic systems the formulas (0.10) and (0.11) is given in M. Matsumura [2] and the scattering theory is developed in K. Mochizuki [5]. In the papers C.H. Wilcox [11], J.R. Schulenberger [6], J.R. Schulenberger and C.H. Wilcox [7], [8] and [9], the formulas (0.10) and (0.11) are given and the scattering theory is developed for uniformly propagative systems whose slowness surface has no parabolic points, that is, the points where the Gaussian curvature vanishes. In the proof of (0.10) and (0.11) the stationary phase method is essentially used for the integral on the slowness surface.

However there are some important systems which are not uniformly propagative, for example electromagnetic wave propagation in crystals. In this case the slowness surface consists of two sheets which intersect at four points with each other. These points are the algebraic singularities of the slowness surface. Moreover there are four circles where the Gaussian curvature vanishes. In the neighborhood of parabolic points the uniformity for η of the constant C of (0.11) cannot be expected. In the neighborhood of singularities the usual stationary phase method cannot be applied. So in this paper we give an expansion formula and an estimate of the remainder term corresponding to (0.10) and (0.11) for a class of systems including the electromagnetic wave propagation in crystals which is sufficient to develop the scattering theory (theorem 7.1). The class will be given in the next section.

By using this expansion formula and this asymptotic estimate the scattering theory can be developed in the method of J.R. Schulenberger [6], J.R. Schulenberger and C.H. Wilcox [7], [8] and [9] for some non-uniformly propagative system Λ of the form

$$\Lambda = E(x)^{-1}(\sum_{j=1}^{n} A_j(x)D_j + B(v)) \quad (x \in \Omega = \boldsymbol{R}^n \setminus \mathcal{O})$$

where $\mathcal{O}$ is compact and $\Lambda = \Lambda^0$ outside a bounded set. Details of the calculas will be given in another paper by the author.

Here we should state the difference between the results of M. Matsumura [3] and those of ours. He has considered the system whose slowness surface consists of some smooth strictly convex surfaces which may intersect with one another, and he get the decaying order of $|x|$ at infinity for each fixed η. But the wave propagation in crystals is not included in this class. Moreover the uniformity for η of the remainder term estimate is not obtained. We shall give some kind of uniformity for η for a class of systems given in the next section.

The paper is organized as follows. In section 1 the assumptions for the slowness surface are given. In section 2 the Green function is represented by the slowness surface integral. A modification of stationary phase method is considered in sections 3~6. In section 3 the case in the neighborhood of stationary point is treated. Section 4 developes the analysis of the slowness surface and the Gauss map, and gives some geometrical properties. In section 5 some estimates with respect to the slowness surface integral is given and some lemmas are prepared, and in section 6 the proof of modified stationary phase method of the slowness surface integral is concluded. The main theorem is proved in section 7.

The author would like to express his sincere gratitude to Professor M. Ikawa for his kind suggestions and constant help.

1. Assumptions for the slowness surface

In this paper some geometrical properties of the slowness surface will be assumed. First we assume that the space dimension n is odd. Under the assumption (0.8) the eigenvalues of $\Lambda^0(\xi)$ can be enumerated for $\xi \neq 0$ as follows:

$$\lambda_\rho(\xi) \geqq \cdots \geqq \lambda_1(\xi) > 0 \ (\equiv \lambda_0(\xi)) > \lambda_{-1}(\xi) \geqq \cdots \geqq \lambda_{-\rho}(\xi)$$

where ρ is independent of ξ ([12, §2]). If l of (0.8) is zero $\lambda_0(\xi)$ does not appear. We put

$$S_k = \{\xi \in \boldsymbol{R}^n ; \lambda_k(\xi) = 1\} \quad (k = 1, 2, \cdots, \rho). \tag{1.1}$$

It immediately follows that

$$S = \bigcup_{k=1}^{\rho} S_k .$$

Since $\lambda_k(\xi)$ is continuous S_k is a continuous $(n-1)$-dimensional surface. These surface may intersect with one another. Put

$$Z_S^{(1)} = \{\xi \in S;\ \xi \in S_j \cap S_k \text{ for some } j \neq k\}.$$

Then $Z_S^{(1)}$ consists of all algebraic singularities of S.

The wave surface is defined as the polar reciprocal with respect to the unit sphere of the slowness surface S ([12, §1]). This means that

$$W = \{x \in \boldsymbol{R}^n;\ \{x \cdot \xi = 1\} \text{ is a tangent plane to } S\}.$$

Let T be the polar reciprocal map from S to W. It means that for $s \in S$

$$T(s) = \frac{N(s)}{s \cdot N(s)} \in W$$

where N denotes an exterior unit normal vector to S at s (the Gauss map). Let $Z_W^{(1)}$ be the set of algebraic singularities of W, and $Z_S^{(2)}$ be the inverse image of $Z_W^{(1)}$ of T:

$$Z_S^{(2)} = T^{-1}(Z_W^{(1)})$$

Similarly

$$Z_W^{(2)} = T(Z_S^{(1)}).$$

If the Gaussian curvature $K(s)$ vanishes at $s \in S$, then $s \in Z_S^{(2)}$. (For the definition of the Gaussian curvature refer to M. Matsumura [3, §5]).

We assume following conditions on the slowness surface.

Si) $Z_S^{(1)}$ is an $(n-d)$-dimensional smooth submanifold of $\boldsymbol{R}^n$ where $d \geqq (n+1)/2$.

Sii) $Z_S^{(2)}$ is an at most $(n-2)$-dimensional smooth submanifold of $\boldsymbol{R}^n$.

Siii) The Gaussian curvature $K(s)$ satisfies

$$|K(s)| \geqq c \operatorname{dist}_S(s, Z_S^{(1)})^{-(d-1)}$$

for some constant c in a neighborhood of $Z_S^{(1)}$.

Siv) In a neighborhood of $Z_S^{(1)}$

$$\operatorname{dist}_S(s, Z_S^{(1)}) \sim \operatorname{dist}_{S^{n-1}}(N(S), N(Z_S^{(1)})).$$

Sv) On each S_k

$$|\lambda_j(s) - \lambda_k(s)| \geqq c \operatorname{dist}_S(s, Z_S^{(1)})$$

for some constant c in a neighborhood of $Z_S^{(1)} \supset S_j \cap S_k$.

Svi) $$Z_S^{(1)} \cap Z_S^{(2)} = \phi.$$

(Throughout the paper dist_X denotes the distance of a metric space X).

The slowness surface of the system of electromagnetic wave propagation in crystals is well-known as the Fresnel surface (cf. C.H. Wilcox [13]). Its figure is illustrated in P. Appel et E. Lacour [1, page 178, 186 and 187]. The surface can be parametrized with the elliptic functions ([1, page 180, (10)]). For the Fresnel surface the conditions Si)$\sim$Svi) can be checked with this parametrization and some fundamental calculas.

2. Representation of the Green function by slowness surface integral

In this section we shall give a representation of the Green function by slowness surface integral. First of all we treat orthogonal projections on eigenspaces corresponding to eigenvalues $\lambda_k(\xi)$ of $\Lambda^0(\xi)$. The orthogonal projection $\hat{P}_k f\xi)$ on eigenspace corresponding to $\lambda_k(\xi)$ is given by

$$(2.1) \qquad \hat{P}_k(\xi) = -\frac{1}{2\pi i}\int_{\gamma_k(\xi)}(\Lambda^0(\xi)-z)^{-1}dz, \quad |k| = \pi(l), \cdots, \rho$$

where $\pi(l)$ is defined by

$$(2.2) \qquad \pi(l) = \begin{cases} 1 & \text{if } l=0 \\ 0 & \text{if } l\neq 0 \end{cases} \quad (l \text{ of } (1.8))$$

and $\gamma_k(\xi)$ by

$$\gamma_k(\xi) = \{z;\ |z-\lambda_k(\xi)| = c_k(\xi)\}$$

with $c_k(\xi)$ so small that γ_k do not intersect with one another. Sets Z_S and Z are defined by $Z_S=Z_S^{(1)}\cup Z_S^{(2)}$ and

$$(2.3) \qquad Z = \{\xi = rs;\ r>0 \text{ and } s\in Z_S\}\ .$$

When $\xi\in\boldsymbol{R}^n\backslash Z$ we can take such $c_k(\xi)$'s since $\lambda_k(\xi)$ are distinct. Similarly Z_W and $\bar{Z}$ are defined by $Z_W=Z_W^{(1)}\cup Z_W^{(2)}$ and

$$(2.4) \qquad \bar{Z} = \{x = rw;\ r>0 \text{ and } w\in Z_W\}\ .$$

Z_S, Z, Z_W and $\bar{Z}$ are all closed null sets. Note that T is bijective and diffeomorphic from $S\backslash Z_S$ to $W\backslash Z_W$.

Now eigenvalues $\lambda_k(\xi)$ and the orthogonal projections $\hat{P}_k(\xi)$ ($|k|=\pi(l)$, 1, $\cdots$, ρ) have following properties [12]:

(2.5) $\lambda_k(\xi)$ is continuous on $\boldsymbol{R}^n\backslash\{0\}$ and real analytic on $\boldsymbol{R}^n\backslash Z$

(2.6) $\lambda_k(\alpha\xi) = \alpha\lambda_k(\xi)$ for all $\alpha>0$, $\xi\in\boldsymbol{R}^n$ and $|k| = 1, \cdots, \rho$

(2.7) $\lambda_k(-\xi) = -\lambda_{-k}(\xi)$ for all $\xi\in\boldsymbol{R}^n$ and $|k| = 1, \cdots, \rho$

and

$$\hat{P}_k(\xi) \text{ is real analytic on } \boldsymbol{R}^n\backslash Z \text{ for } |k|=\pi(l), 1, \cdots, \rho \tag{2.8}$$

$$\hat{P}_k(\alpha\xi) = \hat{P}_k(\xi) \text{ for all } \alpha>0,\ \xi\in\boldsymbol{R}^n\backslash Z \text{ and } |k| = 1, \cdots, \rho \tag{2.9}$$

$$\hat{P}_k(-\xi) = \hat{P}_{-k}(\xi) \text{ for all } \xi\in\boldsymbol{R}^n\backslash Z \text{ and } |k| = \pi(l), 1, \cdots, \rho \tag{2.10}$$

$$\sum_{|k|=\pi(l)}^{\rho} \hat{P}_l P/(\xi) = I \qquad \text{for all } \xi\in\boldsymbol{R}^n\backslash Z \tag{2.11}$$

$$\Lambda^0(\xi)\hat{P}_k(\xi) = \hat{P}_k(\xi)\Lambda^0(\xi) = \lambda_k(\xi)\hat{P}_k(\xi) \tag{2.12}$$

for all $\xi\in\boldsymbol{R}^n\backslash Z$ and $|k|=\pi(l), \cdots, \rho$.

(2.11) and (2.12) imply

$$\Lambda^0(\xi) = \sum_{|k|=1}^{\rho} \lambda_k(\xi)\hat{P}_k(\xi) \qquad \text{for all } \xi\in\boldsymbol{R}^n\backslash Z\,, \tag{2.13}$$

and then we have

$$(\Lambda^0(\xi)-\zeta I)^{-1} = \sum_{|k|=\pi(l)}^{\rho} \frac{\hat{P}_k(\xi)}{\lambda_k(\xi)-\zeta} \tag{2.14}$$

for all $\xi\in\boldsymbol{R}^n\backslash Z$ and $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.

Note that $\hat{P}_0(\xi)$ is real analytic not only for $\xi\in\boldsymbol{R}^n\backslash Z$ but for $\xi\in\boldsymbol{R}^n\backslash\{0\}$. Let a and b be positive numbers with $a<b$. Then $\lambda_1(\xi)$ attains minimum value c_0 if ξ is in the bounded domain $\{a\leqq|\xi|\leqq b\}$ because of its continuity. Then if we define $c_0(\xi)$ with $c_0>c_0(\xi)$ for $\xi\in\{a\leqq|\xi|\leqq b\}$, $\gamma_0(\xi)$ never intersect with another circle $\gamma_k(\xi)$ and $\hat{P}_0(\xi)$ is real analytic in this domain. Since a and b can be chosen arbitrally, $\hat{P}_0(\xi)$ is real analytic for $\xi\in\boldsymbol{R}^n\backslash\{0\}$.

Next we consider the Green function $G(x, \zeta)$ for $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$. From (0.5)

$$G(x, \zeta) = \mathcal{F}^{-1}[(\Lambda^0(\cdot)-\zeta I)^{-1}]\,.$$

Let a, b and ε_0 be given constants with $[a, b]\subset\boldsymbol{R}^1\backslash\{0\}$ and $\varepsilon_0>0$. Here we suppose $\zeta=\lambda\pm i\varepsilon$ where $\lambda\in[a, b]$ and $0<\varepsilon\leqq\varepsilon_0$. In proving the existence of the limit $G_\pm(x, \lambda)$ of (0.6), there are difficulties if some eigenvalue $\lambda_k(\xi)$ of $\Lambda^0(\xi)$ is equal to λ. From (2.6), (2.7) and the definition of S_k, $\lambda_k(\xi)=\lambda$ is equivalent to $\xi=\lambda s$ for $s\in S_k$. So it follows from (0.9) that

$$\begin{cases} a\ C_S^{-1} \leqq |\xi| \leqq |\lambda|\cdot|s| \leqq\ b\ \cdot C_S & (\text{if } 0<a<b) \\ |b|\cdot C_S^{-1} \leqq |\xi| \leqq |\lambda|\cdot|s| \leqq |a|\cdot C_S & (\text{if } a<b<0)\,, \end{cases}$$

where C_S is of (0.9). Put $\phi_1(\xi)\in C_0^\infty(\boldsymbol{R}^n)$ as

$$\phi_1(\xi) = \begin{cases} 1 & \text{if } \ a\cdot C_S^{-1} \leqq |\xi| \leqq b\cdot C_S \\ 0 & \text{if } \ |\xi| \leqq a\cdot C_S^{-1}/2 \ \text{ or } \ b\cdot C_S+1 \leqq |\xi| \end{cases} \tag{2.15}$$

when $0<a<b$. When $a<b<0$, we replace a and b with $|b|$ and $|a|$ respectively in (2.15). And put $\phi_2(\xi)=1-\phi_1(\xi)$. Then

$$
\begin{aligned}
(2.16) \qquad G(x, \zeta) &= \mathcal{F}^{-1}[(\Lambda^0(\cdot)-\zeta I)^{-1}\phi_1(\cdot)] \\
&\quad +\mathcal{F}^{-1}[(\Lambda^0(\cdot)-\zeta I)^{-1}\phi_2(\cdot)] \\
&\equiv G_1(x, \zeta)+G_2(x, \zeta).
\end{aligned}
$$

To begin with we consider $G_2(x, \zeta)$. By the properties of inverse Fourier transformation

$$x^{\alpha}G_2(x, \zeta) = \mathcal{F}^{-1}[(-D_{\xi})^{\alpha}\{(\Lambda^0(\cdot)-\zeta I)^{-1}\phi_2(\cdot)\}]$$

holds for any multi-index $\alpha=(\alpha_1 \cdots \alpha_n)$ where $x^{\alpha}=x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $D_{\xi}^{\alpha}=D_{\xi_1}^{\alpha_1} \cdots D_{\xi_n}^{\alpha_n}$. Since $\lambda_k(\xi)$ is never equal to λ on the support of $\phi_2(\xi)$. and since $\phi_2(\xi) \equiv 1$ if $|\xi|$ is sufficiently large, we have

$$|(-D_{\xi})^{\alpha}\{(\Lambda^0(\xi)-\zeta I)^{-1}\phi_2(\xi)\}| \leqq C_{\alpha}\langle\xi\rangle^{-|\alpha|},$$

where $\langle\xi\rangle$ denotes $(1+|\xi|^2)^{1/2}$ and the constant C_{α} is uniform in

$$\Delta \equiv \{\zeta = \lambda \pm i\varepsilon;\ \lambda \in [a, b],\ \varepsilon \in (0, \varepsilon_0]\}$$

with respect to ζ. Since $\langle\xi\rangle^{-|\alpha|}$ is integrable on $\boldsymbol{R}^n$ for $|\alpha|>n+1$, we have for $l>n+1$ and $x \neq 0$

$$(2.17) \qquad |G_2(x, \zeta)| \leqq C_l|x|^{-l},$$

where C_l is uniform for $\eta=\dfrac{x}{|x|}$ and $\zeta \in \Delta$.

Next we consider $G_1(x, \zeta)$. Since $(\Lambda^0(\xi)-\zeta I)^{-1}\phi_1(\xi)$ has compact support on $\boldsymbol{R}^n$, $G_1(x, \zeta)$ can be regarded as the inverse Fourier transformation of function of L^1. Then by (2.14) we have

$$
\begin{aligned}
(2.18) \qquad G_1(x, \zeta) &= \int_{\boldsymbol{R}^n} e^{ix\xi}(\Lambda^0(\xi)-\zeta I)^{-1}\phi_1(\xi)\,d\!\!\bar{}\,\xi \\
&= \int_{\boldsymbol{R}^n\setminus Z} e^{ix\xi} \sum_{|k|=\pi(l)}^{p} \frac{\hat{P}_k(\xi)}{\lambda_k(\xi)-\zeta}\cdot\phi_1(\xi)\,d\!\!\bar{}\,\xi \\
&= \sum_{|k|=\pi(l)}^{p} \int_{\boldsymbol{R}^n\setminus Z} e^{ix\xi}\frac{\hat{P}_k(\xi)}{\lambda_k(\xi)-\zeta}\cdot\phi_1(\xi)\,d\!\!\bar{}\,\xi \\
&\equiv \sum_{|k|=\pi(l)}^{p} G_{1,k}(x, \zeta)
\end{aligned}
$$

where $d\!\!\bar{}\,\xi=(2\pi)^{-n/2}d\xi$. (Note that Z is a closed null set.) In the case of $k>0$ we make the change of variables ξ to (r, s) for $r>0$ and $s \in S_k\setminus Z$ by $\xi=rs$. Then the n-dimensional volume element of $\boldsymbol{R}^n\setminus Z$ can be written

$$d\xi = r^{n-1}|\nabla\lambda_k(\xi)|^{-1}dr\,dS$$

where $đS=(2\pi)^{-(n-1)}dS$ is the $(n-1)$-dimensional volume element of S_k as a submanifold of $\boldsymbol{R}^n$ and $đr=(2\pi)^{-1}dr$. This change of variable gives

$$G_{1,k}(x,\zeta)=\int_0^\infty\int_{S_k}e^{irx\cdot s}\frac{\hat{P}_k(rs)}{\lambda_k(rs)-\zeta}\cdot\phi_1(rs)r^{n-1}|\nabla\lambda_k(s)|^{-1}đSđr$$

$\nabla\lambda_k(s)$ is normal to S_k at $s\in S_k\backslash Z_S^{(1)}$. The homogeneity of $\lambda_k(\xi)$ gives $\xi\cdot\nabla\lambda_k(\xi)=1$. These facts imply

$$T(s)=(s\cdot N(s))^{-1}N(s)=(s\cdot\nabla\lambda_k(s))^{-1}\nabla\lambda_k(s)=\nabla\lambda_k(s)\,. \tag{2.19}$$

Then from (2.6), (2.9) and (2.19)

$$G_{1,k}(x,\zeta)=\int_0^\infty\frac{r^{n-1}}{r-\zeta}\int_{S_k}e^{irx\cdot s}\hat{P}_k(s)|T(s)|^{-1}\phi_1(rs)đS_kđr \tag{2.20}$$

follows. In the case of $-k$ the change of variables ξ to $-\xi$ gives

$$G_{1,-k}(x,\zeta)=\int_{R^n\backslash Z}e^{-ix\xi}\frac{\hat{P}_{-k}(-\xi)}{\lambda_{-k}(-\xi)-\zeta}\cdot\phi_1(-\xi)đ\xi\,.$$

Recall (2.7) and (2.10). Then the coordinates based on S_k gives

$$G_{1,-k}(x,\zeta)=\int_0^\infty\frac{r^{n-1}}{-r-\zeta}\int_{S_k}e^{-irx\cdot s}\hat{P}_k(s)\cdot\phi_1(-rs)|T(s)|^{-1}đS_kđr\,.$$

and the change of variable r to $-r$ gives

$$G_{1,-k}(x,\zeta)=\int_{-\infty}^0\frac{|r|^{n-1}}{r-\zeta}\int_{S_k}e^{irx\cdot s}\hat{P}_k(s)\phi_1(rs)|T(s)|^{-1}đS_kđr\,. \tag{2.21}$$

Combining (2.20) and (2.21) we have

$$\begin{aligned}&G_{1,k}(x,\zeta)+G_{1,-k}(x,\zeta)\\&=\int_{-\infty}^\infty\frac{|r|^{n-1}}{r-\zeta}\Big(\int_{S_k}e^{ir_x\cdot s}\hat{P}_k(s)\phi_1(rs)|T(s)|^{-1}đS_k\Big)đr\,.\end{aligned} \tag{2.22}$$

In the case of $k=0$ we write

$$G_{1,0}(x,\zeta)=-\frac{1}{\zeta}\int_{R^n}e^{ix\xi}\hat{P}_0(\xi)\phi_1(\xi)đ\xi\,.$$

Since $\hat{P}_0(\xi)$ is homogeneous of degree 0 and analytic in $\boldsymbol{R}^n\backslash\{0\}$, it follows from Theorem 2.16 of [4, page 116] that

$$\mathcal{F}^{-1}[\hat{P}_0]=\int_{R^n}e^{ix\xi}\hat{P}_0(\xi)đ\xi=\mu_0+\text{p.v.}\frac{\Gamma_0(\eta)}{|x|^n}\,,$$

where $\mu_0=\dfrac{1}{\Omega_n}\displaystyle\int_{S^{n-1}}\hat{P}_0(\omega)d\omega$ (Ω_n is the surface area of S^{n-1}), $\eta=\dfrac{x}{|x|}$ and

$\int_{S^{n-1}}\Gamma_0(\eta)d\eta=0$. Hence $G_{1,0}(x, \zeta)=-\frac{1}{\zeta}(\mathcal{F}^{-1}[\hat{P}_0]*\mathcal{F}^{-1}\phi_1)(x)$ gives

$$G_{1,0}(x, \zeta)$$
$$= -\frac{1}{\zeta}\{\mu_0(\mathcal{F}^{-1}\phi_1)(x)+\text{p.v.}\int_{R^n}\frac{\Gamma_0\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n}(\mathcal{F}^{-1}\phi_1)(y)\,dy$$

Since $(\mathcal{F}^{-1}\phi_1)(x)\in\mathcal{S}_x$ we have

$$|G_{1,0}(x, \zeta)|\leqq C_0|x|^{-n}, \tag{2.23}$$

where C_0 is uniform for $\eta=\frac{x}{|x|}$ and $\zeta\in\Delta$.

(2.16) and (2.18) give

$$\begin{aligned}&\sum_{k=1}^{p}(G_{1,k}(x, \zeta)+G_{1,-k}(x, \zeta))\\&=\sum_{k=1}^{p}\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}\left(\int_{S_k}e^{irx\cdot s}\hat{P}_k(s)\phi_1(rs)|T(s)|^{-1}dS_k\right)dr\\&=\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}\left(\int_{S}e^{irx\cdot s}\hat{P}(s)\phi_1(rs)|T(s)|^{-1}dS\right)dr\end{aligned} \tag{2.24}$$

where $\hat{P}(s)$ is the function on $S\backslash Z_S^{(1)}$ which satisfies $\hat{P}(s)=\hat{P}_k(s)$ if $s\in S_k$. Then by putting

$$G_0(x, \zeta) = G_{1,0}(x, \zeta)+G_2(x, \zeta),$$

we have from (2.24) a representation of $G(x, \zeta)$

$$\begin{aligned}&G(x, \zeta)\\&= G_0(x, \zeta)+\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}\int_S e^{irx\cdot s}\hat{P}(s)|T(s)|^{-1}\phi_1(rs)\,dS\,dr\\&= G_0(x, \zeta)+\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}v(rx, r)dr,\end{aligned} \tag{2.25}$$

where

$$v(x, r) = \int_S e^{ix\cdot s}\hat{P}(s)\phi_1(rs)|T(s)|^{-1}dS. \tag{2.26}$$

(2.17) and (2.23) give

$$|G_0(x, \zeta)|\leqq C|x|^{-n}$$

where C is uniform for $\eta=\frac{x}{|x|}$ and $\zeta\in\Delta$. Then it is easy to see that the limit $G_{0,\pm}(x, \lambda)=\lim_{\varepsilon\downarrow 0}G_0(x, \lambda\pm i\varepsilon)$ exists and satisfies

$$|G_{0,\pm}(x, \lambda)|\leqq C|x|^{-n}$$

uniformly in η and ζ. Moreover it is also easy to see that the convergence is uniform with respect to $\lambda\in[a, b]$.

Here we state a theorem which is connected with the asymptotic behavior at infinity of $v(x, r)$. The following four sections are devoted to prove this theorem.

Theorem 2.1. *Let the space dimension n be odd and let* rank $\Lambda^0(\xi)$ *be constant if $\xi\neq 0$. Let l be a constant satisfying*

$$|K(s)|\geqq c\cdot\operatorname{dist}(s, Z_S^{(2)})^l \tag{2.27}$$

for some constant c in the neighbourhood of $Z_S^{(2)}$, whose existence is assured by the analyticity of $K(s)$. Let $\rho(\eta)$ denote the number of points $s\in S$ with $N(s)=\eta$ and let $\{s^{(\gamma)}(\eta);\ \gamma=1, \cdots, \rho(\eta)\}$ denote the set of these points. $p^+(s)$ (resp. $p^-(s)$) *denotes the number of positive* (resp. *negative*) *principal curvatures of S at $s\in S$ and*

$$\psi_\pm(s)=\exp\{\pm\frac{\pi i}{4}(p^+(s)-p^-(s))\}\ .$$

Then under assumptions Si)$\sim$Svi)

$$v(x, r)=\int_S e^{ixs}\hat{P}(s)|T(s)|^{-1}\phi_1(rs)dS$$

can be represented for $\eta\in S^{n-1}\backslash\bar{Z}$ as follows:

$$\begin{aligned}(2.28)\qquad & v(x, r)\\ &=\sum_{\gamma=1}^{\rho(\eta)}(2\pi)^{-(n-1)/2}|x|^{-(n-1)/2}e^{ixs}\psi_+(s)\\ &\qquad\cdot\hat{P}(s)|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(\eta)}\\ &+\sum_{\gamma=1}^{\rho(-\eta)}(2\pi)^{-(n-1)/2}|x|^{-(n-1)/2}e^{ixs}\psi_-(s)\\ &\qquad\cdot\hat{P}(s)|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(-\eta)}\\ &+q(x, r)\end{aligned}$$

where $q(x, r)$ has a compact support with respect to r and satisfies that for any p with $1\leqq p<1+\dfrac{1}{l}$ there exists a positive number $\nu=\nu_p$ such that

$$|q(x, r)|\leq C(\eta)|x|^{-(n-1)/2-\nu} \tag{2.29}$$

and

$$C(\eta)\in L^p(S^{n-1})\,. \tag{2.30}$$

3. Modification of the stationary phase method (1)

Recall a proposition, which is called a stationary phase method.

Proposition 3.1. *Let $h\in C^\infty(U)$ and $g\in C_0^\infty(U)$ where U is a domain of $\boldsymbol{R}^n$.*

1) *If h has no stationary points (the point where $\nabla h=0$) on supp g, then we get*

$$\int_U e^{ith(\sigma)}g(\sigma)d\sigma = O(t^{-\infty}) \quad as\ |t| \to \infty\,.$$

2) *Let h have stationary points on supp g and assume that these points are all non-degenerate. The set of non-degenerate stationary points on supp g is finite in number and we denote these points by $a^{(1)}, \cdots, a^{(\rho)}$. Then we get*

$$\begin{aligned}\int_U e^{ith(\sigma)}g(\sigma)d\sigma = (2\pi)^{N/2}\sum_{\gamma=1}^{\rho} e^{ith\,(a^{(\gamma)})+\frac{\pi i}{4}\mathrm{sign}H(a^{(\gamma)})}\\ \cdot|\mathrm{Hess}\ h(a^{(\gamma)})|^{-1/2}g(a^{(\gamma)})t^{-N/2}\\ +q(t)\,,\end{aligned}$$

where

$$|q(t)|\leqq Ct^{-N/2-1}\,.$$

Proposition 3.1 is applied to prove

Proposition 3.2. *Let S be a C^∞ hypersurface in $\boldsymbol{R}^n$, m a function defined on $S\times\boldsymbol{R}$ with compact support and defined*

$$I(x, r) = \int_S e^{ix\cdot s}m(s, r)dS,\ x\in\boldsymbol{R}^n,\ r\in\boldsymbol{R}\,,$$

where dS is the surface element on S. Assume that the Gaussian curvature $K(s)$ of S does not vanish on support of m. Then the set of points on support of m at which the normal to S is parallel to η is finite in number for each unit vector $\eta\in S^{n-1}$. We denote these points by $s^{(1)}, \cdots, s^{(\rho(\eta))}(\eta)$. Denote by $p^+(s)$ (resp. *$p^-(s)$*) *the number of positive* (resp. *negative*) *principal curvatures at $s\in S$ and*

$$\psi_\pm(s) = \exp\{\pm\frac{\pi i}{4}(p^+(s)-p^-(s))\}\,.$$

Then the asymptotic behavior of $I(x, r)$ at infinity along the ray $\eta=\frac{x}{|x|}$ is given by

$$\begin{aligned}&I(x, r)\\ &= (2\pi)^{(n-1)/2}|x|^{-(n-1)/2}\\ &\quad\cdot\sum_{\gamma=1}^{\rho(\eta)} e^{ix\cdot s}m|(s, r)|K(s)|^{-1/2}\psi_+(s)|_{s=s^{(\gamma)}(\eta)}\\ &+(2\pi)^{(n-1)/2}|x|^{-(n-1)/2}\\ &\quad\cdot\sum_{\gamma=1}^{\rho(-\eta)} e^{ix\cdot s}m(s, r)|K(s)|^{-1/2}\psi_-(s)|_{s=s^{(\gamma)}(-\eta)}\\ &+q(x, r)\,,\end{aligned}$$

where for each non-negative integer l

$$\left|\frac{\partial^l}{\partial|x|^l}q(x, r)\right| \leqq C_l|x|^{-(n+1)/2}$$

with uniform constant C_l for $\eta\in S^{n-1}$ and $r\in\boldsymbol{R}$.

For the proof of the Proposition 3.1 and Proposition 3.2, refer to M. Matsumura [3, §4 and §5].

If the slowness surface S of Λ^0 has no singularities (so-called uniformly propagative) and no parabolic points, that is, the points where $K(s)$ vanishes, we can apply Proposition 3.2 to $v(x, r)$ of (2.26). Then it can be proved that the remainder term of the Green function has the estimate (0.11) ([7]).

But under assumptions Si)$\sim$Svi) S may have some singularities and some parabolic points. Thus we have to modify the stationary phase method to be suitable for our case.

In this section we prove only the following two propositions, the one is related to the case that there are no stationary points in the neighborhood of singularities and the other to the case that the integrand has its support only in a sufficiently small neighborhood of a stationary point.

Proposition 3.3. *Let U be an open domain of $\boldsymbol{R}^N$, M be a $(N-d)$-dimensional submanifold of U where $d\geqq 2$. $h(\sigma)$ and $g(\sigma)$ are given functions with the following properties:*

$$h(\sigma)\in C^\infty(U\backslash M),\ \text{real valued},\ |\nabla h|\geqq c>0\ \text{for any}\ \sigma\in U\backslash M\,, \tag{3.1}$$

$$g(\sigma)\in C^\infty(U\backslash M),\ \text{have compact support in}\ U\,, \tag{3.2}$$

$$|\partial^\alpha g(\sigma)|\leqq C_\alpha \mathrm{dist}_U(\sigma, M)^{-|\alpha|}\ \text{for any}\ \alpha\ (|\alpha|\geqq 0)\,, \tag{3.3}$$

$$|\partial^\alpha h(\sigma)|\leqq C_\alpha \mathrm{dist}_U(\sigma, M)^{-|\alpha|+1}\ \text{for any}\ \alpha\ (|\alpha|\geqq 1) \tag{3.4}$$

for some constants C_α. Then we have

$$\left|\int_U e^{ith(\sigma)}g(\sigma)d\sigma\right|\leqq Ct^{-(d-1)-\nu} \tag{3.5}$$

for some positive number ν and positive constant C.

Proof. Let δ be a positive number. Put

$$U_\delta=\{\sigma\in U;\ \mathrm{dist}_U(\sigma, M)\leqq\delta\}\ . \tag{3.6}$$

Then it is clear that

$$|\partial U_\delta|\sim\delta^{d-1} \tag{3.7}$$

where $|\cdot|$ means the area. Then we calculate as follows:

$$(3.8)\qquad \int_U e^{ith(\sigma)}g(\sigma)d\sigma = \lim_{\delta\downarrow 0}\int_{U\setminus U_\delta} e^{ith(\sigma)}g(\sigma)d\sigma$$

$$= \lim_{\delta\downarrow 0}\int_{U\setminus U_\delta}\frac{1}{it|\nabla h(\sigma)|^2}(\nabla h(\sigma)\cdot\nabla e^{ith(\sigma)})g(\sigma)d\sigma$$

$$= \lim_{\delta\downarrow 0}\{\frac{1}{it}\int_{\partial U_\delta} e^{ith(\sigma)}\frac{\nabla h\cdot n}{|\nabla h|^2}g(\sigma)d\sigma$$

$$-\frac{1}{it}\int_{U\setminus U_\delta} e^{ith(\sigma)}\nabla(\frac{\nabla h}{|\nabla h|^2}g)d\sigma\}$$

$$= \lim_{\delta\downarrow 0}\{\sum_{j=1}^{d-1}(\frac{1}{it})^j\int_{\partial U_\delta} e^{ith(\sigma)}\frac{\nabla h\cdot n}{|\nabla h|^2}(\nabla\cdot\frac{\nabla h}{|\nabla h|^2})^{j-1}g(\sigma)dS$$

$$-(-\frac{1}{it})^{d-1}\int_{U-U_\delta} e^{ith(\sigma)}(\nabla\cdot\frac{\nabla h}{|\nabla h|^2})^{d-1}g(\sigma)d\sigma\}$$

$$\equiv \lim_{\delta\downarrow 0}\{\sum_{j=1}^{d-1}J_{1j}-J_2\}\,.$$

It follows from (3.3), (3.4) and (3.6) that

$$(\text{The integrand of } J_{1j}) = O(\delta^{-(d-2)})\,.$$

Thus (3.7) implies

$$(3.9)\qquad |\sum_{j=1}^{d-1}J_{1j}| \leqq \sum_{j=1}^{d-1}\frac{1}{t^j}\int_{\partial U_\delta}\delta^{-(d-2)}dS = C\sum_{j=1}^{d-1}\frac{1}{t^j}\,\delta^{d-1}\cdot\delta^{-(d-2)}$$

$$= C(\sum_{j=1}^{d-1}t^{-j}) \to 0 \quad\text{as}\quad \delta\to 0$$

with suitable constant C. From (3.3) and (3.4)

$$(\text{The integrand of } J_2) = O(\mathrm{dist}_U(\sigma, M)^{-(d-1)})$$

also follows, and thus it is integrable on U. Then we have

$$(3.10)\qquad \int_U e^{ith(\sigma)}g(\sigma)d\sigma = (-\frac{1}{it})^{d-1}\int_U e^{ith(\sigma)}(\nabla\cdot\frac{\nabla h}{|h\nabla|^2})^{d-1}g(\sigma)d\sigma$$

by making $\delta\to 0$ in (3.8).

Next we consider

$$I \equiv \int_U e^{ith(\sigma)}(\nabla\cdot\frac{\nabla h}{|\nabla h|^2})^{d-1}g(\sigma)d\sigma\,.$$

Take sufficiently large t and put $\varepsilon=t^{-\nu}$. Here we put $\rho\in C^\infty(\boldsymbol{R})$ as

$$\rho(r) = \begin{cases} 1 & |r|\geqq 1 \\ 0 & |r|\leqq 1/2 \end{cases}$$

and put

$$\rho_\varepsilon(\sigma) = \rho(\frac{1}{\varepsilon}\,\mathrm{dist}_U(\sigma, M))\,.$$

Then it follows from (3.3) and (3.4) that

$$|\partial^\alpha \rho_\varepsilon(\sigma)| \leqq C_\alpha \varepsilon^{-|\alpha|},\ |\partial^\alpha g(\sigma)| \leqq C_\alpha \varepsilon^{-|\alpha|},\ |\partial^\alpha h(\sigma)| \leqq C_\alpha \varepsilon^{-|\alpha|+1}$$

on supp $\rho_\varepsilon(\sigma)$. Here we devide I into two parts:

$$I = \int_U e^{ith(\sigma)} (\nabla \cdot \frac{\nabla h}{|\nabla h|^2})^{d-1} \{g(\sigma)\rho_\varepsilon(\sigma)\} d\sigma \tag{3.11}$$

$$+ \int_U e^{ith(\sigma)} (\nabla \cdot \frac{\nabla h}{|\nabla h|^2})^{d-1} \{g(\sigma)\,(1-\rho_\varepsilon(\sigma))\} d\sigma$$

$$\equiv I_1 + I_2\,.$$

For I_1 one more integration by parts gives

$$I_1 = -\frac{1}{it}\int_U e^{ith(\sigma)} \nabla(\frac{\nabla h}{|\nabla h|^2}[(\nabla \cdot \frac{\nabla h}{|\nabla h|^2})^{d-1} g(\sigma)]\rho_\varepsilon(\sigma)) d\sigma$$

and thus

$$|I_1| \leqq C\delta^{-d} t^{-1} = Ct^{-1+d\nu} \quad \text{for some } C\,. \tag{3.12}$$

For I_2 we estimate it with (3.11)

$$|I_2| \leqq C \int_{\mathrm{dist}_U(\sigma, M)} \mathrm{dist}_U(\sigma, M)^{-(d-1)} d\sigma \tag{3.13}$$

$$= C \int_0^\infty r^{-(d-1)} r^{d-1} dr = C\varepsilon$$

$$= Ct^{-\nu}$$

By (3.12) and (3.13) and by putting $\nu = 1/(d+1)$ we have

$$|I| = |I_1 + I_2| \leqq Ct^{-\nu} \quad \text{for some } C\,.$$

Thus (3.5) follows from (3.10) and the above fact. Q.E.D.

In the next proposition we consider an oscillating integral with a parameter and take out a relationship between principal part and the distance from stationary points to singularities of phase functions.

Proposition 3.4. *Let U be an open domain of $\boldsymbol{R}^N$ and $\{h_a\}_{a\in U}$ and $\{g_a\}_{a\in U}$ be families of functions defined on U with the following properties*:

i) *Let* $\delta = \mathrm{dist}(a, \partial U)$, $h_a(\sigma) \in C^\infty(U)$, *real valued and*

$$|\partial^\alpha h_a(\sigma)| \leqq C_\alpha \delta^{-|\alpha|+1} \tag{3.14}$$

for some constant C_α independent of a.

ii) *For any* $a\in U$ $\nabla h_a(a)=0$ *and* $\nabla h_a(\sigma)\neq 0$ *if* $\sigma\neq a$. *Moreover*

$$|\text{Hess } h_a(a)|>c_0\delta^l,\ a\in U \tag{3.15}$$

for some positive constant c_0 *and some positive integer* l *independent of* a.

iii) $g_a(\sigma)\in C_0^\infty(U)$ *and*

$$|\partial^\alpha g_a(\sigma)|\leqq \tilde{C}_\alpha\delta^{-|\alpha|(N+1+l)} \tag{3.16}$$

for some constants C_α *independent of* a.

iv)

$$\operatorname{supp} g_a\subset\{|\sigma-a|<\bar{c}\delta^{N+1+l}\}\ , \tag{3.17}$$

where $\bar{c}$ *is a constant depending only on* N *and* $\{h_a\}$.

Then we have

$$\begin{aligned}\int_U e^{ith_a(\sigma)}g_a(\sigma)d\sigma \\ = (2\pi)^{N/2}e^{ith_a(a)+\frac{\pi i}{4}\operatorname{sign} H_a(a)}|\text{Hess } h_a(a)|^{-1/2}\cdot g_a(a)t^{-N/2}+q_a(t)\end{aligned} \tag{3.18}$$

where $H_a(a)=(\frac{\partial^2 h_a}{\partial\sigma_j\partial\sigma_k}(a))$. *Here* $q_a(t)$ *satisfies that for any positive number* μ *there exist* ν (>0) *such that*

$$|q_a(t)|\leqq C\delta^{-\mu}t^{-N/2-\nu}\cdot|\text{Hess } h_a(a)|^{-1/2} \tag{3.19}$$

for some constant independent of a.

For the proof of Proposition 3.4 we prepare two lemmas related to the phase functions. We write

$$\begin{aligned}h_a(\sigma)-h_a(a) &= \sum_{j,k=1}^N \alpha_{jk}(\alpha_j-a_j)(\alpha_k-a_k) \\ &= \frac{1}{2}\langle H_a(\sigma)(\sigma-a),\sigma-a\rangle\,,\end{aligned} \tag{3.20}$$

where $\langle\ ,\ \rangle$ denotes the inner product of $\boldsymbol{R}^N$ and $H_a(\sigma)=(2\alpha_{jk}(\sigma))$,

$$\alpha_{jk}(\alpha)=\int_0^1(1-\rho)\frac{\partial^2 h_a}{\partial\sigma_j\partial\sigma_k}(a+(\sigma-a)\rho)d\rho\,. \tag{3.21}$$

Then $H_a(\sigma)$ is a real symmetric matrix valued C^∞ function on U. Since

$$\alpha_{jk}(a)=\frac{1}{2}\frac{\partial^2 h_a}{\partial\sigma_j\partial\sigma_k}(a)\,,$$

$H_a(a)=(\frac{\partial^2 h_a}{\partial\sigma_j\partial\sigma_k}(a))$ agrees with the definition above. Put

$$K_a(\sigma)=H_a(\sigma)^{-1}H_a(a)\,. \tag{3.22}$$

Now we shall make a transformation of variables $\sigma \mapsto \Xi$ by

$$\sigma - a = K_a(\sigma)^{1/2}\Xi \,. \tag{3.23}$$

Then the following facts related to this transformation holds.

Lemma 3.5. 1) $c_+^{-1}|\sigma - a| \leqq |\Xi| \leqq c_-|\sigma - a|$.

2) $h_a(\sigma) - h_a(a) = (1/2)\langle H_a(a)\Xi, \Xi\rangle$.

3) *The map* $\sigma \mapsto \Xi$ *is a diffeomorphism from* $\{|\sigma - a| \leqq \bar{c}\delta^{N+1+l}\}$ *to a domain* $\{|\Xi| \leqq c_-\bar{c}\delta^{N+1+l}\}$.

Here $\bar{c}$ *is the same constant of* (3.17). c_+ *and* c_- *are some constants depending only on* N *and* $\{h_a\}$.

Proof. 1) We put

$$E_a(\sigma) = K_a(\sigma) - I = H_a(\sigma)^{-1}(H_a(a) - H_a(\sigma)) \,. \tag{3.24}$$

By (3.14) and (3.21) we have

$$\begin{aligned} &||H_a(a) - H_a(\sigma)|| \\ &= ||\int_0^1 (1-\rho)\, [\frac{\partial^2 h_a}{\partial\sigma_j \partial\sigma_k}(a + (\sigma - a)\, \rho) - \frac{\partial^2 h_a}{\partial\sigma_j \partial\sigma_k}(a)]_{j,k} d\rho|| \\ &= ||\int_0^1 (1-\rho)\, [\int_0^1 \nabla(\frac{\partial^2 h_a}{\partial\sigma_j \partial\sigma_k}(a + (\sigma - a)\rho\theta) d\theta \cdot (\sigma - a)\rho]_{j,k} d\rho|| \\ &\leqq C_3 \delta^{-2} |\rho - a| \,, \end{aligned} \tag{3.25}$$

where $C_3 = \sqrt{N} \max_{|\alpha|=3} C_\alpha$ and $||A|| = \sqrt{\sum_{j,k} |a_{jk}|^2}$. On the other hand (3.14), (3.21) and the definition of inverse matrix give

$$||H_a(\sigma)^{-1}|| \leqq |\det H_a(\sigma)|\, C_2^{N-1} \cdot \delta^{-(N-1)} \,, \tag{3.26}$$

where $C_2 = \sqrt{N} \max_{|\alpha|=2} C_\alpha$. Since $\det X$ for $X = (x_{jk})$ is a polynomial of N^2 variables, it is Lipschitz continuous on the bounded domain $\{||X|| \leqq R\}$, which means that there exists some constant $L = L(R)$ such that

$$|\det X - \det Y| \leqq L||X - Y|| \,. \tag{3.27}$$

(3.14) and (3.21) give

$$||H_a(\sigma)|| \leqq \delta \cdot C_2 \delta^{-1} = C_2 \,.$$

Thus it follows from (3.27) that for $L = L(C_2)$

$$|\det(\delta H_a(\sigma)) - \det(\delta H_a(a))| \leqq L||\delta H_a(\sigma) - \delta H_a(a)|| \,,$$

that is,

$$\delta^N |\det H_a(\sigma) - |\det H_a(a)| \leqq \delta L ||H_a(\sigma) - H_a(a)|| .$$

Thus we get

(3.28) $$|\det H_a(\sigma)| \geqq |\det H_a(a)| - \delta^{-N+1} L ||H_a(\sigma) - H_a(a)|| .$$

Let $\tilde{c}_0 < c_0$ and let

$$|\sigma - a| \leqq C_3^{-1} L^{-1} c_0 \delta^{N+1+l} ,$$

Then it follows from (3.15), (3.25) and (3.28) that

(3.29) $$|\det H_a(\sigma)| \geqq |\det H_a(a)| - \tilde{c}_0 \delta^l \geqq (c_0 - \tilde{c}_0)\delta^l > 0 .$$

Hence from (3.25), (3.26) and (3.29) it follows that

(3.30) $$\begin{aligned} ||E_a(\sigma)|| &\leqq ||H_a(\sigma)^{-1}|| \, ||H_a(a) - H_a(\sigma)|| \\ &\leqq (c_0 - \tilde{c}_0)^{-1} \delta^{-l} \cdot C_2^{N-1} \delta^{-(N-1)} C_3 \delta^{-2} |\sigma - a| \\ &\leqq (c_0 - \tilde{c}_0)^{-1} C_2^{N-1} L^{-1} \tilde{c}_0 \equiv K . \end{aligned}$$

By taking sufficiently large L if necessary we may assume that $K < 1$. Let c_n and d_n be the coefficients in the expansion $(1+\rho)^{1/2} = \sum_{n=0}^{\infty} c_n \rho^n$ and $(1+\rho)^{-1/2} = \sum_{n=0}^{\infty} d_n \rho^n$, respectively. Since the radius of convergence of both of them are 1, (3.30) implies that

(3.31) $$K_a(\sigma)^{1/2} = \sum_{n=0}^{\infty} c_n E_a(\sigma)^n, \; K_a(\sigma)^{-1/2} = \sum_{n=0}^{\infty} d_n E_a(\sigma)^n$$

converge absolutely and

(3.32) $$||K_a(\sigma)^{1/2}|| \leqq \sum_{n=0}^{\infty} |c_n| \, ||E_a(\sigma)||^n \leqq \sum_{n=0}^{\infty} |c_n| K^n \equiv c_+ < \infty$$

(3.33) $$||K_a(\sigma)^{-1/2}|| \leqq \sum_{n=0}^{\infty} |d_n| \, ||E_a(\sigma)||^n \leqq \sum_{n=0}^{\infty} |d_n| K^n \equiv c_- < \infty .$$

Then 1) follows from (3.32) and (3.33).

2) It follows from (3.31) that

(3.34) $$(K_a(\sigma)^{1/2})^{-1} = K_a(\sigma)^{-1/2} ,$$

(3.35) $$K_a(\sigma)^{1/2} K_a(\sigma)^{1/2} = K_a(\sigma)$$

and

(3.36) $$K_a(\sigma)^{-1/2} K_a(\sigma)^{-1/2} = K_a(\sigma)^{-1} .$$

Note that a relation

f3.37) $${}^t K_a(\sigma)^{1/2} H_a(\sigma) = H_a(\sigma) K_a(\sigma)^{1/2}$$

holds. Indeed the definition (3.22) of $K_a(\sigma)$ and the symmetricity of $H_a(\sigma)$ give

$$ {}^tK_a(\sigma)\,H_a(\sigma) = H_a(\sigma)\,K_a(\sigma)\,, \tag{3.38} $$

the definition (3.24) of $E_a(\sigma)$ and (3.38) give

$$ {}^tE_a(\sigma)\,H_a(\sigma) = H_a(\sigma)\,H_a(\sigma)\,, \tag{3.39} $$

and then (3.37) follows from (3.31) and (3.39). By making the transformation of (3.23) in (3.20) and from (3.37) it follows

$$ \begin{aligned} h_a(\sigma)-h_a(a) &= 1/2\langle H_a(\sigma)\,K_a(\sigma)^{1/2}\Xi,\ K_a(\sigma)^{1/2}\Xi\rangle \\ &= 1/2\,\langle H_a(\sigma)K_a(\sigma)\Xi,\ \Xi\rangle = 1/2\,\langle H_a(a)\Xi,\ \Xi\rangle\,. \end{aligned} \tag{3.40} $$

This shows 2) of the lemma.

3) If σ satisfies $|\sigma-a|\leqq C_3^{-1}L^{-1}c_0\delta^{N+1+l}$, we have

$$ \vec{\nabla}_\Xi\sigma = \vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi)\cdot\vec{\nabla}_\Xi\sigma + K_a(\sigma)^{1/2} $$

by differentiating the both side of $\sigma-a=K_a(\sigma)^{1/2}\Xi$ by Ξ. (Here $\vec{\nabla}_\Xi=(\frac{\partial}{\partial\Xi_1}\cdots\frac{\partial}{\partial\Xi_N})$ row vector). Hence

$$ (I-\vec{\nabla}_\sigma(K_a(\sigma))^{1/2}\Xi)\cdot\vec{\nabla}_\Xi\sigma = K_a(\sigma)^{1/2}\,. \tag{3.41} $$

By the way

$$ \frac{\partial}{\partial\sigma_j}K_a(\sigma)^{1/2} = \sum_{n=0}^{\infty}c_n\sum_{k=0}^{n-1}E_a(\sigma)^k\,\frac{\partial E_a(\sigma)}{\partial\sigma_j}\,E_a(\sigma)^{d-1-k} \tag{3.42} $$

follows from (3.31). Since $E_a(\sigma)=K_a(\sigma)-I$,

$$ \begin{aligned} \frac{\partial E_a}{\partial\sigma_j}(\sigma) = \frac{\partial K_a}{\partial\sigma_j}(\sigma) &= \frac{\partial}{\partial\sigma_j}(H_a(\sigma)^{-1}H_a(a)) \\ &= -H_a(\sigma)^{-1}\frac{\partial H_a(\sigma)}{\partial\sigma_j}H_a(\sigma)^{-1}\,H_a(a) \\ &= -H_a(\sigma)^{-1}\frac{\partial H_a(\sigma)}{\partial\sigma_j}K_a(\sigma) \end{aligned} \tag{3.43} $$

follows. By using the estimates (3.14), (3.21), (3.26) and (3.22) to (3.43) we get

$$ \begin{aligned} \left\|\frac{\partial E_a}{\partial\sigma_j}(\sigma)\right\| &\leqq \|H_a(\sigma)^{-1}\|\,\left\|\frac{\partial H_a}{\partial\sigma_j}(\sigma)\right\|\,\|K_a(\sigma)\| \\ &\leqq (c_0-\tilde{c}_0)^{-1}\delta^{-l}C_2^{N-1}\delta^{-(N-1)}C_3\delta^{-2}c_+^2\,. \end{aligned} \tag{3.44} $$

Thus by using (3.30) and (3.45) to (3.43) and by the inequality of 1) we have

$$(3.45)\qquad \begin{aligned} &\|\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi)\| \\ &\leqq \sum_{n=0}^{\infty} n|c_n|K^{n-1}(c_0-\tilde{c}_0)^{-1}C_2^{N-1}C_3c_+^2\cdot\delta^{-(N+1+l)}|\Xi| \\ &\leqq \sum_{n=0}^{\infty} n|c_n|K^{n-1}(c_0-\tilde{c}_0)^{-1}C_2^{N-1}C_3c_+^2\cdot\delta^{-(N+1+l)}c_-|\sigma-a| \\ &\leqq \sum_{n=0}^{\infty} n|c_n|K^{n-1}(c_0-\tilde{c}_0)^{-1}C_2^{N-1}C_3\cdot C_3^{-1}L^{-1}\tilde{c}_0c_+^2c_- \\ &= \sum_{n=0}^{\infty} n|c_n|K^{n-1}(c_0-\tilde{c}_0)^{-1}C_2^{N-1}L^{-1}\tilde{c}_0c_+^2c_- \\ &= \sum_{n=0}^{\infty} n|c_n|K^n\cdot c_+^2c_- . \end{aligned}$$

Since $K<1$, $\sum_{n=0}^{\infty} n|c_n|K^{n-1}$ converges. By the definition of c_n and d_n we get $n|c_n| =2|d_{n-1}|$, and from it

$$(3.48)\qquad \begin{aligned} \|\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi)\| &\leqq 2\sum_{n=1}^{\infty} |d_{n-1}|K^{n-1}\cdot Kc_+^2c_- \\ &= 2c_+^2c_-^2K \end{aligned}$$

follows. By making L sufficiently large in the definition (3.30) of K if necessary, we may assume $2c_+^2c_-^2K<1$. Thus we get

$$\|\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi)\|<1 ,$$

and as a result $(I-\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi))^{-1}$ exists. Hence it follows from (3.42) that

$$\det \vec{\nabla}_\Xi\sigma = (\det K_a(\sigma)^{1/2})\,(\det (I-\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi))^{-1})\neq 0 .$$

Consequently the map $\sigma\mapsto\Xi$ is a diffeomorphism from $\{|\sigma-a|\leqq C_3^{-1}L^{-1}\tilde{c}_0\delta^{N+1+l}\}$ into $\{|\Xi|\leqq c_-C_3^{-1}L^{-1}\tilde{c}_0\delta^{N+1+l}\}$. Then we put $\bar{c}=C_3^{-1}L^{-1}\tilde{c}_0$. Q.E.D.

Lemma 3.6. $|\partial_\Xi^\alpha\sigma(\Xi)|\leqq C'_\alpha\delta^{-(|\alpha|-1)(N+1+l)}$

Proof. It follows from (3.46) that

$$(I-\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi))^{-1} = \sum_{j=1}^{\infty} (\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi))^j$$

and

$$\|(I-\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi))^{-1}\|\leqq \sum_{j=1}^{\infty}(2c_+^2c_-^2K)^j = \frac{1}{1-2c_+^2c_-^2K}$$

Hence it follows from (3.41) that

$$(3.47)\qquad \|\vec{\nabla}_\Xi\sigma\|\leqq\|(I-\vec{\nabla}_\sigma(K_a(\sigma)^{1/2}\Xi))^{-1}\|\cdot\|K_a(\sigma)^{1/2}\|\leqq\frac{c_+}{1-2c_+^2c_-^2K},$$

and (3.47) implies Lemma 3.6 in the case of $|\alpha|=1$. When $|\alpha|\geqq 2$, we differentiate (3.42) by Ξ and estimate it. Details are almost same as the proof of the case of $|\alpha|=1$ by using an inequality

$$\begin{aligned}&\|\frac{\partial}{\partial\sigma_j}K_a(\sigma)^{1/2}\|\\ &\leqq \sum_{n=0}^{\infty} n|c_n|K^{n-1}(c_0-\tilde{c}_0)^{-1}C_2^{N-1}C_3c_+\delta^{-(N+1+l)}\end{aligned}$$

which is obtained by (3.42) and (3.44). Q.E.D.

Proof of Proposition 3.4. The transformation from σ to Ξ gives

$$\begin{aligned}I &\equiv \int_U e^{ith_a(\sigma)}g_a(\sigma)d\sigma \\ &= \int_{|\Xi|\leqq c_-\tilde{c}\delta^{N+1+l}} e^{ith_a(\sigma)+it/2\langle H_a(a)\Xi,\Xi\rangle} g_1(\Xi)d\Xi\end{aligned} \tag{3.48}$$

by Lemma 3.5, where

$$g_1(\Xi)=g_a(\sigma(\Xi))J\left(\frac{\partial\sigma}{\partial E}\right) \quad \text{for} \quad J\left(\frac{\partial\sigma}{\partial\Xi}\right)=|\det\vec{\nabla}_\Xi\sigma|\,. \tag{3.49}$$

Hereafter we denote $\langle x,y\rangle=x\cdot y$ for simplicity. Here g_1 is a function of $C_0^\infty(\boldsymbol{R}^N)$ by putting $g_1=0$ outside the support of it. Then (3.48) can be regarded as the inner product in the sense of distribution. Then we have

$$\begin{aligned}Ie^{-ith_a(a)} &= \int_{\boldsymbol{R}^N} e^{i\frac{t}{2}H_a(a)\Xi\cdot\Xi} g_a(\Xi)d\Xi \\ &= \langle\mathcal{F}^{-1}[e^{i\frac{t}{2}(H_a(a)\cdot)\cdot}],\ \mathcal{F}[g_1]\rangle\end{aligned} \tag{3.50}$$

by Parseval's formula. Here the formula

$$\int_{-\infty}^{\infty} e^{(id/2)\rho^2-i\rho\tau}\,d\rho=\left(\frac{2\pi}{|d|}\right)^{1/2}e^{(i/2d)\tau^2+(\pi i/4)\operatorname{sign} d}$$

for $d\in\boldsymbol{R}\backslash\{0\}$ gives

$$\begin{aligned}&\mathcal{F}^{-1}[e^{i(t/2)(H_a(a)\cdot)\cdot}]\,(y)\\ &=(2\pi)^{-N}\int_{\boldsymbol{R}^N} e^{i(t/2)H_a(a)\Xi\cdot\Xi-iy\Xi}\,d\Xi\\ &=\frac{(2\pi t)^{-N/2}}{|\det H_a(a)|^{1/2}}e^{-(i/2t)H_a(a)^{-1}y\cdot y+(\pi i/4)\operatorname{sign}(H_a(a)t)}\end{aligned}$$

Hence from (3.50) it follows that

$$\begin{aligned}I=(2\pi)^{-N/2}&|\operatorname{Hess} h_a(a)|^{-1/2}e^{ith_a(a)+(\pi i/4)\operatorname{sign}(H_a(a)t)}\\ &t^{-N/2}\int_{\boldsymbol{R}^N}\hat{g}_1(y)\,e^{-(i/2t)H_a(a)^{-1}y\cdot y}dy\,.\end{aligned} \tag{3.51}$$

Then we put

$$\psi(y) = e^{-(i/2)(H_a(a)^{-1}y)\cdot y}$$

and

$$\psi_1(y) = \psi(y) - \psi(0) = \psi(y) - 1 .$$

It is clear that

$$|\psi_1(y)| \leqq 2 . \tag{3.52}$$

On the other hand it follows from

$$\psi_1(y) = \sum_{j=1}^{N}\int_0^1 y_j \frac{\partial\psi}{\partial y_j}(\theta y)d\theta$$

that

$$|\psi_1(y)| \leqq ||H_a(a)||^{-1}|y|^2 \leqq c_0^{-1}C_2^{N-1}\delta^{-(N-1)-l}|y|^2 , \tag{3.53}$$

Thus by an interpolation of (3.52) and (3.53) we get for any ν with $0<\nu<2$

$$|\psi_1(y)| \leqq C_\nu \delta^{-(\nu/2)(N-1+l)}|y|^\nu \tag{3.54}$$

for some constant C_ν. This gives

$$\begin{aligned} &|\int_{R^N} e^{-(i/2t)H_a(a)^{-1}y\cdot y}\hat{g}_1(y)dy - \int_{R^N}\hat{g}_1(y)dy| \\ &\leqq \int_{R^N}|\psi_1(\frac{y}{t^{1/2}})|\cdot|\hat{g}_1(y)|dy \\ &\leqq C_\nu\delta^{-(\nu/2)(N-1+l)}t^{-\nu/2}\int_{R^N}\langle y\rangle^\nu|\hat{g}_1(y)|dy \\ &\leqq C_\nu\delta^{-(\nu/2)(N-1+l)}t^{-\nu/2}(\sup_{y\in R^N}\langle y\rangle^{N+\varepsilon+\nu}|\hat{g}_1(y)|)\cdot\int_{R^N}\langle y\rangle^{-(N+\varepsilon)}dy , \end{aligned} \tag{3.55}$$

where $y=(1+|y|^2)^{1/2}$ and $\varepsilon>0$. Then we consider $\langle y\rangle^{N+\varepsilon+\nu}|g_1(y)|$. Here

$$|\partial_\Xi^\alpha g_1(\Xi)| \leqq C'_\alpha\delta^{-|\alpha|(N+1+l)} \tag{3.56}$$

follows from (3.16), (3.49) and Lemma 3.6. Let m be a non-nagetive integer. Since

$$\begin{aligned} \langle y\rangle^{2m}g_1(y) &= \langle y\rangle^{2m}\int_{R^N}e^{-iy\cdot\Xi}\hat{g}_1(\Xi)d\Xi \\ &= \int_{R^N}\{\langle D_\Xi\rangle^{2m}e^{-iy\cdot\Xi}\}g_1(\Xi)d\Xi \\ &= \int_{R^N}e^{-iy\cdot\Xi}\langle D_\Xi\rangle^{2m}g_1(\Xi)d\Xi , \end{aligned}$$

it follows from (3.17), (3.56) and Lemma 3.5 that

$$\langle y\rangle^{2m}|\hat{g}_1(y)| \leqq \int_{|\Xi|\leqq c-\bar{c}\delta^{N+1+l}} C_{2m}\delta^{-2m(N+1+l)}d\Xi \tag{3.57}$$

$$= C_{2m}\delta^{-(2m-N)(N+1+l)}\,.$$

Let m_1 and m_2 be non-negative integers and let s be a real number satisfying $s=2\theta m_1+2(1-\theta)m_2$. (3.57) for m_1, m_2 are

$$\langle y\rangle^{2m_1}|\hat{g}_1(y)|\leqq C_{2m_1}\delta^{-(2m_1-N)(N+1+l)}$$
$$\langle y\rangle^{2m_2}|\hat{g}_1(y)|\leqq C_{2m_2}\delta^{-(2m_2-N)(N+1+l)}$$

and an interpolation of the above inequalities gives

$$\begin{aligned}\langle y\rangle^{s}|\hat{g}_1(y)| &= (\langle y\rangle^{2m_1}|\hat{g}_1(y)|)^{\theta}(\langle y\rangle^{2m_2}|\hat{g}_1(y)|)^{1-\theta}\\ &\leqq C^{\theta}_{2m_2}C^{1-\theta}_{2m_2}\delta^{-(2m_1-N)(N+1+l)\theta}\delta^{-(2m_2-N)(N+1+l)(1-\theta)}\\ &= C_s\delta^{-(2m_1\theta+2m_2(1-\theta)-N)(N+1+l)}\\ &= C_s\delta^{-(s-N)(N+1+l)}\,.\end{aligned}$$

Especially, we have

$$\langle y\rangle^{N+\varepsilon+\nu}|\hat{g}_1(y)|\leqq C_{N+\varepsilon+\nu}\delta^{-(N+1+l)(\varepsilon+\nu)}$$

by putting $s=N+\varepsilon+\nu$. Thus (3.55) gives

$$\begin{aligned}(3.58)\qquad &\left|\int_{R^N}e^{-(i/2t)H_a(a)^{-1}y\cdot y}\hat{g}_1(y)dy-\int_{R^N}\hat{g}_1(y)dy\right|\\ &\leqq C_{N,\varepsilon,\nu}t^{-\nu/2}\delta^{-(\nu/2)(N-1+l)-(N+1+l)(\varepsilon+\nu)}\\ &=C_{N,\varepsilon,\nu}t^{-\nu/2}\delta^{-(N+1+l)\varepsilon-(3N+1+3l)\nu/2}\,.\end{aligned}$$

From (3.51) we have

$$\begin{aligned}(3.59)\qquad I &= (2\pi)^{-N/2}e^{ith_a(a)+(\pi i/4)\mathrm{sign}H_a(a)}|\mathrm{Hess}\ h_a(a)|^{-1/2}\\ &\quad\cdot t^{-N/2}\cdot\int_{R^N}\hat{g}_1(y)dy+q_a(t)\,,\\ q_a(t) &= (2\pi)^{-N/2}e^{ith_a(a)+(\pi i/4)\mathrm{sign}H_a(a)}|\mathrm{Hess}\ h_a(a)|^{-1/2}\\ &\quad\cdot t^{-N/2}\cdot\int_{R^N}[e^{-(i/2t)H_a(a)^{-1}y\cdot y}-1]\hat{g}_1(y)dy\,.\end{aligned}$$

From (3.58)

$$|q_a(t)|\leqq Ct^{-(N+\nu)/2}\cdot\delta^{-(N+1+l)\varepsilon-(3N+1+3l)\nu/2}|\mathrm{Hess}\ h_a(a)|^{-1/2}$$

follows. Hence for any positive number μ we have (3.19) by taking $\varepsilon>0$ and $\nu>0$ satisfying

$$\mu=(N+1+l)+(3N+1+3l)\nu/2$$

It is easy to verify

$$(3.60)\qquad \sigma(0)=a\quad\text{and}\quad J\left(\frac{\partial\sigma}{\partial\Xi}\right)\Bigg|_{\Xi=0}=1\,,$$

and with (3.60) we get

$$\int_{R^N}\hat{g}_1(y)dy = g_1(0) = g_a(\sigma(0))\cdot J\left(\frac{\partial\sigma}{\partial\Xi}\right)\Big|_{\Xi=0} = g_a(a)\,. \tag{3.61}$$

(3.59) and (3.61) give (3.18). Q.E.D.

4. Calculus of the slowness surface and the Gauss map

In this section we consider the slowness surfaces and the Gauss maps for the investigation of the slowness surface integral (2.26).

First of all we define the map R from S to S^{n-1} as

$$S \ni s \mapsto R(s) = \frac{s}{|s|}\,. \tag{4.1}$$

The surface S_k defined by (1.1) is star-shaped, which means that the intersection of S_k and any half-line $\{r\theta;\ r\theta>0,\ \theta\in S^{n-1}\}$ consists of only one point, and this fact gives that $R|_{S_k}$ is bijective. Clearly $R|_{S_k}$ is continuous. Onth e other hand, since $(R|_{S_k})^{-1}=\dfrac{\theta}{\lambda_k(\theta)}$ for $\theta\in S^{n-1}$, the continuity of λ_k gives that $(R|_{S_k})^{-1}$ is also continuous. Thus $R|_{S_k}$ is a homeomorphism from S_k to S^{n-1}. Since $\lambda_k(\xi)$ is analytic on $\boldsymbol{R}^n\backslash Z$, $S_k\backslash Z_S^{(1)}$ is a real analytic surface. Then we get similarly that $R|_{S_k\backslash Z_S^{(1)}}$ is a diffeomorphism from $S_k\backslash Z_S^{(1)}$ to $S^{n-1}\backslash R(Z_S^{(1)})$.

Next we shall define a covering of S $\{U_{ij}^k\}_{k=1,\cdots,p\,;\,i=1,2,3\,;\,j=1,\cdots,m_i}$ and a partition of unity $\{\psi_{ij}^k\}$ in the following way. Take an open covering $\{U_{ij}^k\}$ of S_k with the properties:

$$Z_S^{(1)}\subset\bigcup_{j=1}^{m_{k1}}U_{1j}^k,\quad Z_S^{(2)}\subset\bigcup_{j=1}^{m_{k2}}U_{2j}^k \tag{4.2}$$

$$Z_S^{(i_1)}\cap U_{i_2j}^k=\phi\quad\text{if}\ \ i_1\neq i_2 \tag{4.3}$$

and

$$S_k=\Big(\bigcup_{j=1}^{m_{k1}}U_{1j}^k\Big)\cup\Big(\bigcup_{j=1}^{m_{k2}}U_{2j}^k\Big)\cup\Big(\bigcup_{j=1}^{m_{k3}}U_{3j}^k\Big)\,. \tag{4.4}$$

Such $\{U_{ij}^k\}$ surely exists on account of Svi). Then map $\{U_{ij}^k\}$ into S^{n-1} by (4.1). Since $R|_{S_k}$ is a homeomorphism, $\{R(U_{ij}^k)\}$ is an open covering of S^{n-1}. Then let $\{\tilde{\psi}_{ij}^k\}$ be a partition of unity with respect to $\{R(U_{ij}^k)\}$. Since $R|_{S_k}$ (resp. $R|_{S_k\backslash Z_S^{(1)}}$) is a homeomorphism (resp. diffeomorphism), $\{\psi_{ij}^k=\tilde{\phi}_{ij}^k\circ R\}$ is a partition of unity with the properties:

$$\text{the support of } \psi_{ij}^k \text{ is contained in } U_{ij}^k \tag{4.5}$$

and

$$\psi_{ij}^k=\tilde{\psi}_{ij}^k\circ R \text{ is continuous (resp. } C^\infty\text{) on } S_k \text{ (resp. } S_k\backslash Z_S^{(1)})\,. \tag{4.6}$$

By putting $\psi_{ij}^k=0$ on S_l if $l\neq k$ we can define a covering of S $\{U_{ij}^k\}_{k=1,\cdots,p\,;\,i=1,}$

${}_{2,3;\, j=1,\cdots,m_k}$ and the partition of unity ψ^k_{ij}, which satisfies (4.2)～(4.6).

Then (2.26) can be represented as

$$v(x, r) = \sum_{k=1}^{\rho} \sum_{i=1}^{3} \sum_{j=1}^{m_{ki}} \int_{U^k_{ij}} e^{ix\cdot s} m(s, r) \psi^k_{ij}(s) dS \tag{4.7}$$
$$\equiv \sum_{k=1}^{\rho} \sum_{i=3}^{3} \sum_{j=1}^{m_{ki}} v^k_{ij}(x, r)\,,$$

where

$$m(s, r) = \hat{P}(s) | T(s) |^{-1} \phi_1(rs)\,. \tag{4.8}$$

Here we introduce a local coordinate system of U^k_{ij}. To begin with we define a coordinate system $(\sigma_1, \cdots, \sigma_{n-1})$ into $R(U^k_{ij})$ in the following way. We may assume that U^k_{ij} is sufficiently small, and so there exists ν such that

$$R(U^k_{ij}) = \{(\xi_1, \cdots, \xi_n);\ \xi_\nu = \sqrt{1-(\xi_1^2+\cdots+\xi_{\nu-1}^2+\xi_{\nu+1}^2+\cdots+\xi_n^2)}$$
$$\text{for } (\xi_1, \cdots, \xi_{\nu-1}, \xi_{\nu+1}, \cdots, \xi_n) \in V \subset \boldsymbol{R}^{n-1}\}\,.$$

Then we define $(\sigma_1, \cdots, \sigma_{n-1}) = (\xi_1, \cdots, \xi_{\nu-1}, \xi_{\nu+1}, \cdots, \xi_n)$. Through $(R|_{S_k})^{-1}$ we introduce the coordinate system $(\sigma_1, \cdots, \sigma_{n-1})$ in U^k_{ij}. $\sigma = (\sigma_1, \cdots, \sigma_{n-1})$ is a map from $R(U^k_{ij})$ into $\boldsymbol{R}^{n-1}$. With this new coordinate v^k_{ij} can be written as

$$v^k_{ij}(x, r) = \int_{\sigma\circ R(U^k_{ij})} e^{ix\cdot s(\sigma)} m(s(\sigma), r) \psi^k_{ij}(s(\sigma)) w(\sigma) d\sigma$$

where $dS = w(\sigma) d\sigma$ and

$$w(\sigma) = \{\sum_{\mu=1}^{n} \left| \frac{\partial(s_1 \cdots \hat{s}_\mu \cdots s_n)}{\partial(\sigma_1 \cdots \sigma_{n-1})} \right|^2\}^{1/2}$$

($\hat{\ }$ means omitted). About $(\sigma_1, \cdots, \sigma_{n-1})$ there are some lemmas.

Lemma 4.1. *$s = s(\sigma_1, \cdots, \sigma_{n-1}) \in U^k_{ij}$ for $s \notin Z^{(1)}_S$ satisfies*

1) $0 < c_1 \leqq \left| \dfrac{\partial s}{\partial \sigma_\mu} \right| \leqq c_2$

for $\mu = 1, \cdots, n-1$ and some constants c_1 and c_2

2) $\|\vec{\nabla}_\sigma s\| \leqq C$ *for some constant C*

3) $\mathrm{dist}_S(s_1, s_2) \sim \mathrm{dist}_V(\sigma_1, \sigma_2)$ *where $s(\sigma_j) = s_j$ and $V = \sigma \circ R(U^k_{ij})$*

4) $\mathrm{dist}_S(s, Z_S) \sim \mathrm{dist}_V(\sigma, Z_0)$,

where $Z_0 = (\sigma \circ R)\ (Z_S \cap U^k_{ij})$

5) $w(\sigma) \geqq c > 0$ *for some constant c.*

In 1)～5) *the constants are uniform for $\sigma \notin Z_0$.*

Proof. $\theta = \theta(\sigma_1, \cdots, \sigma_{n-1}) \in R(U^k_{ij})$ satisfy

$$\theta \cdot \frac{\partial \theta}{\partial \sigma_\mu} = 0 \text{ and } \left|\frac{\partial \theta}{\partial \sigma_\mu}\right| < 1 \tag{4.9}$$

because $\frac{\partial \theta}{\partial \sigma_\mu} = (0, \cdots, \underset{\mu}{1}, \cdots, 0, \underset{\nu}{\frac{\partial \xi_\nu}{\partial \sigma_\mu}}, 0, \cdots, 0)$. Then, since $s = \frac{\theta}{\lambda_k(\theta)}$ we can write

$$\begin{aligned}\frac{\partial s}{\partial \sigma_\mu} &= \frac{1}{\lambda_k(\theta)} \cdot \frac{\partial \theta}{\partial \sigma_\mu} + \sum_{l=1}^{n} \left(-\frac{(\partial_l \lambda_k)(\theta)}{\lambda_k(\theta)^2}\right) \cdot \frac{\partial \theta_l}{\partial \sigma_\mu} \cdot \theta \\ &= \frac{1}{\lambda_k(\theta)} \cdot \frac{\partial \theta}{\partial \sigma_\mu} - \frac{1}{\lambda_k(\theta)^2} \left(\nabla \lambda_k(\theta) \cdot \frac{\partial \theta}{\partial \sigma_\mu}\right)\theta\end{aligned}$$

Hence we get by (4.9)

$$\begin{aligned}\left|\frac{\partial s}{\partial \sigma_\mu}\right|^2 &= \frac{1}{\lambda_k(\theta)^2}\left|\frac{\partial \theta}{\partial \sigma_\mu}\right|^2 + \frac{1}{\lambda_k(\theta)^4}\left(\nabla \lambda_k(\theta) \cdot \frac{\partial \theta}{\partial \sigma_\mu}\right)^2 \\ &\geqq \frac{1}{\lambda_k(\theta)^2} \geqq c_1' > 0\end{aligned} \tag{4.10}$$

for some constant c_1' depending on k.

On the other hand, if we estimate $\frac{\partial s}{\partial \sigma_\mu}$ from above, we get

$$\begin{aligned}\left|\frac{\partial s}{\partial \sigma_\mu}\right|^2 &\leqq \frac{1}{\lambda_k(\theta)^2}\left|\frac{\partial \theta}{\partial \sigma_\mu}\right|^2 + \frac{1}{\lambda_k(\theta)^4} |\nabla \lambda_k(\theta)|^2 \left|\frac{\partial \theta}{\partial \sigma_\mu}\right|^2 \\ &= \frac{1}{\lambda_k(\theta)^2}\left|\frac{\partial \theta}{\partial \sigma_\mu}\right|^2 \left(1 + \frac{1}{\lambda_k(\theta)^2} |\nabla \lambda_k(\theta)|^2\right),\end{aligned}$$

Note that $\frac{1}{\lambda_k(\theta)^2} \leqq c_2'$, $|\nabla \lambda_k(\theta)|^2 \leqq c_3'$ and $\left|\frac{\partial \theta}{\partial \sigma_\mu}\right|^2 = 1 + \left(\frac{\partial \xi_\nu}{\partial \sigma_\mu}\right)^2 \leqq c_4'$ for some constants c_2', c_3' and c_4' depending on U_{ij}^k. Then

$$\left|\frac{\partial s}{\partial \sigma_\mu}\right|^2 \leqq c_2'' \tag{4.11}$$

for some constant c_2'' determined by c_2', c_3' and c_4'.

(4.10) and (4.11) give

$$0 < c_1 \leqq \left|\frac{\partial s}{\partial \sigma_\mu}\right| \leqq c_2$$

by putting $c_1 = \sqrt{c_1'}$ and $c_2 = \sqrt{c_2''}$. Thus 1) is proved.

2) is clear from 1).

Next we prove 3). For any curve $s(t)$ on U_{ij}^k with $s(0) = s_1$ and $s(1) = s_2$ it follows that

$$\int_0^1 \left|\frac{ds}{dt}\right|^2 dt = \int_0^1 \left|\sum_{\mu=1}^{n} \frac{\partial s}{\partial \sigma_\mu} \cdot \frac{d\sigma_\mu}{dt}\right|^2 dt \leqq c_2^2 \int_0^1 \sum_{\mu=1}^{n} \left(\frac{d\sigma_\mu}{dt}\right)^2 dt.$$

By taking inferimum with respect to $s(t)$ we have

(4.12) $$\mathrm{dist}_S(s_1, s_2) \leqq c_2\, \mathrm{dist}_V(\sigma_1, \sigma_2)\,.$$

On the other hand

(4.13) $$\mathrm{dist}_{S_k}(s_1, s_2) \geqq |s_2 - s_1| = \left|\frac{\theta_2}{\lambda_k(\theta_2)} - \frac{\theta_1}{\lambda_k(\theta_1)}\right|,$$

where $|\cdot|$ denotes the norm of $\boldsymbol{R}^n$ and $\theta_j = R(s_j)$. We may assume $\lambda_k(\theta_2)/\lambda_k(\theta_1) \geqq 1$. Then

$$\left|\theta_2 - \frac{\lambda_k(\theta_2)}{\lambda_k(\theta_1)}\theta_1\right| \geqq |\theta_2 - \theta_1|\,.$$

Hence it follows that

(4.14) $$\left|\frac{\theta_2}{\lambda_k(\theta_2)} - \frac{\theta_1}{\lambda_k(\theta_1)}\right| = \frac{1}{\lambda_k(\theta_2)}\left|\theta_2 - \frac{\lambda_k(\theta_2)}{\lambda_k(\theta_1)}\theta_1\right|$$
$$\geqq \frac{1}{\lambda_k(\theta_2)}|\theta_2 - \theta_1| \geqq c|\theta_2 - \theta_1| \geqq c'|\sigma_2 - \sigma_1|$$

for some constant c' depending on i, j and k of U^k_{ij}. (4.13) and (4.14) give

(4.15) $$\mathrm{dist}_S(s_1, s_2) \geqq c'\, \mathrm{dist}_V(\sigma_2, \sigma_1)\,.$$

Then 3) follows from (4.12) and (4.15).

Let s_0 be a point of Z_S satisfying

$$\mathrm{dist}_S(s, Z_S) = \mathrm{dist}_S(s, s_0)\,.$$

It follows from (4.15) that

(4.16) $$\mathrm{dist}_S(s, Z_S) \geqq c'\, \mathrm{dist}_V(\sigma, \sigma_0) \geqq c'\, \mathrm{dist}_V(\sigma, Z_0)\,,$$

where $\sigma_0 = \sigma(s_0) \in Z_0$. Similarly it follows from (4.12) that

(4.17) $$\mathrm{dist}_V(\sigma, Z_0) \geqq c_2^{-1}\, \mathrm{dist}_S(s, Z_S)\,.$$

(4.16) and (4.17) give 4).

5) immediately follows from the definition of the coordinates. Q.E.D.

To estimate the derivatives of $s(\sigma)$ we must consider derivatives of $\lambda_k(\xi)$:

Lemma 4.2. 1)

$$\partial_\mu \hat{P}_k(\xi) = \sum_{l \neq k} \left\{\frac{\hat{P}_l(\xi) A_\mu \hat{P}_k(\xi) + \hat{P}_k(\xi) A_\mu \hat{P}_l(\xi)}{\lambda_k(\xi) - \lambda_l(\xi)}\right\}$$

for $\xi \in \boldsymbol{R}^n \setminus Z$ *where* $\partial_\mu = \dfrac{\partial}{\partial \xi_\mu}$

2) $$|\partial_\xi^\alpha \lambda_k(\xi)| \leqq C_\alpha\, \mathrm{dist}(\xi, Z^{(1)})^{-|\alpha|+1}$$

for some constant C_α *and* $\xi\in\boldsymbol{R}^n\backslash Z$ where

$$Z^{(1)}=\{\xi=rs;\ r>0,\ s\in Z_S^{(1)}\}.$$

Proof. 1) (2.23) gives

$$\Lambda^0(\xi)\hat{P}_k(\xi)=\lambda_k(\xi)\hat{P}_k(\xi)\,. \tag{4.18}$$

If we differentiate the both side of (4.18) by ξ_μ we get

$$\partial_\mu\Lambda^0(\xi)\cdot\hat{P}_k(\xi)+\Lambda^0(\xi)\cdot\partial_\mu\hat{P}_k(\xi)=\partial_\mu\lambda_k(\xi)\cdot\hat{P}_k(\xi)+\lambda_k(\xi)\cdot\partial_\mu\hat{P}_k(\xi)\,.$$

Hence

$$\partial_\mu\Lambda^0(\xi)\cdot\hat{P}_k(\xi)-\partial_\mu\lambda_k(\xi)\cdot\hat{P}_k(\xi)=-(\Lambda^0(\xi)-\lambda_k(\xi))\cdot\partial_\mu\hat{P}_k(\xi)\,. \tag{4.19}$$

Since $\Lambda^0(\xi)=\sum_{j=1}^n\xi_jA_j$, $\partial_\mu\Lambda^0(\xi)=A_\mu$ holds. Then by multiplying $\hat{P}_l(\xi)$ for $l\neq k$ from the left to (4.19) we get

$$\hat{P}_l(\xi)A_\mu\hat{P}_k(\xi)=-(\lambda_l(\xi)-\lambda_k(\xi))\hat{P}_l(\xi)\cdot\partial_\mu\hat{P}_k(\xi)\,.$$

Hence

$$\hat{P}_l(\xi)\cdot\partial_\mu\hat{P}_k(\xi)=\frac{\hat{P}_l(\xi)A_\mu\hat{P}_k(\xi)}{\lambda_k(\xi)-\lambda_l(\xi)}\,. \tag{4.20}$$

On the other hand by differentiating the both side of $\hat{P}_k(\xi)^2=\hat{P}_k(\xi)$ we get

$$\hat{P}_k(\xi)\cdot\partial_\mu\hat{P}_k(\xi)+\partial_\mu\hat{P}_k(\xi)\cdot\hat{P}_k(\xi)=\partial_\mu\hat{P}_k(\xi)\,.$$

Then

$$\hat{P}_k(\xi)\cdot\partial_\mu\hat{P}_k(\xi)=\partial_\mu\hat{P}_k(\xi)(I-\hat{P}_k(\xi))=\partial_\mu\hat{P}_k(\xi)\cdot\sum_{l\neq k}\hat{P}_l(\xi) \tag{4.21}$$

follows. (4.20) and (4.21) give

$$\partial_\mu\hat{P}_k(\xi)=\sum_{l\neq k}\frac{\hat{P}_l(\xi)A_\mu\hat{P}_k(\xi)}{\lambda_k(\xi)-\lambda_l(\xi)}+\partial_\mu\hat{P}_k(\xi)\sum_{l\neq k}\hat{P}_l(\xi)\,. \tag{4.22}$$

In the same way as in the proof of (4.20) we get

$$\partial_\mu\hat{P}_k(\xi)\cdot\hat{P}_l(\xi)=\frac{\hat{P}_k(\xi)A_\mu\hat{P}_l(\xi)}{\lambda_k(\xi)-\lambda_l(\xi)}\,.$$

Then

$$\partial_\mu\hat{P}_k(\xi)\sum_{l\neq k}\hat{P}_k(\xi)=\sum_{l\neq k}\frac{\hat{P}_k(\xi)A_\mu\hat{P}_l(\xi)}{\lambda_k(\xi)-\lambda_l(\xi)} \tag{4.23}$$

follows. (4.22) and (4.23) give

$$\partial_\mu\hat{P}_k(\xi)=\sum_{l\neq k}\left\{\frac{\hat{P}_l(\xi)A_\mu\hat{P}_k(\xi)+\hat{P}_k(\xi)A_\mu\hat{P}_l(\xi)}{\lambda_k(\xi)-\lambda_l(\xi)}\right\}.$$

2) By (4.19) we get

$$\partial_\mu \lambda_k(\xi)\cdot\hat{P}_k(\xi) = A_\mu \hat{P}_k(\xi) + (\Lambda^0(\xi)-\lambda_k(\xi))\cdot\partial_\mu \hat{P}_k(\xi)\,. \tag{4.24}$$

If we multiply $\hat{P}_k(\xi)$ from the left to (4.24),

$$\partial_\mu \lambda_k(\xi)\cdot\hat{P}_k(\xi) = \hat{P}_k(\xi) A_\mu \hat{P}_k(\xi) \tag{4.25}$$

follows. Since $\Lambda^0(\xi)$ is a hermitian matrix there exists a unitary matrix $U(\xi)$ such that

$$\Lambda^0(\xi) = U(\xi)^* \begin{bmatrix} \lambda_\rho(\xi) I_{m_\rho} & & & & 0 \\ & \ddots & & & \\ & & \lambda_k(\xi) I_{m_k} & & \\ & & & \ddots & \\ 0 & & & & \lambda_{-\rho}(\xi) I_{m_{-\rho}} \end{bmatrix} U(\xi)\,,$$

where each m_k denotes the multiplicity of $\lambda_k(\xi)$, and this gives

$$\hat{P}_k(\xi) = U(\xi)^* \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & I_{m_k} & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} U(\xi)\,. \tag{4.26}$$

It follows from (4.25) and (4.26) that

$$\begin{aligned} \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & \partial_\mu \lambda_k(\xi) I_{m_k} & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} &= U(\xi)\partial_\mu \lambda_k(\xi)\hat{P}_k(\xi) U(\xi)^* \\ &= U(\xi)(\hat{P}_k(\xi) A_\mu \hat{P}_k(\xi)) U(\xi)^*\,. \end{aligned}$$

Thus we get

$$\begin{aligned} |\partial_\mu \lambda_k(\xi)| &= \frac{1}{\sqrt{m_k}} \left\| \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & \partial_\mu \lambda_k(\xi) I_{m_k} & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} \right\| \\ &= \frac{1}{\sqrt{m_k}} \|\hat{P}_k(\xi) A_\mu \hat{P}_k(\xi)\| \\ &\leqq \frac{C}{\sqrt{m_k}} \|A_\mu\| \end{aligned} \tag{4.27}$$

for some constant C, and $\partial_\mu \lambda_k(\xi)$ is bounded. This fact is already known in another way. See C.H. Wilcox [12, § 3]. Next we differentiate (4.25) by ξ_ν. Then we get

$$\partial_\nu\partial_\mu\lambda_k(\xi)\cdot\hat{P}_k(\xi)+\partial_\mu\lambda_k(\xi)\cdot\partial_\nu\hat{P}_k(\xi)$$
$$=\partial_\nu\hat{P}_k(\xi)\cdot A_\mu\hat{P}_k(\xi)+\hat{P}_k(\xi)A_\mu\cdot\partial_\nu\hat{P}_k(\xi)\,.$$

Then

$$\begin{aligned}&\partial_\nu\partial_\mu\lambda_k(\xi)\cdot\hat{P}_k(\xi)\\ &=\partial_\nu\hat{P}_k(\xi)A_\mu\hat{P}_k(\xi)+\hat{P}_k(\xi)A_\mu\cdot\partial_\nu\hat{P}_k(\xi)-\partial_\mu\lambda_k(\xi)\cdot\partial_\nu\hat{P}_k(\xi)\\ &\equiv J(\xi)\,.\end{aligned}$$

In the same way as in the proof of (4.27) we get

$$|\partial_\nu\partial_\mu\lambda_k(\xi)|\leqq\frac{1}{\sqrt{m_k}}\|J(\xi)\|\,.$$

The result of 1) and the boundedness of $\partial_\nu\lambda_k(\xi)$ and $\hat{P}_k(\xi)$ give

$$\|J(\xi)\|\leqq c\sum_{l\neq k}|\lambda_k(\xi)-\lambda_l(\xi)|^{-1}\,.$$

Thus it follows from the assumption Sv) that

$$|\partial_\nu\partial_\mu\lambda_k(\xi)|\leqq c\ \mathrm{dist}(\xi,Z^{(1)})^{-1}$$

for some constant c. By repeating this process we get

$$|\partial^\alpha\lambda_k(\xi)|\leqq C_\alpha\ \mathrm{dist}_{R^n}(\xi,Z^{(1)})^{-|\alpha|+1}$$

for any α. Q.E.D.

With Lemma 4.2 we prove

Lemma 4.3. 1) $|\partial_\sigma^\alpha s(\sigma)|\leqq C_\alpha$ *for* $\sigma\in(\sigma\circ R)(U_{ij}^k)$ *with* $i=2$ *or* 3

2) $$|\partial_\sigma^\alpha s(\sigma)|\leqq C_\alpha\ \mathrm{dist}_V(\sigma,Z_0)^{-|\alpha|+1}$$

for $\sigma\in V\equiv(\sigma\circ R)(U_{1j}^k)$.

Proof. 1) is clear from the smoothness of U_{ij}^k for $i=2$ and 3.
To prove 2), note that $s=\theta/\lambda_k(\theta)$ for $\theta=\theta(\sigma)\in R(U_{ij}^k)$. Then we get

$$\frac{\partial^2 s}{\partial\sigma_\mu\partial\sigma_\nu}=-\frac{\theta}{\lambda_k(\theta)^2}\sum_{a=1}^n\sum_{b=1}^n\frac{\partial^2\lambda_k}{\partial\xi_a\partial\xi_b}\cdot\frac{\partial\theta_a}{\partial\sigma_\mu}\cdot\frac{\partial\theta_b}{\partial\sigma_\nu}+(\text{bounded terms})\,. \tag{4.28}$$

From Lemma 4.2 2)

$$\left|\frac{\partial^2\lambda_k}{\partial\xi_a\partial\xi_b}(\theta)\right|\leqq C_2\ \mathrm{dist}_{R^n}(\theta,Z^{(1)})^{-1} \tag{4.29}$$

follows. From the definition of the coordinate $(\sigma_1,\cdots,\sigma_{n-1})$

$$\mathrm{dist}_{R^n}(\theta,Z^{(1)})\sim\mathrm{dist}_V(\sigma,Z_0) \tag{4.30}$$

and

$$\left|\frac{\partial\theta_a}{\partial\sigma_\mu}\right| \leqq \text{Const}. \tag{4.31}$$

follow. Thus we get

$$\left|\frac{\partial^2 s}{\partial\sigma_\mu\partial\sigma_\nu}(\sigma)\right| \leqq C_2' \operatorname{dist}_V(\sigma, Z_0)^{-1} \qquad \text{for} \quad \sigma\in(\sigma\circ R)(U_{1j}^k)$$

by applying (4.29), (4.30) and (4.31) to (4.28). In the same way as above we get

$$|\partial_\sigma^\alpha s(\sigma)| \leqq C_\alpha \operatorname{dist}_V(\sigma, Z_0)^{-|\alpha|+1}. \qquad \text{Q.E.D.}$$

Here we consider

$$m(s(\sigma), r)\psi_{ij}^k(s(\sigma))w(\sigma) = \hat{P}(s(\sigma))\,|T(s(\sigma))|^{-1}\phi_1(rs(\sigma))\psi_{ij}^k(s(\sigma))w(\sigma).$$

To begin with we treat $\hat{P}(s)$. A derivative of $\hat{P}(s(\sigma))$ with respect to σ_μ is

$$\partial_{\sigma_\mu}(P(s(\sigma))) = \sum_{\nu=1}^{n-1}(\partial_\nu\hat{P}_k)(s(\sigma))\frac{\partial s_\nu}{\partial\sigma_\mu}. \tag{4.32}$$

Thus we have from (4.32) by using Lemma 4.2 1), the assumption Sv) and Lemma 4.3

$$|\partial_{\sigma_\mu}(\hat{P}(s(\sigma)))| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-1}$$

for $\sigma\in(\sigma\circ R)(U_{1j}^k)$. In the same way we have

$$|\partial_\sigma^\alpha(\hat{P}(s(\sigma)))| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-|\alpha|} \tag{4.33}$$

for $\sigma\in V=(\sigma\circ R)(U_{1j}^k)$. Next we treat $|T(s(\sigma))|^{-1}=|\nabla\lambda_k(s(\sigma))|^{-1}$. Then Lemma 4.1 4), Lemma 4.2 1) and Lemma 4.3 give

$$|\partial_\sigma^\alpha|T(s(\sigma))|^{-1}| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-|\alpha|} \tag{4.34}$$

for $\sigma\in V=(\sigma\circ R)(U_{1j}^k)$. For $\phi_1(rs(\sigma))$, the smoothness of ϕ_1 and Lemma 4.3 give

$$|\partial_\sigma^\alpha(\phi_1(rs(\sigma)))| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-|\alpha|+1} \tag{4.35}$$

for $\sigma\in V=(\sigma\circ R)(U_{1j}^k)$. Since ϕ_1 has compact support, the constant c of (4.35) is independent of r. For $w(\sigma)$, applying Lemma 4.1 5) and Lemma 4.3 to an equality

$$\partial_{\sigma_\mu}w(\sigma) = \sum\frac{(\text{at most first order derivative of } s)}{w(\sigma)}\cdot\frac{\partial^2 s}{\partial s_\mu\partial s_\nu},$$

we have

$$|\partial_{\sigma_\mu} w(\sigma)| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-1}.$$

Similary we have

$$(4.36) \qquad |\partial_\sigma^\alpha w(\sigma)| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-|\alpha|}$$

for $\sigma \in V=(\sigma\circ R)(U_{1j}^k)$. $\psi_{ij}^k(s(\sigma))$ is smooth with respect to σ because $\psi_{ij}^k(s(\sigma)) =\tilde{\psi}_{ij}^k(\theta(\sigma))$ and both of θ and $\tilde{\psi}_{ij}^k$ are smooth. Then by summing up above facts we have

$$(4.37) \qquad |\partial_\sigma^\alpha\{m(s(\sigma), r)\psi_{ij}^k(s(\sigma))w(\sigma)\}| \leqq c \operatorname{dist}_V(\sigma, Z_0)^{-|\alpha|}$$

for $\sigma \in V=(\sigma\circ R)(U_{1j}^k)$ where c is independent of r.

We are now considering to apply the stationary phase method and a modification of it to the integral (2.26). Recall that (2.26) has a decomposition (4.7), where each $v_{ij}^k(x, r)$ can be written

$$(4.38) \qquad v_{ij}^k(x, r) = \int_{(\sigma\circ R)(U_{1j}^k)} e^{i|x|\eta\cdot s(\sigma)} m(s(\sigma), r)\psi_{ij}^k(s(\sigma))w(\sigma)d\sigma .$$

Then we look for the stationary points of the phase function $\eta\cdot s(\sigma)$ for $\eta=x/|x|$. Its gradient is

$$\vec{\nabla}_\sigma(\eta\cdot s(\sigma)) = \vec{\eta}\cdot\vec{\nabla}_\sigma s(\sigma),$$

where $\rightarrow$ denotes the row vector and $\vec{\nabla}_\sigma=\left(\frac{\partial}{\partial\sigma_1}, \cdots, \frac{\partial}{\partial\sigma_{n-1}}\right)$. Here note that the column vectors of

$$\vec{\nabla}_\sigma s(\sigma) = \begin{pmatrix} \frac{\partial s_1}{\partial\sigma_1} & \cdots & \frac{\partial s_1}{\partial\sigma_{n-1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial s_n}{\partial\sigma_1} & \cdots & \frac{\partial s_n}{\partial\sigma_{n-1}} \end{pmatrix}$$

construct the basis of the tangent plane of S_k at $s(\sigma)$. Hence

$$\vec{\eta}\cdot\vec{\nabla}_\sigma s(\sigma) = 0$$

is equivalent to the fact that η is normal to S_k at $s(\sigma)$, namely, $s(\sigma)$ is a stationary point of $\eta\cdot(\sigma)(\vec{\nabla}(\eta\cdot s(\sigma))=0)$ if and only if η is normal to S_k at $s(\sigma)$ $(\pm\eta=N(s(\sigma))$, where N denotes the Gauss map). Then the next problem is to search for s with $\eta=N(s)$ on the slowness surface S when $\eta\in S^{n-1}$ is given, in other words to determine the inverse image of the Gauss map N.

The following facts about the Gauss map on the slowness surfaces are already known ([12, § 6]).

Let $\rho(\eta)$ for $\eta\in S^{n-1}\backslash\bar{Z}$ denotes the number of points in which a ray $\{x=rs; r>0\}$ meets the wave surface W, and

$$\tilde{\rho} = \max_{\eta \in S^{n-1}} \rho(\eta) .$$

($\bar{Z}$ is defined in (2.4)). Let

$$\Omega^{\beta} = \{\eta \in S^{n-1} \backslash \bar{Z};\ \rho(\eta) = \beta\} \qquad (\beta = 1, 2, \cdots, \tilde{\rho}) .$$

Then S has a decomposition

$$S = (\bigcup_{\beta=1}^{\tilde{\rho}} \bigcup_{\gamma=1}^{\beta} S^{\beta\gamma}) \cup Z_S$$

which is disjoint union, and for each Ω^{β} there exists a diffeomorphism from Ω^{β} to $S^{\beta\gamma}$

$$s^{\beta\gamma} \colon \Omega^{\beta} \to S^{\beta\gamma}$$

which satisfies

$$N \circ s^{\beta\gamma} = id \qquad \text{on} \quad \Omega^{\beta}$$

for $\beta=1, \cdots, \tilde{\rho}$ and $\gamma=1, \cdots, \beta$. This shows that if $\eta \in \Omega^{\beta}$ the number of $s \in S$ with $\eta = N(s)$ is β and these points s are represented as $s^{\beta\gamma}(\eta)$. Since $\eta \in \Omega^{\beta}$ means $\rho(\eta)=\beta$, these can also be written as $s^{\rho(\eta)\gamma}(\eta)$ $(\gamma=1, 2, \cdots, \rho(\eta))$.

Thus when $\eta \in S^{n-1} \backslash \bar{Z}$ is given there exists a unique point s in $S^{\beta\gamma}$ which satisfies

$$\eta = N(s) .$$

If σ is a stationary point of $\eta \cdot s(\sigma)$, σ satisfies $N(s(\sigma))=\eta$ or $N(s(\sigma))=-\eta$. Then there exists at most one such point in U_{ij}^{k} for $i=1$ and 3 since U_{ij}^{k} 's are sufficiently small. On the other hand U_{2j}^{k} intersects with $Z_S^{(2)}$ which forms the boundaries of $S^{\beta\gamma}$'s, and $Z_S^{(2)}$ is an at most $(n-1)$-dimensional smooth submanifold by Sii). Thus U_{2j}^{k} can be represented as

$$U_{2j}^{k} = (S^{\beta\gamma} \cap U_{2j}^{k}) \cup (S^{\beta'\gamma'} \cap U_{2j}^{k}) \cup (Z_S^{(2)} \cap U_{2j}^{k}) \tag{4.39}$$

for some $(\beta, \gamma) \neq (\beta', \gamma')$, which is a disjoint union. This fact means that $(\sigma \circ R)(U_{2j}^{k})$ may have two stationary points.

In the rest of this section we introduce local coordinate systems in S_{η}^{n-1} and give the relation between the coordinate of S and that of S_{η}^{n-1} through the diffeomorphism N.

The local coordinate $\bar{\sigma}$ of S_{η}^{n-1} is introduced in the same way to σ. Hereafter we denote $(\sigma \circ R)(U_{ij}^{k})$ by V_i. Note that V_i depends on not only i but also j and k. For $i=1$, $V_1 \backslash Z_0$ is diffeomorphic to $U_{1j}^{k} \backslash Z_S^{(1)}$ by $\sigma \circ R$. On the other hand, since the Gauss map N is a diffeomorphism from $S^{\beta\gamma}$ to $\Omega^{\beta} \subset S^{n-1}$ for any β and γ, $U_{1j}^{k} \backslash Z_S^{(1)}$ is diffeomorphic to $N(U_{1j}^{k} \backslash Z_S^{(1)})$. Hence $V_1 \backslash Z_0$ is diffeomorphic to $(\bar{\sigma} \circ N)(U_{1j}^{k} \backslash Z_S^{(1)})$ which we denote by $\bar{V}_1$. Note that $(\bar{\sigma} \circ N) \cdot$

$(Z_S^{(1)})\subset\partial\bar{V}_1$. For $i=2$ $V_2\backslash Z_0$ is diffeomorphic to $U_{2j}^k\backslash Z_S^{(2)}$ but is not always diffeomorphic to $N(U_{2j}^k\backslash Z_S^{(2)})$ because U_{2j}^k may have the decomposition (4.39). However, since N is a diffeomorphism from $S^{\beta\gamma}$ to Ω^β, $S^{\beta\gamma}\cap U_{2j}^k$ is diffeomorphic to $N(S^{\beta\gamma}\cap U_{2j}^k)$. Thus if we write $V_2^1=(\sigma\circ R)(U_{2j}^k\cap S^{\beta\gamma})$ and $V_2^2=(\sigma\circ R)(U_{2j}^k\cap S^{\beta'\gamma'})$, then each V_2^μ is diffeomorphic to its image of $\bar{\sigma}\circ N\circ R^{-1}\circ\sigma^{-1}$ which we denote by $\bar{V}_2^\mu$.

We introduce the coordinate $\bar{\sigma}$ to have good properties, and we can write

$$dS^{n-1}=\bar{w}(\bar{\sigma})d\bar{\sigma}\,, \tag{4.40}$$

where $\bar{w}(\bar{\sigma})$ is a bounded function with its derivatives on $\bar{V}_1$ (or $\bar{V}_2^\mu$). On the other hand, the following two equalities are already known:

$$dS=w(\sigma)d\sigma \tag{4.41}$$

and

$$dS^{n-1}=|K(s)|\,dS\,. \tag{4.42}$$

Then (4.40), (4.41) and (4.42) give

$$\bar{w}(\sigma)d\bar{\sigma}=|K(s(\sigma))|\,w(\sigma)d\sigma\,.$$

Thus the Jacobian of the map from V_1 (V_2^μ) to $\bar{V}_1$ ($\bar{V}_2^\mu$) can be represented as

$$J\left(\frac{\partial\sigma}{\partial\bar{\sigma}}\right)=w(\sigma)^{-1}|K(s(\sigma))|^{-1}\bar{w}(\bar{\sigma})\,. \tag{4.43}$$

For V_1 it follows from the assumption Siii) and Lemma 4.1 5) that

$$J\left(\frac{\partial\sigma}{\partial\bar{\sigma}}\right)\leqq\text{Const. }\mathrm{dist}_S(s(\sigma),\,Z_S^{(1)})^{d-1}\,. \tag{4.44}$$

The assumption Siv) gives

$$\mathrm{dist}_S(s(\sigma),\,Z_S^{(1)})\sim\mathrm{dist}_{S^{n-1}}(N(s(\sigma)),\,N(Z_S^{(1)}))\,. \tag{4.45}$$

In a similar way to the proof of Lemma 4.1 3) and 4) we can prove

$$\mathrm{dist}_{S^{n-1}}(\eta_1,\,\eta_2)\sim\mathrm{dist}_{\bar{V}_1}(\bar{\sigma}_1,\,\bar{\sigma}_2) \tag{4.46}$$

and

$$\mathrm{dist}_{S^{n-1}}(\eta,\,N(Z_S^{(1)}))\sim\mathrm{dist}_{\bar{V}}(\bar{\sigma},\,\partial\bar{V}_1)\,. \tag{4.47}$$

Then by summing up (4.44), (4.45), (4.35) and Lemma 4.1 4) we have

$$\mathrm{dist}_{V_1}(\sigma,\,Z_0)\sim\mathrm{dist}_{\bar{V}_1}(\bar{\sigma},\,\bar{V}_1) \tag{4.48}$$

and

$$J\left(\frac{\partial\sigma}{\partial\bar{\sigma}}\right)\leqq Const.\ \mathrm{dist}_{\bar{V}_1}(\bar{\sigma}, \partial\bar{V}_1)^{d-1}. \tag{4.49}$$

5. Calculus of the slowness surface integral

Now we begin to prove Theorem 2.1 from the preparation as mentioned above. We estimate each $v_{ij}^k(x, r)$ of the decomposition (4.7) of the slowness surface integral $v(x, r)$.

When $x=|x|\eta$ for $\eta\in S^{n-1}\backslash\bar{Z}$ is given, each (i, j, k) satisfies either of following six cases:

i)–a) $i=1$ and one of $s^{(\gamma)}(\pm\eta)$'s belong to U_{1j}^k
i)–b) $i=1$ and none of $s^{(\gamma)}(\pm\eta)$'s belong to U_{1j}^k
ii)–a) $i=2$ and some of $s^{(\gamma)}(\pm\eta)$'s belong to U_{2j}^k
ii)–b) $i=2$ and none of $s^{(\gamma)}(\pm\eta)$'s belong to U_{2j}^k
iii)–a) $i=3$ and one of $s^{(\gamma)}(\pm\eta)$'s belong to U_{3k}^k
iii)–b) $i=3$ and none of $s^{(\gamma)}(\pm\eta)$'s belong to U_{3j}^k,

where $s^{(\gamma)}(\eta)$ denotes $s^{\rho(\eta)\gamma}(\eta)$ for $\gamma=1, 2, \cdots, \rho(\eta)$. In the case of ii)–b) and iii)–b) we estimate v_{ij}^k by using Porposition 3.1 1), in the case of iii)–a) by Proposition 3.1 2) (or Proposition 3.2) and in the case of i)–b) by Proposition 3.3. In the case of i)–a) and ii)–a) we need more precious considerations. We devide v_{ij}^k into two parts, and estimate the principal part of them by using Proposition 3.4. To estimate the other part we shall prepare some lemmas in section 5, and the desired estimate will be shown in section 6.

In the case of ii)–b) and iii)–b) Proposition 3.1 1) can be applied to the integral (4.38) and

$$v_{ij}^k(x, r)=O(|x|^{-\infty}) \text{ uniformly for } \eta=\frac{x}{|x|} \text{ and } r \tag{5.1}$$

follows. In the case of iii)–a) usual stationary phase method Proposition 3.1 2) (or Proposition 3.2) can be applied to (4.38) because there are neither singularities nor parabolic points in U_{3j}^k. Then it follows that

$$\begin{aligned} v_{3j}^k(x, r) &= (2\pi)^{-(n-1)/2}e^{ix\cdot s}|x|^{-(n-1)/2} \\ &\quad \cdot\psi_{\pm}(s)m(s, r)|K(s)|^{-1/2}|_{s=s^{(\gamma)}(\pm\eta)}+q_{3j}^k(x, r) \end{aligned} \tag{5.2}$$

for some $s^{(\gamma)}(\pm\eta)$ contained in U_{3j}^k, where

$$q_{3j}^k(x, r) = O(|x|^{-n/2}) \text{ uniformly for } \eta=\frac{x}{|x|} \text{ and } r\,. \tag{5.3}$$

Remark that the uniformity in (5.1) and (5.2) follows from the uniformity of the derivatives of $s(\sigma)$ and $m(s(\sigma), r)$ for η and r.

In the case of i)–b) we apply Proposition 3.3 to the integral $v_{1j}^k(x, r)$. Lemma 4.3 2) gives

$$(5.4) \qquad |\partial_\sigma^\alpha(\eta\cdot s(\sigma))| \leqq C_\alpha \operatorname{dist}_{V_1}(\sigma, Z_0)^{-|\alpha|+1} \qquad \text{for} \quad \sigma\in V_1,$$

which is the condition (3.4) for the phase function in Proposition 3.3. On the other hand the condition (3.3) of Proposition 3.3 has already verified as (4.37). Thus we get

$$(5.5) \qquad |v_{1j}^k(x, r)| \leqq C|x|^{-(n-1)/2-\nu}$$

for some positive constants ν and C independent of $\eta=\dfrac{x}{|x|}$ and r.

Note that Proposition 3.3 and Proposition 3.4 can be extended for g and g_a depending on r under the condition that they and their derivatives have uniform estimates for r.

In the cases of i)–a) and ii)-a) a more precise consideration is needed. $s_{1\eta}$ and $s_{2\eta}$ denote $s^{(\gamma)}(\eta)$ or $s^{(\gamma)}(-\eta)$ which are contained U_{ij}^k for $i=1$ or 2. Notations $V_i=(\sigma\circ R)(U_{ij}^k)$ and $a_\mu=(\sigma\circ R)(s_{\mu\eta})$ will also be used. We put $\rho\in C_0^\infty(\boldsymbol{R}^1)$ as

$$\rho(r) = \begin{cases} 0 & |r|\geqq 1 \\ 1 & |r|\leqq 1/2\,. \end{cases}$$

Then $v_{ij}^k(x, r)$ can be written as

$$\begin{aligned}(5.6) \qquad & v_{ij}^k(x, r) \\ & = \sum_{\mu=1}^{\mu(\eta)}\int_{V_i} e^{ix\cdot s(\sigma)}m(s(\sigma), r)\psi_{ij}^k(s(\sigma))\rho(|\sigma-a_\mu|/\delta^{n+l})w(\sigma)d\sigma \\ & \quad +\int_{V_i} e^{ix\cdot s(\sigma)}m(s(\sigma), r)\psi_{ij}^k(s(\sigma))\prod_{\mu=1}^{\eta(\mu)}(1-\rho(|\sigma-a_\mu|/\delta^{n+l}))w(\sigma)d\sigma \\ & \equiv \sum_{\mu=1}^{\mu(\eta)} J_{1\mu}+J_2\,,\end{aligned}$$

where $\delta=\min_\mu \operatorname{dist}_{V_i}(a_\mu, \partial U_{ij}^k\cup Z_0)$ and $\mu(\eta)=1$ for $i=1$ and $=1$ or 2 for $i=2$.

First we consider $J_{1\mu}$. For simplicity the index μ will be omitted, for example $J_{1\mu}=J_1$, $a_\mu=a$ and so on. To consider J_1, we shall introduce another coordinate $\tilde\sigma=(\tilde\sigma_1, \cdots, \tilde\sigma_{n-1})$ in the neighborhood of s_η in the following way. Take an orthogonal matrix $T=T_\eta$ with the property ${}^t(T\eta)=(0, \cdots, 0, 1)$, write ${}^t(Ts)=(\tilde s_1, \cdots, \tilde s_n)$ for $s\in S$ and define $(\tilde\sigma_1, \cdots, \tilde\sigma_{n-1})=(\tilde s_1, \cdots, \tilde s_{n-1})$. This coordinate system is well-defined in

$$(5.7) \qquad U_\eta\equiv\{s\in S;\ \operatorname{dist}_S(s, s_\eta)<R \operatorname{dist}_S(s_\eta, Z_S)\}$$

if the constant R is sufficiently small. With this coordinate $\eta\cdot s(\tilde\sigma)=\tilde s_n(\tilde\sigma)$ and the eigenvalues of the Hesse matrix of $\tilde{s}_n(\tilde{\sigma})$ coinside with the principal curvatures at s_η. Moreover we have

$$\mathrm{Hess}(\eta \cdot s(\tilde{\sigma}))|_{s=s_\eta} = K(s_\eta) \tag{5.8}$$

and

$$dS_k = (1+|\nabla_{\tilde{\sigma}} \tilde{s}_n(\tilde{\sigma})|^2)^{1/2} d\tilde{\sigma} . \tag{5.9}$$

(For details refer to M. Matsumura [3, § 5]). Here some lemmas about $\tilde{\sigma}$ will be prepared. Note that T_η can be written as

$$T_\eta = \begin{bmatrix} t_1 \\ \vdots \\ t_{n-1} \\ {}^t\eta \end{bmatrix},$$

where $\{t_1, \cdots, t_{n-1}\}$ spans the tangent plane at s_η.

Lemma 5.1. $$\left\| \frac{\partial \sigma_\nu}{\partial \tilde{\sigma}_\mu} \right\| \leqq C$$

for some constant C and for any ν and μ.

Proof. Note that

$$\left[\frac{\partial \sigma_\nu}{\partial \tilde{\sigma}_\mu}\right] = \left[\frac{\partial \tilde{\sigma}_\mu}{\partial \sigma_\nu}\right]^{-1} \quad \text{i.e.} \quad \vec{\nabla}_{\tilde{\sigma}} \sigma = (\vec{\nabla}_\sigma \tilde{\sigma})^{-1} .$$

It follows that

$$\frac{\partial \tilde{\sigma}_\mu}{\partial \sigma_\nu} = \frac{\partial}{\partial \sigma_\nu}(Ts)_\mu = \left(T \frac{\partial s}{\partial \sigma_\mu}\right)_\mu = t_\mu \frac{\partial s}{\partial \sigma_\mu} .$$

Thus

$$\frac{\partial \tilde{\sigma}}{\partial \sigma_\nu} = \begin{bmatrix} t_1 \\ \vdots \\ t_{n-1} \end{bmatrix} \frac{\partial s}{\partial \sigma_\nu},$$

that is,

$$\vec{\nabla}_\sigma \tilde{\sigma} = \begin{bmatrix} t_1 \\ \vdots \\ t_{n-1} \end{bmatrix} \vec{\nabla}_\sigma s .$$

Then

$$T(\vec{\nabla}_\sigma s, \eta) = \left[\begin{array}{c|c} \vec{\nabla}_\sigma \tilde{\sigma} & 0 \\ \hline * & 1 \end{array}\right].$$

Hence

$$\det(\vec{\nabla}_\sigma \tilde{\sigma}) = \det T \cdot \cdot \det(\vec{\nabla}_\sigma s, \eta) . \tag{5.10}$$

On the other hand η can be written as

$$\eta=\cos(\eta, N(s))N(s)+c_1\frac{\partial s}{\partial\sigma_1}+\cdots+c_{n-1}\frac{\partial s}{\partial\sigma_{n-1}}$$

for some constants $c_1, \cdots, c_{n-1}$ because $\left\{\frac{\partial s}{\partial\sigma_1}, \cdots, \frac{\partial s}{\partial\sigma_{n-1}}\right\}$ span the tangent plane of S at $s=s(\sigma)$. The equality $\det(\vec{\nabla}_\sigma s, N(s))=(-1)^{n+2}w(\sigma)$ is easily obtained (refer to [3, § 5]). Then

$$\begin{aligned}&\det(\vec{\nabla}_\eta s, \eta)\\ &=\cos(\eta, N(s))\det(\vec{\nabla}_\sigma s, N(s))+\sum_{\mu=1}^{n-1}c_\mu\det\left(\vec{\nabla}_\sigma s, \frac{\partial s}{\partial\sigma_\mu}\right)\\ &=\cos(\eta, N(s))\det(s\vec{\nabla}_\sigma, N(s))\\ &=\cos(\eta, N(s))w(\sigma)\,.\end{aligned}$$

By making R of (5.7) sufficiently small, we may assume that $\cos(\eta, N(s))>1/2$. Thus by Lemma 4.1 5) we have

$$|\det(\vec{\nabla}_\sigma s, \eta)|\geqq c/2>0\,. \tag{5.11}$$

Hence (5.10) and (5.11) imply

$$|\det\vec{\nabla}_\sigma\tilde{\sigma}|\geqq c/2>0\,.$$

Lemma 4.1 1) gives

$$\left|\frac{\partial\tilde{\sigma}_\mu}{\partial\sigma_\nu}\right|=\left|t_\mu\frac{\partial s}{\partial\sigma_\nu}\right|\leqq Const\,.$$

Thus it follows that

$$\|(\vec{\nabla}_\sigma\tilde{\sigma})^{-1}\|=\left\|\left(\frac{polyn.\ of\ \frac{\partial\tilde{\sigma}_\mu}{\partial\sigma_\nu}}{\det\vec{\nabla}_\sigma\tilde{\sigma}}\right)\right\|\leqq Const\,. \qquad \text{Q.E.D.}$$

Lemma 5.2. 1) $\quad \mathrm{dist}_{V_\eta}(\sigma_1, \sigma_2)\sim\mathrm{dist}_{\tilde{V}_\eta}(\tilde{\sigma}_1, \tilde{\sigma}_2)\,,$

where V_η and $\tilde{V}_\eta$ denote $(\sigma\circ R)(U_\eta)$ and $\tilde{\sigma}(U_\eta)$ respectively.

2) $$\mathrm{dist}_{V_\eta}(\sigma, Z_0)\sim\mathrm{dist}_{\tilde{V}_\eta}(\tilde{\sigma}, \tilde{Z}_0)\,,$$

where $\tilde{Z}_0=\tilde{\sigma}(Z_S\cap U_\eta)$.

Proof. The boundedness of $\nabla_{\tilde{\sigma}}\sigma$ and $\nabla_\sigma\tilde{\sigma}$ has already proved in Lemma 5.1. 1) is clear from this.

Here we prove 2). There exists $\sigma_0\in Z_0$ such that

$$\mathrm{dist}_{V_\eta}(\sigma, Z_0)=\mathrm{dist}_{V_\eta}(\sigma, \sigma_0)\,.$$

Then 1) gives

$$\operatorname{dist}_V(\sigma, \sigma_0) \geqq \tilde{c}^{-1} \operatorname{dist}_{\tilde{V}_\eta}(\tilde{\sigma}, \tilde{\sigma}_0) \qquad (\tilde{\sigma}_0 = \tilde{\sigma}(\sigma_0) \in \tilde{Z}_0)$$
$$\geqq \tilde{c}^{-1} \operatorname{dist}_{\tilde{V}_\eta}(\tilde{\sigma}, \tilde{Z}_0)\,.$$

In the same way

$$\operatorname{dist}_{\tilde{V}_\eta}(\tilde{\sigma}_\eta, \tilde{Z}_0) \geqq c^{-1} \operatorname{dist}_{V_\eta}(\sigma_0, Z_0)$$

follows. Q.E.D.

Lemma 5.3. $\quad |\partial_{\tilde{\sigma}}^{\alpha} \sigma| \leqq C_\alpha \operatorname{dist}(\tilde{\sigma}, \tilde{Z}_0)^{-|\alpha|+1}$

for some constants C_α.

Proof. The case of $|\alpha|=1$ is already proved in Lemma 5.1. Here we only prove the case of $|\alpha|=2$. When $|\alpha|>2$ the proof is almost same as the following. Differentiate an equality

$$I = \vec{\nabla}_{\tilde{\sigma}} \sigma \cdot \vec{\nabla}_\sigma \tilde{\sigma} \tag{5.12}$$

by $\tilde{\sigma}$ and we have

$$0 = \frac{\partial}{\partial \tilde{\sigma}_\nu} \vec{\nabla}_{\tilde{\sigma}} \sigma \cdot \vec{\nabla}_{\tilde{\sigma}} \tilde{\sigma} + \vec{\nabla}_\sigma \tilde{\sigma} \cdot \frac{\partial}{\partial \tilde{\sigma}_\nu} \vec{\nabla}_\sigma \tilde{\sigma}$$
$$= \frac{\partial}{\partial \tilde{\sigma}_\nu} \vec{\nabla}_{\tilde{\sigma}} \sigma \cdot \vec{\nabla}_\sigma \tilde{\sigma} + \vec{\nabla}_{\tilde{\sigma}} \sigma \Big(\sum_{\mu=1}^{n-1} \frac{\partial \sigma_\mu}{\partial \tilde{\sigma}_\nu} \cdot \frac{\partial}{\partial \sigma_\mu} \vec{\nabla}_\sigma \tilde{\sigma} \Big)\,.$$

Then

$$\frac{\partial}{\partial \tilde{\sigma}_\nu} \vec{\nabla}_{\tilde{\sigma}} \sigma = - \vec{\nabla}_{\tilde{\sigma}} \sigma \Big(\sum_{\mu=1}^{n-1} \frac{\partial \sigma_\mu}{\partial \tilde{\sigma}_\nu} \cdot \frac{\partial}{\partial \sigma_\mu} \vec{\nabla}_\sigma \tilde{\sigma} \Big) \vec{\nabla}_{\tilde{\sigma}} \sigma\,. \tag{5.13}$$

By use of a relation

$$\frac{\partial^2 \tilde{\sigma}_\mu}{\partial \tilde{\sigma}_\nu \partial \tilde{\sigma}_\rho} = t_\mu \frac{\partial^2 s}{\partial \sigma_\nu \sigma \partial_\rho}\,,$$

it follows from Lemma 4.3 2) and Lemma 5.1 that

$$\left| \frac{\partial^2 \sigma}{\partial \tilde{\sigma}_\nu \partial \tilde{\sigma}_\mu} \right| \leqq \text{Const.} \operatorname{dist}_{V_\eta}(\sigma, Z_0)^{-1}$$

from Lemma 5.2 2)

$$\leqq \text{Const.} \operatorname{dist}_{\tilde{V}_\eta}(\tilde{\sigma}, \tilde{Z}_0)^{-1}\,.$$

Q.E.D.

Since the support of $\rho(|\sigma - a|/\delta^{n+l})$ is contained in V_η, we can make a change of variables σ to $\tilde{\sigma}$ in J_1. Then we get by using (5.9)

$$J_1=\int_{\tilde V} e^{i|x|\tilde s_n(\tilde\sigma)} m(s(\tilde\sigma), r)\psi^k_{ij}(s(\tilde\sigma))\rho(|\sigma(\tilde\sigma)-a|/\delta^{n+l})(1+|\nabla_{\tilde\sigma}\tilde s_n(\tilde\sigma)|^2)^{1/2}\,d\tilde\sigma\,.$$

The estimations (4.37), (5.4) and Lemma 5.3 imply

$$|\partial^\alpha_{\tilde\sigma}\tilde s_n(\tilde\sigma)|\leqq C_\alpha\,\mathrm{dist}_{\tilde V_\eta}(\tilde\sigma, \tilde Z_0)^{-|\alpha|+1}$$

and

$$|\partial^\alpha_{\tilde\sigma} m(s(\tilde\sigma),r)|\leqq C_\alpha\,\mathrm{dist}_{\tilde V_\eta}(\tilde\sigma, \tilde Z_0)^{-|\alpha|}\,.$$

Lemma 5.2 implies

$$\mathrm{dist}_{V_\eta}(a, Z_0)\sim\mathrm{dist}_{\tilde V_\eta}(\tilde a, \tilde Z_0)$$

where $\tilde a$ denotes $\tilde\sigma(s_\eta)$. Then it follows that

$$\mathrm{dist}_{\tilde V_\eta}(\tilde\sigma, \tilde Z_0)\sim\mathrm{dist}_{\tilde V_\eta}(\tilde a, \tilde Z_0)\sim\delta$$

on support of $\rho(|\sigma(\cdot)-a|/\delta^{n+l})$. This fact implies

$$|\partial^\alpha_{\tilde\sigma}\rho(|\sigma(\tilde\sigma)-a|/\delta^{n+l})|\leqq C_\alpha\delta^{-|\alpha|(n+l)}\,.$$

Summing up these facts, we have

$$|\partial^\alpha_{\tilde\sigma}\tilde s_n(\tilde\sigma)|\leqq C_\alpha\delta^{-|\alpha|+1}$$

and

$$\begin{aligned}&|\partial^\alpha_{\tilde\sigma}[m(s(\tilde\sigma), r)\psi^k_{ij}(s(\tilde\sigma))\rho(|\sigma(\tilde\sigma)-a|/\delta^{n+l})(1+|\nabla_{\tilde\sigma}\tilde s_n(\tilde\sigma)|^2)^{1/2}]|\\ &\leqq C_\alpha\delta^{-|\alpha|(n+l)}\,.\end{aligned}$$

We apply Proposition 3.4 to the integral J_1 by setting $N=n-1$, $a=a$, $h_a=s_n$ and so on. Note that the conditions corresponding to (3.14) and (3.16) are verified in the above estimates. Then we have

$$\begin{aligned}J_1=(2\pi)^{-(n-1)/2}e^{i|x|\eta\cdot s}\psi_{\underset{(-)}{+}}(s)|K(s)|^{-1/2}m(s, r)&\\ \cdot\psi^k_{ij}(s)|x|^{-(n-1)/2}|_{s=s_\eta}+\bar q^k_{ij}(x, r)\,,&\end{aligned}\tag{5.14}$$

where

$$|\bar q^k_{ij}(x, r)|\leqq C\delta^{-\mu}|x|^{-(n-1)/2-\nu}\tag{5.15}$$

for any positive number μ and for some positive number ν and C independent of δ and r.

In the rest of this section we prepare some fundamental facts which are needed to estimate J_2.

Lemma 5.4. *Let $a=(\sigma\circ R)(s_\eta)$ be a stationary point. Let $\rho\in C^\infty_0(\boldsymbol{R}^1)$ be a function with*

$$\rho(r)=\begin{cases}1 & for \quad |r|\leqq 1/2\\ 0 & for \quad |r|\geqq 1\,.\end{cases}$$

$V_i=(\sigma\circ R)(U^k_{ij})$, $\delta=\mathrm{dist}_{V_i}(a,\ \partial V\cup Z_0)$ *and* $N=n-1$. *Put* $h_a(\sigma)=\eta\cdot s(\sigma)=N(s(a))\cdot s(\sigma)$. *Then*

1) *there exists a transformation* $\sigma\mapsto\Xi$ *in* $\{|\sigma-a|\leqq c_1\delta^{N+1+l}\}$ *for some constant* c_1 *which satisfies the following properties*:

i) *the image of* $\{|\sigma-a|\leqq c_1\delta^{N+1+l}\}$ *is contained in* $\{|\Xi|\leqq c_2\delta^{N+1+l}\}$, *and*

$$c_3^{-1}|\sigma-a|\leqq|\Xi|\leqq c_3|\sigma-a|$$

for some constants c_2 *and* c_3.

ii)

$$h_a(\sigma)=\frac{1}{2}(\lambda_1\Xi_1^2+\cdots+\lambda_N\Xi_N^2)+h_a(a)$$

where $\lambda_1\cdots\lambda_N$ *are the principal curvatures of* S *at* s_η.

2)

$$\int_{B_a(c_1)}|\,|\nabla h_a(\sigma)|^{-N+m}d\sigma\leqq C_m|\lambda_1\cdots\lambda_{N-m}|^{-1}\delta^{m(N+1+l)}\,, \tag{5.16}$$

where

$$B_a(c_1)=\{(c_1/2)\delta^{N+1+l}\leqq|\sigma-a|\leqq c_1\delta^{N+1+l}\}\,. \tag{5.17}$$

Proof. 1) First of all we make the transformation $V\in\sigma\mapsto\tilde\sigma\in V$. By Lemma 5.2 1) $\{|\sigma-a|\leqq c_1\delta^{N+1+l}\}$ is mapped into $\{|\tilde\sigma-\tilde a|\leqq\tilde c_1\delta^{N+1+l}\}$ for some c_1. As we have already remarked the eigenvalues of Hesse matrix of $h_a(\tilde\sigma)=\eta\cdot s(\tilde\sigma)$ coinside the principal curvatures at s_η. So we make a transformation $\tilde\sigma\mapsto\tilde\Xi$ as

$$\tilde\sigma-a=K_a(\tilde\sigma)^{1/2}\tilde\Xi\,.$$

From Lemma 3.5 it is well-defined. We make one more transformation $\tilde\Xi\mapsto\Xi$ as $P\Xi=\tilde\Xi$, where P is an orthogonal matrix satisfying

$${}^tPH_a(a)P=\mathrm{diag}(\lambda_1\cdots\lambda_N)\quad\left(H_a(a)=\left(\frac{\partial^2h_a}{\partial\sigma_j\partial\sigma_k}(a)\right)\right).$$

Then the assertions of lemma follow from Lemma 3.5.

2) Lemma 3.6 and Lemma 5.1 imply that $J\left(\dfrac{\partial\sigma}{\partial\Xi}\right)$ is bounded. $B_a(C_1)$ is mapped into $B\equiv\{(c_1c_3^{-1}2)\delta^{N+1+l}\leqq|\Xi|\leqq c_2\delta^{N+1+l}\}$ from the property i) of transformation. Then we make a transformation $\sigma\to\Xi$ in the integral (5.16), and we have

$$\int_{B_a(c_1)} |\nabla h_a(\sigma)|^{-N+m} d\sigma \leqq C \int_B (\lambda_1^2 \Xi_1^2 + \cdots + \lambda_N^2 \Xi_N^2)^{-(N-m)/2} d\Xi$$
$$= C' \int_{(c_1 c_3^{-1}/2)\delta^{N+1+l}}^{c_1 \delta^{N+1+l}} r^{-N+m} r^{N-1} dr \int_0^{\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} (\lambda_1^2 \cos^2\theta_1$$
$$+ \lambda_2^2 \sin^2\theta_1 \cos^2\theta_2 + \cdots + \lambda_N^2 \sin^2\theta_1 \cdots \sin^2\theta_{N-1})^{-(N-m)/2}$$
$$\cdot \sin^{N-2}\theta_1 \cdots \sin\theta_{N-2} d\theta_1 \cdots d\theta_{N-1}$$
$$= C'' 2^{2N-3} \int_0^{\pi/2} \cdots \int_0^{\pi/2} (\cdots) d\theta_1 \cdots d\theta_{N-1} \int (\cdots) dr\,.$$

Integration by r can easily be calculated and it is $C\delta^{m(N+1+l)}$ for $m \neq 0$ or $C|\log \delta|$ for $m=0$. Integrations by $\theta_1, \cdots, \theta_{N-1}$ are calculated as follows. In the case of $m=0$,

$$\int_0^{\pi/2} \cdots \int_0^{\pi/2} (\lambda_1^2 \cos^2\theta_1 + \cdots + \lambda_N \sin^2\theta_1 \cdots \sin^2\theta_{N-1})^{-N/2}$$
$$\cdot \sin^{N-2}\theta_1 \cdots \sin\theta_{N-2} d\theta_1 \cdots d\theta_{N-1}$$
$$= \int_0^{\pi/2} \cdots \int_0^{\pi/2} \Big[\int_0^{\pi/2} (\lambda_1^2 \cos^2\theta_1 + (\lambda_2^2 \cos^2\theta_2 + \lambda_3^2 \sin^2\theta_3 \cos^2\theta_3 + \cdots$$
$$+ \lambda_N^2 \sin^2\theta_2 \cdots \sin^2\theta_{N-1}) \sin^2\theta_1)^{N/2}) \sin^{N-2}\theta_1 d\theta_1 \Big]$$
$$\cdot \sin^{N-3}\theta_2 \cdots \sin\theta_{N-2} d\theta_2 \cdots d\theta_{N-1}$$
$$= \int_0^{\pi/2} \cdots \int_0^{\pi/2} (\lambda_2^2 \cos^2\theta_2 + \cdots + \lambda_N^2 \sin^2\theta_2 \cdots \sin^2\theta_{N-1})^{-N/2+1/2}$$
$$\cdot \sin^{N-3}\theta_2 \cdots \sin\theta_{N-2} d\theta_2 \cdots d\theta_{N-1} |\lambda_1|^{-1} \int_0^{\infty} x^{N-2}(1+x^2)^{N/2} dx$$
$$= \cdots = \mathit{Const.} |\lambda_1 \cdots \lambda_N|^{-1}\,.$$

In the case of $m>0$ the calculus is almost the same. But last step only N–m times of iterations are needed. Thus the proof of lemma is complete. Q.E.D.

From Lemma 5.4 1)

$$\lambda_j = \frac{\partial^2}{\partial \Xi_j^2} h_a(\sigma(\Xi))$$

follows. Then it is easy to verify

$$|\lambda_j| \leqq C\delta^{-(N+1+l)} \qquad \text{for} \quad j=1, 2, \cdots, N,$$

where C is independent of a.

When s_η is in the neighborhood of $Z_S^{(1)}$, the assumption Siii) gives

$$|\lambda_1 \cdots \lambda_N| \leqq \mathit{Const.}\, \delta^{-(d-1)}\,.$$

Then each λ_j has the polynomial order of δ:

$$\lambda_j \sim \delta^{M_j} \qquad \text{for some } M_j \text{ (maybe negative)}\,. \tag{5.18}$$

When s_η is in the neighborhood of $Z_S^{(2)}$

$$|\lambda_1 \cdots \lambda_N| = |K(s_\eta)| \leqq Const\ \delta^l .$$

by (2.27). Then (5.18) also holds for this case with different M_j.

Next we state a relation between the metric of $V_i \subset \boldsymbol{R}_\sigma^{n-1}$ and that of $\bar{V}_i \subset \boldsymbol{R}_{\bar{\sigma}}^{n-1}$.

Lemma 5.5. *Let $a \in V_i \backslash Z_0 = (\sigma \circ R)(U_{ij}^k)$. s_0 and b denote $(\sigma \circ R)^{-1}(a)$ and $\bar{\sigma} \circ N \circ (\sigma \circ R)^{-1}(a)$ respectively.*

If $|\sigma - a| \leqq \bar{c}\delta^{N+1+l}$, then $|\bar{\sigma} - b| \geqq C|\lambda_q||\sigma - a|$ for some positive constant C where $\lambda_1, \cdots, \lambda_{n-1}$ denote the principal curvatures at s_0 and $|\lambda_q| = \min\{|\lambda_1|, \cdots, |\lambda_{n-1}|\}$.

Proof. Since $|\sigma - a| \leqq \bar{c}\delta^{N+1+l}$, we can introduce the local coordinate $\tilde{\sigma}$. By use of this coordinate unit normal at $s(\tilde{\sigma})$ can be written as

$$N(s(\tilde{\sigma})) = \frac{1}{(1+|\nabla_{\tilde{\sigma}}\tilde{s}_n|^2)^{1/2}}\,{}^t\!\left(-\frac{\partial \tilde{s}_n}{\partial \tilde{\sigma}_1}, \cdots, -\frac{\partial \tilde{s}_n}{\partial \tilde{\sigma}_{n-1}}, 1\right)$$

and

$$N(s_0) = (0, \cdots, 0, 1) .$$

Then

$$\begin{aligned} |N(s) - N(s_0)|^2 &= \frac{|\nabla_{\tilde{\sigma}} s_n|^2 + (1 - (|\nabla_{\tilde{\sigma}}\tilde{s}_n|^2 + 1)^{1/2})^2}{1 + |\nabla_{\tilde{\sigma}}\tilde{s}_n|^2} \\ &= \frac{2(1 + |\nabla_{\tilde{\sigma}}\tilde{s}_n|^2)^{1/2}((1 + |\nabla_{\tilde{\sigma}}\tilde{s}_n|^2)^{1/2} - 1)}{1 + |\nabla_{\tilde{\sigma}}\tilde{s}_n|^2} \\ &\geqq c'|(1 + |\nabla_{\tilde{\sigma}}\tilde{s}_n|)^{1/2} - 1| \quad (\text{for some } c') . \end{aligned}$$

By the Taylor expansion of $(1+x^2)^{1/2}$ we have

$$|N(s) - N(s_0)|^2 \geqq c''|\nabla_{\tilde{\sigma}}\tilde{s}_n|^2$$

for some constant $c'' > 0$. Here we make a change of variables $\tilde{\sigma} \to \Xi$ once more. By this coordinate

$$\nabla_{\Xi}\tilde{s}_n = \vec{\nabla}_{\Xi}\tilde{\sigma} \cdot \nabla_{\tilde{\sigma}}\tilde{s}_n$$

and

$$|\nabla_{\Xi}\tilde{s}_n|^2 = \lambda_1^2\Xi_1^2 + \cdots + \lambda_{n-1}^2\Xi_{n-1}^2$$

hold. Hence by noting (3.47), we have

$$\text{(5.19)} \qquad Const.\ |\nabla_{\tilde{\sigma}}\tilde{s}_n| \geqq |\nabla_{\Xi} s_n| \geqq |\lambda_q||\Xi|$$

Lemma 5.4 1) and (5.19) give

$$|N(s)-N(s_0)| \geqq C|\lambda_q||\sigma-a|$$

for some positive constant C. Thus the conclusion of lemma follows from (4.46). Q.E.D.

Lemma 5.5 shows that the intersection of $\{|\sigma-a| \leqq \bar{c}\delta^{N+1+l}\}$ and the inverse image of $\{|\bar{\sigma}-b| \leqq c_0|\lambda_q|\delta^{N+1+l}\}$ by $\bar{\sigma}\circ N\circ R^{-1}\circ\sigma^{-1}$ is included in $\{|\sigma-a| \leqq c_1\delta^{N+1+l}\}$ for some constants c_0 and c_1. We may assume that c_0 and c_1 are sufficiently small and that $\{|a-\sigma| \leqq c_1\delta^{N+1+l}\}$ is included in $\{|\sigma-a| \leqq \bar{c}\delta^{N+1+l}\}$. Then the connectedness of $U \equiv (\bar{\sigma}\circ N\circ R^{-1}\circ\sigma^{-1})^{-1}\{|\bar{\sigma}-b| \leqq c_0|\lambda_q|\cdot\delta^{N+1+l}\}$ gives

$$U \subset \{|\sigma-a| \leqq c_1\delta^{N+1+l}\}.$$

Moreover this fact implies that if $|\sigma-a| \geqq c_1\delta^{N+1+l}$, then $|\bar{\sigma}-b| \geqq c_0|\lambda_q|\delta^{N+1+l}$.

Last of this section we estimate the gradient of the phase function $\eta\cdot s(\sigma)$ from below.

Lemma 5.6. *Let $s=s(\sigma)\in U^k_{ij}$ and $\eta\in S^{n-1}$. For any orthogonal basis $\{e_1(\sigma), \cdots, e_{n-1}(\sigma)\}$ of tangent plane of S_k at $s=s(\sigma)$, we have*

$$|\nabla_\sigma(\eta\cdot s(\sigma))| \geqq C|\vec{\eta}(e_1, \cdots, e_{n-1})|$$

where C is independent of σ.

Proof. Recall that

$$\nabla_\sigma(\eta\cdot s(\sigma)) = \vec{\eta}\cdot\vec{\nabla}_\sigma s(\sigma)$$

and

$$\vec{\nabla}_\sigma s(\sigma) = \begin{pmatrix} \dfrac{\partial s_1}{\partial\sigma_1} & \cdots & \dfrac{\partial s_1}{\partial\sigma_{n-1}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial s_n}{\partial\sigma_1} & \cdots & \dfrac{\partial s_n}{\partial\sigma_{n-1}} \end{pmatrix}$$

and that the column vectors of $\vec{\nabla}_\sigma s(\sigma)$ span the tangent plane of S_k at $s=s(\sigma)$. Denoting by $P=P(\sigma)$ an $(n-1)\times(n-1)$-matrix which transforms $\left\{\dfrac{\partial s}{\partial\sigma_1} \cdots \dfrac{\partial s}{\partial\sigma_{n-1}}\right\}$ to $\{e_1(\sigma) \cdots e_{n-1}(\sigma)\}$:

$$\left(\frac{\partial s}{\partial\sigma_1} \cdots \frac{\partial s}{\partial\sigma_{n-1}}\right)P = (e_1 \cdots e_{n-1}), \tag{5.20}$$

we have

$$\nabla_\sigma(\eta\cdot s(\sigma)) = \vec{\eta}(e_1 \cdots e_{n-1})P^{-1}.$$

Since $N(s(\sigma))$ is a unit normal of S_k at $s(\sigma)$, $\left\{\dfrac{\partial s}{\partial\sigma_1} \cdots \dfrac{\partial s}{\partial\sigma_{n-1}} N(s(\sigma))\right\}$ is a basis of

$\boldsymbol{R}^n$. Then we multiply the inverse matrix of $\left(\frac{\partial s}{\partial\sigma_1}\cdots\frac{\partial s}{\partial\sigma_{n-1}}N(s(\sigma))\right)$ to the both side of (5.20) from left and we get

$$\begin{bmatrix} I_{n-1} \\ 0\cdots 0\end{bmatrix} P=\left(\frac{\partial s}{\partial\sigma_1}\cdots\frac{\partial s}{\partial\sigma_{n-1}}N(s(\sigma))\right)^{-1}(e_1\cdots e_{n-1})\,. \tag{5.21}$$

The left-hand side of (5.21) is a product of $n\times(n-1)$-matrix and $(n-1)\times(n-1)$-matrix, and the right-hand side is a product of $n\times n$-matrix and $n\times(n-1)$-matrix. Since $\det\left(\frac{\partial s}{\partial\sigma_1}\cdots\frac{\partial s}{\partial\sigma_{n-1}}N(s(\sigma))\right)=(-1)^{n+2}w(\sigma)$, it follows from Lemma 4.1 5) that $\det\left(\frac{\partial s}{\partial\sigma_1}\cdots\frac{\partial s}{\partial\sigma_{n-1}}N(s(\sigma))\right)^{-1}$ is bounded with respect to σ. Then (4.11) implies

$$||P||\leqq c_0$$

for some constant c_0 which is independent of σ and the choice of the basis $\{e_1\cdots e_{n-1}\}$. Thus

$$|\vec{\eta}(e_1\cdots e_{n-1})|=|\nabla_\sigma(\eta\cdot s(\sigma))P|\leqq||P||\,|\nabla_\sigma(\eta\cdot s(\sigma))|$$

follows. Hence we have

$$|\nabla_\sigma(\eta\cdot s(\sigma))|\geqq||P||\,|\vec{\eta}(e_1\cdots e_{n-1})|\geqq c_0^{-1}|\vec{\eta}(e_1\cdots e_{n-1})|\,.$$

Q.E.D.

$\vec{\eta}(e_1\cdots e_{n-1})$ represents the projection of η to the tangent plane of S_k at $s(\sigma)$. Thus

$$|\vec{\eta}(e_1\cdots e_{n-1})|=\sin\theta_0$$

where θ_0 denotes the angle between η and $N(s(\sigma))$. Since

$$\theta_0=\mathrm{dist}_{S^{n-1}}(\eta, N(s(\sigma)))\,,$$

we have from Lemma 5.6

$$|\nabla_\sigma(\eta\cdot s(\sigma))|\geqq c\ \mathrm{dist}_{S^{n-1}}(\eta, N(s(\sigma))) \tag{5.22}$$

for some constant $c>0$. Then we have from (4.46) and (5.22)

$$|\nabla_\sigma(\eta\cdot s(\sigma))|\geqq c\ \mathrm{dist}_{\boldsymbol{R}^{n-1}}(\bar{\sigma}, b)=c|\bar{\sigma}-b|\,, \tag{5.23}$$

where $b=(\bar{\sigma}\circ N)(s_\eta)$. ($s_\eta$ denotes $s^{(\gamma)}(\eta)$ or $s^{(\gamma)}(-\eta)$.)

6. Modification of the stationary phase method (2)

The purpose of this section is to give a stationary phase estimate of J_2

of (5.6). First we treat the case of $i=1$:

$$J_2=\int_{V_1} e^{i|x|\eta\cdot s(\sigma)} m(s(\sigma), r)\psi^k_{ij}(s(\sigma))(1-\rho(|\sigma-a|/\delta^{n+l}))w(\sigma)d\sigma .$$

Here we put $h_a(\sigma)=\eta\cdot s(\sigma)=N(s(a))\cdot s(\sigma)$, $g_a(\sigma)=m(s(\sigma), r)\psi^k_{ij}(s(\sigma))w(\sigma)$, $|x|=t$, $N=n-1$, $V=V_1$ and $M=Z_0=(\sigma\circ R)(U^k_{ij}\cap Z^{(1)}_S)$. By Si) M is an $(n-1)$-dimensional submanifold of V. Clearly

$$\nabla h_a(a)=0 \quad \text{and} \quad \nabla h_a(\sigma)\neq 0 \qquad \text{if} \quad \sigma\neq a . \tag{6.1}$$

(4.37) and (5.4) give

$$|\partial^\alpha g_a(\sigma)|\leqq C_\alpha \operatorname{dist}(\sigma, M)^{-|\alpha|} \qquad \text{for} \quad |\alpha|\geqq 0 \tag{6.2}$$

and

$$|\partial^\alpha h_a(\sigma)|\leqq C_\alpha \operatorname{dist}(\sigma, M)^{-|\alpha|+1} \qquad \text{for} \quad |\alpha|>0 . \tag{6.3}$$

The transformation $\bar{\sigma}\circ N\circ(\sigma\circ R)^{-1}$: $V\backslash M\ni\sigma\mapsto\bar{\sigma}\in\bar{V}\subset \boldsymbol{R}^N$ satisfies

$$J\left(\frac{\partial\sigma}{\partial\bar{\sigma}}\right)\leqq c_0 \operatorname{dist}(\bar{\sigma}, \partial\bar{V})^{d-1},$$

$$|\nabla h_a(\sigma(\bar{\sigma}))|\geqq c_1|\bar{\sigma}-b| \qquad \text{where} \quad a\to b$$

and

$$\operatorname{dist}(\sigma, M)\sim\operatorname{dist}(\bar{\sigma}, \bar{V})$$

for some constants c_0 and c_1 because of Siv), (4.55) and Lemma 5.6. Then we prove

Proposition 6.1. *Under the above situation there exist some positive number ν and C_m independent of a such that*

$$\left|\int_V e^{ith_a(\sigma)} g_a(\sigma)(1-\rho(|\sigma-a|/\bar{c}\delta^{N+1+l}))d\sigma\right|\leqq C_m t^{-(m-1)-\nu}\delta^{-\mu}$$

for any natural number m with $m\leqq d$ and $m-1\leqq N/2$ and for any positive number μ.

Proof. By a similar way as in the proof of (3.10) we get

$$\begin{aligned} I &\equiv \int_V e^{ith_a(\sigma)} g_a(\sigma)(1-\rho(|\sigma-a|/\bar{c}\delta^{N+1+l})d\sigma \\ &= (-1/it)^{m-1}\int_V e^{ith_a(\sigma)}\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)^{m-1} \\ &\qquad \cdot[g_a(\sigma)(1-\rho(|\sigma-a|/\bar{c}\delta^{N+1+l}))]d\sigma \\ &= (-1/it)^{m-1}\int_V e^{ith_a(\sigma)}\left[\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)^{m-1}\tilde{g}_a(\sigma)\right](1-\rho_\delta(\sigma))d\sigma \end{aligned} \tag{6.4}$$

$$+(-1/it)^{m-1}\int_V e^{ith_a(\sigma)}\left(\left[\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)^{m-1}\tilde{g}_a(\sigma)\right]\rho_\varepsilon(\sigma)d\sigma$$
$$\equiv I_1+I_2,$$

where $\tilde{g}_a(\sigma)=g_a(\sigma)(1-\rho(|\sigma-a|/\bar{c}\delta^{N+1+l}))$ and $\rho_\varepsilon(\sigma)=\rho(\mathrm{dist}(\sigma, M)/\varepsilon)$. Here we put $\varepsilon=\delta t^{-\nu_0}$ $(\nu_0>0)$ for sufficiently large fixed t. By integrating by parts once more, we get

$$I_1=(-1/it)^{m-1}\int_V e^{ith_a(\sigma)}\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)\left\{\left[\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)^{m-1}\tilde{g}_a(\sigma)\right](1-\rho_\varepsilon(\sigma))\right\}d\sigma.$$

Recall that

$$\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}f=\frac{\nabla h_a}{|\nabla h_a|^2}\cdot\nabla f+\frac{\Delta h_a}{|\nabla h_a|^2}\cdot f+\nabla\left(\frac{1}{|\nabla h_a|^2}\right)\cdot\nabla h_a\cdot f,\right. \tag{6.5}$$

and that $|\nabla h_a|^{-2}$ appears in the estimate by one operation of $\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)$. On the other hand for $\alpha\neq0$

$$\mathrm{supp}\ \partial_\sigma^\alpha[\rho(|\sigma-a|/\bar{c}\delta^{N+1+l})]\subset B_a(\bar{c}), \tag{6.6}$$

where $B_a(\bar{c})$ is a set defined by (5.17) by replacing c_1 with $\bar{c}$, and on this set it follows from (6.2) and (6.3) that

$$|\partial^\beta g_a(\sigma)|\leqq C_\beta\delta^{-|\beta|}\quad\text{and}\quad|\partial^\beta h_a(\sigma)|\leqq C_\beta\delta^{-|\beta|+1} \tag{6.7}$$

for $\beta\neq0$. Moreover since t is sufficiently large, $\rho_\varepsilon(\sigma)\equiv1$ on this set. Then, when we write

$$I_1=(-1/it)^m\int_V e^{ith_a(\sigma)}(1-\rho(|\sigma-a|/\bar{c}\delta^{N+1+l}))$$
$$\cdot\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)\left\{\left[\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)^{m-1}g_a(\sigma)\right](1-\rho_\varepsilon(\sigma))\right\}d\sigma$$
$$+(-1/it)^m\int_V e^{ith_a(\sigma)}\sum_{\alpha\neq0}C_\alpha\partial^\alpha[\rho(|\sigma-a|/\bar{c}\delta^{N+1+l})](\cdots)d\sigma$$
$$\equiv I_{11}+I_{12},$$

we get from (6.5), (6.6) and (6.7)

$$|I_{12}|\leqq C'_m t^{-m}\int_{B_a(\bar{c})}\sum_{\alpha\neq0}\delta^{-|\alpha|(N+1+l)}|\nabla h_a(\sigma)|^{-2m+|\alpha|}\delta^{m-|\alpha|}d\sigma$$
$$\leqq C'_m\delta^{-m(N+1+l)}t^{-m}\int_{B_a(\bar{c})}\sum_{\alpha\neq0}|\nabla h_a(\sigma)|^{-2m+|\alpha|}d\sigma.$$

It follows from Lemma 5.5 and Lemma 5.6 that

$$|\nabla h_a(\sigma)|\leqq C|\lambda_q||\sigma-a|,$$

where $|\lambda_q|=\min\{|\lambda_1|\cdots|\lambda_N|\}$. Thus

$$(6.8)\qquad \begin{aligned}|I_{12}| &\leqq C''_m t^{-m}\delta^{-m(N+1+l)}\int_{B_a(\bar{c})}\sum_{\alpha\neq 0}|\sigma-a|^{-2m+|\alpha|}d\sigma\\ &\leqq C''_m t^{-m}\delta^{-m(N+1+l)}\delta^{-2m(N+1+l)}\int_{B_a(\bar{c})}d\sigma\cdot|\lambda_q|^{-2m}\\ &=C'''_m t^{-m}\delta^{-3m(N+1+l)+(N+1+l)}|\lambda_q|^{-2m}.\end{aligned}$$

Then we get from (5.18)

$$(6.9)\qquad |I_{12}|\leqq C_m t^{-m}\delta^{-(3m-N)(N+1+l)-2mM_q}.$$

On the other hand

$$(6.10)\qquad \begin{aligned}&|\partial^\alpha\rho_\varepsilon(\sigma)|\leqq C_\alpha\varepsilon^{-|\alpha|},\quad |\partial^\alpha g_a(\sigma)|\leqq C_\alpha\varepsilon^{-|\alpha|}\\ &\text{and}\quad |\partial^\alpha h_a(\sigma)|\leqq C_\alpha\varepsilon^{-|\alpha|+1}\end{aligned}$$

follows from (6.2) and (6.3). Then we get

$$(6.11)\qquad \begin{aligned}|I_{11}| &\leqq C'_m t^{-m}\varepsilon^{-m}\int_V|\nabla h_a(\sigma)|^{-2m}d\sigma\\ &\leqq C''_m t^{-m}\varepsilon^{-m}\int_V|\sigma-a|^{-m}d\sigma\cdot|\lambda_q|^{-2m}\\ &=C'''_m t^{-m}\varepsilon^{-m}\int_{\bar{c}\delta^{N+1+l}/2}^{\infty}\rho^{-2m+N-1}d\rho\cdot|\lambda_q|^{-2m}\\ &=C_m t^{-m}\varepsilon^{-m}\delta^{-(2m-N)(N+1+l)-2mM_q}\\ &=C_m t^{-m+m\nu_0}\delta^{-m-(2m-N)(N+1+l)-2mM_q}.\end{aligned}$$

For I_2 of (6.4) note that

$$\operatorname{supp}\partial^\alpha[\rho(|\sigma-a|/\bar{c}\delta^{N+1+l})]\cap\operatorname{supp}\rho_\varepsilon=\phi\qquad\text{for}\quad \alpha\neq 0$$

when $t>2^{\nu_0}$. Thus

$$(6.12)\qquad \begin{aligned}|I_2| &\leqq C't^{-(m-1)}\int_{V_\varepsilon}|\nabla h_a(\sigma)|^{-2(m-1)}\operatorname{dist}(\sigma, M)^{-m+1}d\sigma\\ &(\text{here } V_\varepsilon=\{\operatorname{dist}(\sigma, M)\leqq\varepsilon/2\})\\ &\leqq C''t^{-(m-1)}\int_{V_\varepsilon}|\sigma-a|^{-2(m-1)}|\lambda_q|^{-2(m-1)}\cdot\operatorname{dist}(\sigma, M)^{-m+1}d\sigma\\ &\leqq C'''t^{-(m-1)}\delta^{-2(m-1)(1+M_q)}\int_0^\varepsilon r^{-(m-1)+d-1}dr\\ &\leqq C'''t^{-(m-1)}\delta^{-2(m-1)(1+M_q)}\varepsilon\\ &=C'''t^{-(m-1)-\nu_0}\delta^{-2(m-1)(1+M_q)+1}.\end{aligned}$$

By putting $\nu_0=1/(m+1)$ and by summing up (6.9), (6.11) and (6.12) we get

$$(6.13)\qquad |I|=|I_{11}+I_{12}+I_2|\leqq C_m t^{-(m-1)-\nu_0}\delta^{-A},$$

where $A=\max\{(3m-N)(N+1+l)+2mM_q,\ m+(2m-1)(N+1+l)+2mM_q,\ 2(m-1)(1+M_q)+1\}$.

Next we look for another estimate of I without using the decomposition of (6.4), and we shall make an interpolation of two estimates. We put

$$\begin{aligned} I &= (-1/it)^{m-1}\int_V e^{ith_a(\sigma)}(1-\rho(|\sigma-a|/\bar{c}\delta^{N+1+l}))\cdot\left(\nabla\cdot\frac{\nabla h_a}{|\nabla h_a|^2}\right)^{m-1} g_a(\sigma)d\sigma \\ &\quad +(-1/it)^{m-1}\int_V e^{ith_a(\sigma)}\sum_{\alpha\neq 0}(-\partial^\alpha[\rho(|\sigma-a|/\bar{c}\delta^{N+1+l})])\cdot(\cdots)d\sigma \\ &\equiv K_1+K_2 \end{aligned}$$

and calculate as the following way.

$$(6.14)\qquad \begin{aligned} |K_2| &\leqq C'_{m-1}t^{-(m-1)}\int_{B_a(\bar{c})}\sum_{\alpha\neq 0}\delta^{-|\alpha|(N+1+l)} \\ &\qquad \cdot|\nabla h_a(\sigma)|^{-2(m-1)+|\alpha|}\delta^{-(m-1)+|\alpha|}d\sigma \end{aligned}$$

by using Lemma 5.4

$$\leqq C''_{m-1}t^{-(m-1)}\sum_{\alpha\neq 0}\delta^{-(m-1)+|\alpha|}|\lambda_1\cdots\lambda_{N-|\alpha|}|^{-1}.$$

Note $|\lambda_1\cdots\lambda_N|\geqq Const.\ \delta^{-(d-1)}$ and (5.18). Then it is easy to verify

$$|\lambda_1\cdots\lambda_{N-|\alpha|}|\leqq c\delta^{(1-|\alpha|/N)(d-1)}.$$

Hence from (6.14) we get

$$(6.15)\qquad \begin{aligned} |K_2| &\leqq C_{m-1}t^{-(m-1)}\sum_{\alpha\neq 0}\delta^{-(m-1)+|\alpha|}\delta^{(1-|\alpha|/N)(d-1)} \\ &= C_{m-1}t^{-(m-1)}\sum_{\alpha\neq 0}\delta^{(d-m)+(1-1/N)|\alpha|} \\ &\leqq C_{m-1}t^{-(m-1)} \end{aligned}$$

(here note $d\geqq m$). Next we estimate K_1 by changing variables σ to $\bar{\sigma}$. As we have remarked after Lemma 5.5, $\Omega\equiv\{|\sigma-a|\geqq\bar{c}\delta^{N+1+l}/2\}$ is mapped into $\{|\bar{\sigma}-b|\geqq\tilde{c}|\lambda_q|\delta^{N+1+l}\}$ for some constant $\tilde{c}$, and by (5.18) into $\bar{\Omega}\equiv\{|\bar{\sigma}-b|\geqq\tilde{c}\delta^{M_q+(N+1+l)}\}$. Thus

$$(6.16)\qquad \begin{aligned} |K_1| &\leqq C'_{m-1}t^{-(m-1)}\int_\Omega|\nabla h_a|^{-2(m-1)}\operatorname{dist}(\sigma, M)^{-(m-1)}d\sigma \\ &\leqq C'_{m-1}t^{-(m-1)}\int_{\bar{\Omega}\cap\bar{V}}|\bar{\sigma}-b|^{-2(m-1)}\operatorname{dist}(\bar{\sigma}, \partial\bar{V})^{-(m-1)+(d-1)}d\bar{\sigma}. \end{aligned}$$

Here note that for sufficiently large M, $\bar{\Omega}\cap\bar{V}\subset\{M\geqq|\bar{\sigma}-b|\geqq\tilde{c}\delta^{M_q+(N+1+l)}\}$. Then since $d\geqq m$, we have

$$|K_1|\leqq C''_{m-1}t^{-(m-1)}\int_{\bar{\Omega}\cap\bar{V}}|\bar{\sigma}-b|^{-2(m-1)}d\bar{\sigma}$$

$$=C''_{m-1}t^{-(m-1)}\int_{\tilde{c}\delta^{M_q+(N+1+l)}}^{M} r^{-2(m-1)+N-1}dr\,.$$

Since $N-2(m-1)\geqq 0$,

$$|K_1|\leqq C_{m-1}t^{-(m-1)}|\log\delta|\,.$$

Thus from (6.15) and (6.16),

$$|I|=|K_1+K_2|\leqq C_{m-1}t^{-(m-1)}|\log\delta| \tag{6.17}$$

follows.

Hence it follows from (6.13) and (6.17) that

$$|I|\leqq C_m^{\theta}C_{m-1}^{1-\theta}t^{-(m-1)-\theta\nu_0}\delta^{-\theta A}|\log\delta|^{1-\theta}$$

for any θ with $0<\theta<1$. Then if we put $\theta<\mu/A$, we have the conclusion of Proposition 6.1 for $\nu=\theta\nu_0$. Q.E.D.

We can get the estimate of J_2 for $i=1$ from Proposition 6.1. Since n is odd, $(n-1)/2$ is an integer. So, if we apply Proposition 6.1 for $m-1=(n-1)/2$, then we have

$$|J_2|\leqq C|x|^{-(n-1)/2-\nu}\delta^{-\mu} \tag{6.18}$$

for any $\mu>0$ and for some $\nu>0$.

By (5.14), (5.15) and (6.18) we have in the case of i)–a)

$$\begin{aligned} v_{ij}^k(x, r) &= (2\pi)^{-(n-1)/2}e^{i|x|\eta\cdot s}\psi_{\substack{+\\(-)}}(s)|K(s)|^{-1/2} \\ &\quad\cdot m(s, r)\psi_{1j}^k(s)|x|^{-(n-1)/2}|_{s=s_\eta}+q_{1j}^k(x, r)\,, \end{aligned} \tag{6.19}$$

where

$$|q_{1j}^k(x, r)|\leqq C\delta^{-\mu}|x|^{-(n-1)/2-\nu} \tag{6.20}$$

for any $\mu>0$ and for some $\nu>0$.

Next we consider the case of $i=2$:

$$J_2=\int_{V_2}e^{i|x|\eta\cdot s(\sigma)}m(s(\sigma), r)\psi_{2j}^k(s(\sigma))\prod_{\mu=1}^{\mu(\eta)}(1-\rho(|\sigma-a_\mu|/\tilde{c}\delta^{n+l}))w(\sigma)d\sigma\,.$$

Here we put $h_\eta(\sigma)=\eta\cdot s(\sigma)$, $g(\sigma)=m(s(\sigma), r)\psi_{2j}^k(s(\sigma))w(\sigma)$, $|x|=t$, $N=n-1$, $M=Z_0=(\sigma\circ R)\,(U_{2j}^k\cap Z_S)$, $V=V_2$ and $\prod_{\mu=1}^{\mu(\eta)}(1-\rho(|\sigma-a_\mu|/\tilde{c}\delta^{n+l}))=\phi_\eta(\sigma)$. (Note that a_μs' depend only on η). By Sii) M is an at most $(N-1)$-dimensional submanifold. Let V^1 and V^2 be domains with the properties $V=V^1\cup V^2\cup M$ and $a_\mu\in V^\mu$ for $\mu=1, 2$. Clearly

$$\nabla h_\eta(a_\mu)=0 \quad\text{for}\quad \mu=1, 2 \quad\text{and}\quad \nabla h_\eta(\sigma)\neq 0 \quad\text{for}\quad \sigma\neq a_\mu \tag{6.21}$$

(6.22) $\quad |\partial^\alpha h_\eta(\sigma)| \leqq C_\alpha$

(6.23) $\quad |\partial^\alpha g(\sigma)| \leqq C_\alpha$

and

(6.24) $\quad |\partial^\alpha \phi_\eta(\sigma)| \leqq C_\alpha \delta^{-(N+1+l)|\alpha|}$,

where C_α's are independent of η. Then we prove the following Proposition.

Proposition 6.2. *Under the above situation there exists some function* $\Phi_\mu(\eta)$ *with*

$$\Phi_\mu(\eta) \in L^{p'}(S_\eta^{n-1})$$

such that

$$\left|\int_V e^{ith_\eta(\sigma)} g(\sigma)\phi_\eta(\sigma)d\sigma\right| \leqq t^{-(m-1)-\nu}\delta^{-\mu}\Phi_\mu(\eta)$$

for any positive number μ, *for any* p' *with* $1 \leqq p' < 1+1/l$ *and for any natural number* m *with* $m-1 \leqq N/2$.

Proof. By integration by parts we have

$$\begin{aligned} I &\equiv \int_V e^{ith_\eta(\sigma)} g(\sigma)\phi_\eta(\sigma)d\sigma \\ &= (-1/it)^n \int_V e^{ith_\eta(\sigma)} \left(\nabla\cdot\frac{\nabla h_\eta}{|\nabla h_\eta|^2}\right)^n (g(\sigma)\phi_\eta(\sigma))d\sigma \end{aligned}$$

for any n.

In the case of $n=m$, we get from (6.24) and Lemma 5.6

$$\begin{aligned} |I| &\leqq C'_m t^{-m} \int_{V\cap \operatorname{supp}\phi_\eta} |\nabla h_\eta|^{-2m} d\sigma \cdot \delta^{-2m(N+1+l)} \\ &\leqq C'_m t^{-m} \sum_{\mu=1}^{\mu(\eta)} \int_{\Omega_\mu} |\nabla h_\eta|^{-2m} d\sigma \cdot \delta^{-2m(N+1+l)}, \end{aligned}$$

where $\Omega_\mu = \{|\sigma - a_\mu| \geqq \bar{c}\delta^{N+1+l}/2\}$. Since $|\nabla h_\eta(\sigma)| \geqq c|\lambda_q||\sigma - a_\mu|$ by Lemma 5.4 and Lemma 5.5 where $|\lambda_q| = \max\{|\lambda_1|, \cdots, |\lambda_N|\}$, we get

$$\begin{aligned} |I| &\leqq C''_m t^{-m}\delta^{-2m(N+1+l)}|\lambda_q|^{-2m} \sum_{\mu=1}^{\mu(\eta)} \int_{\Omega_\mu} |\sigma - a_\mu|^{-2m} d\sigma \\ &\leqq C'''_m t^{-m}\delta^{-2m(N+1+l)}|\lambda_q|^{-2m} \int_{\bar{c}\delta^{N+1+l}}^\infty r^{-2m+N-1}dr \\ &= C''''_m t^{-m}\delta^{-2m(N+1+l)}\cdot\delta^{-(2m-N)(N+1+l)}|\lambda_q|^{-2m}. \end{aligned}$$

Then from (5.18)

(6.25) $$|I| \leqq C_m \delta^{-(4m-N)(N+1+l)-2mM_q} t^{-m}$$

follows.

In the case of $n=m-1$, we write

$$I=(-1/it)^{m-1}\int_V e^{ith_\eta(\sigma)}\phi_\eta(\sigma)\Big(\nabla\cdot\frac{\nabla h_\eta}{|\nabla h_\eta|^2}\Big)^{m-1}g(\sigma)d\sigma$$
$$+(-1/it)^{m-1}\int_V e^{ith_\eta(\sigma)}\sum_{\alpha\neq 0}\partial^\alpha\phi_\eta(\sigma)\cdot(\cdots)d\sigma$$
$$\equiv I_1+I_2\,.$$

To estimate I_2, note (6.22), (6.23) and (6.24). In the same way as in the proof of (6.14) we can get

$$(6.26)\qquad \begin{aligned}|I_2|&\leqq C'_{m-1}t^{-(m-1)}\sum_{\alpha\neq 0}|\lambda_1\cdots\lambda_{N-|\alpha|}|^{-1}\\ &\leqq C_{m-1}t^{-(m-1)}|\lambda_1\cdots\lambda_N|^{-1}\\ &=C_{m-1}t^{-(m-1)}|K(s_\eta)|^{-1}\,.\end{aligned}$$

Next we estimate I_1. It follows from (6.24) and Lemma 5.6 that

$$\begin{aligned}|I_1|&\leqq C'_{m-1}t^{-(m-1)}\int_{V\cap\operatorname{supp}\phi_\eta}|\nabla h_\eta|^{-2(m-1)}d\sigma\\ &\leqq C''_{m-1}t^{-(m-1)}\sum_{\rho=1}^{\mu(\eta)}\int_{\tilde V_\mu}|\bar\sigma-b|^{-2(m-1)}J\Big(\frac{\partial\sigma}{\partial\bar\sigma}\Big)d\bar\sigma\,,\end{aligned}$$

where $\tilde V_\mu=(\bar\sigma\circ N\circ R^{-1}\circ\sigma^{-1})(V_\mu\cap\operatorname{supp}\phi_\eta)$. As we have remarked after Lemma 5.5,

$$\tilde V_\mu\subset\{|\bar\sigma-b|\geqq\tilde c|\lambda_q|\delta^{N+1+l}\}\,.$$

Thus

$$|I_1|\leqq C'''_{m-1}t^{-(m-1)}\int_{\tilde\Omega}|\bar\sigma-b|^{-2(m-1)}J\Big(\frac{\partial\sigma}{\partial\bar\sigma}\Big)d\bar\sigma\,,$$

where $\tilde\Omega=\{\tilde c\delta^{M_q+(N+1+l)}\leqq|\bar\sigma-b|\leqq M\}$ for some sufficiently large M. Then for arbitrary positive number ε we have

$$(6.27)\qquad |I_1|\leqq C_{m-1}t^{-(m-1)}\delta^{-(M_q+N+1+l)}\int_{R^N}|\bar\sigma-b|^{-2(m-1)+\varepsilon}J\Big(\frac{\partial\sigma}{\partial\bar\sigma}\Big)d\bar\sigma\,,$$

where $J\Big(\dfrac{\partial\sigma}{\partial\bar\sigma}\Big)$ is extended as zero for large $\bar\sigma$. Since $m-1\leqq N/2$, $|\cdot-b|^{-2(m-1)+\varepsilon}\in L^1(\boldsymbol{R}^N_{\bar\sigma})$, and (2.27) and (4.43) imply $J\Big(\dfrac{\partial\sigma}{\partial\bar\sigma}\Big)\in L^{p'}(\boldsymbol{R}^N_{\bar\sigma})$ for any p' with $1\leqq p'<1/l+1$. Thus

$$\Psi_\varepsilon(b)\equiv\int_{R^N}|\bar\sigma-b|^{-2(m-1)+\varepsilon}J\Big(\frac{\partial\sigma}{\partial\bar\sigma}\Big)d\bar\sigma\in L^1*L^{p'}\subset L^{p'}\,.$$

Since $dS^{n-1}=w(\bar\sigma)d\bar\sigma$ with $w(\bar\sigma)$ and $w(\bar\sigma)^{-1}$ bounded, the integrability is conserved under change of variables. On the other hand (2.27) imply $|K(s_\eta)|^{-1} \in L^{p'}(S_\eta^{n-1})$. Hence by putting $\psi_\varepsilon(\eta)=C_m[|K(s_\eta)|^{-1}+\Psi_\varepsilon(\bar{\sigma}(\eta))]$, we have from (6.26) and (6.27)

$$|I|\leqq t^{-(m-1)}\delta^{-(Mq+N+1+l)\varepsilon}\psi_\varepsilon(\eta), \tag{6.28}$$

where $\psi_\varepsilon(\eta)\in L^{p'}(S_\eta^{n-1})$ for arbitrary ε and p' with $\varepsilon>0$ and $1\leqq p'<1/l+1$.

Then it follows from (6.25) and (6.28) that for any θ with $0<\theta<1$

$$|I|\leqq C_m^\theta t^{-(m-1)-\theta}\delta^{-A\theta}\delta^{-(M_q+N+1+l)\varepsilon(1-\theta)}\psi_\varepsilon(\eta)^{1-\theta},$$

where $A=-(4m-N)(N+1+l)-2mM_q$. Thus, if we take μ and make θ and ε sufficiently small depending on it, we get the conclusion of lemma with $\Phi_\varepsilon(\eta)=\psi_\varepsilon(\eta)^{1-\theta}$. Q.E.D.

We can get the estimate of J_2 for $i=2$ from Proposition 6.2. Since n is odd, we can apply it for $m-1=(n-1)/2$. Then we have

$$|J_2|\leqq|x|^{-(n-1)/2-\nu}\delta^{-\mu}\Phi_\mu(\eta) \tag{6.29}$$

for any μ and some $\Phi_\mu(\eta)\in L^{p'}(S_\eta^{n-1})$ with $1\leqq p'<1/l+1$.

By (5.14), (5.15) and (6.29) we have in the case of ii)–a)

$$\begin{aligned} v_{2j}^k(x,r) = {} & (2\pi)^{-(n-1)/2}e^{i|x|\eta\cdot s}\psi_{\overset{+}{(-)}}(s)|K(s)|^{-1/2} \\ & \cdot m(s,r)\psi_{2j}^k(s)|x|^{-(n-1)/2}|_{s=s_\eta}+q_{2j}^k(x,r), \end{aligned} \tag{6.30}$$

where

$$|q_{2j}^k(x,r)|\leqq\delta^{-\mu}\Phi_\mu(\eta)|x|^{-(n-1)/2-\nu} \tag{6.31}$$

for any $\mu>0$ and for some $\nu>0$ and $\Phi_\mu(\eta)\in L^{p'}(S_\eta^{n-1})$ with $1\leqq p'<1/l+1$. Then for any p with $1\leqq p<1/l+1$ we make μ and p' satisfy $1/p'+\mu/l=1/p$. (It is possible if μ is sufficiently small). Note that Lemma 4.3 3) implies $\mathrm{dist}_S(s_\eta, Z_S)\sim\mathrm{dist}_V(a, Z_0)=\delta$. Then $\delta^{-\mu}\in L^{l/\mu}(S_\eta^{n-1})$ because

$$\begin{aligned} \int(\delta^{-\mu})^{l/\mu}dS_\eta^{n-1} &\leqq Const.\int\mathrm{dist}(s_\eta, Z_S)^{-l}dS_\eta^{n-1} \\ &= Const.\int\mathrm{dist}(s, Z_S)^{-l}|K(s)|dS \\ &\leqq Const.\int dS. \end{aligned}$$

Hence $\Phi_\mu(\eta)\delta^{-\mu}\in L^p(S_\eta^{n-1})$.

Then the decomposition (4.7) and estimates (5.1), (5.2), (5.3), (6.19), (6.20) and (6.30), (6.31) imply the conclusion of the Theorem 2.1.

There are some corollaries to Theorem 2.1.

Corollary 6.3. *Under the same conditions of Theorem* 2.1 *it follows that for any natural number m*

(6.32)
$$
\begin{aligned}
&D^m_{|x|}v(x, r)\\
&=\sum_{\gamma=1}^{\rho(\eta)}(2\pi)^{-(n-1)/2}|x|^{-(n-1)/2}(\eta\cdot s)^m e^{i|x|\eta\cdot s}\psi_+(s)\\
&\quad\cdot\hat{P}(s)|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(\eta)}\\
&\quad+\sum_{\gamma=1}^{\rho(-\eta)}(2\pi)^{-(n-1)/2}|x|^{-(n-1)/2}(\eta\cdot s)^m e^{i|x|\eta\cdot s}\psi_-(s)\\
&\quad\cdot\hat{P}(s)|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(-\eta)}+q_m(x, r),
\end{aligned}
$$

where $q_m(x, r)$ satisfies (2.29) *and* (2.30).

Proof. By differentiating $v(x, r)$ by $|x|$ we can write

$$D^m_{|x|}v(x, r)=\int_S e^{ix\cdot s}(\eta\cdot s)^m\hat{P}(s)|T(s)|^{-1}\phi_1(rs)dS$$

and in the same way as in the proof of Theorem 2.1 we get the conclusion. Q.E.D.

Corollary 6.4. *Not only $q(x, r)$ of Theorem* 2.1 *but $D^m_{|x|}q(x, r)$ and $D^m_r q(x, r)$ also satisfy* (2.29) *and* (2.30).

Proof. (6.32) implies

$$
\begin{aligned}
q(x, r)&=v(x, r)-\{\sum_{\gamma=1}^{\rho(\eta)}(2\pi)^{-(n-1)/2}|x|^{-(n-1)/2}\\
&e^{ix\cdot s}\psi_+(s)\hat{P}(s)|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(\eta)}\\
&+\sum_{\gamma=1}^{\rho(-\eta)}(2\pi)^{-(n-1)/2}|x|^{-(n-1)/2}e^{ix\cdot s}\psi_-(s)\hat{P}(s)|T(s)|^{-1}\\
&\cdot|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(\eta)}\}\,.
\end{aligned}
$$

Then Corollary 6.3 implies

$$D^m_{|x|}q(x, r)=q_m(x, r)+C_m(\eta)|x|^{-(n+1)/2},$$

where

$$
\begin{aligned}
C_m(\eta)&=\sum_{\gamma=1}^{\rho(\eta)}(2\pi)^{-(n-1)/2}(-(n-1)/2)e^{ix\cdot s}\psi_+(s)\hat{P}(s)\\
&\cdot|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(\eta)}+\sum_{\gamma=1}^{\rho(-\eta)}(2\pi)^{-(n-1)/2}\\
&(-(n-1)/2)e^{ix\cdot s}\psi_-(s)\hat{P}(s)|T(s)|^{-1}|K(s)|^{-1/2}\phi_1(rs)|_{s=s^{(\gamma)}(-\eta)}\,.
\end{aligned}
$$

Clearly $C_m(\eta)\in L^2(S^{n-1}_\eta)$. Hence $D^m_{|x|}q(x, r)$ satisfies (2.29) and (2.30). For $D^m_r q(x, r)$ the proof is almost the same. Q.E.D.

7. Asymptotic behavior at infinity of the Green function

In this section we give the following theorem which is our main purpose of this paper.

Theorem 7.1. *Under conditions* Si)$\sim$Svi) *we have for any* p *with* $1 \leqq p < 1/l+1$ *and for some* $v = v_p > 0$

$$
\begin{aligned}
(7.1) \qquad & G(x, \zeta) \\
&= \sum_{\gamma=1}^{\rho(\pm\eta)} (2\pi)^{-(n-1)/2} e^{\pm i\zeta|x||T(s)|^{-1}} |x|^{-(n-1)/2} \\
&\quad \cdot |\lambda|^{(n-1)/2} |K(s)|^{-1/2} |T(s)|^{-1} \hat{P}(s) \psi_{\mathrm{sign}(\pm\lambda)}(s) \\
&\quad \cdot C_\gamma(\eta, \zeta, |x|)|_{s=s^{(\gamma)}(\pm\eta)} + q(x, \zeta)
\end{aligned}
$$

when $\zeta = \lambda \pm i\varepsilon$ *for* $\lambda \in \boldsymbol{R}^1 \backslash \{0\}$ *and* $\varepsilon > 0$ *where* C_γ'*s are bounded functions and* $q(x, \zeta)$ *satisfies*

$$
(7.2) \qquad |q(x, \zeta)| \leqq C(\eta) |x|^{-(n-1)/2-v}
$$

and

$$
(7.3) \qquad C(\eta) \in L^p(S_\eta^{n-1}).
$$

Moreover $G(x, \lambda \pm i\varepsilon)$ *converges as* $\varepsilon \downarrow 0$ *uniformly for* $\lambda \in [a, b] \subset \boldsymbol{R}^1 \backslash \{0\}$ *for any* $a, b \in \boldsymbol{R}^1$, *and the limit* $G_\pm(x, \lambda)$ *can be represented as*

$$
\begin{aligned}
& G_\pm(x, \lambda) \\
&= \sum_{\gamma=1}^{\rho(\pm\eta)} (2\pi)^{-(n-1)/2} e^{\pm i\lambda|x||T(s)|^{-1}} |x|^{-(n-1)/2} |\lambda|^{(n-1)/2} \\
&\quad \cdot |K(s)|^{-1/2} |T(s)|^{-1} \hat{P}(s) \psi_{\mathrm{sign}(\pm\lambda)}(s)|_{s=s^{(\gamma)}(\pm\eta)} \\
&\quad + q_\pm(x, \lambda).
\end{aligned}
$$

Here $q_\pm$ *also satisfies*

$$
(7.4) \qquad |q_\pm(x, \lambda)| \leqq C(\eta) |x|^{-(n-1)/2-v}
$$

and

$$
(7.5) \qquad C(\eta) \in L^p(S_\eta^{n-1}).
$$

Proof. We suppose $\zeta \in \Delta \equiv \{\zeta = \lambda \pm i\varepsilon;\ \lambda \in [a, b] \text{ and } \varepsilon \in (0, \varepsilon_0]\}$. $G(x, \zeta)$ can be written as (2.25):

$$
G(x, \zeta) = G_0(x, \zeta) + \int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r-\zeta} v(rx, r) dr.
$$

Recall that $G_0(x, \zeta)$ satisfies the estimate

$$
|G_0(x, \zeta)| \leqq C |x|^{-n}
$$

for some constant C independent of $\zeta \in \Delta$. So we may consider only

$$\int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r-\zeta} v(rx, r)dr . \tag{7.6}$$

Applying theorem 2.1 to (7.6), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r-\zeta} v(rx, r)dr \\
&= \sum_{\gamma=1}^{\rho(\eta)} a_{\gamma}^{+}(\eta)\int_{-\infty}^{\infty} \frac{|r|^{(n-1)/2}}{r-\zeta} \phi_1(rs^{(\gamma)}(\eta))e^{ir|x|\eta\cdot s^{(\gamma)}(\eta)} \\
&\quad \cdot \psi_{\mathrm{sign}\, r}(s^{(\gamma)}(\eta))dr \cdot |x|^{-(n-1)/2} + \sum_{\gamma=1}^{\rho(-\eta)} a_{\gamma}^{-}(\eta)\int_{-\infty}^{\infty} \frac{|r|^{(n-1)/2}}{r-\zeta} \\
&\quad \cdot \phi_1(rs^{(\gamma)}(-\eta))e^{ir|x|\eta\cdot s^{(\gamma)}(-\eta)}\psi_{\mathrm{sign}(-r)}(s^{(\gamma)}(-\eta))dr \\
&\quad \cdot |x|^{-(n-1)/2} + \int_{-\infty}^{\infty} \frac{|r|^{(n-1)/2}}{r-\zeta} q(rx, r)dr \\
&\equiv I_1 + I_2 + I_3 ,
\end{aligned}$$

where

$$a_{\gamma}^{\pm}(\eta) = (2\pi)^{-(n-1)/2} |K(s)|^{-1/2} |T(s)|^{-1} \hat{P}(s)|_{s=s^{(\gamma)}(\pm\eta)} .$$

Note the relations

$$\int_{-\infty}^{\infty} \frac{e^{ir\rho}}{r-\zeta} dr = \pm iY(\pm\rho)e^{i\zeta\rho}$$

for $\zeta=\lambda\pm i\varepsilon$ ($\varepsilon>0$) where $Y(\rho)$ is the Heaviside function. Hereafter we assume $\zeta=\lambda+i\varepsilon$ for simplicity. In the case of $\zeta=\lambda-i\varepsilon$ the proof is almost the same. Then by Perseval's formula we can get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|r|^{(n-1)/2}}{r-\zeta} \phi_1(rs^{(\gamma)}(\pm\eta))e^{ir|x|\eta\cdot s^{(\gamma)}(\pm\eta)}dr \\
&= \int_{-\infty}^{\infty} iY(|x|\eta\cdot s^{(\gamma)}(\pm\eta)-\rho)e^{i\zeta(|x|\eta\cdot s^{(\gamma)}(\pm\eta)-\rho)} \\
&\qquad \cdot \hat{f}_{\pm}(\rho, s^{(\gamma)}(\pm\eta))d\rho ,
\end{aligned} \tag{7.7}$$

where

$$f_{\pm}(r, s) = |r|^{(n-1)/2}\phi_1(rs)\psi_{\mathrm{sign}(\pm r)}(s) \tag{7.8}$$

and

$$\hat{f}_{\pm}(\rho, s) = \int_{-\infty}^{\infty} e^{-ir\rho} |r|^{(n-1)/2}\phi_1(rs)\psi_{\mathrm{sign}(\pm r)}(s)dr . \tag{7.9}$$

By (7.7) we get

$$\begin{aligned}
I_1 = \sum_{\gamma=1}^{\rho(\eta)} a_{\gamma}^{+}(\eta) |x|^{-(n-1)/2} &\int_{-\infty}^{|x|\eta\cdot s^{(\gamma)}(\eta)} \\
&e^{i\zeta(|x|\eta\cdot s^{(\gamma)}(\eta)-\rho)} \hat{f}_{\pm}(\rho, s^{(\gamma)}(\eta))d\rho
\end{aligned} \tag{7.10}$$

and

$$(7.11)\qquad I_2=\sum_{\gamma=1}^{\rho(-\eta)} a_\gamma^-(\eta)\,|x|^{-(n-1)/2}\int_{-\infty}^{|x|\eta\cdot s^{(\gamma)}(-\eta)} e^{i\zeta(|x|\eta\cdot s^{(\gamma)}(-\eta)-\rho)}\hat{f}_\pm(\rho,\, s^{(\gamma)}(-\eta))d\rho\,.$$

From the definition of $s^{(\gamma)}$

$$(7.12)\qquad N(s^{(\gamma)}(\pm\eta))=\pm\eta$$

holds. On the other hand it holds that for any $s\in S_k\backslash Z_S^{(1)}$

$$(7.13)\qquad s\,N(s)=\frac{s\cdot\nabla\lambda_k(s)}{|\nabla\lambda_k(s)|}=\frac{\lambda_k(s)}{|\nabla\lambda_k(s)|}=|\nabla\lambda_k(s)|^{-1}>0\,.$$

Thus (7.12), (7.13) and the definition of T imply

$$(7.14)\qquad \pm\eta\cdot s^{(\gamma)}(\pm\eta)=N(s^{(\gamma)}(\pm\eta))\cdot s^{(\gamma)}(\pm\eta)=|T(s^{(\gamma)}(\pm\eta))|^{-1}\,.$$

Then by the change of variable from ρ to $-\rho$ in the integral (7.11) we get

$$(7.15)\qquad I_2=\sum_{\gamma=1}^{\rho(-\eta)} a_\gamma^-(\eta)\,|x|^{-(n-1)/2}\cdot\int_{|x|(-\eta)s^{(\gamma)}(-\eta)}^{\infty} e^{i\zeta(|x|\eta\cdot s^{(\gamma)}(-\eta)+\rho)}\hat{f}_\pm(-\rho,\, s^{(\gamma)}(-\eta))d\rho\,,$$

By (7.14) we have

$$=\sum_{\gamma=1}^{\rho(-\eta)} a_\gamma^-(\eta)\,|x|^{-(n-1)/2}\cdot\int_{|x||T(s^{(\gamma)}(-\eta))|^{-1}}^{\infty} e^{i\zeta(-|x||T(s^{(\gamma)}(-\eta))|^{-1}+\rho)}\cdot\hat{f}_\pm(-\rho,\, s^{(\gamma)}(-\eta))d\rho\,.$$

Since $s^{(\gamma)}(\pm\eta)\in S$ and S is bounded, it follows for any integer $m\leqq(n+1)/2$ and for $\rho\neq0$ that

$$(7.16)\qquad |\hat{f}_\pm(\rho,\, s^{(\gamma)}(\pm\eta))|\leqq C_m|\rho|^{-m}\,,$$

where C_m is independent of η. Hence

$$(7.17)\qquad \left|\int_{|x|\cdot|T(s^{(\gamma)}(-\eta))|^{-1}}^{\infty} e^{i\zeta\rho}\hat{f}_\pm(-\rho,\, s^{(\gamma)}(-\eta))d\rho\right| \leqq\int_{|x|\cdot|T(s^{(\gamma)}(-\eta))|^{-1}}^{\infty} e^{-\varepsilon\rho}|\rho|^{-2}d\rho \leqq \mathit{Const.}\ |T(s^{(\gamma)}(-\eta))|\,|x|^{-1}\,.$$

(0.9) and (2.19) imply

$$C^{-1}\leqq|T(s)|=|\nabla\lambda_k(s)|\leqq C$$

for $s\in S_k\backslash Z_S^{(1)}$ and some constant C. Then by (7.15) and (7.17) we have

$$|I_2|\leqq Const.\sum_{\gamma=1}^{\rho(-\eta)}a_\gamma^-(\eta)|x|^{-(n+1)/2}. \tag{7.18}$$

Here $C_2(\eta)\equiv\sum_{\gamma=1}^{\rho(-\eta)}a_\gamma^-(\eta)$ belongs to $L^2(S^{n-1})$. Thus it follows from (7.10) and (7.18) that

$$\begin{aligned}I_1+I_2=&\sum_{\gamma=1}^{\rho(\eta)}a_\gamma^+(\eta)e^{i\zeta|x||T(s^{(\gamma)}(\eta))|^{-1}}|x|^{-(n-1)/2}\\&\cdot\int_{-\infty}^{|x||T(s^{(\gamma)}(\eta))|^{-1}}e^{i\zeta\rho}\hat{f}_\pm(\rho,s^{(\gamma)}(\eta))d\rho\\&+q_0(x,\zeta),\end{aligned} \tag{7.19}$$

where

$$|q_0(x,\zeta)|\leqq C_2(\eta)|x|^{-(n+1)/2} \tag{7.20}$$

for some $C_2(\eta)\in L^2(S^{n-1})$ independent of $\zeta\in\Delta$.

Next the rest I_3 will be considered. Since ϕ_1 has compact support, (2.28) implies

$$\operatorname*{supp}_r q(rx,r)\subset[-A,A]$$

for some constant A. Then we can write

$$\begin{aligned}I_3=&\int_{-A}^{A}\frac{|r|^{n-1}q(rx,r)-|\lambda|^{n-1}q(\lambda x,\lambda)}{r-\zeta}dr\\&+|\lambda|^{n-2}q(\lambda x,\lambda)\int_{-A}^{A}\frac{dr}{r-\zeta}\\\equiv&\ I_{31}+I_{32}.\end{aligned} \tag{7.21}$$

(2.29) of Theorem 2.1 gives

$$|I_{32}|\leqq|\lambda|^{n-1}C(\eta)|\lambda x|^{-(n-1)/2-\nu}\left|\int_{-A}^{A}\frac{dr}{r-\zeta}\right|.$$

Note the equation

$$\int_{-R}^{R}\frac{dr}{r-i\varepsilon}=2i\arctan R/\varepsilon.$$

Then $\left|\int_{-A}^{A}\frac{dr}{r-\zeta}\right|$ is uniformly bounded with respect to $\zeta\in\Delta$. Hence it follows that

$$|I_{32}|\leqq\tilde{C}(\eta)|x|^{-(n-1)/2-\nu}, \tag{7.22}$$

where $\tilde{C}(\eta)$ is independent of $\zeta\in\Delta$ and satisfies (2.30). Next I_{31} is considered. Since

$$
\begin{aligned}
&|r|^{n-1}q(rx, r)-|\lambda|^{n-1}q(\lambda x, \lambda)\\
&=\int_0^1 \frac{d}{d\tilde{r}}\{|\tilde{r}|^{n-1}q(\tilde{r}x, \tilde{r})\}\,|_{\tilde{r}=\lambda+\theta(r-\lambda)}d\theta\cdot(r-\lambda\\
&=\int_0^1\{(n-1)|\tilde{r}|^{n-2}q(\tilde{r}x, \tilde{r})+|\tilde{r}|^{n-1}\frac{\partial q}{\partial|x|}(\tilde{r}x, \tilde{r})|x|\\
&\quad+|\tilde{r}|^{n-1}\frac{\partial q}{\partial\tilde{r}}(\tilde{r}x, \lambda)\}\,|_{\tilde{r}=\lambda+\theta(r-\lambda)}d\theta\cdot(r-\lambda),
\end{aligned}
$$

it follows from Corollary 6.4 that

$$
\begin{aligned}
&||r|^{n-1}q(rx, r)-|\lambda|^{n-1}q(\lambda x, \lambda)|\\
&\leqq \tilde{C}(\eta)|x|^{-(n-1)/2-\nu+1}|r-\lambda|,
\end{aligned}
\tag{7.23}
$$

where $\tilde{C}(\eta)$ is independent of $\zeta\in\Delta$ and satisfies (2.30). On the other hand it also follows that

$$
\begin{aligned}
&||r|^{n-1}q(rx, r)-|\lambda|^{n-1}q(\lambda x, \lambda)|\\
&\leqq \tilde{C}(\eta)|x|^{-(n-1)/2-\nu}.
\end{aligned}
\tag{7.24}
$$

Hence by (7.23) and (7.24)

$$
\begin{aligned}
&||r|^{n-1}q(rx, r)-|\lambda|^{n-1}q(\lambda x, \lambda)|\\
&\leqq \tilde{C}(\eta)|x|^{-(n-1)/2-(\nu-\theta)}|r-\lambda|^{\theta}
\end{aligned}
\tag{7.25}
$$

for any θ which satisfies $0<\theta<1$ and $\nu-\theta>0$, where $\tilde{C}(\eta)$ is independent of $\zeta\in\Delta$ and satisfies (2.30). Thus I_{31} is estimated by (7.25) as

$$
\begin{aligned}
|I_{31}| &\leqq \int_{-A}^{A}\frac{|r-\lambda|^{\theta}}{|r-\zeta|}dr\cdot\tilde{C}(\eta)|x|^{-(n-1)/2-(\nu-\theta)}\\
&\leqq C(\eta)|x|^{-(n-1)/2-(\nu-\theta)}.
\end{aligned}
\tag{7.26}
$$

Then (7.20), (7.22) and (7.26) imply (7.1), (7.2) and (7.3) of theorem.

Next we shall consider the behavior of $G(x, \zeta)$ as $\varepsilon\downarrow 0$. Note that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{|x||T(s^{(\gamma)}(\eta))|^{-1}} e^{-i\zeta\rho}\hat{f}_+(\rho, s^{(\gamma)}(\eta))d\rho\\
&=\int_{-\infty}^{|x||T(s^{(\gamma)}(\eta))|^{-1}}\lim_{\varepsilon\downarrow 0} e^{-i\zeta\rho}\hat{f}_+(\rho, s^{(\gamma)}(\eta))d\rho\\
&=\int_{-\infty}^{|x||T(s^{(\gamma)}(\eta))|^{-1}} e^{-i\lambda\rho}\hat{f}_+(\rho, s^{(\gamma)}(\eta))d\rho\\
&=\int_{-\infty}^{\infty}-\int_{|x||T(s^{(\gamma)}(\eta))|^{-1}}^{\infty}\\
&\equiv J_1-J_2.
\end{aligned}
\tag{7.27}
$$

In the same way as in the proof of (7.18) we can get

$$|\int_{|x||T(s^{(\gamma)}(\omega))|^{-1}}^{\infty} e^{-i\lambda\rho}\hat{f}_+(\rho, s^{(\gamma)}(\eta))d\rho| \leqq Const.\ |x|^{-1}.$$

Thus

$$(7.28)\qquad |\sum_{\gamma=1}^{\rho(\eta)} a_+^{\gamma}(\eta)|x|^{-(n-1)/2}e^{i\zeta|x||T(s^{(\gamma)}(\eta))|^{-1}}J_2| \leqq C_2(\eta)|x|^{-(n+1)/2}$$

for some $C_2(\eta)\in L^2(S^{n-1})$ independent of $\lambda\in[a, b]$. On the other hand it follows from (7.8), (7.9) and the definition of ϕ_1 that

$$(7.29)\qquad \begin{aligned}\int_{-\infty}^{\infty} e^{i\lambda\rho}\hat{f}_+(\rho, s^{(\gamma)}(\eta))d\rho &= f_+(\lambda, s^{(\gamma)}(\eta))\\ &= |\lambda|^{(n-1)/2}\phi_1(\lambda s^{(\gamma)}(\eta))\psi_{\mathrm{sign}\lambda}(s^{(\gamma)}(\eta))\\ &= |\lambda|^{(n-1)/2}\psi_{\mathrm{sign}\lambda}(s^{(\gamma)}(\eta)).\end{aligned}$$

Then (7.27), (7.28) and (7.29) give

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\sum_{\gamma=1}^{\rho(\eta)} a_\gamma^+(\eta)e^{i\zeta|x||T(s^{(\gamma)}(\eta))^{-1}}|x|^{-n(n-1)/2}\\ &\quad\cdot\int_{-\infty}^{|x||T(s^{(\gamma)}(\eta))|^{-1}} e^{-i\zeta\rho}\hat{f}_+(\rho, s^{(\gamma)}(\eta))d\rho\\ &=\sum_{\gamma=1}^{\rho(\eta)} a_\gamma^+(\eta)e^{i\lambda|x||T(s^{(\gamma)}(\eta))|^{-1}}|\lambda|^{(n-1)/2}|x|^{-(n-1)/2}\\ &\quad+q_1(x, \lambda),\end{aligned}$$

where

$$|q_1(x, \lambda)| \leqq C_2(\eta)|x|^{-(n+1)/2}$$

for some $C_2(\eta)\in L^2(S^{n-1})$ independent of $\lambda\in[a, b]$. It is easy to show the existence of the limit of $q(x, \lambda+i\varepsilon)$. In fact $q(x, \lambda+i\varepsilon)=I_2+I_3$ holds and it is clear for I_2 from (7.15) and (7.20) and for I_3 from (7.21). Then we put $q_+(x, \lambda)=q(x, \lambda+i0)+q_1(x, \lambda)$. This completes the proof of Theorem 7.1.

Q.E.D.

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