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THEOREMS ON LARGE DEVIATIONS FOR A FAMILY OF STOCHASTIC PROCESSES CONVERGING TO A MARKOV PROCESS

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Introduction

In their powerful consecutive works [5]–[7], [9], [12], Donsker and Varadhan have developed the theory of large deviations for Markov processes. Applying their fundamental theorems, they also obtained several remarkable results [8], [10], [11], [13] concerning the large-time asymptotics for certain Markov processes. Although each theorem in their works on general theory involves the probabilities of large deviations for a single Markov process, it is quite natural in some applications (see [11], [19], [16]) to consider those for a family of stochastic processes. In the present paper we study the theory of large deviations for a certain family of stochastic processes converging to a Markov process, which corrseponds to the general theory in [12]. As its applications we prove some theorems on the Chung type laws of iterated logarithm, generalizing some reuslts in [10].

Let X be a Polish space and let Ω [resp. Ω^+] denote the space of X-valued right-continuous functions on $(-\infty, +\infty)$ [resp. $[0, +\infty)$] without discontinuities of the second kind, endowed with the Skorohod type topology. Let θ_t denote the shift operator on Ω, i.e., $\theta_t\omega=\omega(t+\cdot)$. For any $t>0$ define $p_t\colon \Omega^+\to\Omega$ by $(p_t\omega)(s)=\omega(s)$, $0\leqq s<t$, and $(p_t\omega)(s+t)=(p_t\omega)(s)$, $-\infty<s<\infty$. We can define for any $t>0$ and $\omega\in\Omega^+$

$$(1)\qquad R_{t,\omega}(A)=R_t(\omega, A)=\frac{1}{t}\int_0^t \chi_A(\theta_s p_t\omega)\,ds,\ A\subset\Omega\,.$$

Let $\mathcal{M}_S(\Omega)$ denote the space of all probability measures Q on Ω such that $Q\circ\theta_t^{-1}=Q$, $-\infty<t<\infty$, i.e., the space of all stationary processes on X. Note that $R_t(\omega, \cdot)\in\mathcal{M}_S(\Omega)$. For any $Q\in\mathcal{M}_S(\Omega)$ we denote the one-dimensional marginal of Q by $q[Q]$.

Let $\{x(t)\}$ be a homogeneous Markov process on X. In [12] Donsker and Varadhan give the definition of the *entropy* function $H(Q)$, $Q\in\mathcal{M}_S(\Omega)$, associated

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with the Markov process $\{x(t)\}$ and prove that, under suitable hypotheses, $H(\cdot)$ governs the rate at which Prob $\{R_t(x(\cdot), \cdot)\in A\}\to 0$ as $t\to\infty$ for suitable $A\subset\mathcal{M}_S(\Omega)$ (see Theorems 1.1 and 1.2).

In principle, our main results (Theorems 2.1 and 2.2) can be stated as follows: Let $\{x(t)\}$ be as above. Suppose we are given a family of Markov processes $\{\tilde{x}^\varepsilon(t)\}$, $\varepsilon>0$, on some space $\tilde{X}$ and a mapping π from $\tilde{X}$ onto X. Let

$$x^\varepsilon(t) = \pi[\tilde{x}^\varepsilon(t)],\ t\geqq 0\,. \tag{2}$$

Suppose that the family of porcesses $\{x^\varepsilon(t)\}$ converges in law on Ω^+ to $\{x(t)\}$ as $\varepsilon\downarrow 0$. (Precisely, we should mention about the starting points of the processes $\{\tilde{x}^\varepsilon(t)\}$ and $\{x(t)\}$, but we do not go into details here; see $(\mathbf{A}_1^\varepsilon)$, $(\mathbf{A}_2^\varepsilon)$ and $(\mathbf{B}^\varepsilon)$ in Section 2.) Then we have

$$\varlimsup_{\substack{t\to\infty\\ \varepsilon\downarrow 0}} \frac{1}{t}\log \text{Prob}\ \{R_t(x^\varepsilon(\cdot), \cdot)\in A\}\leqq -\inf_{Q\in A} H(Q) \tag{3}$$

for any weakly closed set $A\subset\mathcal{M}_S(\Omega)$ such that $\{q[Q];\ Q\in A\}$ is tight, and

$$\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}} \frac{1}{t}\log \text{Prob}\ \{R_t(x^\varepsilon(\cdot), \cdot)\in A\}\geqq -\inf_{Q\in A} H(Q) \tag{4}$$

for any weakly open set $A\subset\mathcal{M}_S(\Omega)$. (In the trivial case that $\tilde{x}^\varepsilon(t)=x(t)$ for every $\varepsilon>0$ we recover the results in [12].)

Such theorems are precisely stated in Section 2 and the proofs of them are given in Sections 3 and 4.

In Sectoin 5 some examples of $\{x^\varepsilon(t)\}$, $\varepsilon>0$, are given. We give here two typical ones in the simplest forms to illustrate the feature of our results.

Example 1 (Example 5.2). Consider the process of the form

$$X(t) = \int_0^t F(Y(s))ds\,. \tag{5}$$

Here $\{Y(t)\}$ is a "strongly" ergodic Markov process on an auxiliary space S, and $F(y)$ is a suitable function on S satisfying the *centering condition:*

$$\int_S F(y)\,\bar{P}(dy) = 0\,, \tag{6}$$

where $\bar{P}$ denotes the invariant probability measure of $\{Y(t)\}$. Let $x^\varepsilon(t)=\varepsilon X(t/\varepsilon^2)$ and $y^\varepsilon(t)= Y(t/\varepsilon^2)$ so that $(x^\varepsilon(t), y^\varepsilon(t))$ forms a Makrov process on $\boldsymbol{R}^1\times S$ for each $\varepsilon>0$. It is known [22] that $\{x^\varepsilon(t)\}$ converges to a Brownian motion $\{x(t)\}$ as $\varepsilon\downarrow 0$. For the family $\{x^\varepsilon(t)\}$, $\varepsilon>0$, the assertions (3) and (4) hold with $H(Q)$ being the entropy function for $\{x(t)\}$. In this example we take

$$X=\boldsymbol{R}^1,\ \tilde{X}=X\times S,\ \pi(x, y) = x \text{ and } \tilde{x}^\varepsilon(t) = (x^\varepsilon(t), y^\varepsilon(t))\,.$$

EXAMPLE 2 (Example 5.3). Let $a(x)$ be a periodic measurable function on $\boldsymbol{R}^1$ such that $0<\nu\leqq a(x)\leqq\nu^{-1}<\infty$ for some constant $\nu>0$. Let $\{X(t)\}$ be the diffusion process associated with

$$A=\frac{1}{2}\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) \tag{7}$$

and let $x^{\varepsilon}(t)=\varepsilon X(t/\varepsilon^2)$. It is known [15] that there exists a Brownian motion $\{x(t)\}$ such that $\{x^{\varepsilon}(t)\}$ converges in law on Ω^+ to $\{x(t)\}$; the limiting Brownian motion is called the *homogenized* process for the family $\{x^{\varepsilon}(t)\}$, or simply, for $\{X(t)\}$. The assertions (3) and (4) hold for $\{x^{\varepsilon}(t)\}$, $\varepsilon>0$, with the entropy function $H(Q)$ for the homogenized process $\{x(t)\}$. Here we take $\tilde{X}=X=\boldsymbol{R}^1$, $\pi(x)=x$ and $\tilde{x}^{\varepsilon}(t)=x^{\varepsilon}(t)$.

Note that $\overline{\lim}_{\varepsilon\downarrow 0}\overline{\lim}_{t\to\infty} f(\varepsilon, t)\leqq\overline{\lim}_{\substack{t\to\infty\\ \varepsilon\downarrow 0}} f(\varepsilon, t)$ for any function $f(\varepsilon, t)$, $\varepsilon>0$, $t>0$. Thus, if we define $\bar{J}^{\varepsilon}(A)=\overline{\lim}_{t\to\infty}\dfrac{1}{t}\log \operatorname{Prob}\{R_t(x^{\varepsilon}(\cdot), \cdot)\in A\}$ and $\underline{J}^{\varepsilon}(A)=\underline{\lim}_{t\to\infty}\dfrac{1}{t}\log\operatorname{Prob}\{R_t(x^{\varepsilon}(\cdot), \cdot)\in A\}$, then we have, from (3) and (4), as $\varepsilon\downarrow 0$

$$\bar{J}^{\varepsilon}(A)\leqq-\inf_{Q\in A} H(Q)+o(1) \text{ for } A \text{ as in (3)}, \tag{8}$$

$$\underline{J}^{\varepsilon}(A)\geqq-\inf_{Q\in A} H(Q)+o(1) \text{ for } A \text{ as in (4)}. \tag{9}$$

This means that probabilities of large deviations for $\{x^{\varepsilon}(t)\}$ with $\varepsilon>0$ fixed can be also controled by the entropy function $H(Q)$, but only approximately. (Of course, the error terms $o(1)$ in (8) and (9) are not uniform for $A\subset\mathcal{M}_S(\Omega)$.)

Let $\Psi(Q)$ be a bounded (weakly) continuous function on $\mathcal{M}_S(\Omega)$. In case that X is compact, we have as a corollary to (3) and (4)

$$\lim_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log E\{e^{t\Psi(R_t(x^{\varepsilon}(\cdot),\cdot))}\}=\sup_{Q\in\mathcal{M}_S(\Omega)}[\Psi(Q)-H(Q)]. \tag{10}$$

As the above argument, we have from (10)

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\overline{\lim_{t\to\infty}}\frac{1}{t}\log E\{e^{t\Psi(R_t(x^{\varepsilon}(\cdot),\cdot))}\}&=\lim_{\varepsilon\downarrow 0}\underline{\lim}_{t\to\infty}\frac{1}{t}\log E\{\,\text{〃}\,\}\\ &=\sup_{Q\in\mathcal{M}_S(\Omega)}[\Psi(Q)-H(Q)].\end{aligned}$$

Let

$$L_t(\omega, A)=\frac{1}{t}\int_0^t \chi_A(\omega(s))\,ds,\ A\subset X. \tag{11}$$

In [9], Donsker and Varadhan define the *I-function* $I(\mu)$, $\mu\in\mathcal{M}(X)$ (the space of probability measures on X) for a Markov process $\{x(t)\}$ and prove that $I(\mu)$ governs the rate at which $\operatorname{Prob}\{L_t(x(\cdot), \cdot)\in A\}\to 0$ as $t\to\infty$ for suitable $A\subset\mathcal{M}$

(X) as $H(Q)$ did for $R_t(x(\cdot), \cdot)$. Such results for $L_t(x(\cdot), \cdot)$ are the corollaries to those for $R_t(x(\cdot), \cdot)$ since $q[R_t(\omega, \cdot)]=L_t(\omega, \cdot)$ and $I(\mu)=\inf_{Q\,;\,q[Q]=\mu} H(Q)$ as was pointed out in [12]. Similarly, we have as corollaries to (3) and (4)

$$\overline{\lim_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}}\ \frac{1}{t}\log \operatorname{Prob}\{L_t(x^{\varepsilon}(\cdot), \cdot)\in A\}\leqq -\inf_{\mu\in A} I(\mu) \tag{12}$$

for any weakly compact set $A\subset\mathcal{M}(X)$, and

$$\underline{\lim_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}}\ \frac{1}{t}\log \operatorname{Prob}\{L_t(x^{\varepsilon}(\cdot), \cdot)\in A\}\geqq -\operatorname{Inf}_{\mu\in A} I(\mu) \tag{13}$$

for any weakly open set $A\subset\mathcal{M}(X)$. In some applications the results for $L_t(x^{\varepsilon}(\cdot), \cdot)$, like (12) and (13), will be convenient. In fact, we will give the applications of (12) and (13) in Section 6.

In Section 6, the Chung type laws of iterated logarithm are proved for a certain class of homogenizable (see (**H**) in Section 6) processes on $\boldsymbol{R}^d$. For example, we prove that for the process $\{X(t)\}$ in Example 1

$$\underline{\lim_{t\to\infty}}\left(\frac{\log\log t}{t}\right)^{1/2}\sup_{0\leqq s\leqq t}|X(s)| = \frac{\sqrt{a}\,\pi}{\sqrt{8}}\ \text{a.s.}, \tag{14}$$

where $a>0$ denotes the variance of the limiting Brownian motion $\{x(t)\}$ in Example 1. (See Theorem 6.8 and Remark 6.1.)

In the proof of (14) we apply (12) and (13) to the family of processes $x^{\varepsilon}(t)=\varepsilon X(t/\varepsilon^2)$, $\varepsilon>0$. In [16], Jain has proved (14) in the case that $X(t)$ $(t=0, 1, \cdots)$ are the sums of independent identically distributed random variables in the domain of attraction of the normal distribution, by using large deviation theorems for a family of Markov chains converging to a Brownian motion, which are also developed by himself. Thus our results like (14) in Section 6 can be considered as analogues of the results by Jain for continuous time processes, but it is the advantage of our results that $\{X(t)\}$ in (14) is not necessarily Markovian itself. Further, we note that in [10] theorems like (12) and (13) for the family of the processes $x^{\varepsilon}(t)=\varepsilon X(t/\varepsilon^2)$ have been already used implicitly in case that $\{X(t)\}$ is itself a Brownian motion. But in that case $\{x^{\varepsilon}(t)\}$ is the same in law as $\{X(t)\}$ for every $\varepsilon>0$ and so theorems on large deviations for the single process $\{X(t)\}$ were sufficient. (See Remark 6.2 for some other results in [10] and [16].)

Finally, we make a brief comment on another application to the problem of the Wiener sausage (see [17], [8]). For $\delta>0$, $t\geqq 0$ and $\omega\in\Omega^+$ (with $X=\boldsymbol{R}^d$) consider the set $C_t^{\delta}(\omega)=\{y\in\boldsymbol{R}^d;\ |\omega(s)-y|<\delta$ for some $s\in[0, t]\}$. Let $\{X(t)\}$ be a d-dimensional Brownian motion. It is proved in [8] that for any $\nu>0$

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}}\log E\{e^{-\nu|C_t^{\delta}(X(\cdot))|}\}=-k(\nu) \tag{15}$$

with $k(\nu)=\inf_{f\geq 0,\int f dx=1}[\nu|\{f>0\}|+I(fdx)]$, where $|\cdot|$ denotes the d-dimensional volume and $I(fdx)$ denotes the I-function for the Brownian motion $\{X(t)\}$. Formula (15) is also known ([11], [19]) for a certain class of processes with independent increments. By using the results of the present paper, we can prove (15) when $\{X(t)\}$ is a homogenizable diffusion process with periodic coefficients (for example, the process $\{X(t)\}$ in Example 2). The details of this part will appear elsewhere.

1. Notations and preliminaries

In the present paper we will use the following notations. Let $(E, \mathcal{E})$ be any measurable space and let $\mathcal{F}$ be any sub σ-field of $\mathcal{E}$. We denote by $B(\mathcal{F})$ the set of all bounded $\mathcal{F}$-measurable functions on E and by $\mathcal{M}(\mathcal{F})$ the set of all probability measures on $(E, \mathcal{F})$. If E is a Polish space, then we denote by $C(E)$ the set of all bounded continuous functions on E and by $\mathcal{B}(E)$ the topological σ-filed in E. In that case we write $B(E)$ and $\mathcal{M}(E)$ for $B(\mathcal{B}(E))$ and $\mathcal{M}(\mathcal{B}(E))$, respectively, and we assume that $\mathcal{M}(E)$ is endowed with the weak topology so that $\mathcal{M}(E)$ is itself a Polish space.

Let X be a complete separable metric space with metric d. Let I denote either of $(-\infty, +\infty)$, $[0, +\infty)$ and $[0, T]$ $(T>0)$ and let $D(I\to X)$ denote the space of all X-valued right continuous functions on I without discontinuities of the second kind. We now recall the definition of the topology on $D(I\to X)$ of the Skorohod type (see [3], [18], [25]). For $\{\omega_n\}\subset D(I\to X)$ and $\omega\in D(I\to X)$ $\{\omega_n\}$ is said to converge to ω in $D(I\to X)$ and we write $\omega_n\to\omega$ in $D(I\to X)$ if there exists a sequence $\{\lambda_n\}$ of strictly increasing continuous mappings from I onto itself such that

$$(1.1)\qquad \sup_{t\in I_T}[d(\omega_n(\lambda_n(t)), \omega(t))+|\lambda_n(t)-t|]\to 0 \text{ as } n\to\infty$$

for every finite $T>0$, where $I_T=I\cap[-T, T]$. It is known that $D(I\to X)$ with the above convergence is a Polish space. In the following we will write, for simplicity, Ω, Ω^+ and $D[0, T]$ $(T>0)$ for $D((-\infty, +\infty)\to X)$, $D([0, +\infty)\to X)$ and $D([0, T]\to X)$, respectively. For $-\infty\leqq s\leqq t\leqq+\infty$, we denote by $\mathcal{F}_t^s$ the σ-field in Ω generated by $\{\omega(\sigma);\ \sigma\in[s, t]\cap(-\infty, +\infty)\}$. It is known that $\mathcal{B}(\Omega)=\mathcal{F}_{+\infty}^{-\infty}$ and we can naturally identify $\mathcal{B}(\Omega^+)$ and $\mathcal{B}(D[0, T])$ with $\mathcal{F}_\infty^0$ and $\mathcal{F}_T^0$, respectively. On the other hand, we have natural inclusion relations:

$$(1.2)\qquad \mathcal{M}(\Omega)\hookrightarrow\mathcal{M}(\mathcal{F}_\infty^0)\hookrightarrow\mathcal{M}(\mathcal{F}_T^0)\,,$$

which are defined by the restriction mappings $\mu \rightsquigarrow \mu|_{\mathcal{F}_\infty^0}$ for $\mu\in\mathcal{M}(\Omega)$ and $\mu \rightsquigarrow \mu|_{\mathcal{F}_T^0}$ for $\mu\in\mathcal{M}(\mathcal{F}_\infty^0)$, respectively. Thus we can think as

$$(1.3)\qquad \mathcal{M}(\Omega)\subset\mathcal{M}(\Omega^+)\subset\mathcal{M}(D[0, T])\,.$$

In the present paper we repeatedly use such relations without any specifications.

We define the shift operator θ_t on Ω by $\theta_t\omega=\omega(t+\cdot)$. We also use the same notation θ_t for the shift operator on Ω^+ if $t\geqq 0$. Let $\mathcal{M}_S(\Omega)$ denote the set of all $\mu\in\mathcal{M}(\Omega)$ which are stationary, i.e., $\mu\circ\theta_t^{-1}=\mu$ for all $t\in\boldsymbol{R}^1$. $\mathcal{M}_S(\Omega)$ is a closed subspace of $\mathcal{M}(\Omega)$.

In the rest of this section we will recall the results of Donsker and Varadhan [12] for the comparison to our results and for later use. We will mainly follow the notations in [12]. Let $\{P_x\}$ be a homogeneous Markov process with sample space Ω^+. We will always impose the following hypothesis.

(A) $x\rightsquigarrow P_x$ is a continuous mapping from X to $\mathcal{M}(\Omega^+)$.

We now define the *entropy* $H(Q)$ of $Q\in\mathcal{M}_S(\Omega)$ with respect to $\{P_x\}$ by

$$H(Q)=\int_\Omega h(\omega)\,Q(d\omega)\,, \tag{1.4}$$

where

$$h(\omega)=\sup_{\Phi\in B(\mathcal{F}_1^0)}\Big[\int\Phi dQ_\omega-\log\int e^\Phi\,dP_{\omega(0)}\Big] \tag{1.5}$$

with Q_ω denoting the regular conditional probability distribution of Q given $\mathcal{F}_0^{-\infty}$. It is known that $H(Q)$ is a lower semicontinuous affine functional on $\mathcal{M}_S(\Omega)$ with values in $[0,+\infty]$. We also define for $t>0$ and $Q\in\mathcal{M}_S(\Omega)$

$$\bar{H}(t,Q)=\sup_{\Phi\in B(\mathcal{F}_t^0)}\Big[\int\Phi dQ-\int\Big(\log\int e^\Phi dP_{\omega(0)}\Big)\,Q(d\omega)\Big]\,. \tag{1.6}$$

Later we will use the following relation [12; Theorem 3.6]:

$$H(Q)=\lim_{t\to\infty}\bar{H}(t,Q)/t=\sup_{t>0}\bar{H}(t,Q)/t\,. \tag{1.7}$$

For any $t>0$ define $p_t\colon\Omega^+\to\Omega$ by

$$\begin{aligned}(p_t\omega)(s)&=\omega(s)\quad\text{for}\quad 0\leqq s<t\,,\\ (p_t\omega)(s+t)&=(p_t\omega)(s)\quad\text{for all}\quad s\in(-\infty,+\infty)\,,\end{aligned} \tag{1.8}$$

and for any $B\in\mathcal{B}(\Omega)$ define

$$R_{t,\omega}(B)=\frac{1}{t}\int_0^t\chi_B(\theta_s p_t\omega)\,ds\,. \tag{1.9}$$

This gives the mapping $\omega\rightsquigarrow R_{t,\omega}$ from Ω^+ into $\mathcal{M}_S(\Omega)$ which is $\mathcal{F}_t^0$-measurable for each $t>0$. Thus we can define

$$\Gamma_{t,x}(A)=P_x(\omega\in\Omega^+\colon R_{t,\omega}\in A)\quad\text{for}\quad A\in\mathcal{B}(\mathcal{M}_S(\Omega))\,. \tag{1.10}$$

The one dimensional marginal of any $Q\in\mathcal{M}_S(\Omega)$ is denoted by $q[Q]$, which defines the mapping q: $\mathcal{M}_S(\Omega)\to\mathcal{M}(X)$, i.e., $q[Q]=Q(\omega(0)\in\cdot)$. Further, the family $\{q[Q];\ Q\in A\}$ is denoted by $q[A]$ for any $A\subset\mathcal{M}_S(\Omega)$.

The following theorem gives the asymptotic upper bound for $\Gamma_{t,x}$ as $t\to\infty$.

Theorem 1.1 ([12; Lemma 4.4]). *Suppose* (**A**) *is satisfied. Let A be a closed subset of $\mathcal{M}_S(\Omega)$ such that $q[A]$ is a tight family of $\mathcal{M}(X)$. Then*

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\sup_{x\in X}\Gamma_{t,x}(A)\leqq-\inf_{Q\in A}H(Q)\,. \tag{1.11}$$

Next we will be concerned with the lower bound for $\Gamma_{t,x}$. Let $p(t, x, dy)$ denote the transition probability of $\{P_x\}$. We will impose the following hypothesis.

(**B**) (i) There exists a σ-finite measure α on X and a function $p(x, y)$ on $X\times X$ such that $p(1, x, dy)=p(x, y)\,\alpha(dy)$ and $p(x, y)>0$ for α-almost all $y\in X$ for all $x\in X$.

(ii) $p(1, x, B)$ is continuous in $x\in X$ for every $B\in\mathcal{B}(X)$.

Theorem 1.2 ([12; Theorem 5.5]). *Suppose* (**A**) *and* (**B**) *are satisfied. Let $Q\in\mathcal{M}_S(\Omega)$ be such that $H(Q)<\infty$ and let N be a neighborhood of Q in $\mathcal{M}_S(\Omega)$. Let K_1 be a compact set in X such that $\alpha(K_1)>0$ and let K be any compact set in X. Then*

$$\underline{\lim_{t\to\infty}}\frac{1}{t}\log\inf_{x\in K}P_x(R_{t,\omega}\in N,\ \omega(t)\in K_1)\geqq-H(Q)\,. \tag{1.12}$$

In particular, if G is an open set in $\mathcal{M}_S(\Omega)$ and K is any compact set in X, then

$$\underline{\lim_{t\to\infty}}\frac{1}{t}\log\inf_{x\in K}\Gamma_{t,x}(G)\geqq-\inf_{Q\in G}H(Q)\,. \tag{1.13}$$

REMARK 1.1. The bound (1.12) follows immediately from the proof of Theorem 5.5 in [12]. What is stated in Theorem 5.5 in [12] is (1.13), while (1.12) asserts a little more than it. Later, in the proof of Theorem 2.2 below, we will use (1.12) rather than (1.13).

We now recall the definition of the I-function for $\{P_x\}$ defined in [9] and its relation to the entropy function $H(Q)$.

Let $\{T_t\}$ denote the semigroup on $C(X)$ corresponding to $\{P_x\}$ and let

$$I_t(\mu)=\sup_{u\in C(X)_+}\int_X\log\left(\frac{u}{T_tu}\right)(x)\,\mu\,(dx),\ \mu\in\mathcal{M}(X)\,, \tag{1.14}$$

where $C(X)_+$ denotes the set of all $u\in C(X)$ such that $\inf_{x\in X}u(x)>0$. It is easy to see that $I_t(\mu)$ is subadditive in $t>0$ for each $\mu\in\mathcal{M}(X)$, and so we can define

(1.15) $$I(\mu)=\lim_{t\to 0}\frac{1}{t}I_t(\mu)=\sup_{t>0}\frac{1}{t}I_t(\mu),\ \mu\in\mathcal{M}(X)\,.$$

Following [9] (see Remark 1.2 below), we call $I(\mu)$ the I-function for $\{P_x\}$. It is easy to see that $I(\mu)$ is a convex, lower semicontinuous function on $\mathcal{M}(X)$ with values in $[0, +\infty]$.

It was shown in [12; Theorem 6.1] that for any $\mu\in\mathcal{M}(X)$

(1.16) $$I(\mu)=\inf_{Q:[Q]=\mu}H(Q)\,,$$

which we will refer to as the *contraction principle*.

REMARK 1.2. Let L be the infinitesimal generator of $\{T_t\}$ having $\mathcal{D}(L)$ as its domain. Then we can easily obtain

(1.17) $$I(\mu)=\sup_{u\in\mathcal{D}(L)_+}\int_X\left(\frac{-Lu}{u}\right)(x)\,\mu(dx),\ \mu\in\mathcal{M}(X)\,,$$

where $\mathcal{D}(L)_+=\mathcal{D}(L)\cap C(X)_+$, provided $\mathcal{D}(L)_+$ is rich enough in the sense that, in (1.14), $C(X)_+$ can be replaced by $\mathcal{D}(L)_+$. In [9], the I-function $I(\mu)$ for $\{P_x\}$ was defined by (1.17).

2. Main results

In this section we will state our main results without proof. Recall that $\{P_x\}$ is a homogeneous Markov process with sample space $\Omega^+=D([0,\infty)\to X)$ and that $H(Q)$ denotes the entropy of $Q\in\mathcal{M}_S(\Omega)$ with respect to $\{P_x\}$. Suppose we are given a family of homogeneous Markov processes $\tilde{M}^\varepsilon=(\{\tilde{x}_t^\varepsilon\}_{t\geq 0}, \{\tilde{P}_{\tilde{x}}^\varepsilon\}_{\tilde{x}\in\tilde{X}})$, $\varepsilon>0$, on a measurable space $\tilde{X}$ and a measurable mapping π from $\tilde{X}$ onto X. Here $\tilde{x}_t^\varepsilon$ is a mapping from the underlying probability space $\tilde{\Omega}$ to $\tilde{X}$ such that $\tilde{x}_t^\varepsilon(\tilde{\omega})$ is jointly measurable in $(t,\tilde{\omega})\in[0,\infty)\times\tilde{\Omega}$. Further, suppose that there exists a family $\{\tilde{\theta}_t\}_{t\geq 0}$ of measurable mappings of $\tilde{\Omega}$ into itself such that $\tilde{x}_s^\varepsilon\circ\tilde{\theta}_t=\tilde{x}_{t+s}^\varepsilon$, $s\geqq 0$, $t\geqq 0$. We write $\tilde{x}^\varepsilon(t,\tilde{\omega})$ for $\tilde{x}_t^\varepsilon(\tilde{\omega})$ and let

(2.1) $$x^\varepsilon(t,\tilde{\omega})=\pi(\tilde{x}^\varepsilon(t,\tilde{\omega})),\ t\geqq 0\,,$$

Although we do not assume any sample properties of the process $\{\tilde{x}^\varepsilon(t,\tilde{\omega})\}$ itself, we assume that for each $\varepsilon>0$

(2.2) $$x^\varepsilon(\cdot,\tilde{\omega})\in\Omega^+\ \tilde{P}_{\tilde{x}}^\varepsilon\text{—a.s. for every }\ \tilde{x}\in\tilde{X}\,.$$

Define a family $\{P_{\tilde{x}}^\varepsilon\}_{\tilde{x}\in\tilde{X}}$, $\varepsilon>0$, of measures on Ω^+ by

(2.3) $$P_{\tilde{x}}^\varepsilon(B)=\tilde{P}_{\tilde{x}}^\varepsilon(\tilde{\omega}: x^\varepsilon(\cdot,\tilde{\omega})\in B),\ B\in\mathcal{M}(\Omega^+)\,.$$

REMARK 2.1. Throughout the paper, π and even $\tilde{X}$ may depend on $\varepsilon>0$.

In fact, in Section 6, we will meet the case that π actually depends on $\varepsilon>0$.

We make a hypothesis.

($\mathbf{A}_1^\varepsilon$) If $\varepsilon_n \downarrow 0$ and $\pi(\tilde{x}_n)\to x\in X$, then

$$P_{\tilde{x}_n}^{\varepsilon_n}\to P_x \text{ in } \mathcal{M}(\Omega^+) \quad \text{as} \quad n\to\infty\,. \tag{2.4}$$

REMARK 2.2. It is easy to see that ($\mathbf{A}_1^\varepsilon$) implies (**A**). Thus ($\mathbf{A}_1^\varepsilon$) as well as (**A**) implies the Feller property of $\{P_x\}$ and hence that P_x has no fixed discontinuities for any $x\in X$. Therefore, by [18; Theorem 3] we can replace (2.4) by

$$P_{\tilde{x}_n}^{\varepsilon_n}\to P_x \text{ in } \mathcal{M}(D[0,T]) \text{ as } n\to\infty \text{ for every } T<\infty\,. \tag{2.4}'$$

Recall $R_{t,\omega}$ is defined by (1.9) and define

$$\Gamma_{t,\tilde{x}}^\varepsilon(A)=P_{\tilde{x}}^\varepsilon(\omega\colon R_{t,\omega}\in A) \quad \text{for} \quad A\in\mathcal{B}(\mathcal{M}_S(\Omega))\,. \tag{2.5}$$

We are interested in the asymptotic behavior of $\Gamma_{t,\tilde{x}}^\varepsilon$ as $t\to\infty$ and $\varepsilon\downarrow 0$ simultaneously.

First, we will be concerned with the upper bound for $\Gamma_{t,\tilde{x}}^\varepsilon$. In addition to ($\mathbf{A}_1^\varepsilon$), we will impose the following hypothesis.

($\mathbf{A}_2^\varepsilon$) For any $\varepsilon>0$ and any compact set $K\subset X$, $\{P_{\tilde{x}}^\varepsilon;\ \tilde{x}\in\pi^{-1}K\}$ is tight as a family of measures on $D[0, 1]$.

Theorem 2.1. *Suppose* ($\mathbf{A}_1^\varepsilon$) *and* ($\mathbf{A}_2^\varepsilon$) *are satisfied. Let A be a closed subset of $\mathcal{M}_S(\Omega)$ such that $q[A]$ is a tight family in $\mathcal{M}(X)$. Then*

$$\varlimsup_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\sup_{\tilde{x}\in\tilde{X}}\Gamma_{t,\tilde{x}}^\varepsilon(A)\leqq-\inf_{Q\in A}H(Q)\,. \tag{2.6}$$

The proof will be given in Section 3.

REMARK 2.3. If X is compact, then the hypothesis that $q[A]$ is tight in Theorem 2.1 is automatically satisfied.

Next we are concerned with the lower bound for $\Gamma_{t,\tilde{x}}^\varepsilon$. We assume that ($\mathbf{A}_1^\varepsilon$) and (**B**) are satisfied. Recall that $p(t, x, dy)$ denotes the transition probability of $\{P_x\}$ and $\alpha(dy)$ denotes the reference measure in (**B**). Let $\tilde{p}^\varepsilon(t, \tilde{x}, d\tilde{y})$ denote the transition probability of $\{\tilde{P}_{\tilde{x}}^\varepsilon\}$. We will impose either of the following two hypotheses.

($\mathbf{B}^\varepsilon$) There exists a compact set $K_1\subset X$ such that $\alpha(K_1)>0$ and that if $\varepsilon_n\downarrow 0$ and $\pi(\tilde{x}_n)\to x\in X$, then

(2.7) $$\lim_{n\to\infty}\tilde{p}^{\varepsilon_n}(1, \tilde{x}_n, \pi^{-1}K_1)=p(1, x, K_1)\,.$$

(C) There exists a relatively compact open set $G_1\subset X$ such that $\alpha(G_1)>0$.

Remark 2.4. Hypothesis (C) is automatically satisfied if X is locally compact.

Theorem 2.2. *Suppose* ($\mathbf{A}_1^\varepsilon$) *and* (**B**) *are satisfied. Further, suppose either* ($\mathbf{B}^\varepsilon$) *or* (**C**) *is satisfied. Let A be an open subset of $\mathcal{M}_S(\Omega)$ and let K be a compact set in X. Then*

(2.8) $$\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\inf_{\tilde{x}\,:\,\pi(\tilde{x})\in K}\Gamma^\varepsilon_{t,\tilde{x}}(A)\geqq-\inf_{Q\in A}H(Q)\,.$$

The proof will be given in Section 4.

Remark 2.5. Consider the trivial case that $\tilde{X}=X$, $\pi(x)=x$ and $P^\varepsilon_x=P_x$ for every $\varepsilon>0$. Then (**A**) implies ($\mathbf{A}_1^\varepsilon$) and ($\mathbf{A}_2^\varepsilon$), and also (**B**) implies ($\mathbf{B}^\varepsilon$). Thus Theorems 2.1 and 2.2 are generalizations of Theorems 1.1 and 1.2, respectively.

Remark 2.6. Let $C([0, 1]\to X)$ [resp. $C([0, \infty)\to X)$] denote the space of all continuous functions from [0, 1] [resp. $[0, \infty)$] to X with the topology of uniform convergence on [0, 1] [resp. on every finite interval $I\subset[0, \infty)$]. Suppose that

(2.9) $$P^\varepsilon_{\tilde{x}}(C[0, \infty)\to X))=1 \text{ for all } \tilde{x}\in\tilde{X} \text{ and all } \varepsilon>0\,.$$

Then it is not difficult to see that ($\mathbf{A}_1^\varepsilon$) is satisfied if and only if ($\mathbf{A}_1^\varepsilon$) with Ω^+ replaced by $C([0, \infty)\to X)$ is satisfied (in that case P_x is necessarily supported on $C([0, \infty)\to X)$, and that ($\mathbf{A}_2^\varepsilon$) is satisfied if and only if ($\mathbf{A}_2^\varepsilon$) with $D[0, 1]$ replaced by $C([0, 1]\to X)$ is satisfied. This is because $C([0, \infty)\to X)$ and $C([0, 1]\to X)$ are closed subspaces of Ω^+ and $D[0, 1]$, respectively.

The following is a corollary to Theorems 2.1 and 2.2; the proof is established by the argument in [27; Section 3].

Corollary 2.1. *Suppose that* ($\mathbf{A}_1^\varepsilon$), ($\mathbf{A}_2^\varepsilon$) *and* (**B**) *are satisfied and that X is compact. Let Ψ be a real-valued bounded continuous function on $\mathcal{M}_S(\Omega)$. Then uniformly for $\tilde{x}\in\tilde{X}$*

(2.10) $$\lim_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log E^\varepsilon_{\tilde{x}}[e^{t\Psi(R_{t,\omega})}]=\sup_{Q\in\mathcal{M}_S(\Omega)}[\Psi(Q)-H(Q)]\,,$$

where $E^\varepsilon_{\tilde{x}}$ denotes the expectation with respect to $P^\varepsilon_{\tilde{x}}$.

In some applications (see, for example, Section 6), it will be convenient to restate the above results in a special case. For $t>0$ and $\omega\in\Omega^+$ define $L_t(\omega, \cdot)\in\mathcal{M}(X)$ by

(2.11) $$L_t(\omega, B) = \frac{1}{t}\int_0^t \chi_B(\omega(s))\,ds,\ B\in\mathcal{B}(X)$$

and for any Borel set $A\subset\mathcal{M}(X)$ define

(2.12) $$Q^{\varepsilon}_{t,\tilde{x}}(A) = P^{\varepsilon}_{\tilde{x}}(\omega\colon L_t(\omega,\cdot)\in A),\ t>0,\ \tilde{x}\in\tilde{X},\ \varepsilon>0\,.$$

We make a hypothesis.

($\mathbf{A}^{\varepsilon}_0$) If $\varepsilon_n\downarrow 0$ and $\pi(\tilde{x}_n)\to x\in X$, then for any $t>0$

(2.13) $$P^{\varepsilon_n}_{\tilde{x}_n}(\omega\colon \omega(t)\in dy)\to p(t,x,dy) \text{ in } \mathcal{M}(X)\,.$$

(This is the same as saying that if $\varepsilon_n\downarrow 0$ and $\pi(\tilde{x}_n)\to x\in X$, then $P^{\varepsilon_n}_{\tilde{x}_n}\to P_x$ in the sense of convergence of all finite dimensional distributions, which is obviously weaker than ($\mathbf{A}^{\varepsilon}_1$).)

Recall that $I(\mu)$ denotes the I-function for $\{P_x\}$ (see (1.15)).

Theorem 2.3. *Suppose* ($\mathbf{A}^{\varepsilon}_0$) *is satisfied. Let A be a compact subset of $\mathcal{M}(X)$. Then*

(2.14) $$\varlimsup_{\substack{t\to\infty\\ \varepsilon\downarrow 0}} \frac{1}{t}\log \sup_{\tilde{x}\in\tilde{X}} Q^{\varepsilon}_{t,\tilde{x}}(A) \leqq -\inf_{\mu\in A} I(\mu)\,.$$

This is a corollary to Theorem 2.1 if ($\mathbf{A}^{\varepsilon}_1$) and ($\mathbf{A}^{\varepsilon}_2$) are assumed since $L_t(\omega,\cdot)=q[R_{t,\omega}]$ and since the contraction principle (1.16) holds. But we can prove Theorem 2.3 if only ($\mathbf{A}^{\varepsilon}_0$) is assumed, by a slight modification of the proof of Theorem 2.1. The details will be given in Section 3.

The following is an immediate corollary to Theorem 2.2.

Theorem 2.4. *Suppose* ($\mathbf{A}^{\varepsilon}_1$), (**B**) *and either of* ($\mathbf{B}^{\varepsilon}$) *and* (**C**) *are satisfied. Let A be an open subset of $\mathcal{M}(X)$. Then for any compact set $K\subset X$*

(2.15) $$\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}} \frac{1}{t}\log \inf_{\tilde{x}\,:\,\pi(\tilde{x})\in K} Q^{\varepsilon}_{t,\tilde{x}}(A) \geqq \inf_{\mu\in A} I(\mu)\,.$$

3. Upper bounds

In this section we will prove Theorems 2.1 and 2.3. We follow the proof in [12] with necessary modification. Throughout the section, we will drop the "ε" from the notations $\tilde{x}^{\varepsilon}(t,\tilde{\omega})$ and $x^{\varepsilon}(t,\tilde{\omega})$.

Let $T>0$ and let $\Phi\in B(\mathcal{F}^0_T)$. In this section we use the following notations.

(3.1) $$\tilde{\phi}^{\varepsilon}(\tilde{x}) = \log\int_{\Omega} e^{\Phi(\omega)}\,P^{\varepsilon}_{\tilde{x}}(d\omega),\ \varepsilon>0,\ \tilde{x}\in\tilde{X}\,.$$

(3.2) $$\tilde{\Phi}^{\varepsilon}(\tilde{\omega})=\Phi(x(\cdot,\tilde{\omega}))-\tilde{\phi}^{\varepsilon}(\tilde{x}(0,\tilde{\omega})),\ \varepsilon>0,\ \tilde{\omega}\in\tilde{\Omega}\,.$$

Lemma 3.1. *For any $t>0$, any $\tilde{x}\in\tilde{X}$ and any $\varepsilon>0$ we have*

(3.3) $$\int_{\tilde{\Omega}}\exp\left\{\frac{1}{T}\int_0^t\tilde{\Phi}^{\varepsilon}(\tilde{\theta}_s\tilde{\omega})\,ds\right\}\tilde{P}^{\varepsilon}_{\tilde{x}}(d\tilde{\omega})\leqq 1\,.$$

Proof. Since $\int e^{\tilde{\Phi}^{\varepsilon}}d\tilde{P}^{\varepsilon}_{\tilde{x}}=1$ for every $\tilde{x}\in\tilde{X}$, the proof is the same as that of Lemma 4.1 in [12]. Q.E.D.

The mapping $\pi\colon\tilde{X}\to X$ induces one from $\mathcal{M}(\tilde{X})$ into $\mathcal{M}(X)$, in a natural manner; for this we use the same notation, π, i.e.,

(3.4) $$\pi\tilde{\lambda}=\tilde{\lambda}\circ\pi^{-1}\quad\text{for}\quad\tilde{\lambda}\in\mathcal{M}(\tilde{X})\,.$$

Let $\|\Phi\|_\infty=\sup_{\omega\in\Omega}|\Phi(\omega)|$.

Lemma 3.2. *Let A be a Borel subset of $\mathcal{M}_S(\Omega)$. Then we have*

(3.5) $$\int_A\exp\left\{\frac{t}{T}\int_\Omega\Phi(\omega)\,Q(d\omega)\right\}\Gamma^{\varepsilon}_{t,\tilde{x}}(dQ)$$
$$\leqq\exp\{2\|\Phi\|_\infty+\frac{t}{T}\sup_{\tilde{\lambda}\,:\,\pi\tilde{\lambda}\in q[A]}\int_{\tilde{X}}\tilde{\phi}^{\varepsilon}\,d\tilde{\lambda}\}$$

for every $t>0$, every $\tilde{x}\in\tilde{X}$ and every $\varepsilon>0$.

Proof. Since $x(\cdot,\tilde{\theta}_t\tilde{\omega})=x(\cdot+t,\tilde{\omega})=\theta_t[x(\cdot,\tilde{\omega})]$, we have

$$\frac{1}{t}\int_0^t\tilde{\Phi}^{\varepsilon}(\tilde{\theta}_s\tilde{\omega})\,ds$$
$$=\frac{1}{t}\int_0^t\Phi(\theta_s[x(\cdot,\tilde{\omega})])\,ds-\frac{1}{t}\int_0^t\tilde{\phi}^{\varepsilon}(\tilde{x}(s,\tilde{\omega}))\,ds$$
$$\geqq\int_\Omega\Phi\,dR_{t,x(\cdot,\tilde{\omega})}-\frac{2T}{t}\|\Phi\|_\infty-\int_{\tilde{X}}\tilde{\phi}^{\varepsilon}d\tilde{\lambda}_{t,\tilde{\omega}}\,,$$

where $\tilde{\lambda}_{t,\tilde{\omega}}=\frac{1}{t}\int_0^t\chi.\,(\tilde{x}(s,\tilde{\omega}))\,ds\in\mathcal{M}(\tilde{X})$. Thus we have, from Lemma 3.1,

$$\int_A\exp\left\{\frac{t}{T}\int_\Omega\Phi(\omega)\,Q(d\omega)\right\}\Gamma^{\varepsilon}_{t,\tilde{x}}(dQ)$$
$$\leqq\exp\{2\|\Phi\|_\infty+\frac{t}{T}\sup_{\tilde{\lambda}\in\tilde{\Lambda}}\int_{\tilde{X}}\tilde{\phi}^{\varepsilon}d\tilde{\lambda}\}\,,$$

where $\tilde{\Lambda}=\{\tilde{\lambda}_{t,\tilde{\omega}};\,R_{t,x(\cdot,\tilde{\omega})}\in A\}\subset\mathcal{M}(\tilde{X})$. Since

$$\pi\tilde{\lambda}_{t,\tilde{\omega}}=\frac{1}{t}\int_0^t\chi.\,(x(s,\tilde{\omega}))\,ds=q[R_{t,x(\cdot,\tilde{\omega})}]\,,$$

we have $\pi\tilde{\Lambda}\subset q[A]$, and hence (3.5). Q.E.D.

Let $\mathcal{C}_T=\{\Phi(\omega)=f(\omega(t_1), \cdots, \omega(t_n));\ 0\leqq t_1<\cdots<t_n\leqq T,\ f\in C(X^n),\ n=1, 2, \cdots\}$, where X^n denotes the n-fold product of X and let $\mathcal{D}_T=\{\Phi\in\mathcal{C}_T;\ \int_{\Omega^+} e^{\Phi}dP_x \leqq 1$ for every $x\in X\}$.

Lemma 3.3. *For every $T>0$ and $Q\in\mathcal{M}_S(\Omega)$ we have*

$$\bar{H}(T,Q)=\sup_{\Phi\in\mathcal{C}_T}\left[\int\Phi\,dQ-\int\left(\log\int e^{\Phi}dP_{\omega(0)}\right)Q(d\omega)\right] \tag{3.6}$$

$$=\sup_{\Phi\in\mathcal{D}_T}\int\Phi\,dQ\,,$$

where $\bar{H}(t,Q)$ is given in (1.6).

Proof. The first equality follows from (1.6) by a standard argument. The second one is immediate since $\Phi(\omega)-\log\int_{\Omega^+}e^{\Phi}dP_{\omega(0)}$ belongs to $\mathcal{D}_T$ for any $\Phi\in\mathcal{C}_T$. Q.E.D.

Lemma 3.4. *Let $\Phi\in\mathcal{D}_T$ and let $\tilde{\phi}^{\varepsilon}(\tilde{x})$ be given by* (3.1). *Let Λ be a tight family in $\mathcal{M}(X)$. Assuming* ($\mathbf{A}_0^{\varepsilon}$), *we have*

$$\varlimsup_{\varepsilon\downarrow 0}\ \sup_{\tilde{\lambda}:\pi\tilde{\lambda}\in\Lambda}\int_{\tilde{X}}\tilde{\phi}^{\varepsilon}(\tilde{x})\,\tilde{\lambda}(d\tilde{x})\leqq 0\,. \tag{3.7}$$

Proof. Let $\tilde{\Lambda}=\pi^{-1}\Lambda$. Given $\delta>0$, there exists a compact set $K\subset X$ such that $\pi\tilde{\lambda}(K^c)=\tilde{\lambda}(\pi^{-1}K^c)\leqq\delta$ for every $\tilde{\lambda}\in\tilde{\Lambda}$. Thus we have

$$\begin{aligned} I&=\sup_{\tilde{\lambda}\in\tilde{\Lambda}}\int_{\tilde{X}}\tilde{\phi}^{\varepsilon}d\tilde{\lambda}\\ &\leqq \sup_{\tilde{x}\in\pi^{-1}K}\tilde{\phi}^{\varepsilon}(\tilde{x})+\sup_{\tilde{\lambda}\in\tilde{\Lambda}}\int_{\pi^{-1}K^c}\tilde{\phi}^{\varepsilon}d\tilde{\lambda}\\ &\equiv I_1+I_2\,. \end{aligned}$$

Noting that $\tilde{\phi}^{\varepsilon}(\tilde{x})\leqq||\Phi||_\infty$, we have $I_2\leqq\delta||\Phi||_\infty$. We next choose $\varepsilon_n\downarrow 0$ and $\tilde{x}_n\in\pi^{-1}K$ such that

$$\varlimsup_{\varepsilon\downarrow 0} I_1=\lim_{n\to\infty}\tilde{\phi}^{\varepsilon_n}(\tilde{x}_n)\,.$$

We can assume $\pi(\tilde{x}_n)$ converges to some $x\in K$. By ($\mathbf{A}_0^{\varepsilon}$) we have

$$\lim_{n\to\infty}\tilde{\phi}^{\varepsilon_n}(\tilde{x}_n)=\log\int e^{\Phi}\,dP_x\,, \tag{3.8}$$

which is non positive since $\Phi\in\mathcal{D}_T$. Thus we have $\varlimsup_{\varepsilon\downarrow 0} I\leqq\varlimsup_{\varepsilon\downarrow 0} I_2\leqq\delta||\Phi||_\infty$ for arbitrary $\delta>0$. Q.E.D.

In the following we set

(3.9) $$J(A)=\overline{\lim_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}}\frac{1}{t}\log\sup_{\tilde{x}\in\tilde{X}}\Gamma^{\varepsilon}_{t,\tilde{x}}(A)\quad\text{for } A\in\mathcal{B}(\mathcal{M}_S(\Omega))\,.$$

Further, for any finite family $\{A_j\}_{j=1}^{l}$ of subsets of $\mathcal{M}_S(\Omega)$ we set

(3.10) $$F(\{A_j\}_{j=1}^{l})=\inf_{1\leqq j\leqq l}\sup_{T>0}\sup_{\Phi\in\mathcal{D}_T}\inf_{Q\in A_j}\frac{1}{T}\int_{\Omega}\Phi(\omega)\,Q(d\omega)\,.$$

Lemam 3.5. *Let A be a Borel subset of $\mathcal{M}_S(\Omega)$ such that $q[A]$ is a tight family in $\mathcal{M}(X)$. Assuming* ($\mathbf{A}_0^{\varepsilon}$), *we have*

(3.11) $$J(A)\leqq -F(\{A_j\}_{j=1}^{l})$$

for any finite Borel covering $\{A_j\}_{j=1}^{l}$ of A in $\mathcal{M}_S(\Omega)$.

Proof. It follows from (3.5) that for any $\Phi\in\mathcal{D}_T$

$$\Gamma^{\varepsilon}_{t,\tilde{x}}(A)$$
$$\leqq \exp\{2\|\Phi\|_{\infty}+\frac{t}{T}\sup_{\tilde{\lambda}\,:\,\pi\tilde{\lambda}\in q[A]}\int_{\tilde{X}}\tilde{\phi}^{\varepsilon}d\tilde{\lambda}\}\cdot\exp\{-\frac{t}{T}\inf_{Q\in A}\int_{\Omega}\Phi(\omega)\,Q(d\omega)\}\,.$$

Since $q[A]$ is tight, we have, from Lemma 3.4,

$$J(A)\leqq -\inf_{Q\in A}\frac{1}{T}\int\Phi(\omega)\,Q(d\omega)\,,$$

and thus

$$J(A)\leqq -\sup_{T>0}\sup_{\Phi\in\mathcal{D}_T}\inf_{Q\in A}\frac{1}{T}\int_{\Omega}\Phi(\omega)\,Q(d\omega)\,.$$

Since this holds for any Borel set $A\subset\mathcal{M}_S(\Omega)$ such that $q[A]$ is tight, we have, using the relation $J(B\cup C)=\max\,\{J(B), J(C)\}$,

$$\begin{aligned}J(A)&\leqq \max_{1\leqq j\leqq l}J(A\cap A_j)\\ &\leqq -\inf_{1\leqq j\leqq l}\sup_{T>0}\sup_{\Phi\in\mathcal{D}_T}\inf_{Q\in A\cap A_j}\frac{1}{T}\int_{\Omega}\Phi(\omega)\,Q(d\omega)\end{aligned}$$

for any Borel covering $\{A_j\}_{j=1}^{l}$ of A. This proves (3.11). Q.E.D.

Lemma 3.6. *Let A be a compact set in $\mathcal{M}_S(\Omega)$. For any $\delta>0$ there exists a finite open covering $\{G_j\}_{j=1}^{l}$ of A in $\mathcal{M}_S(\Omega)$ such that*

(3.12) $$F(\{G_j\}_{j=1}^{l})\geqq \inf_{Q\in A}H(Q)-\delta\,.$$

In particular, for any compact set A in $\mathcal{M}_S(\Omega)$ we have

(3.13) $$\sup F(\{A_j\}_{j=1}^{l})\geqq \inf_{Q\in A}H(Q)\,,$$

where the supremum is taken over all finite Borel coverings $\{A_j\}_{j=1}^{l}$ *of* A.

This lemma is deduced from (1.7) and Lemma 3.3. The proof is the same as in [12; Lemma 4.3] only except that we use Lemma 3.3 in place of Lemma 3.7 in [12], and so omitted.

Corollary 3.1. *Suppose* $(\mathbf{A}_0^{\varepsilon})$ *is satisfied. Then for any compact set* $A\subset \mathcal{M}_S(\Omega)$

$$J(A) \leqq -\inf_{Q\in A} H(Q)\,. \tag{3.14}$$

This immediately follows from Lemmas 3.5 and 3.6.

Proof of Theorem 2.1. First we choose $\varepsilon_k \downarrow 0$ so that

$$J(A)=\varlimsup_{\substack{t\to\infty\\ k\to\infty}} \frac{1}{t}\log \sup_{\tilde{x}\in\tilde{X}} \Gamma_{t,\tilde{x}}^{\varepsilon_k}(A)\,.$$

In the rest we fix $\{\varepsilon_k\}$ and write $P_{\tilde{x}}^{k}$ and $\Gamma_{t,\tilde{x}}^{k}$ for $P_{\tilde{x}}^{\varepsilon_k}$ and $\Gamma_{x,\tilde{x}}^{\varepsilon_k}$ respectively. Since $q[A]$ is tight there exists for any n, a compact set $K_n\subset X$ such that $\mu(K_n)\geqq 1-1/n$ for all $\mu\in q[A]$. We can show, by a standard argument, that $(\mathbf{A}_1^{\varepsilon})$ and $(\mathbf{A}_2^{\varepsilon})$ imply that $\{P_{\tilde{x}}^{k};\ \tilde{x}\in\pi^{-1}K_n,\ k=1, 2, \cdots\}$ is tight in $\mathcal{M}(D[0, 1])$ for each n. Therefore, given $\eta_n\downarrow 0$, there exist compact sets $C_n\subset D[0, 1]$ such that $P_{\tilde{x}}^{k}(C_n)\geqq 1-\eta_n$ for all $\tilde{x}\in\pi^{-1}K_n$ and all k. Here and after we think of C_n as a subset of Ω (i.e., $C_n\in\mathcal{F}_1^0$). Thus we can show as in the proof of Lemma 4.4 in [12; p. 197] that for all k, all $\tilde{x}\in\tilde{X}$ and all $\nu>0$

$$\int \exp\{\nu\chi_{K_n}(\omega(0))\,\chi_{C_n^c}(\omega)\}\,P_{\tilde{x}}^{k}(d\omega)\leqq 1+\eta_n(e^{\nu}-1)\,. \tag{3.15}$$

Let $\Phi(\omega)=\nu\chi_{K_n}(\omega(0))\,\chi_{C_n^c}(\omega)$ and write $\tilde{\phi}^k(\tilde{x})$ for $\tilde{\phi}^{\varepsilon}(\tilde{x})$ defined by (3.1) with $\varepsilon=\varepsilon_k$. Then we have from (3.15), $\tilde{\phi}^k(\tilde{x})\leqq\log[1+\eta_n(e^{\nu}-1)]$ and thus, using Lemma 3.1 with $T=1$, we get

$$\int \exp\{\nu\int_0^t \chi_{K_n}(\omega(s))\,\chi_{C_n^c}(\theta_s\omega)\,ds\}\,P_{\tilde{x}}^{k}(d\omega)\leqq \exp\{t\log[1+\eta_n(e^{\nu}-1)]\}\,. \tag{3.16}$$

Let $\lambda>0$ be fixed. We can see as in [12; pp. 197–198] that (3.16) with $\nu=\lambda n^2$ and $\eta_n=\exp(-\lambda n^2)$ implies

$$\varlimsup_{\substack{t\to\infty\\ k\to\infty}} \frac{1}{t}\log\sup_{\tilde{x}\in\tilde{X}} \Gamma_{t,\tilde{x}}^{k}(A\cap A_t^c)\leqq \log 2-\lambda\,, \tag{3.17}$$

where $A_t=\{Q\in\mathcal{M}_S(\Omega);\ Q(C_n^c)\leqq 1/t+2/n \text{ for all } n\}$. Let $A_\infty=\bigcap_{t>0}A_t$. Then A_∞ is compact and hence so is $A\cap A_\infty$. Thus, by Lemma 3.6, for any $\delta>0$ there exists a finite open covering $\{G_j\}_{j=1}^{l}$ of $A\cap A_\infty$ in $\mathcal{M}_S(\Omega)$ such that

$F(\{G_j\}_{j=1}^l)\geqq \inf_{Q\in A\cap A_\infty} H(Q)-\delta$. Let $G=\bigcup_{j=1}^l G_j$. Since $\{G_j\}_{j=1}^l$ is also an open covering of $A\cap G$ and since $q[A\cap G]$ is tight in $\mathcal{M}(X)$, we have, from Lemma 3.5, $J(A\cap G)\leqq -F(\{G_j\}_{j=1}^l)$ and hence

$$J(A\cap G)\leqq -\inf_{Q\in A} H(Q)+\delta\,. \tag{3.18}$$

On the other hand, by Lemma 4.5 in [12], there exists a t_0 such that $t\geqq t_0$ implies $A\cap A_t\subset G$. Thus we have

$$\begin{aligned}\Gamma^k_{t,\tilde{x}}(A) &= \Gamma^k_{t,\tilde{x}}(A\cap A_t)+\Gamma^k_{t,\tilde{x}}(A\cap A_t^c)\\ &\leqq \Gamma^k_{t,\tilde{x}}(A\cap G)+\Gamma^k_{t,\tilde{x}}(A\cap A_t^c)\,.\end{aligned}$$

Therefore, from (3.17) and (3.18), we have

$$\begin{aligned}J(A) &\leqq \max\{J(A\cap G), \log 2-\lambda\}\,,\\ &\leqq \max\{-\inf_{Q\in A} H(Q)+\delta, \log 2-\lambda\}\,,\end{aligned}$$

and hence, letting $\delta\to 0$ and $\lambda\to\infty$, we have the theorem. Q.E.D.

Proof of Theorem 2.3. We prove (2.14) under the hypothesis ($\mathbf{A}_0^\varepsilon$) by modifying the above argument. Let A be a compact subset of $\mathcal{M}(X)$. Since $Q^\varepsilon_{t,\tilde{x}}(A)=\Gamma^\varepsilon_{t,\tilde{x}}(q^{-1}A)$, we have to prove

$$J(q^{-1}A)\leqq -\inf_{\mu\in A} I(\mu)\,. \tag{3.19}$$

For any finite family $\{B_j\}_{j=1}^l$ of subsets of $\mathcal{M}(X)$ define

$$F'(\{B_j\}_{j=1}^l)=\inf_{1\leqq j\leqq l}\sup_{t>0}\sup_{u\in C(X)_+}\inf_{\mu\in B_j}\int_X \log\left(\frac{u}{T_t u}\right)(x)\mu(dx)\,. \tag{3.20}$$

Then we can prove

$$\sup F'(\{B_j\}_{j=1}^l)\geqq \inf_{\mu\in A} I(\mu)\,, \tag{3.21}$$

where the supremum is taken over all finite Borel coverings $\{B_j\}_{j=1}^l$ of A in $\mathcal{M}(X)$. This is the analogue of (3.13) for $I(\mu)$ and deduced from (1.14) and (1.15). On the other hand, we have for any $B_j\subset\mathcal{M}(X)$, $j=1, \cdots, l$,

$$F'(\{B_j\}_{j=1}^l)\leqq F(\{q^{-1}B_j\}_{j=1}^l)\,, \tag{3.22}$$

which follows from the fact that if $u\in C(X)_+$ and if $q[Q]=\mu$, then $\Phi(\omega)=\log\dfrac{u(\omega(t))}{T_t u(\omega(0))}\in\mathcal{D}_t$ and $\int\Phi(\omega)\,Q(d\omega)=\int\log\left(\frac{u}{T_t u}\right)(x)\,\mu(dx)$. Therefore, we have, from Lemma 3.5,

$$J(q^{-1}A)\leqq -F(\{q^{-1}B_j\}_{j=1}^l)\leqq -F'(\{B_j\}_{j=1}^l)$$

for any Borel covering $\{B_j\}_{j=1}^{l}$ of A in $\mathcal{M}(X)$, which combined with (3.21) proves (3.19). Q.E.D.

4. Lower bounds

In this section we give the proof of Theorem 2.2 and, in the course of the proof, we also get an auxiliary estimate (Corollary 4.1), which will be used in Section 6. As in the previous section, we will drop the "ε" from $\tilde{x}^{\varepsilon}(t, \tilde{\omega})$ and $x^{\varepsilon}(t, \tilde{\omega})$.

We need some lemmas. The first one is elementary.

Lemma 4.1. *Let $T \geqq 0$, $s>0$ and $t>0$. Let $||\mu||_{\mathcal{F}_T^0}$ denote the total variation of any signed measure μ on Ω relative to $\mathcal{F}_T^0$. Then we have*

$$(4.1) \qquad ||R_{t,\omega}-\frac{1}{n(t)}\sum_{i=1}^{n(t)} R_{s,\theta_{(i-1)s}\omega}||_{\mathcal{F}_T^0} \leqq \frac{2s}{t}+\frac{2T}{s},$$

where $n(t)=[t/s]+1$ and we have

$$(4.2) \qquad ||R_{s,\omega}-R_{s-1,\omega}||_{\mathcal{F}_T^0} \leqq \frac{2T+2}{s}$$

if $s>1$.

The proof is omitted.

Note that, by the definition of $R_{t,\omega}$, we can think of $\omega \rightsquigarrow R_{t,\omega}$ as a mapping from $D[0, t]$ to $\mathcal{M}_S(\Omega)$ for each $t>0$.

Lemma 4.2. *The mapping $\omega \rightsquigarrow R_{t,\omega}$ from $D[0, t]$ into $\mathcal{M}_S(\Omega)$ is continuous for each $t>0$.*

Proof. Let $\omega_n \rightarrow \omega$ in $D[0, t]$ as $n \rightarrow \infty$. Then it is easy to see that $p_t\omega_n \rightarrow p_t\omega$ in Ω as $n \rightarrow \infty$, where p_t is defined by (1.8). Let $F \in C(\Omega)$ be arbitrary. Then, since $\theta_s \colon \Omega \rightarrow \Omega$ is continuous for every $s \in \boldsymbol{R}^1$, we have

$$\lim_{n\to\infty} \frac{1}{t}\int_0^t F(\theta_s p_t \omega_n)\, ds = \frac{1}{t}\int_0^t F(\theta_s p_t \omega)\, ds ,$$

which completes the proof. Q.E.D.

The following two lemmas will give the main estimates which reduce the desired estimate (2.8) to the known one (1.12). In both lemmas $(\mathbf{A}_1^{\varepsilon})$ will be assumed. As for $(\mathbf{B}^{\varepsilon})$ and $(\mathbf{C})$, they will correspond to the respective lemmas.

Lemma 4.3. *Suppose that $(\mathbf{A}_1^{\varepsilon})$ is satisfied and that there exists a compact set $K_1 \subset X$ satisfying (2.7). Let $Q \in \mathcal{M}_S(\Omega)$ and let N be a neighborhood of Q in $\mathcal{M}_S(\Omega)$. Then there exist an $s_0>0$ and a neighborhood N' of Q in $\mathcal{M}_S(\Omega)$ such that* *for any compact set* $K \subset X$ *and any* $s \geqq s_0$

$$\varliminf_{\substack{t\to\infty \\ \varepsilon\downarrow 0}} \frac{1}{t} \log \inf_{\tilde{x}\in\pi^{-1}K} \Gamma^{\varepsilon}_{t,\tilde{x}}(N) \tag{4.3}$$

$$\geqq \frac{1}{s} \log \inf_{x\in K\cup K_1} P_x(R_{s,\omega}\in N',\ \omega(s)\in K_1)\,.$$

Proof. First we note that there exist $\delta>0$, $T>0$ and $\Phi_l \in C(D[0, T])$ with $\sup_{\omega} |\Phi_l(\omega)| \leqq 1$ $(l=1, 2, \cdots, m)$ such that

$$N(\delta) \equiv \{Q' \in \mathcal{M}_S(\Omega);\ \max_{1\leqq l\leqq m} |\int \Phi_l\, dQ' - \int \Phi_l\, dQ| < \delta\} \subset N\,. \tag{4.4}$$

Taking account of Lemma 4.1, we choose $s_0>0$ such that $(2T+2)/s_0<\delta/4$ and fix any $s\geqq s_0$, and then we take $t_0>0$ such that $2s/t_0+2T/s<\delta/4$. Then it follows from (4.1) and (4.4) that for any $t\geqq t_0$

$$\{\omega;\ R_{t,\omega}\in N\} \supset \{\omega;\ R_{s,\theta_{(i-1)s}\omega} \in N\left(\frac{3}{4}\delta\right),\ \omega(is)\in K_1,\ i=1, \cdots, n(t)\}\ ,$$

where $n(t)=[t/s]+1$. Here we have used the convexity of $N\left(\frac{3}{4}\delta\right)$. Thus, by the Markovian property of $\{\tilde{P}^{\varepsilon}_{\tilde{x}}\}$, we have

$$\inf_{\tilde{x}\in\pi^{-1}K} \Gamma^{\varepsilon}_{t,\tilde{x}}(N) \geqq \{\inf_{\tilde{x}\in\pi^{-1}K'} P^{\varepsilon}_{\tilde{x}}(R_{s,\omega}\in N\left(\frac{3}{4}\delta\right),\ \omega(s)\in K_1)\}^{n(t)}\ ,$$

where $K'=K\cup K_1$. Since $\lim_{t\to\infty} n(t)/t=1/s$, we have

$$\varliminf_{\substack{t\to\infty \\ \varepsilon\downarrow 0}} \frac{1}{t} \log \inf_{\tilde{x}\in\pi^{-1}K} \Gamma^{\varepsilon}_{t,\tilde{x}}(N) \geqq \frac{1}{s} \log I_1\,, \tag{4.5}$$

where $I_1 = \varliminf_{\varepsilon\downarrow 0} \inf_{\tilde{x}\in\pi^{-1}K'} P^{\varepsilon}_{\tilde{x}}(R_{s,\omega}\in N\left(\frac{3}{4}\delta\right),\ \omega(s)\in K_1)$. There exist $\varepsilon_n \downarrow 0$ and $\tilde{x}_n \in\pi^{-1}K'$ such that

$$I_1 = \lim_{n\to\infty} P^n(R_{s,\omega}\in N\left(\frac{3}{4}\delta\right),\ \omega(s)\in K_1)\,,$$

where $P^n = P^{\varepsilon_n}_{\tilde{x}_n}$. Since K' is compact, we can assume $\pi(\tilde{x}_n)$ converges to some $x\in K'$ so that $P^n\to P_x$ in $\mathcal{M}(\Omega^+)$ by $(\mathrm{A}^{\varepsilon}_1)$. Using (4.2), we have

$$\begin{aligned} I_1 &\geqq \varliminf_{n\to\infty} P^n(R_{s-1,\omega}\in N(\delta/2),\ \omega(s)\in K_1) \\ &= \varliminf_{n\to\infty} \tilde{E}^n[\chi_{\{R_{s-1,x(\cdot,\tilde{\omega})}\in N(\delta/2)\}} \tilde{p}^{\varepsilon_n}(1, \tilde{x}(s-1, \tilde{\omega}), \pi^{-1}K_1)]\,, \end{aligned}$$

where $\tilde{E}^n$ denotes the expectation with respect to $\tilde{P}^{\varepsilon_n}_{\tilde{x}_n}$. We now claim that

(4.6) $$\lim_{n\to\infty} \tilde{E}^n[|\tilde{p}^{\varepsilon_n}(1, \tilde{x}(s-1, \tilde{\omega}), \pi^{-1}K_1)-p(1, x(s-1, \tilde{\omega}), K_1)|] = 0,$$

which will be proved later. Assuming (4.6), we have

$$I_1 \geqq \varliminf_{n\to\infty} \int F(\omega)\, P^n(d\omega),$$

where $F(\omega)=\chi_{\{R_{s-1,\omega}\in N(\delta/2)\}}\, p(1, \omega(s-1), K_1)$. It follows from Lemma 4.2 that $F(\omega)$ as a function on $D[0, s-1]$ is lower semicontinuous. Therefore, we have

$$I_1 \geqq \int F(\omega)\, P_x(d\omega) = P_x(R_{s-1,\omega}\in N(\delta/2), \omega(s)\in K_1)$$

since $P^n \to P_x$ in $\mathcal{M}(D[0, s-1])$ (see Remark 2.2). Using (4.2) again, we have $I_1 \geqq P_x(R_{s,\omega}\in N(\delta/4), \omega(s)\in K_1)$, proving (4.3) with $N'=N(\delta/4)$.

It only remains to prove (4.6). Let $\mu^n(B)=P^{\varepsilon_n}_{\tilde{x}_n}(\omega(s-1)\in B)$ for $B\in\mathcal{B}(X)$. Since $\mu^n \to p(s-1, x, \cdot)$ in $\mathcal{M}(X)$ by $(\mathbf{A}_1^\varepsilon)$, $\{\mu^n\}$ is tight, that is, for any $\eta>0$ there exists a compact set $L\subset X$ such that $\sup_n \mu^n(L^c)<\eta$. Thus, denoting the expectation in (4.6) by I_2, we have

(4.7) $$I_2 \leqq 2\mu^n(L^c)+\sup_{\tilde{x}\in\pi^{-1}L} |\tilde{p}^{\varepsilon_n}(1, \tilde{x}, \pi^{-1}K_1)-p(1, \pi(\tilde{x}), K_1)| \equiv J_1+J_2.$$

We choose $\tilde{x}_n\in\pi^{-1}L$ so that

(4.8) $$\varlimsup_{n\to\infty} J_2 = \lim_{n\to\infty} |\tilde{p}^{\varepsilon_n}(1, \tilde{x}_n, \pi^{-1}K_1)-p(1, \pi(\tilde{x}_n), K_1)|.$$

Since L is compact, we can assume $\pi(\tilde{x}_n)$ converges to some $x\in X$. Note that (2.7) implies that $x \rightsquigarrow p(1, x, K_1)$ is continuous. Thus from (2.7) we have

$$\varlimsup_{n\to\infty} J_2 = \lim_{n\to\infty} |\tilde{p}^{\varepsilon_n}(1, \tilde{x}_n, \pi^{-1}K_1)-p(1, x, K_1)| = 0.$$

Therefore, we have $\varlimsup_{n\to\infty} I_2 \leqq \varlimsup_{n\to\infty} J_1 \leqq 2\eta$, proving (4.6). Q.E.D.

In the following lemma we give an estimate which is stronger than what we need for the proof of Theorem 2.2. This is because we need it (or rather its corollary) for the application in Section 6.

For any open set $G\subset X$ define

(4.9) $$\tau_G^*(\omega) = \inf\{t\geqq 0; \omega(t)\notin G \text{ or } \omega(t-)\notin G\}, \quad \omega\in\Omega^+$$

with the convention $\omega(0-)=\omega(0)$. It is easy to see that τ_G^* is lower semicontinuous on Ω^+.

Lemma 4.4. *Suppose that $(\mathbf{A}_1^\varepsilon)$ is satisfied and that there exists a relatively compact open set $G_1\subset X$. Let $Q\in\mathcal{M}_S(\Omega)$ and let N be a neighborhood of Q in $\mathcal{M}_S(\Omega)$. Then there exist an $s_0>0$ and a neighborhood N' of Q in $\mathcal{M}_S(\Omega)$ such that for* *any open set $G\subset X$, any compact set $K\subset X$ and any $s\geqq s_0$*

$$\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\inf_{\tilde{x}\in\pi^{-1}K}P_{\tilde{x}}^{\varepsilon}(R_{t,\omega}\in N,\ t<\tau_G^*(\omega))$$
$$\geqq\frac{1}{s}\log\inf_{x\in K\cup\bar{G}_1}P_x(R_{s,\omega}\in N',\ s<\tau_G^*(\omega),\ \omega(s)\in G_1)\,. \tag{4.10}$$

Proof. First we take a neighborhood $N(\delta)$ of Q in $\mathcal{M}_S(\Omega)$ of the form (4.4) such that $N(\delta)\subset N$. Then, noting that

$$\{\omega;\ t<\tau_G^*(\omega)\}\supset\{\omega;\ s<\tau_G^*(\theta_{(i-1)s}\omega),\ i=1,\cdots,n(t)\}\ ,$$

where $n(t)=[t/s]+1$, we can show as in the proof of Lemma 4.3 that there exists an $s_0>0$ such that for all $s\geqq s_0$

$$\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\inf_{\tilde{x}\in\pi^{-1}K}P_{\tilde{x}}^{\varepsilon}(R_{t,\omega}\in N,\ t<\tau_G^*(\omega))\geqq\frac{1}{s}\log I_1, \tag{4.11}$$

where

$$I_1=\varliminf_{\varepsilon\downarrow 0}\inf_{\tilde{x}\in\pi^{-1}K'}P_{\tilde{x}}^{\varepsilon}(R_{s,\omega}\in N(\delta/2),\ s<\tau_G^*(\omega),\ \omega(s)\in G_1)$$

with $K'=K\cup\bar{G}_1$. We can choose $\varepsilon_n\downarrow 0$ and $\tilde{x}_n\in\pi^{-1}K'$ such that

$$I_1=\lim_{n\to\infty}P^n(R_{s,\omega}\in N(\delta/2),\ s<\tau_G^*(\omega),\ \omega(s)\in G_1)\,,$$

where $P^n=P_{\tilde{x}_n}^{\varepsilon_n}$. Since K' is compact, we can assume $\pi(\tilde{x}_n)$ converges to some $x\in K'$ so that $P^n\to P_x$ in $\mathcal{M}(\Omega^+)$, and hence in $\mathcal{M}(D[0,s])$. Thus, noting that $\{\omega;\ R_{s,\omega}\in N(\delta/2),\ s<\tau_G^*(\omega),\ \omega(s)\in G_1\}$ as a subset of $D[0,s]$ is open, we have

$$I_1\geqq P_x(R_{s,\omega}\in N(\delta/2),\ s<\tau_G^*(\omega),\ \omega(s)\in G_1)\,,$$

proving (4.10). Q.E.D.

Recall that $L_t(\omega,\cdot)$ is defined by (2.11). The following will be used in Section 6.

Corollary 4.1. *Suppose that $(\mathbf{A}_1^{\varepsilon})$ is satisfied and that there exists a relatively compact open set $G_1\subset X$. Let $\mu\in\mathcal{M}(X)$ and let V be a neighborhood of μ in $\mathcal{M}(X)$. Then there exist an $s_0>0$ and a neighborhood V' of μ in $\mathcal{M}(X)$ such that for any open set $G\subset X$, any compact set $K\subset X$ and any $s\geqq s_0$*

$$\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\inf_{\tilde{x}\in\pi^{-1}K}P_{\tilde{x}}^{\varepsilon}(L_t(\omega,\cdot)\in V,\ t<\tau_G^*(\omega))$$
$$\geqq\frac{1}{s}\log\inf_{x\in K\cup\bar{G}_1}P_x(L_s(\omega,\cdot)\in V',\ s<\tau_G^*(\omega),\ \omega(s)\in G_1)\,. \tag{4.12}$$

Proof. First we note that there exist $\delta>0$ and $f_l \in C(X)$ with $\sup_{x\in X}|f_l(x)| \leqq 1$, $l=1, \cdots, m$, such that

$$V(\delta)\equiv\{\mu'\in\mathcal{M}(X); \max_{1\leqq l\leqq m}|\int f_l\, d\mu'-\int f_l\, d\mu|<\delta\}\subset V.$$

Then $q^{-1}V(\delta)$ is of the form (4.4) with $\Phi_l(\omega)=f_l(\omega(0))$. As in the proof of Lemma 4.4, we can prove (4.10) with $N=q^{-1}V$ and $N'=q^{-1}V(\delta/2)$, which means (4.12) since $L_t(\omega, \cdot)=q[R_{t,\omega}]$. Q.E.D.

Proof of Theorem 2.2. Let A be an open set in $\mathcal{M}_S(\Omega)$ and let $Q\in A$ be arbitrary. We always assume $(\mathbf{A}_1^\varepsilon)$ and $(\mathbf{B})$, and besides we first assume $(\mathbf{B}^\varepsilon)$. Then by Lemma 4.3 we have, for sufficiently large $s>0$,

$$(4.13)\qquad I\equiv\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\inf_{\tilde{x}\in\pi^{-1}K}\Gamma_{t,\tilde{x}}^{\varepsilon}(A)$$
$$\geqq\frac{1}{s}\log\inf_{x\in K'}P_x(R_{s,\omega}\in N,\ \omega(s)\in K_1),$$

where N is a neighborhood of Q in $\mathcal{M}_S(\Omega)$ such that $N\subset A$ and K' and K_1 are compact sets in X such that $\alpha(K_1)>0$. Letting $s\to\infty$, we have, from Theorem 1.2, $I\geqq -H(Q)$. Since $Q\in A$ is arbitrary, we have (2.8). Next we assume $(\mathbf{C})$ in place of $(\mathbf{B}^\varepsilon)$. Let $G_1\subset X$ be a relatively compact open set with $\alpha(G_1)>0$. Then there eixsts a compact set $K_2\subset G_1$ such that $\alpha(K_2)>0$. Thus (4.13) with $K_1=K_2$ follows from Lemma 4.4 with $G=X$. This suffices as above. Q.E.D.

5. Examples

In this section we will give some examples of $\{P_{\tilde{x}}^{\varepsilon}\}$ and $\{P_x\}$ satisfying our hypotheses. In all the examples below, we will take $\mathbf{R}^d$ or $\mathbf{T}^d=\mathbf{R}^d/\mathbf{Z}^d$ (d-dimensional torus) as X so that $(\mathbf{C})$ will be automatically satisfied. Further, $P_{\tilde{x}}^{\varepsilon}$ will be supported on $C([0, \infty])\to X)$ and so Remark 2.6 will be useful.

EXAMPLE 5.1. This example is a special case of the transport process (see [22], [2] and [21]). Let S be a Polish space. We will take $X=\mathbf{R}^d$, $\tilde{X}=\mathbf{R}^d\times S$ and $\pi(x, y)=x$, $(x, y)\in\mathbf{R}^d\times S$. Let $q(y)\geqq 0$ be a bounded continuous function on S and let $k(y, dz)$ be a probability kernel on $(S, \mathcal{B}(S))$ such that $k(y, A)$ is continuous in y for every $A\in\mathcal{B}(S)$. Define a bounded operator Q on $C(S)$ by

$$(5.1)\qquad Qf(y)=q(y)\int_S[f(z)-f(y)]\,k(y, dz).$$

Let $F(x, y)=(F^1(x, y), \cdots, F^d(x, y))$ be a bounded continuous function on $\mathbf{R}^d\times S$ with values in $\mathbf{R}^d$ such that each $F^i(x, y)$ has its x-derivatives of all orders which are bounded on $\mathbf{R}^d\times S$. For each $\varepsilon>0$ define

(5.2) $$L^\varepsilon = \frac{1}{\varepsilon^2} Q + \frac{1}{\varepsilon} \sum_{i=1}^{d} F^i(x, y) \frac{\partial}{\partial x^i} .$$

This is the infinitesimal generator of a Markov process $\{(x^\varepsilon(t), y^\varepsilon(t))\}$. Indeed, $\{(x^\varepsilon(t), y^\varepsilon(t))\}$ can be constructed as follows. (It is only for simplicity that Q is assumed to be independent of $x \in \boldsymbol{R}^d$. In general, Q may depend on $x \in \boldsymbol{R}^d$ in a suitable manner as in the references cited above.)

It is easy to see that Q is itself the infinitesimal generator of a Feller Markov process on S which we refer to as Q-process. The Q-process is a pure jump process and we normalize it so that its sample paths are right continuous. Let $\{Y^y(t)\}$ denote the Q-process starting from $Y^y(0) = y \in S$ and let $y^\varepsilon(t) = Y^y(t/\varepsilon^2)$, $t \geqq 0$, $\varepsilon > 0$. Let $x^\varepsilon(t)$ be the unique solution of the following ordinary differential equation

(5.3) $$\frac{dx^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} F(x^\varepsilon(t), y^\varepsilon(t)) ,$$
$$x^\varepsilon(0) = x \in \boldsymbol{R}^d .$$

Then it is easy to see that for each $\varepsilon > 0$ the family of processes $\{(x^\varepsilon(t), y^\varepsilon(t))\}$ constitutes a Feller Markov process generated by L^ε. It is shown in [22] that under certain hypotheses the process $\{x^\varepsilon(t)\}$ converges to a diffusion process on $\boldsymbol{R}^d$ as $\varepsilon \downarrow 0$. Such a limit theorem corresponds to $(\mathbf{A}_1^\varepsilon)$.

We now specify the hypotheses on L^ε and the limiting diffusion process. We assume that there exist constants q_1 and q_2 such that

(5.4) $$0 < q_1 \leqq q(y) \leqq q_2 < \infty$$

and that there exists a reference probability measure $\phi(dz)$ on S such that $k(y, dz) = k(y, z)\, \phi(dz)$ with density $k(y, z)$ satisfying

(5.5) $$0 < k_1 \leqq k(y, z) \leqq k_2 < \infty ,$$

where k_1 and k_2 are constants. Let $P(t, y, dz)$ denote the transition probability of the Q-process. We can see that (5.4) and (5.5) imply that there exists a unique invariant probability measure $\bar{P}(dz)$ such that

(5.6) $$\sup_{y \in S, A \in \mathcal{B}(S)} |P(t, y, A) - \bar{P}(A)| \leqq e^{-ct}$$

for sufficiently large $t > 0$ for some constant $c > 0$. Thus we can define the *recurrent potential kernel*

(5.7) $$G_0(y, A) = \int_0^\infty [P(t, y, A) - \bar{P}(A)]\, dt, \; y \in S, \; A \in \mathcal{B}(S)$$

so that the *Fredholm alternative* for Q holds in the sense that the equation

(5.8) $$-Qg(y) = f(y), \; y \in S$$

has a bounded continuous solution $g(y)$ if and only if $f(y)$ is a bounded continuous function such that

$$\int_S f(y)\,\bar{P}(dy) = 0\,. \tag{5.9}$$

If this is the case, then any solution $g(y)$ of (5.8) is given by

$$g(y) = \int_S f(z)\,G_0(y, dz) + \text{a constant}\,. \tag{5.10}$$

We assume that $F^i(x, y)$, $i=1, \cdots, d$, satisfy the *centering condition*

$$\int_S F^i(x, y)\,\bar{P}(dy) = 0 \text{ for all } x \in \boldsymbol{R}^d,\ i = 1, \cdots, d\,, \tag{5.11}$$

so that the equations

$$-Q\chi^i(x, y) = F^i(x, y),\ i = 1, \cdots, d\,, \tag{5.12}$$

have the bounded continuous solutions $\chi^i(x, y)$, $i=1, \cdots, d$, such that their x-derivatives of all orders are bounded on $\boldsymbol{R}^d \times S$. Let

$$\begin{aligned} a^{ij}(x) &= \int_S [\chi^i F^j + \chi^j F^i]\,(x, y)\,\bar{P}(dy)\,, \\ b^i(x) &= \int_S [\sum_{k=1}^{d} \frac{\partial \chi^i}{\partial x^k} F^k]\,(x, y)\,\bar{P}(dy) \end{aligned} \tag{5.13}$$

and let

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x^i}\,. \tag{5.14}$$

Since $a^{ij}(x)$ and $b^i(x)$ are bounded smooth functions having bounded derivatives, the martingale problem for L is well-posed and the associated family of solutions $\{P_x\}$ constitutes a diffusion process on $\boldsymbol{R}^d$. Here P_x are probability measures on $C([0, \infty) \to \boldsymbol{R}^d)$ as usual. (See [26].)

Let $P^{\varepsilon}_{x,y}$ denote the measure on $C([0, \infty) \to \boldsymbol{R}^d)$ induced by the process $\{x^{\varepsilon}(t)\}$ defined by (5.3) for the initial value $(x^{\varepsilon}(0), y^{\varepsilon}(0)) = (x, y) \in \boldsymbol{R}^d \times S$. We now see that $(\mathbf{A}^{\varepsilon}_1)$ and $(\mathbf{A}^{\varepsilon}_2)$ are satisfied for $\{P^{\varepsilon}_{x,y}\}$, $\varepsilon > 0$, and $\{P_x\}$ (see Remark 2.6). We can show as in [22] that if $\varepsilon_n \downarrow 0$ and $x_n \to x \in \boldsymbol{R}^d$ and if $y_n \in S$ are arbitrary, then

$$P^{\varepsilon_n}_{x_n, y_n} \to P_x \text{ in } \mathcal{M}(C([0, \infty) \to \boldsymbol{R}^d)) \quad \text{as } n \to \infty\,, \tag{5.15}$$

which implies $(\mathbf{A}^{\varepsilon}_1)$. Though in [22] (5.15) is proved only in the case that $(x_n, y_n) \equiv (x, y)$, we can establish (5.15) by a minor modification of the proof in [22]. We omit the details. On the other hand, $(\mathbf{A}^{\varepsilon}_2)$ immediatley follows from (5.3).

As for (**B**), this is satisfied if L is uniformly elliptic, i.e., there exists a constant $\nu > 0$ such that

$$(5.16)\qquad \sum_{i,j=1}^{d} a^{ij}(x)\xi^i \xi^j \geqq \nu|\xi|^2,\ \xi=(\xi^1, \cdots, \xi^d)\in \boldsymbol{R}^d,\ x\in \boldsymbol{R}^d .$$

Indeed, (**B**) (i) follows from the fact [14] that the fundamental solution for $\partial/\partial t - L$ is strictly poistive, and (**B**) (ii), which is the strong Feller property of $\{P_x\}$, is known [26; Theorem 7.2.4].

Example 5.2. This is a special case of Example 5.1. Let $\{Y^y(t)\}$ be the Q-process in Example 5.1 starting from $y\in S$ and let

$$(5.17)\qquad X^{x,y}(t) = x+\int_0^t F(Y^y(s))\, ds ,$$

where $F(y)=(F^1(y), \cdots, F^d(y))$ is a bounded continuous function on S with values in $\boldsymbol{R}^d$ satisfying the centering condition

$$(5.18)\qquad \int_S F^i(y)\, \bar{P}(dy) = 0,\ i = 1, \cdots, d .$$

For each $\varepsilon>0$ and each $(x, y)\in \boldsymbol{R}^d\times S$ define

$$(5.19)\qquad \begin{aligned} x^\varepsilon(t) &= \varepsilon X^{x/\varepsilon, y}(t/\varepsilon^2),\ t\geqq 0 ,\\ y^\varepsilon(t) &= Y^y(t/\varepsilon^2),\ t\geqq 0 . \end{aligned}$$

Then it is easy to see that $(x^\varepsilon(t), y^\varepsilon(t))$ is a Markov process starting from $(x, y)\in \boldsymbol{R}^d\times S$ generated by

$$(5.20)\qquad L^\varepsilon = \frac{1}{\varepsilon^2} Q+\frac{1}{\varepsilon}\sum_{i=1}^{d} F^i(y)\frac{\partial}{\partial x^i} .$$

This is a special case of (5.2) ($F(x, y)=F(y)$). In this case L becomes

$$(5.21)\qquad L = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} \quad \text{with}$$

$$(5.22)\qquad a^{ij} = \int_S [\chi^i F^j + \chi^j F^i]\,(y)\, \bar{P}(dy) ,$$

where $\chi^i(y), i=1, \cdots, d$, are the solutions of

$$(5.23)\qquad -Q\chi^i(y) = F^i(y),\ i = 1, \cdots, d .$$

Thus the diffusion process $\{P_x\}$ in this case is a Brownian motion with covariance matrix (a^{ij}).

As before, ($\mathbf{A}_1^\varepsilon$) and ($\mathbf{A}_2^\varepsilon$) are satisfied for $\{x^\varepsilon(t)\}$, $\varepsilon>0$, and $\{P_x\}$. Moreover, if (a^{ij}) is positive definite, then (**B**) is satisfied. In this connection, we note that for each $\xi=(\xi^1, \cdots, \xi^d)\in \boldsymbol{R}^d$ $(\xi\neq 0)$

$$(5.24)\qquad \sum_{i=1}^{d} F^i(\cdot)\,\xi^i \not\equiv 0$$

implies that

$$\sum_{i,j=1}^{d} a^{ij} \xi^i \xi^j > 0 . \tag{5.25}$$

To see this, using (5.22) and (5.23), and denoting $\sum_{i=i}^{d} \chi^i(y) \xi^i$ by $\chi^{\xi}(y)$, we observe

$$\begin{aligned}\sum_{i,j=1}^{d} a^{ij} \xi^i \xi^j &= \int_S [-2\chi^{\xi} Q\chi^{\xi}](y) \bar{P}(dy) \\ &= \int_S [Q(\chi^{\xi})^2 - 2\chi^{\xi} Q\chi^{\xi}](y) \bar{P}(dy) \\ &= \int_S [q(y) \int_S (\chi^{\xi}(z) - \chi^{\xi}(y))^2 k(y, z) \phi(dz)] \bar{P}(dy) .\end{aligned} \tag{5.26}$$

Suppose $\sum_{i,j=1}^{d} a^{ij} \xi^i \xi^j = 0$. Then there exists a $y \in S$ such that $\chi^{\xi}(\cdot) = \chi^{\xi}(y)$ ϕ-a.e. Thus we have $\sum_{i=1}^{d} F^i(\cdot) \xi^i \equiv -Q\chi^{\xi}(\cdot) \equiv 0$, which contradicts (5.24). Consequently, **(B)** is satisfied if (5.24) holds for any $(\xi^1, \cdots, \xi^d) \neq 0$.

REMARK 5.1. In Examples 5.1 and 5.2 the Q-process can be replaced by another ergodic Markov process for which the Fredholm alternative holds. For example, we can take a Brownian motion on a torus ($S = \boldsymbol{T}^m$) in place of the Q-process.

REMARK 5.2. In Examples 5.1 and 5.2 we have taken $\pi(x, y) = x$ and $\tilde{x}^{\varepsilon}(t) = (x^{\varepsilon}(t), y^{\varepsilon}(t)) = (\varepsilon X^{x/\varepsilon, y}(t/\varepsilon^2), Y^y(t/\varepsilon^2))$. But we can also take $\pi_{\varepsilon}(x, y) = \varepsilon x$ (see Remark 2.1) and $\tilde{x}^{\varepsilon}(t) = (X^{x,y}(t/\varepsilon^2), Y^y(t/\varepsilon^2))$. We will meet the latter case in the proof of Theorem 6.8.

EXAMPLE 5.3. We consider the homogenization problem for a diffusion process with random stationary coefficients. This problem has been discussed in [23], [24], [20], etc. We follow the results by Osada [20]. We begin with the abstract framework, which covers the cases of periodic ([1]) and almost periodic coefficients (see [23]); in particular, Example 2 in Introduction is a special case of this example.

Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ be a probability space and let $\{T_x\}$, $x \in \boldsymbol{R}^d$, be a d-dimensional stationary ergodic flow on $\hat{\Omega}$. Let $L^2(\hat{\Omega})$ be the real L^2-space on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and let $\{U_x\}$ denote the strongly continuous unitary group on $L^2(\hat{\Omega})$ induced by $\{T_x\}$, i.e., $U_x f(\hat{\omega}) = f(T_x \hat{\omega})$, $x \in \boldsymbol{R}^d$, $\hat{\omega} \in \hat{\Omega}$, $f \in L^2(\hat{\Omega})$. For each $i = 1, \cdots, d$, let D_i denote the infinitesimal generator of $\{U_x\}$ in the i-th direction with domain $\mathcal{D}(D_i)$, i.e.,

$$D_i f(\hat{\omega}) = \frac{\partial}{\partial x^i} (U_x f)(\hat{\omega}) |_{x=0} , \tag{5.27}$$

where the differentiation is in the $L^2(\hat{\Omega})$-sense. Let $H^1(\hat{\Omega}) = \bigcap_{i=1}^{d} \mathcal{D}(D_i)$.

Let $\hat{a}^{ij}(\hat{\omega})$ and $\hat{b}^i(\hat{\omega})$, $i, j = 1, \cdots, d$, be real valued measurable functions. We assume that there exist constants $\nu>0$ and $M>0$, and functions $\hat{c}^{ij}\in H^1(\hat{\Omega})$, $i, j=1, \cdots, d$, such that

(i) $\sum_{i,j=1}^{d} \hat{a}^{ij}(\hat{\omega})\,\xi^i\,\xi^j \geqq \nu|\xi|^2$ for all $\xi=(\xi^1, \cdots, \xi^d)\in \boldsymbol{R}^d$ and $|\hat{a}^{ij}(\hat{\omega})|\leqq M$.

(ii) $\hat{b}^i(\hat{\omega})=\sum_{j=1}^{d} D_j\,\hat{c}^{ij}(\hat{\omega})$ and $|\hat{c}^{ij}(\hat{\omega})|\leqq M$.

(iii) $\sum_{i=1}^{d} D_i\,\hat{b}^i=0$ in the generalized sense, i.e.,

$$\int_{\hat{\Omega}} \sum_{i=1}^{d} \hat{b}^i(\hat{\omega})\,D_i\phi(\hat{\omega})\,\hat{P}(d\hat{\omega})=0 \text{ for all } \phi\in H^1(\hat{\Omega}).$$

Consider the formal differential operator

$$A^{\hat{\omega}}=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x^j}\left(a^{ij}(x,\hat{\omega})\frac{\partial}{\partial x^i}\right)+\sum_{i=1}^{d} b^i(x,\hat{\omega})\frac{\partial}{\partial x^i}, \tag{5.28}$$

where $a^{ij}(x,\hat{\omega})=\hat{a}^{ij}(T_x\hat{\omega})$ and $b^i(x,\hat{\omega})=\hat{b}^i(T_x\hat{\omega})$. It is known (see [20] and Remark 5.3 below) that for $\hat{P}$-almost all $\hat{\omega}$ there exists a unique fundamental solution $p^{\hat{\omega}}(t,x,y)$ for $\partial/\partial t-A^{\hat{\omega}}$ (in the weak sense) having the following global estimates:

$$C_1 g(\gamma_1 t,x,y)\leqq p^{\hat{\omega}}(t,x,y)\leqq C_2 g(\gamma_2 t,x,y) \tag{5.29}$$

for all $(t,x,y)\in(0,+\infty)\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where $g(t,x,y)=(2\pi t)^{-d/2}\exp\{-|x-y|^2/t\}$ and C_1, C_2, γ_1 and γ_2 are positive constants depending only on ν, M and d. Further, for any $T>0$

$$|p^{\hat{\omega}}(t,x,y)-p^{\hat{\omega}}(t',x',y')|\leqq K(|x-x'|^{\alpha}+|y-y'|^{\alpha}+|t-t'|^{\alpha/2}) \tag{5.30}$$

for all (t,x,y), $(t',x',y')\in(T,+\infty)\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where K and α are positive constants depending only on T, ν, M and d.

Let $\hat{\Omega}_0\subseteq\hat{\Omega}$ be such that $\hat{P}(\hat{\Omega}_0)=1$ and that (5.29) and (5.30) hold for every $\hat{\omega}\in\hat{\Omega}_0$. For any $\hat{\omega}\in\hat{\Omega}_0$ we can construct diffusion measures $P_x^{\hat{\omega}}$ on $C([0,\infty)\to\boldsymbol{R}^d)$ having $p^{\hat{\omega}}(t,x,y)$ as their transition density functions relative to the Lebesgue measure. We refer to $\{P_x^{\hat{\omega}}\}$ as $A^{\hat{\omega}}$-diffusion process.

Let $\varepsilon>0$ and let $P_x^{\hat{\omega},\varepsilon}$ denote the measure on $C([0,\infty)\to\boldsymbol{R}^d)$ induced from $P_{x/\varepsilon}^{\hat{\omega}}$ by

$$x^{\varepsilon}(t,\omega)=\varepsilon\omega(t/\varepsilon^2),\ t\geqq 0,\ \omega\in C([0,\infty)\to\boldsymbol{R}^d). \tag{5.31}$$

Then $\{P_x^{\hat{\omega},\varepsilon}\}$ forms the diffusion process associated with

$$A^{\hat{\omega},\varepsilon}=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x^j}\left(a^{ij}\left(\frac{x}{\varepsilon},\hat{\omega}\right)\frac{\partial}{\partial x^i}\right)+\frac{1}{\varepsilon}\sum_{i=1}^{d} b^i\left(\frac{x}{\varepsilon},\hat{\omega}\right)\frac{\partial}{\partial x^i}. \tag{5.32}$$

It is shown in [20] that there exists $\hat{\Omega}_1\subset\hat{\Omega}_0$ with $\hat{P}(\hat{\Omega}_1)=1$ such that for all $\hat{\omega}\in\hat{\Omega}_1$

(5.33) $$P_0^{\hat{\omega},\varepsilon}\to P_0 \text{ in } \mathcal{M}(C([0,\infty)\to\boldsymbol{R}^d)) \text{ as } \varepsilon\downarrow 0,$$

where P_0 is the Wiener measure with the covariance matrix (q^{ij}) defined below starting at the origin. We now define

(5.34) $$q^{kl}=\frac{1}{2}\sum_{i,j=1}^{d}\int_{\hat{\Omega}}[(\delta_i^k+\psi_i^k)(\hat{a}^{ij}+\hat{a}^{ji})(\delta_j^l+\psi_j^l)](\hat{\omega})\hat{P}(d\hat{\omega}),$$

where δ_i^k denotes Kronecker's delta and $\psi_i^k\in L^2(\hat{\Omega})$ are the unique solutions of the equations (5.35)–(5.37) below (see Proposition 3.1 in [20]):

(5.35) $$\int_{\hat{\Omega}}\Big[-\frac{1}{2}\sum_{i,j=1}^{d}\hat{a}^{ij}(\delta_i^k+\psi_i^k)D_j\phi+\sum_{i=1}^{d}\hat{b}^i(\delta_i^k+\psi_i^k)\phi\Big](\hat{\omega})\hat{P}(d\hat{\omega})=0,$$

$k=1,\cdots,d$ for all $\phi\in H^1(\hat{\Omega})$.

(5.36) $$\int_{\hat{\Omega}}[\psi_i^k D_j\phi](\hat{\omega})\hat{P}(d\hat{\omega})=\int_{\hat{\Omega}}[\psi_j^k D_i\phi](\hat{\omega})\hat{P}(d\hat{\omega}),\qquad i,j,k=1,\cdots,d,$$

for all $\phi\in H^1(\hat{\Omega})$.

(5.37) $$\int_{\hat{\Omega}}\psi_i^k(\hat{\omega})\hat{P}(d\hat{\omega})=0,\qquad i,k=1,\cdots,d.$$

Let

(5.38) $$L=\frac{1}{2}\sum_{i,j=1}^{d}q^{ij}\frac{\partial^2}{\partial x^i\partial x^j}$$

and let $\{P_x\}$ denote the diffusoin process (Brownian motion) generated by L. We refer to $\{P_x\}$ as the *homogenized* process for the family $\{P_x^{\hat{\omega},\varepsilon}\}$, or simply, for the family $\{P_x^{\hat{\omega}}\}$.

We now verify that $(\mathbf{A}_1^\varepsilon)$, $(\mathbf{A}_2^\varepsilon)$ and $(\mathbf{B})$ are satisfied for $\{P_x^{\hat{\omega},\varepsilon}\}$ and $\{P_x\}$ for $\hat{P}$-almost all $\hat{\omega}\in\hat{\Omega}$ (see Remark 2.6). First, we note that for any $\hat{\omega}\in\hat{\Omega}_0$ and any $\varepsilon>0$ the coefficients of $A^{\hat{\omega},\varepsilon}$ satisfy (i)–(iii) with the same ν and M as those for $A^{\hat{\omega}}$, so that the fundamental solution $p^{\hat{\omega},\varepsilon}(t,x,y)$ of $\partial/\partial t-A^{\hat{\omega},\varepsilon}$ satisfies (5.29) and (5.30) with the same constants as before. Thus there exists a constant $C>0$ such that

(5.39) $$\sup_{x\in\boldsymbol{R}^d,\,\varepsilon>0,\,\hat{\omega}\in\hat{\Omega}_0}E_x^{\hat{\omega},\varepsilon}[|\omega(t)-\omega(s)|^4]\leqq C|t-s|^2,\ t,s\geqq 0,$$

where $E_x^{\hat{\omega},\varepsilon}$ denotes the expectation with respect to $P_x^{\hat{\omega},\varepsilon}$. This implies that for any $R<\infty$

(5.40) $$\{P_x^{\hat{\omega},\varepsilon};\ |x|\leqq R,\ \varepsilon>0,\ \hat{\omega}\in\hat{\Omega}_0\}\quad\text{is tight}.$$

Next note that (5.33) and the estimates (5.29) and (5.30) for $p^{\hat{\omega},\varepsilon}(t,x,y)$ imply that

$$p^{\hat{\omega},\varepsilon}(1, 0, x) \to p(1, 0, x) \quad \text{uniformly on } \boldsymbol{R}^d \text{ as } \varepsilon \downarrow 0, \tag{5.41}$$

where $p(t, x, y)$ denotes the transition density function of $\{P_x\}$. Let $f \in C(\boldsymbol{R}^d)$ and take $\varepsilon_n \downarrow 0$. Let $\hat{\omega} \in \hat{\Omega}_1$ be fixed and define

$$u^n(x) = E_x^{\hat{\omega},\varepsilon_n}[f(\omega(t))] \quad \text{and} \quad u(x) = E_x[f(\omega(t))], \qquad x \in \boldsymbol{R}^d,$$

where E_x denotes the expectation with respect to P_x. It follows from the estimates (5.29) and (5.30) for $p^{\hat{\omega},\varepsilon}(t, x, y)$ that the family $\{u^n(x)\}$ is uniformly equicontinuous on $\boldsymbol{R}^d$. On other hand, (5.33) implies that for any $g \in C(\boldsymbol{R}^d)$

$$\int p^{\hat{\omega},\varepsilon_n}(1, 0, x) g(x) u^n(x) dx \to \int p(1, 0, x) g(x) u(x) dx \qquad \text{as } n \to \infty. \tag{5.42}$$

It is easy to see that $u^n \to u$ uniformly on every compact subset of $\boldsymbol{R}^d$ as $n \to \infty$. Hence, taking $x_n \to x \in \boldsymbol{R}^d$, we have $u^n(x_n) \to u(x)$ as $n \to \infty$, which means that

$$P_{x_n}^{\hat{\omega},\varepsilon_n}(\omega(t) \in dy) \to P_x(\omega(t) \in dy) \qquad \text{in } \mathcal{M}(\boldsymbol{R}^d). \tag{5.43}$$

This implies that every finite dimensional distribution of $P_{x_n}^{\hat{\omega},\varepsilon_n}$ converges to that of P_x. Thus $(\mathbf{A}_1^\varepsilon)$ and $(\mathbf{A}_2^\varepsilon)$ are immediate from (5.40). As for (**B**), this is satisfied since (q^{ij}) is positive definite (see [23; Remark 3]).

REMARK 5.3. In [20] some restrictions on the coefficients of $A^{\hat{\omega}}$ have been imposed in addition to (i)–(iii) in Example 5.3. In fact, $\hat{a}^{ij} = \hat{a}^{ji}$ and the smoothness of $a^{ij}(x, \hat{\omega})$ and $b^i(x, \hat{\omega})$ in x have been assumed in [20]. However, Osada has recently proved that such restrictions can be removed (private communication). See also [15] for the case where the coefficients are periodic but are not necessarily smooth nor even continuous.

REMARK 5.4. Let $\hat{m}(\hat{\omega})$ be a measurable function such that $0 < m_1 \leqq \hat{m}(\hat{\omega}) \leqq m_2 < \infty$ for some constants m_1 and m_2 and let $m(x, \hat{\omega}) = \hat{m}(T_x \hat{\omega})$. Consider the formal operator $B^{\hat{\omega}} = m(x, \hat{\omega})^{-1} A^{\hat{\omega}}$. In [20] the homogenization problem for the diffusion process associated with $B^{\hat{\omega}}$ is also discussed, and it is shown that the homogenized process for $B^{\hat{\omega}}$-diffusion process is the Brownian motion generated by $(\int \hat{m} d\hat{P})^{-1} L$. Let $B^{\hat{\omega},\varepsilon} = m(x/\varepsilon, \hat{\omega})^{-1} A^{\hat{\omega},\varepsilon}$. It can be verified that our hypotheses are satisfied for $B^{\hat{\omega},\varepsilon}$-diffusion processes by an argument similar to that in Example 5.3.

EXAMPLE 5.4. We will make new examples by projecting the processes on $\boldsymbol{R}^d$ in the previous examples onto the torus $\boldsymbol{T}^d$. Let $\hat{\pi}$ denote the canonical mapping from $\boldsymbol{R}^d$ onto $\boldsymbol{T}^d = \boldsymbol{R}^d / \boldsymbol{Z}^d$. Let $(\{x(t)\}, \{P_x\})$ be a non-degenerate Brownian motion on $\boldsymbol{R}^d$. Then $\{\hat{\pi}(x(t))\}_{t \geqq 0}$ forms a Brownian motion on $\boldsymbol{T}^d$, which we denote by $(\{\hat{x}(t)\}, \{\hat{P}_{\hat{x}}\})$ $(\hat{x}\in \boldsymbol{T}^d)$. Let π be a measurable mapping from a measurable space $\tilde{X}$ to $\boldsymbol{R}^d$ and let $(\{\tilde{x}^\varepsilon(t)\}, \{\tilde{P}^\varepsilon_{\tilde{x}}\})$, $\varepsilon>0$, be a family of Markov processes on $\tilde{X}$ satisfying (2.2). Suppose $(\mathbf{A}^\varepsilon_1)$ and $(\mathbf{A}^\varepsilon_2)$ are satisfied for the family of processes $\{\pi(\tilde{x}^\varepsilon(t))\}$, $\varepsilon>0$, and $\{P_x\}$. Then $(\mathbf{A}^\varepsilon_1)$ and $(\mathbf{A}^\varepsilon_2)$ are also satisfied for the family $\{\hat{\pi}\circ\pi(\tilde{x}^\varepsilon(t))\}$, $\varepsilon>0$, of processes on $\boldsymbol{T}^d$ and $\{\hat{P}_{\hat{x}}\}$. To see this, let $\hat{\pi}_*$ denote the mapping from $D(I\to\boldsymbol{R}^d)$ to $D(I\to\boldsymbol{T}^d)$ induced by $\hat{\pi}$: $\boldsymbol{R}^d\to\boldsymbol{T}^d$, where $I=[0,\infty)$ or $[0,1]$. Since $\hat{\pi}$ is uniformly continuous, it is easy to see that $\hat{\pi}_*$ is continuous. Therefore, $\mu_n\to\mu$ in $\mathcal{M}(D([0,\infty)\to\boldsymbol{R}^d))$ implies $\hat{\mu}_n\to\hat{\mu}$ in $\mathcal{M}(D([0,\infty)\to\boldsymbol{T}^d))$, and if $\Lambda\subset\mathcal{M}(D([0,1]\to\boldsymbol{R}^d))$ is tight, then $\{\hat{\mu}; \mu\in\Lambda\}$ is tight in $\mathcal{M}(D([0,1]\to\boldsymbol{T}^d))$, where $\hat{\mu}$ denotes the measure induced by $\hat{\pi}_*$ from any μ. This proves the above assertion. Further, we can easily see that (B) is satisfied for $\{\hat{P}_{\hat{x}}\}$.

We can take for $\{\tilde{x}^\varepsilon(t)\}$ the Markov process generated by L^ε in Example 5.2, or the diffusion process associated with $A^{\hat{\omega},\varepsilon}$ in Example 5.3. Further, we note that the above procedure can be applied to the Markov process generated by L^ε in Example 5.1 provided $F(x,y)$ in (5.2) is periodic in $x\in\boldsymbol{R}^d$ with period one. In that case coefficients $a^{ij}(x)$ and $b^i(x)$ of L in (5.14) are also periodic, so that if $\{x(t)\}$ denotes the L-diffusion process, then $\{\hat{\pi}(x(t))\}$ forms a Markov process on $\boldsymbol{T}^d$.

Finally, note that we can repeat a similar argument by taking a one-dimensional projection $x\rightsquigarrow\langle\xi, x\rangle$, $x\in\boldsymbol{R}^d$ instead of the above $\hat{\pi}$, where $\xi\in\boldsymbol{R}^d$ $(\xi\neq 0)$ and $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. We omit the details.

6. Application to the Chung type laws of iterated logarithm

In this section we will apply the results in Section 2 to the laws of iterated logarithm of the Chung type [4] (see also [10], [16]) for a certain class of stochastic processes, which contains the processes in Examples 5.2 and 5.3.

We start with a general setting. Let $X=\boldsymbol{R}^d$ and let $\tilde{X}$ be a measurable space. Suppose we are given a measurable mapping π from $\tilde{X}$ onto X, and a Markov process $(\{\tilde{x}(t)\}_{t\geq 0}, \{\tilde{P}_{\tilde{x}}\}_{\tilde{x}\in\tilde{X}})$ on $\tilde{X}$ such that

(6.1) $\quad x(\cdot)\equiv\pi(\tilde{x}(\cdot))\in\Omega^+\ (=D([0,\infty)\to X))$ $\tilde{P}_{\tilde{x}}$-a.s. for every $\tilde{x}\in\tilde{X}$.

Let $P_{\tilde{x}}$ denote the distribution of $\{x(\cdot)\}$ on Ω^+ under $\tilde{P}_{\tilde{x}}$. For any $\varepsilon>0$ define

$$x^\varepsilon(t)=\varepsilon x(t/\varepsilon^2)=\varepsilon\pi[\tilde{x}(t/\varepsilon^2)]\,,\quad t\geq 0 \tag{6.2}$$

and denote by $P^\varepsilon_{\tilde{x}}$ the distribution of $\{x^\varepsilon(\cdot)\}$ on Ω^+ under $\tilde{P}_{\tilde{x}}$. Note that $x^\varepsilon(0)=\varepsilon\pi(\tilde{x})$ $\tilde{P}_{\tilde{x}}$-a.s., i.e., $P^\varepsilon_{\tilde{x}}(\omega(0)=\varepsilon\pi(\tilde{x}))=1$. We assume that the family of the processes $P_{\tilde{x}}$, $\tilde{x}\in\tilde{X}$, on $\boldsymbol{R}^d$ is *homogenizable* in the following sense:

(H) If $\varepsilon_n\downarrow 0$ and if $\tilde{x}_n\in\tilde{X}$ are such that $\varepsilon_n\pi(\tilde{x}_n)\to x\in\boldsymbol{R}^d$, then

$$P^{\varepsilon_n}_{\tilde{x}_n}\to P_x \quad\text{in}\quad \mathcal{M}(\Omega^+)\quad\text{as}\quad n\to\infty, \tag{6.3}$$

where P_x denotes the d-dimensional Wiener measure with covariance matrix (a^{ij}) starting from $x \in \boldsymbol{R}^d$.

Then $(\mathbf{A}_1^\varepsilon)$ is satisfied for $\{P_{\tilde{x}}^\varepsilon\}$ and $\{P_x\}$; we take $\tilde{x}^\varepsilon(t) = \tilde{x}(t/\varepsilon^2)$ and $\pi_\varepsilon(\tilde{x}) = \varepsilon\pi(\tilde{x})$ in the context of Section 2. (See Remark 2.1.)

Let M be the space of all measures μ on X such that $\mu(X) \leqq 1$, endowed with the vague topology. Let $Q_{t,\tilde{x}}^\varepsilon(A)$ be defined by (2.12) for any Borel set $A \subset \mathcal{M}(X)$. Since $\mathcal{M}(X)$ is continuously included in M, we can also define $Q_{t,\tilde{x}}^\varepsilon(A)$ for any Borel subset $A \subset M$ so that $Q_{t,\tilde{x}}^\varepsilon(A) = Q_{t,\tilde{x}}^\varepsilon(A \cap \mathcal{M}(X))$. Let $I(\mu)$ be the I-function for $\{P_x\}$ (see (1.15)). We can also define $I(\mu)$ for any $\mu \in M$ so that $I(\mu)$ is homogeneous of degree one and lower semicontinuous on M.

Lemma 6.1. *Suppose that* (**H**) *is satisfied and that if $\varepsilon_n \downarrow 0$ and if $\tilde{x}_n \in \tilde{X}$ are such that $|\varepsilon_n \pi(\tilde{x}_n)| \to \infty$, then for each $t > 0$*

$$\lim_{n\to\infty} P_{\tilde{x}_n}^{\varepsilon_n}(|\omega(t)| \leqq R) = 0 \quad \textit{for all} \quad R < \infty. \tag{6.4}$$

Then for any vaguely closed set $A \subset M$

$$\varlimsup_{\substack{t\to\infty \\ \varepsilon\downarrow 0}} \frac{1}{t} \log \sup_{\tilde{x}\in\tilde{X}} Q_{t,\tilde{x}}^\varepsilon(A) \leqq -\inf_{\mu\in A} I(\mu)\,. \tag{6.5}$$

Proof. Let $X' = X \cup \{\infty\}$ be the one-point compactification of X. We also adjoin an extra point $\tilde{\infty}$ to $\tilde{X}$ so that $\{\tilde{P}_{\tilde{x}}\}$ is a Markov process on $\tilde{X}' = \tilde{X} \cup \{\tilde{\infty}\}$ such that $\tilde{P}_{\tilde{\infty}}(\tilde{x}(t) = \tilde{\infty},\ t \geqq 0) = 1$. Define $P_{\tilde{\infty}}^\varepsilon$ and P_∞ by $P_{\tilde{\infty}}^\varepsilon(\omega(t) = \infty, t \geqq 0) = 1$ and $P_\infty(\omega(t) = \infty, t \geqq 0) = 1$ so that $\{P_{\tilde{x}}^\varepsilon\}_{\tilde{x}\in\tilde{X}'}$ is compatible with (6.2) by the convention $\infty = \varepsilon\pi(\tilde{\infty})$ and that $\{P_x\}_{x\in X'}$ is a Markov process on X'. Note that M can be naturally identified with $\mathcal{M}(X')$ as a topological space. Thus $Q_{t,\tilde{x}}^\varepsilon(A)$ defined above can be identified with those defined for the extended $\{P_{\tilde{x}}^\varepsilon\}_{\tilde{x}\in\tilde{X}'}$. Moreover, $I(\mu)$ as a function on M can be identified with $I(\mu)$ defined for $\{P_x\}_{x\in X'}$. This can be easily seen from (1.14), (1.15) and the fact that $p(t, \infty, dy) = \delta_\infty(dy)$ for all $t > 0$. Thus (6.5) follows immediately from Theorem 2.3 if $(\mathbf{A}_0^\varepsilon)$ is satisfied for $\{P_{\tilde{x}}^\varepsilon\}_{\tilde{x}\in\tilde{X}'}$ and $\{P_x\}_{x\in X'}$; note that in $(\mathbf{A}_0^\varepsilon)$ we take $\pi_\varepsilon(\tilde{x}) = \varepsilon\pi(\tilde{x})$ (see Remark 2.1). Let $\varepsilon_n \downarrow 0$ and let $\tilde{x}_n \in \tilde{X}'$ be such that $\varepsilon_n\pi(\tilde{x}_n) \to x$ in X'. We have to show that

$$P_{\tilde{x}_n}^{\varepsilon_n}(\omega(t) \in dy) \to p(t, x, dy) \quad \text{in} \quad \mathcal{M}(X')\,. \tag{6.6}$$

This follows from (**H**) if $x \in X$. On the other hand, if $x = \infty$, then (6.6) means that

$$P_{\tilde{x}_n}^{\varepsilon_n}(\omega(t) \in dy) \to \delta_\infty(dy) \quad \text{in} \quad \mathcal{M}(X')\,,$$

which follows immediately from (6.4). Q.E.D.

Let

$$(6.7) \qquad \varepsilon(t) = \left(\frac{\log\log t}{t}\right)^{1/2}, \quad t>0$$

and for $t>0$ and $\omega\in\Omega^+$ define

$$(6.8) \qquad \hat{L}_t(\omega, B) = \frac{1}{t}\int_0^t \chi_B(\varepsilon(t)\omega(s))ds, \quad B\in\mathcal{B}(X).$$

Theorem 6.1. *Suppose* (H) *and* (6.4) *are satisfied. Then for any* $\tilde{x}\in\tilde{X}$

$$(6.9) \qquad \bigcap_{T>0}\overline{\bigcup_{t\geqq T}\{\hat{L}_t(\omega, \cdot)\}} \subset C \quad P_{\tilde{x}}\text{-a.s.},$$

where the closure is with respect to the vague topology and

$$(6.10) \qquad C = \{\mu\in M;\ I(\mu)\leqq 1\}.$$

Proof. We follow the proof in [10; Theorem 2.8]. Let G be a neighborhood of C in M and let N be an open neighborhood of C such that $C\subset N\subset \bar{N}\subset G$. Set $\theta=\inf_{\mu\in N^c} I(\mu)$. Since $I(\mu)$ is lower semicontinuous and N^c is compact in M, we have $\theta>1$. Choose θ' and k so that $1<\theta'<\theta$, $0<k<1$ and $k\theta'>1$. Let $t_n=\exp(n^k)$, $n=1, 2, \cdots$. Noting that

$$\hat{L}_t(x(\cdot), B) = L_{\log\log t}(x^{\varepsilon(t)}(\cdot), B), \quad B\subset X,$$

we have

$$P_{\tilde{x}}(\hat{L}_t(\omega, \cdot)\in N^c) = Q^{\varepsilon(t)}_{\log\log t, \tilde{x}}(N^c).$$

By Lemma 6.1 we have

$$Q^{\varepsilon(t)}_{\log\log t, \tilde{x}}(N^c) \leqq \exp\{-(\log\log t)\theta'\}$$

for sufficiently large $t>0$. Thus for sufficiently large n, we have

$$P_{\tilde{x}}(\hat{L}_{t_n}(\omega, \cdot)\in N^c) \leqq \exp\{-k\theta'\log n\} = n^{-k\theta'}$$

so that $\sum_{n=1}^\infty P_{\tilde{x}}(\hat{L}_{t_n}(\omega, \cdot)\in N^c)<\infty$. By the Borel-Cantelli lemma we have for $P_{\tilde{x}}$-almost all ω

$$(6.11) \qquad \hat{L}_{t_n}(\omega, \cdot)\notin N^c \quad \text{for only finitely many } n.$$

Take ω such that (6.11) holds. Then we can show, as in the proof of Theorem 2.8 in [10], that there exists a $T>0$ such that $\hat{L}_t(\omega, \cdot)\in G$ for all $t\geqq T$. Since G is arbitrary, we have (6.9). Q.E.D.

Recall $\tau^*_G(\omega)$ is defined by (4.9) for any open set $G\subset X$ and any $\omega\in\Omega^+$. Let $B(x; R)=\{y\in X;\ |x-y|<R\}$, $x\in X$, $R>0$ and write $B(R)$ for $B(0; R)$.

Theorem 6.2. *Suppose that* (H) *is satisfied and that* (a^{ij}) *in* (H) *is positive* *definite. Further, suppose that* $P_{\tilde{x}_0}$ *satisfies*

$$P_{\tilde{x}_0}(\tau^*_{B(\pi(\tilde{x}_0)\,;\,R)}(\omega)\leqq t) = O(t/R^2) \tag{6.12}$$

as $t/R^2\to 0$ *and* $t/R\to\infty$. *Let* $\mu\in\mathcal{M}(X)$ *be such that* $\mu(B(a))=1$ *for some* $a>0$ *and* $I(\mu)<1$. *Let* $a'>a$ *and let* N *be a (weak) neighborhood of* μ *in* $\mathcal{M}(X)$. *Define*

$$E_t = \{\omega;\ \hat{L}_t(\omega,\ \cdot)\in N,\ \tau^*_{B(\pi(\tilde{x}_0)\,;\,a'/\varepsilon(t))}(\omega)>t\}\ .$$

Then, for $P_{\tilde{x}_0}$*-almost all* ω, *there exists a sequence* $t_n\to\infty$ *such that* $\omega\in E_{t_n}$.

Proof. We follow the proof of Theorem 2.15 in [10]. Let N_1 be a weak neighborhood of μ such that $N_1\subset\bar{N}_1\subset N$. Let $\theta=I(\mu)$ (<1) and choose θ', k and k' such that $\theta<\theta'<1$, $k'\theta'<1$ and $k'>k>1$. Let $t_n=\exp(n^k)$ and let $\varepsilon_n=\varepsilon(t_n)$, $n=1, 2, \cdots$. Using the notation

$$\tilde{L}_{t_{n-1},t_n}(\omega,\ B) = \frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\chi_B(\varepsilon_n\omega(s))ds\,,$$

we have

$$||\tilde{L}_{t_{n-1},t_n}(\omega,\ \cdot)-\hat{L}_{t_n}(\omega,\ \cdot)||\leqq 2t_{n-1}/t_n\to 0$$

as $n\to\infty$, where $||\cdot||$ denotes the variation norm. Thus, for sufficiently large n,

$$\{\omega;\ \tilde{L}_{t_{n-1},t_n}(\omega,\ \cdot)\in N_1\}\subset\{\omega;\ \hat{L}_{t_n}(\omega,\ \cdot)\in N\}\ . \tag{6.13}$$

Let $x_0=\pi(\tilde{x}_0)$ and let $B'(R)=B(x_0;R)$, $R>0$. Define

$$F_n = \{\omega;\ \tilde{L}_{t_{n-1},t_n}(\omega,\ \cdot)\in N_1,\ \tau^*_{B'(a/\varepsilon_n)}>t_{n-1},\ \tau^*_{B'(a'/\varepsilon_n)}>t_n\}\ .$$

Then, noting (6.13), we have $\overline{\lim_n}\,F_n\subset\overline{\lim_n}\,E_{t_n}$. Thus, it suffices to show

$$P_{\tilde{x}_0}(\overline{\lim_n}\,F_n) = 1\ . \tag{6.14}$$

Let $\tilde{\mathcal{F}}_n$ be the σ-field generated by the process $\{\tilde{x}(t)\}$ up to time t_n. Then, using Markovian property of $\{\tilde{P}_{\tilde{x}}\}$, we have

$$\tilde{P}_{\tilde{x}_0}(x(\cdot)\in F_n\,|\,\tilde{\mathcal{F}}_{n-1}) = \chi_{A_{n-1}}(x(\cdot))\tilde{P}_{\tilde{x}(t_{n-1})}(x(\cdot)\in H_n)\,, \tag{6.15}$$

where $A_{n-1}=\{\omega;\ \tau^*_{B'(a/\varepsilon_n)}(\omega)>t_{n-1}\}$ and $H_n=\{\omega;\ \tau^*_{B'(a'/\varepsilon_n)}(\omega)>t_n-t_{n-1}$, $L'_{t_n-t_{n-1},t_n}(\omega,\ \cdot)\in N_1\}$. Here we have used the notation

$$L'_{s,t}(\omega,\ B) = \frac{1}{s}\int_0^s\chi_B(\varepsilon(t)\omega(u))du\ .$$

Thus we get

$$(6.16)\qquad \tilde{P}_{\tilde{x}_0}(x(\cdot)\in F_n\,|\,\tilde{\mathcal{F}}_{n-1})\geqq \chi_{A_{n-1}}(x(\cdot))I_n\,,$$

where $I_n=\inf_{\tilde{x}\in\tilde{B}_n}P_{\tilde{x}}(H_n)$ with $\tilde{B}_n=\pi^{-1}B'(a/\varepsilon_n)$. By the definition of $P^{\varepsilon}_{\tilde{x}}$, we have

$$I_n=\inf_{\tilde{x}\in\tilde{B}_n}P^{\varepsilon_n}_{\tilde{x}}(\tau^*_{B(\varepsilon_n x_0\,;\,a')}>s_n,\ L_{s_n}(\omega,\ \cdot)\in N_1)\,,$$

where $s_n=\varepsilon_n^2(t_n-t_{n-1})$. Choose r and r' such that $a<r<r'<a'$ and let $K=\overline{B(r)}$ and $G=B(r')$. Using the notation $\pi_\varepsilon(\tilde{x})=\varepsilon\pi(\tilde{x})$, we have for sufficiently large n

$$\tilde{B}_n=\pi^{-1}_{\varepsilon_n}B(\varepsilon_n x_0;\,a)\subset\pi^{-1}_{\varepsilon_n}K\quad\text{and}\quad G\subset B(\varepsilon_n x_0;\,a')\,.$$

Noting the above relations, we get

$$\varliminf_{n\to\infty}\frac{1}{s_n}\log I_n\geqq\varliminf_{\substack{t\to\infty\\ \varepsilon\downarrow 0}}\frac{1}{t}\log\inf_{\tilde{x}\in\pi_\varepsilon^{-1}K}P^{\varepsilon}_{\tilde{x}}(L_t(\omega,\ \cdot)\in N_1,\ \tau^*_G>t)\,.$$

Since $(\mathbf{A}^{\varepsilon}_1)$ is satisfied for $\{P^{\varepsilon}_{\tilde{x}}\}$ and $\{P_x\}$, it follows from Corollary 4.1 that there exists an s_0 and a neighborhood N_2 of μ such that for all $s\geqq s_0$

$$\varliminf_{n\to\infty}\frac{1}{s_n}\log I_n\geqq\frac{1}{s}\log\inf_{x\in K}P_x(L_s(\omega,\ \cdot)\in N_2,\ \tau^*_G>s,\ \omega(s)\in G_1)\,,$$

where $G_1=B(r)$. Letting $s\to\infty$, we can show that

$$(6.17)\qquad \varliminf_{n\to\infty}\frac{1}{s_n}\log I_n\geqq -I(\mu)\,.$$

This follows from Corollary 4.3 in [16], which is a modification of Theorem 8.1 in [9]. Since $I(\mu)=\theta<\theta'$ and since $s_n/k\log n\to 1$ as $n\to\infty$, we have, for sufficiently large n,

$$(6.18)\qquad \begin{aligned} I_n&\geqq\exp\{-s_n\theta'\}\\ &\geqq\exp\{-k'\theta'\log n\}=n^{-k'\theta'}\,. \end{aligned}$$

It follows from (6.16) and (6.18) that there exists an n_0 such that

$$(6.19)\qquad \sum_{n=n_0}^{\infty}\tilde{P}_{\tilde{x}_0}(x(\cdot)\in F_n\,|\,\tilde{\mathcal{F}}_{n-1})\geqq\sum_{n=n_0}^{\infty}\chi_{A_{n-1}}(x(\cdot))n^{-k'\theta'}\quad \tilde{P}_{\tilde{x}_0}\text{-a.s.}$$

We can deduce (6.14) from (6.19) by the argument in [10; p. 731] if we show that

$$(6.20)\qquad \sum_n P_{\tilde{x}_0}(A_n^c)<\infty\,.$$

Note that

$$P_{\tilde{x}_0}(A_n^c)=P_{\tilde{x}_0}(\tau^*_{B(x_0\,;\,a/\varepsilon_{n+1})}\leqq t_n)\,.$$

Since $t_n\varepsilon_{n+1}^2 \leqq \exp(-n^{k-1})k\log(n+1) \to 0$ and $t_n\varepsilon_{n+1} \to \infty$ as $n \to \infty$, we have, from (6.12),

$$P_{\tilde{x}_0}(A_n^c) = O(t_n\varepsilon_{n+1}^2) \leqq O(\exp(-n^{k-1})k\log(n+1))$$

as $n \to \infty$, which proves (6.20). This completes the proof. Q.E.D.

In the following, we fix the $\tilde{x}_0$ in Theorem 6.2. Recall C is defined in (6.10).

Theorem 6.3. *Under the same hypotheses as in Theorem* 6.2 *it holds that*

$$(6.21) \qquad \bigcap_{T>0} \overline{\bigcup_{t \geqq T} \{\hat{L}_t(\omega, \cdot)\}} \supset C \quad P_{\tilde{x}_0}\text{-a.s.}$$

This theorem can be deduced from Theorem 6.2 as in [10] and so the proof is omitted.

Combining Theorems 6.1 and 6.3, we have the following.

Theorem 6.4. *Suppose all the hypotheses in Theorems* 6.1 *and* 6.3 *are satisfied. Then*

$$(6.22) \qquad \bigcap_{T>0} \overline{\bigcup_{t \geqq T} \{\hat{L}_t(\omega, \cdot)\}} = C \quad P_{\tilde{x}_0}\text{-a.s.}$$

This has an immediate corollary.

Corollary 6.1. *If Φ is a lower semicontinuous function on M in the vague topology, then*

$$(6.23) \qquad \overline{\lim_{t\to\infty}}\, \Phi(\hat{L}_t(\omega, \cdot)) \geqq \sup_{\mu \in C} \Phi(\mu) \quad P_{\tilde{x}_0}\text{-a.s.},$$

and if Φ is an upper semicontinuous function on M in the vague topology, then

$$(6.24) \qquad \overline{\lim_{t\to\infty}}\, \Phi(\hat{L}_t(\omega, \cdot)) \leqq \sup_{\mu \in C} \Phi(\mu) \quad P_{\tilde{x}_0}\text{-a.s.}$$

In the following, if $L = \frac{1}{2} \sum_{i,j=1}^d a^{ij} \partial^2/\partial x^i \partial x^j$, where (a^{ij}) is a positive definite symmetric matrix, then λ_L denotes the smallest eigenvalue of $-L$ for the d-dimensional ball of unit radius with the Dirichlet condition.

Theorem 6.5. *Suppose that* (**H**) *is satisfied with (a^{ij}) being positive definite. Further, suppose that* (6.4) *and* (6.12) *are satisfied. Then for any $l>0$ there exists a constant k_l such that*

$$(6.25) \qquad \overline{\lim_{t\to\infty}}\, \frac{1}{t} \int_0^t \chi_{[0,l]}\left(\left(\frac{\log\log t}{t}\right)^{1/2} |\omega(s)|\right) ds = k_l \quad P_{\tilde{x}_0}\text{-a.s.},$$

and that $k_l = 1$ if and only if $l \geqq \sqrt{\lambda_L}$.

We now state the Chung type law of iterated logarithm for the process $P_{\tilde{x}_0}$ before proving Theorem 6.5.

Theorem 6.6. *Suppose all the hypotheses in Theorem* 6.5 *are satisfied. Then*

$$\varlimsup_{t\to\infty}\left(\frac{\log\log t}{t}\right)^{1/2}\sup_{0\leq s\leq t}|\omega(s)|=\sqrt{\lambda_L}\quad P_{\tilde{x}_0}\text{-a.s.} \tag{6.26}$$

This can be deduced from Theorems 6.2 and 6.5. See [16; Example 6.4] for the details.

REMARK 6.1. In this section we can take a norm $|\cdot|$ in $\boldsymbol{R}^d$ arbitrarily, but it must be fixed throughout. Note that the definition of the "ball" depends on the choice of the norm, and hence so does λ_L. If, in particular, $d=1$ and $L=\frac{a}{2}d^2/dx^2$, then $\lambda_L=a\pi^2/8$.

REMARK 6.2. By a similar argument, we can also prove the analogue of (6.26) for a suitable process $\{x(t)\}$ such that $\{r(\lambda)x(t/\lambda)\}$ converges to a symmetric stable process as $\lambda\to 0$, where $r(\lambda)$ is some regularly varying function as $\lambda\downarrow 0$ of index $1/\alpha$ with α being the index of the limiting stable process. In that case $(\log\log t/t)^{1/2}$ in (6.26) should be replaced by $r(\log\log t/t)$. We note that the analogous results for a symmetric stable process and for sums of independent identically distributed random variables in the domain of attraction of a stable distribution have been already known [10], [16].

Proof of Theorem 6.5. Let $\Phi_l(\mu)=\mu(\overline{B(l)})$ and $\Phi_l'(\mu)=\mu(B(l))$, $l>0$. By applying (6.24) to Φ_l and (6.23) to Φ_l', we can prove (6.25) with

$$k_l=\sup_{\mu\in C}\Phi_l(\mu)=\sup_{\mu\in C}\Phi_l'(\mu);$$

the second equality follows from the fact that any $\mu\in C$ is absolutely continuous with respect to the Lebesgue measure (see below). It only remains to prove

$$k_l=1\quad\text{if and only if}\quad l\geqq\sqrt{\lambda_L}\,. \tag{6.27}$$

It is known [6; Theorem 5] that $I(\mu)<\infty$ if and only if μ is absolutely continuous with respect to the Lebesgue measure dx and

$$\int\sum_{i=1}^{d}\left|\frac{\partial\varphi}{\partial x^i}\right|^2dx<\infty\,,$$

where $\varphi=\sqrt{d\mu/dx}$ and $\partial/\partial x^i$ is in the generalized sense. Moreover, in that case we have

$$(6.28)\qquad I(\mu)=\frac{1}{2}\int_{R^d}\sum_{i,j=1}^{d} a^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}dx\,.$$

Therefore, if $\lambda(l)$ denotes the smallest eigenvalue of $-L$ for the ball $B(l)$ with the Dirichlet condition, then we have

$$(6.29)\qquad \lambda(l)=\inf\{I(\mu);\ \mu\in\mathcal{M}(\boldsymbol{R}^d),\ \mu(\overline{B(l)})=1\}\ ,$$

which is the classical variational formula for $\lambda(l)$. Further, it is well known that

$$(6.30)\qquad \lambda(l)=\lambda(1)/l^2=\lambda_L/l^2\,.$$

Now let $l\geqq\sqrt{\lambda_L}$, so that $\lambda(l)\leqq 1$. Let φ be the normalized ($\int\varphi^2dx=1$) eigenfunction corresponding to $\lambda(l)$ and let $\mu_\varphi(dx)=\varphi(x)^2dx$. Then we have, from (6.28),

$$I(\mu_\varphi)=\int(-L\varphi)\varphi dx=\lambda(l)\leqq 1\,,$$

so that $\mu_\varphi\in C$. Therefore, we have

$$k_l\geqq\Phi_l(\mu_\varphi)=\int_{B(l)}\varphi^2dx=1\,,$$

which means $k_l=1$. On the other hand, let $l<\sqrt{\lambda_L}$. Since $\Phi_l(\mu)$ is upper semicontinuous and since C is compact, there exists a $\mu\in C$ such that

$$k_l=\Phi_l(\mu)=\mu(\overline{B(l)})\,.$$

Thus, if $k_l=1$, then (6.29) implies $\lambda(l)\leqq I(\mu)$. But since $\lambda(l)=\lambda_L/l^2>1$, we have $I(\mu)>1$, which contradicts the fact that $\mu\in C$. Hence we get $k_l<1$, proving (6.27). Q.E.D.

In the following we prove the Chung type laws of iterated logarithm for the processes in Examples 5.2 and 5.3.

Theorem 6.7. *Let* $(\hat\Omega,\hat{\mathcal{F}},\hat P)$, $\{P_x^{\hat\omega}\}$ *and* L *be as in Example* 5.3. *Then for* $\hat P$*-almost all* $\hat\omega\in\hat\Omega$

$$(6.31)\qquad P_x^{\hat\omega}\Big(\omega:\ \varliminf_{t\to\infty}\Big\{\frac{\log\log t}{t}\Big\}^{1/2}\sup_{0\leqq s\leqq t}|\omega(s)|=\sqrt{\lambda_L}\Big)=1$$

for all $x\in\boldsymbol{R}^d$.

Proof. It suffices to check (6.4) and (6.12). But (6.4) is the same as saying that if $\varepsilon_n\downarrow 0$ and if $|x_n|\to\infty$, then for each $t>0$

$$\lim_{n\to\infty}P_{x_n}^{\hat\omega,\varepsilon_n}(|\omega(t)|\leqq R)=0\quad\text{for all}\quad R<\infty$$

in the notation in Example 5.3, which is immediate from the estimate (5.29) for $p^{\hat{\omega},\varepsilon}(t, x, y)$. Next note that

$$P_x^{\hat{\omega}}(\tau^*_{B(x\,;\,R)}\leqq t) = P_x^{\hat{\omega}}(\sup_{0\leqq s\leqq t} |\omega(s)-x| \geqq R) \leqq \frac{f(t, R/2)}{1-f(t, R/2)},$$

where $f(t, R/2) = \sup_{0\leqq s\leqq t, x} P_x^{\hat{\omega}}(|\omega(s)-x| \geqq R/2)$. Hence (6.12) is also immediate from (5.29). Q.E.D.

Theorem 6.8. *Let $\{Y^y(t)\}$ be the Q-process on a Polish space S in Example 5.2 starting at $y\in S$, and let $F(y)=(F^1(y), \cdots, F^d(y))$ be a bounded continuous function on S with values in $\boldsymbol{R}^d$ satisfying the centering condition* (5.18). *Let $L=\frac{1}{2}\sum a^{ij}\partial^2/\partial x^i\partial x^j$ be defined by* (5.21) *and assume that (a^{ij}) is positive definite. Then for all $y\in S$*

$$\varlimsup_{t\to\infty} \left(\frac{\log\log t}{t}\right)^{1/2} \sup_{0\leqq s\leqq t} \left|\int_0^t F(Y^y(s))ds\right| = \sqrt{\lambda_L} \quad \text{a.s.} \tag{6.32}$$

Proof. Let

$$X^{x,y}(t) = x + \int_0^t F(Y^y(s))ds, \quad t\geqq 0, \quad (x, y)\in \boldsymbol{R}^d\times S. \tag{6.33}$$

Then $(X^{x,y}(t), Y^y(t))$ forms a Markov process on $\boldsymbol{R}^d\times S$ starting at $(x, y)\in\boldsymbol{R}^d \times S$. For any $\varepsilon>0$ define

$$\begin{aligned} x^\varepsilon(t) &= \varepsilon X^{x,y}(t/\varepsilon^2) \\ y^\varepsilon(t) &= Y^y(t/\varepsilon^2). \end{aligned} \tag{6.34}$$

Then $(x^\varepsilon(t), y^\varepsilon(t))$ forms the Markov process generated by L^ε in (5.20) starting at $(\varepsilon x, y)$ (see Remark 5.2). Let $P^\varepsilon_{x,y}$ denote the measure on Ω^+ induced by the process $\{x^\varepsilon(\cdot)\}$ in (6.34) and let $P_{x,y}$ denote that by $\{X^{x,y}(\cdot)\}$. It suffices to check (6.4) and (6.12) for $\{P^\varepsilon_{x,y}\}$, $\{P_{x,y}\}$ and $\pi(x, y)=x$. We claim that there exist $C>0$ and $\varepsilon_0>0$ such that for any $\varepsilon\in(0, \varepsilon_0)$ and any $t>0$

$$\sup_{x,y} P^\varepsilon_{x,y}(\tau^*_{B(\varepsilon x\,;\,1)}\leqq t)\leqq C(t+\varepsilon). \tag{6.35}$$

We assume (6.35) for a while. If $\varepsilon_n\downarrow 0$, $|\varepsilon_n x_n|\to\infty$, and $y_n\in S$ are arbitrary, then for any $t>0$ and any $R>0$

$$\begin{aligned} &\varlimsup_{n\to\infty} P^{\varepsilon_n}_{x_n,y_n}(|\omega(t)|\leqq R) \\ &\leqq \varlimsup_{n\to\infty} P^{\varepsilon_n}_{x_n,y_n}(|\omega(t)-\varepsilon_n x_n|\geqq R_n) \quad (R_n=|\varepsilon_n x_n|-R) \\ &\leqq \varlimsup_{n\to\infty} P^{\varepsilon_n/R_n}_{x_n,y_n}(|\omega(t/R_n^2)-\varepsilon_n x_n/R_n|\geqq 1) \\ &\leqq \varlimsup_{n\to\infty} P^{\varepsilon_n/R_n}_{x_n,y_n}(\tau^*_{B(\varepsilon_n x_n/R_n\,;\,1)}\leqq t/R_n^2) \end{aligned}$$

$$\leqq \lim_{n\to\infty} C(t/R_n^2+\varepsilon_n/R_n)=0\,,$$

which proves (6.4), and we have

$$\begin{aligned} P_{x,y}(\tau^*_{B(x\,;\,R)}\leqq t) &\leqq P^{1/R}_{x,y}(\tau^*_{B(x/R\,;\,1)}\leqq t/R^2)\\ &= O(t/R^2)+O(1/R)\\ &= O(t/R^2)\end{aligned}$$

as $t/R^2\to 0$ and $t/R\to\infty$, proving (6.12).

It remains to prove (6.35). Recall that

$$L^\varepsilon=\frac{1}{\varepsilon^2}Q+\frac{1}{\varepsilon}\sum_{i=1}^{d}F^i(y)\frac{\partial}{\partial x^i}\,.$$

We first note that for any bounded smooth function $\varphi(x)$ on $\boldsymbol{R}^d$ having bounded derivatives of all orders we can find for each $\varepsilon>0$ a function $\varphi^\varepsilon(x, y)$ such that

$$\sup_{x,y}|\varphi^\varepsilon(x,y)-\varphi(x)|\leqq C\varepsilon(1+\varepsilon)\,, \tag{6.36}$$

$$\sup_{x,y}|L^\varepsilon\varphi^\varepsilon(x,y)-L\varphi(x)|\leqq C\varepsilon \tag{6.37}$$

for some constant C independent of $\varepsilon>0$. Such estimates are given in [22], but we give here the proof in our special case for the convenience of the reader. For simplicity, let $d=1$. We find $\varphi^\varepsilon(x, y)$ in the form

$$\varphi^\varepsilon(x,y)=\varphi(x)+\varepsilon\psi_1(x,y)+\varepsilon^2\psi_2(x,y)\,, \tag{6.38}$$

where $\psi_1(x,y)$ and $\psi_2(x,y)$ are bounded functions on $\boldsymbol{R}^d\times S$. Observe that

$$\begin{aligned} L^\varepsilon\varphi^\varepsilon(x,y) &= \frac{1}{\varepsilon}[Q\psi_1(x,y)+F(y)\varphi'(x)]\\ &\quad +[Q\psi_2(x,y)+F(y)\frac{\partial}{\partial x}\psi_1(x,y)]\\ &\quad +\varepsilon F(y)\frac{\partial}{\partial x}\psi_2(x,y)\,.\end{aligned} \tag{6.39}$$

Let $\chi(y)$ be the solution of $-Q\chi=F$ (see (5.23)) and let $\psi_1(x,y)=\varphi'(x)\chi(y)$ so that $Q\psi_1+F\varphi'=0$. Then, noting that

$$\int_S F(y)\frac{\partial}{\partial x}\psi_1(x,y)\bar{P}(dy)=\int_S F(y)\chi(y)\varphi''(x)\bar{P}(dy)=L\varphi(x)\,,$$

we can see that

$$\psi_2(x,y)=\int_S[F(z)\frac{\partial}{\partial x}\psi_1(x,z)-L\varphi(x)]G_0(y,dz)$$

solves $-Q\psi_2=F\frac{\partial}{\partial x}\psi_1-L\varphi$ (see (5.10)), so that

$$L^\varepsilon\varphi^\varepsilon(x, y)-L\varphi(x) = \varepsilon F(y)\frac{\partial}{\partial x}\psi_2(x, y)\,.$$

Since, ψ_1, ψ_2 and $\frac{\partial}{\partial x}\psi_2$ are bounded, we obtain (6.36) and (6.37).

Let $\tilde{P}^\varepsilon_{x,y}$ denote the probability measure governing the process $(x^\varepsilon(t), y^\varepsilon(t))$ in (6.34) starting from $(\varepsilon x, y)$, and $\tilde{E}^\varepsilon_{x,y}$ denote the expectation with respect to $\tilde{P}^\varepsilon_{x,y}$. Since $(x^\varepsilon(t), y^\varepsilon(t))$ is a Markov process generated by L^ε,

$$\varphi^\varepsilon(x^\varepsilon(t), y^\varepsilon(t))-\varphi^\varepsilon(\varepsilon x, y)-\int_0^t L^\varepsilon\varphi^\varepsilon(x^\varepsilon(s), y^\varepsilon(s))ds$$

is a martingale under $\tilde{P}^\varepsilon_{x,y}$, so that

$$\tilde{E}^\varepsilon_{x,y}[\varphi^\varepsilon(x^\varepsilon(t\wedge\tau), y^\varepsilon(t\wedge\tau))] = \varphi^\varepsilon(\varepsilon x, y)+\tilde{E}^\varepsilon_{x,y}\Big[\int_0^{t\wedge\tau} L^\varepsilon\varphi^\varepsilon(x^\varepsilon(s), y^\varepsilon(s))ds\Big]\,,$$

where $\tau=\tau^*_{B(\varepsilon x\,;\,1)}(x^\varepsilon(\cdot))$. It follows from (6.36) and (6.37) that

$$\begin{aligned}&\tilde{E}^\varepsilon_{x,y}[\varphi(x^\varepsilon(t\wedge\tau))]\\ &\leqq\varphi(\varepsilon x)+2C\varepsilon(1+\varepsilon)+t\big(\sup_{\xi\in B(\varepsilon x\,;\,1)}|L\varphi(\xi)|+C\varepsilon\big)\,.\end{aligned}$$

If we take $\varphi(\cdot)$ so that $\varphi(\xi)=|\xi-\varepsilon x|^2$ for $\xi\in B(\varepsilon x; 1)$ and $\varphi\geqq 0$, then we have

$$\tilde{P}^\varepsilon_{x,y}(\tau\leqq t)\leqq 2C(1+\varepsilon)\varepsilon+\big(\sum_i a^{ii}+C\varepsilon\big)t\,,$$

which proves (6.35). Q.E.D.

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